First-price auctions with resale: the case of many bidders*

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Abstract

If agents engage in resale, it changes bidding in the initial auction. Resale offers extra incentives for bidders with lower valuations to win the auction. However, if resale markets are not frictionless, then use values affect bidding incentives, and stronger bidders still win the initial auction more often than weaker ones. I consider a first price auction followed by a resale market with frictions, and confirm the above statements. While intuitive, our results differ from the two bidder case of Hafalir and Krishna (2008): the two bidders win with equal probabilities regardless of their use values. The reason is that they face a common (resale) price at the relevant margin, a property that fails with more than two bidders. Numerical simulations show that asymmetry in winning probabilities increases in the number of bidders, and in large markets resale loses its effect on allocations. We also show in an example that the revenue advantage of first price auctions over second price auctions is positive, but decreasing in the number of bidders.

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1 Introduction

In many markets agents may engage in resale activities after an auction is run. The presence of resale opportunities allow efficiency enhancing trades to take place after the auction. It also affects the way bidders behave in the initial auction and the probability with which each bidder wins the auction. This second effect is the topic of our paper. Intuitively, resale should favor bidders with lower valuations, since the possibility of resale offers them extra incentives (beyond the use value of the object) to win the auction. On the other hand, bidders with higher valuations may depress their bids counting on the possibility of buying the object at the resale stage. These observations suggest that buyers with low value are more likely to win the initial auction if resale is possible than when it is not.

The way resale markets operate is crucial, if we were to fully understand *how strongly* the possibility of resale affects bidding (and allocation) in the initial auction. If the resale

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markets were frictionless (perfectly competitive), then agents took the (resale) market price of the object as given. In this case, it is only the resale price that affects willingness to pay, and thus each bidder should bid similarly and win the auction with the same probability regardless of their valuations. However, very often resale markets are not frictionless, because of the small number of buyers, institutional details, or frictions arising from asymmetric information. In this case the resale stage takes the form of multilateral bargaining under incomplete information, and strategic considerations between the different potential buyers are present. Therefore, one cannot appeal to the price mechanism to obtain a competitive resale price that is independent of the identity and strategies of the bidders in the auction and at the resale stage. The final allocation then is not necessarily the efficient one, and it may depend on who won the initial auction. The only way that a bidder with a high valuation can guarantee to obtain the object for sure is to win the auction itself. Therefore, use values affect bidding incentives, and one may expect that a stronger bidder have an incentive to bid more and win the initial auction more often than weaker ones.

The above discussion implies that when resale markets operate with frictions we may expect an allocation that is in between the allocations obtaining with perfect resale markets and no resale markets at all. In this paper I study such a situation. A single object is sold to privately informed buyers using a first price auction without revealing bids. After the auction a resale market opens where the same buyers participate as in the auction. Confirming the intuition from above, I formally show that resale indeed allows a weaker bidder to win more often than without resale, but less often than a stronger bidder if there are more than two bidders in the auction. Since asymmetric information introduces frictions into the resale market, it is not surprising that use values play a role for the bidders and thus stronger bidders are more likely to win than weaker ones. Despite being intuitive, our result is in contrast with two papers that consider a similar setup to ours with two buyers.[1] Garratt and Troger (2006) consider a setup with a pure speculator (no use value) and a genuine buyer, while Hafalir and Krishna (2008) consider the more general case of two genuine bidders. They show that regardless of the exact distribution of valuations for the two genuine buyers, both produce the same bid distribution and both win the auction with a 50% probability. Although, the resale markets of those papers are clearly not frictionless (because of the small number of buyers and the fact that bargaining is under incomplete information), they achieve the result that each bidder wins with equal probability regardless of their use values, as it would be expected with a *frictionless* resale market.

The logic behind this symmetrization result is that although the resale price is endogenously determined together with the bidding strategies in the auction, but from the point of view of the relevant "marginal types" it is exogenous and common to the two bidders. Therefore, the relevant types are price takers at the margin, and they face a common price, so the resale market behaves as if it were frictionless at the margin. To gain intuition, suppose that one of the bidders is weaker in the sense that he is more likely to have a low valuation. Such a weak bidder bids more aggressively than the strong bidder and thus may win the object even if his valuation is lower than that of his rival. Therefore, he has a profitable resale opportunity at the resale stage. If he wins the auction by a small margin, then his take it or leave it resale offer r will be accepted by the strong bidder with probability 1, and his utility is equal to r.[2] Therefore, his gain from winning at the margin, his *effective*

[1] The papers below also study other auction formats, like second price auctions, and other questions like revenue consequences, that are not addressed here.

[2] The reason is that the weak bidder offers such a resale price in equilibrium that is accepted by at least some types of the strong bidder. Therefore, the type of the strong bidder who lost by a small margin against

valuation is equal to r. A similar observation applies to the strong bidder: if he loses the auction by a small margin, then he buys the object for sure at the resale price r. Therefore, the two bidders have the same marginal gains from winning (effective valuations) r, which leads to bid symmetrization.

Our paper shows that this logic fails when there are more than two bidders, and weaker bidders win the auction with a lower probability than stronger bidders. The argument relies on the fact that a common price for the marginal types does not exist anymore, and consequently the effective valuations of the bidders are not equalized. More precisely, I assume that there are n_w weak bidders and n_s strong bidders with their valuations distributed according to distribution functions F_w and F_s respectively, with F_s first order stochastically dominating F_w.[3] I also assume that the bidder who won the initial auction makes the resale offer which takes the form of a second price auction with a reserve price.[4] The key intuition for the failure of symmetrization is that the effective valuations of different bidders are not equalized any more. More specifically, strong bidders have higher effective valuation for the object. To build intuition, consider the case of several weak bidders and only one strong bidder. A weak bidder will still sell the object in the resale stage if he beats the strong bidder in the initial auction by a small margin and thus he still gains the resale price in this case. However, when he beats another weak bidder by a small margin, then he may not sell the object in the resale stage, because his resale offer may exceed the valuation of the strong bidder. When he does not sell the object in the resale stage, his effective valuation is equal to his use value for the object, which is less than the resale revenue he expects in case of a resale. Therefore, his expected effective value for the object is between r and the use value, and thus it is strictly lower than r. The effective valuation of the strong buyer is equal to r, since if he loses by a small margin then he still buys at the resale stage for sure. Combining the two observations yields that the effective valuation of the strong bidder is higher than that of a weak bidder, and thus intuition suggests that the strong bidder produces a more aggressive bid distribution than the weak bidders. Section 4 confirms this intuition formally for the case of several strong bidders as well.

I also show that when there are two weak and one strong bidder, under further assumptions on the distribution functions, resale acts toward symmetrization even if does not go all the way. More precisely, I show that weak bidders are more likely to win the auction if resale is allowed than in the benchmark case with no resale. The intuition is simple: while effective valuations are not equalized when there are more than two bidders, but (as we saw above) the effective valuation of a weak (strong) bidder is higher (lower) than his use value and thus the weak bidder wins the initial auction with a higher probability than in the case without resale.

It is also interesting how the number of bidders affect the probability with which a weak or a strong bidder wins the initial auction. We know from Hafalir and Krishna (2008) that when there is one weak and one strong bidder, then they each have a 50% chance of winning in the initial auction when resale is allowed and the strong bidder wins with more than 50% probability when resale is not allowed. I construct a measure for symmetrization for the case when there are $n_w > 1$ weak bidders and one strong bidder. Let $\pi_s^r(n_w)$ denote the probability that the strong bidder wins the initial auction when resale is allowed

the particular type of the weak bidder should definitely accept the resale offer, otherwise no type would.

[3] To facilitate analysis, I adopt the assumption of Maskin and Riley (2000) that states that F_s/F_w is increasing. This assumption, reverse hazard rate dominance, is stronger than first order stochastic dominance.

[4] As I discuss it later, conducting such a resale auction is optimal for the winner of the initial auction on the equilibrium path.

and $\pi_s^n(n_w)$ when resale is not allowed. Let $\rho(n_w) = \frac{\pi_s^r(n_w) - \frac{1}{n_w+1}}{\pi_s^n(n_w) - \frac{1}{n_w+1}}$ measure the amount of symmetrization that takes place compared to the case without resale. The Hafalir and Krishna result can be rewritten as $\rho(1) = 0$, i.e. there is complete symmetrization with two bidders. I show that when $F_s(x) = x$ and $F_w(x) = \sqrt{x}$, then function ρ is increasing in n_w and gets close to 1 when n_w is relatively large. This result implies that the more bidders there are, the more similar the allocation of the initial auction to the auction with no resale is. Indeed, our conjecture is that resale becomes ineffective in the limit and $\lim_{n_w \to \infty} \rho = 1$ holds.[5] This result indicates that resale is less important in larger markets, because the outcome of the initial auction tends to be more efficient as the market size grows. A similar result was established for large auctions *without* resale by Swinkels (2001) who argues that asymptotically a first price auction becomes efficient, because the environment of each bidder becomes and thus bidders with the same valuation behave similarly regardless of the ex-ante type distributions.[6]

Hafalir and Krishna (2008) proves that if resale is allowed then a first price auction yields a higher revenue than a second price auction when there are two bidders. The key intuition is that resale symmetrizes the valuations of the two bidders: weak bidders may resell and thus they have more incentive to bid than in an auction without resale, while strong bidders may delay buying until the resale stage, which depresses their bids in the original auction. This symmetrization of valuations makes competition more intense, and increases revenues. The same effect is still present when there are more than two bidders, but it is weaker, because (effective) valuations are not completely symmetric when resale is allowed. Therefore, one may expect that resale increases revenues (as compared to the no resale benchmark), but this increase is decreasing in the number of bidders. To confirm this intuition, I introduce a measure for how much resale increases revenues. Numerical calculations (I use the example described in the previous paragraph) show that the increase in revenues caused by resale is positive, but decreasing in the number of bidders, as we expect it based on the above discussion.

The literature on auctions with resale is still relatively small. Zheng (2002) asks under what conditions the Myerson's auction can be an equilibrium outcome with resale if the initial seller can choose his mechanism as he wishes. Hafalir and Krishna (2009) analyzes revenue and efficiency in a first price auction using their first paper. None of these papers analyze a full blown asymmetric information model of a first price auction when there are more than two bidders. Cheng and Tan (2009) show that a two bidder private values auction with resale can be analyzed as a common value auction with no resale. They also show that their argument could be extended to more than two bidders, but do not consider the question of bid symmetrization. In a work independent from ours, Lebrun (2009) considers the case of many bidders studying a more specific case than ours and addressing only some of our questions.[7] He assumes that there is only one strong and n

[5] The intuition is that when there are many bidders all bidders bid close to their valuation and thus the initial auction is already efficient and thus resale loses its bite. As I discuss it later, this intuition carries over also to the case where $n_s > 1$ or the number of objects grows with the number of bidders and thus the bid functions do not converge to the valuations.

[6] Intuitively, this insight carries over to the case with resale, since if the outcome becomes efficient in the limit, then it is a self-fulfilling prophecy that resale does not matter in the limit. However, it is beyond the scope of this paper to establish this result in the context of general asymmetric first price auctions with resale.

[7] This is only true for our setup of first price auctions with no bid disclosure. He also considers other common auctions and bid revelation protocols that our paper does not cover.

weak bidders. He shows existence and uniqueness of equilibrium under similar distributional assumptions to ours. He also shows that the strong bidder is more likely to win the auction than any of the weak bidders, but does not consider comparative statics results in the number of bidders or the question whether resale achieves some symmetrization as I do it in our paper. Moreover, our paper also provides an intuition for the results by introducing the concept of effective valuation. Cheng and Tan (2009) related auctions with resale to common value auctions. Our concept of effective valuation has a very similar meaning to their common value component introduced by the resale opportunity. There is also a literature that considers the case where asymmetric information plays a smaller role. For example, Gupta and Lebrun (1999) assume that after the auctions valuations are revealed. Haile (2003) assumes that ex-ante bidders are symmetric, but after the auctions each receives a further shock affecting his valuation, which is the source of resale in their model. Finally, Jehiel (2010) considers a problem in mechanism design using his concept the analogy-based expectation equilibrium where the bidders (perhaps erroneously) believe that they face the same bid distribution from the other bidders. Under this assumptions he provides similar results to Hafalir and Krishna (2008) including that first price auction revenue dominates a second price auction in this situation.

The rest of the paper is organized as follows. In Section 2 I setup the model and define equilibrium. Section 3 contains existence and uniqueness results and characterizes the equilibrium. Section 4 provides results related to the question of symmetrization, while Section 5 discusses comparative statics results as the market size changes. The two Appendices contain some proofs.

2 Setup and equilibrium

Hafalir and Krishna (2008) study auctions with resale when there are two bidders, I extend their work by considering the case of n bidders. Assume that there is an indivisible object and there are n risk neutral bidders whose valuations are distributed independently according to distribution functions $F_1, F_2, ..., F_n$ that admit strictly positive and continuous density functions $f_1, f_2, ..., f_n$. For simplicity I assume that there are n_s strong and n_w weak bidders, with distribution functions F_s and F_w that have common support $[0, 1]$. While the non-symmetrization result would continue to hold if all n bidders have different ex-ante value distributions, but the model remains intractable if one wishes to obtain further results. The common support assumption is only for convenience, most of our results would go through even if this assumption was dropped. To be able to benchmark our results with standard asymmetric auctions without resale[8], I assume that $F_s(x)/F_w(x)$ is strictly increasing in x. I also assume that F_s satisfies the regularity condition of Myerson (1981), i.e. $x - \frac{1-F_s(x)}{f_s(x)}$ is increasing in x. This assumption ensures that the resale problem of the monopolist behaves in a tractable manner.[9]

The timing of the game is simple: first there is a first price auction where the bids (including the winning bid) are not revealed.[10] Then the winner of this auction may resell the object to one of the other $n-1$ bidders. I assume that at the resale stage the current

[8] See Maskin and Riley (2000).

[9] Interestingly, we do not need to make the regularity condition that the virtual utility of the *weak* bidder, $x - \frac{1-F_w(x)}{f_w(x)}$ is increasing in x. The reason is that weak bidders do not buy on the resale market in equilibrium, while the out-of-equilibrium case (see Case 4 in the Appendix) where a weak bidder may buy in the resale stage can be handled without this assumption. For the details see Case 4 in the Appendix.

[10] Given our resale mechanism, the results would not change if the winning bid of the original auction was

owner (the winner of the initial auction) conducts a second price auction with an optimally chosen reserve price, and then the game ends without further resale possibilities. Let us discuss our assumption about the resale process. First, the assumption that the winner of the auction makes a resale offer to the other bidders is the most natural when there are multiple losers who consider buying at the resale stage. In contrast, the monopsony case of Hafalir and Krishna (2008) where the loser makes an offer is less natural, because the owner of the object can always contact the other losers to obtain a better resale price. This suggests that the winner of the initial auction should be able to offer a mechanism that makes the losers compete for the right to buy the object at the resale stage. Second, under our assumptions a second price mechanism with an appropriate reserve price is optimal for the seller at the resale stage on the equilibrium path. Finally, having a second price auction at the resale stage simplifies our analysis significantly. However, it is also important to note that the main results of the paper would continue to hold qualitatively for a much broader set of bargaining procedures. In particular, no reasonable bargaining protocol would yield bid symmetrization when there are more than two bidders.[11]

If a bidder with type y owns the object at the end of the game (after the resale market has closed) and his overall payment was m, then his utility is $y - m$. If a bidder does not own the object, and his overall payment in the game was m (possibly negative), then his utility is $-m$.

Our equilibrium concept is Perfect Bayesian equilibrium. In such an equilibrium each bidder places a bid b and offers a reserve price r (if he won the initial auction), such that no other pair $(\widetilde{b}, \widetilde{r})$ would yield a higher expected utility, given the strategies of the other players. Note, that the definition of the equilibrium already assumes that if a buyer with type y did not buy in the original auction and he faces a reserve price $r \leq y$, then he participates in the resale mechanism and uses his dominant strategy, i.e. bids y in the second price resale auction. I consider an equilibrium where each strong bidder uses strictly increasing and continuous strategy $b_s : [0,1] \to \mathbb{R}_0^+$ and each weak bidder employs strictly increasing and continuous strategy $b_w : [0,1] \to \mathbb{R}_0^+$ in the initial auction stage. Moreover, I assume that the bidders have the same support for bidding, i.e. $b_w(0) = b_s(0) = \underline{b}$ and $b_w(1) = b_s(1) = \overline{b}$. I call such an equilibrium a *regular equilibrium.* It is then easy to prove that $\underline{b} = 0$ must hold in a regular equilibrium, otherwise the bidders with the lowest valuations would make negative payoffs. Since increasing functions are almost everywhere differentiable, the bid functions are almost everywhere differentiable and we can characterize the equilibrium as a solution to a system of ordinary differential equations. For simplicity we consider an equilibrium bid function that is everywhere differentiable.

3 Equilibrium analysis

To characterize the equilibrium I start the analysis with the resale stage taking the bid functions (b_w, b_s) as given. First, I study the case when each buyer used the equilibrium bid in the initial auction. As we will see, this case will pin down the equilibrium reserve price uniquely under the assumption of monotone virtual utilities. The first Lemma shows that at any given bid level only one side can be a seller at the resale stage and he will sell to the

revealed. If some of the losing bids would be revealed, then bidders may have an incentive to signal their type to mislead the seller at the resale mechanism, and a pure strategy monotone equilibrium may not exist anymore.

[11] In fact the author has not been able to identify any protocol that would yield such result.

other group of bidders:

Lemma 1 *Suppose that $b_w(y) = b > (<) b_s(y)$ for some y, b. Then it is optimal for a strong (weak) buyer with type y not to offer the good for resale, but a weak (strong) buyer with type y makes a resale offer that is accepted with positive probability by a strong (weak) buyer. When $b_w(y) = b_s(y)$ neither the strong, nor the weak bidder with type y has a profitable resale opportunity, so it is optimal for such a bidder not to make a resale offer at all.*

Proof. If $b_s(y) < b_w(y)$, then under our our assumptions there exists $x > y$ and $z < y$ such that $b_s(x) = b_w(y)$ and $b_s(z) = b_w(y)$. Therefore, upon winning a weak buyer with type y knows that the type of the n_s strong buyers are less than x. Since $x > y$ holds, the winner of the auction has a profitable resale opportunity by offering an auction with any reserve price $r \in (y, x)$. Upon winning the initial auction, a strong buyer with type y knows that the weak buyers' type are less than $z < y$ and the other strong buyers' type are less than y, so resale cannot be conducted profitably. The case when $b_w(y) = b_s(y)$ can be handled similarly. ■

It is a well known result in the literature (see Maskin and Riley (2000)) that when $n_s = n_w = 1$ and there is no resale, then for all $y \in (0, 1)$

$$b_w(y) > b_s(y).$$

Hafalir and Krishna showed that in the same setup when resale is allowed a similar result is still true. This leads to the conjecture that for an arbitrary number of bidders with resale it holds that $b_w(y) > b_s(y)$. I first explore this possibility in my analysis. Then Lemma 1 implies that in equilibrium the strong buyers do not make resale offers and the weak buyers do, and they resell the object with positive probability to the strong buyers. Using the notation of the proof of Lemma 1 a weak buyer with type y faces n_s strong buyers with valuations on $[0, x]$. Let $r(y)$ be the optimally chosen reserve price at the resale stage by a weak buyer with type y.

Lemma 2 *Suppose that for all y it holds that $b_w(y) \geq b_s(y)$. Under the monotone virtual utility assumption the equilibrium reserve price is unique. Moreover, running a second price auction with an optimal reserve price is optimal for a weak bidder with type y who bid $b_w(y)$ in the initial auction. The optimal reserve price $r(y)$ is characterized by*

$$r(y) - \frac{F_s(x) - F_s(r)}{f_s(r)} = y,$$

where $x = b_s^{-1}(b_w(y)) \geq y$.

Proof. First, note that the winner of the initial auction faces n_s strong buyers with valuations on $[0, x]$, i.e. he solves for an optimal auction for the case of symmetric bidders with independent private values. As Myerson (1981) has shown the optimal auction is a second price auction with an optimally chosen reserve price, which yields the second result. Moreover, the optimal reserve price does not depend on the number of bidders (n_s) and thus it is the same as in the $n_s = 1$ case. However, this is the monopoly case of Hafalir and Krishna (2008) who show that a unique optimal reserve price exists when $n_s = 1$, which concludes the first result of the Lemma. They show that $r(y)$ solves

$$\max_r (F_s(x) - F_s(r))r + F_s(r)y,$$

with first order condition

$$r - \frac{F_s(x) - F_s(r)}{f_s(r)} = y. \tag{1}$$

■

The (unique) equilibrium reserve price is described by equation (1). Since $x = b_s^{-1}(b_w(y))$ is determined by functions b_w, b_s, therefore there is a unique optimal reserve price given the bid functions b_w, b_s. The rest of the analysis uses the calculated $r(y)$ function to derive a necessary first order condition for a regular equilibrium in the original auction. Let $\alpha_s(b)$ and $\alpha_w(b)$ denote the equilibrium inverse bid functions. Under the assumption that $b_w(y) > b_s(y)$ for all $y \in (0,1)$ it holds for all $b \in (0, \overline{b})$ that $\alpha_s(b) > \alpha_w(b)$. Moreover, $\alpha_s(\overline{b}) = \alpha_w(\overline{b})$ and $\alpha_s(0) = \alpha_w(0)$.

Suppose that a strong bidder with type $\alpha_s(b)$ considers a small deviation from his equilibrium bid in the initial auction. First, assume that he bids $\widehat{b} < b$, i.e. he deviates downward with his bid. I consider a small enough deviation, such that $r(\alpha_w(\widehat{b})) < \alpha_s(b)$ and thus our deviating strong bidder still buys with positive probability from weak bidders in the resale stage. Since, for now, I concentrate on the case where the strong bidders never make resale offers, thus the deviating strong bidder cannot buy at the resale stage from other strong bidders. Moreover, with bid $\widehat{b}$ the deviating bidder wins exactly against other strong bidders with types lower $\alpha_s(\widehat{b})$ and thus his type $\alpha_s(b)$ is higher than the possible types of any losing (strong) bidders. This implies that he is not reselling the object to other strong bidders either. To summarize this discussion: if $\widehat{b} < b$ and $b - \widehat{b}$ is small enough, then the deviating strong bidder may buy from a weak bidder at the resale stage, but does not transact with other strong bidders. The deviating bidder wins the initial auction with probability $F_s^{n_s-1}(\alpha_s(\widehat{b}))F_w^{n_w}(\alpha_w(\widehat{b}))$ and in this case his utility is $\alpha_s(b) - \widehat{b}$, since he is not going to resell the object. If he loses the initial auction, but he is the highest type among the strong bidders and a weak bidder wins whose type is less than $r^{-1}(\alpha_s(b))$, then he buys the object in the resale stage. In the case he is able to buy his payment is equal to $\max\{r(x), v_s^2\}$, where x is the type of the winning weak bidder and $v_s^2 \le \alpha_s(b)$ is the second highest type among all the strong bidders, i.e. the highest type among the *other* strong bidders. Also, let $\widetilde{U}_s(\alpha_s(b), x)$ denote the expected utility of a strong bidder with type $\alpha_s(b)$ if the auction was won by a weak bidder with type x and the strong bidder with type $\alpha_s(b)$ buys the object in the resale stage. Formally,

$$\widetilde{U}_s(\alpha_s(b), x) = \alpha_s(b) - E[\max\{r(x), v_s^2\} \mid v_s^2 \le \alpha_s(b)], \tag{2}$$

where E stands for the expected value operator.

The utility of the deviating strong buyer can thus be written as

$$U_s^{down}(\alpha_s(b), \widehat{b}) = F_s^{n_s-1}(\alpha_s(\widehat{b}))F_w^{n_w}(\alpha_w(\widehat{b}))[\alpha_s(b) - \widehat{b}] +$$

$$+ F_s^{n_s-1}(\alpha_s(b)) \int_{\alpha_w(\widehat{b})}^{r^{-1}(\alpha_s(b))} n_w F_w^{n_w-1}(x) f_w(x) \widetilde{U}_s(\alpha_s(b), x) dx.$$

The bidder then solves $\max\limits_{\widehat{b} \le b} U_s^{down}(\alpha_s(b), \widehat{b})$. The first order condition for optimum at $\widehat{b} = b$ becomes then

$$n_w F_w^{n_w-1}(\alpha_w(b)) f_w(\alpha_w(b)) \alpha_w'(b) F_s^{n_s-1}(\alpha_s(b))[\alpha_s(b) - b - \widetilde{U}_s(\alpha_s(b), \alpha_w(b))] +$$

$$+ (n_s - 1) F_s^{n_s-2}(\alpha_s(b)) f_s(\alpha_s(b)) \alpha_s'(b) F_w^{n_w}(\alpha_w(b))[\alpha_s(b) - b] \ge F_s^{n_s-1}(\alpha_s(b)) F_w^{n_w}(\alpha_w(b)).$$

Note, that the first order condition takes the form of an inequality, because the set of admissible choices here is all bids less than b. Using (2) and defining

$$\widetilde{r}(b) = E[\max\{r(\alpha_w(b)), v_s^2\} \mid v_s^2 \leq \alpha_s(b)]$$

yields that

$$n_w F_w^{n_w-1}(\alpha_w(b)) f_w(\alpha_w(b)) \alpha_w'(b) F_s^{n_s-1}(\alpha_s(b))[\widetilde{r}(b) - b] +$$
$$+(n_s-1) F_s^{n_s-2}(\alpha_s(b)) f_s(\alpha_s(b)) \alpha_s'(b) F_w^{n_w}(\alpha_w(b))[\alpha_s(b) - b] \geq F_s^{n_s-1}(\alpha_s(b)) F_w^{n_w}(\alpha_w(b)). \tag{3}$$

The interpretation of $\widetilde{r}$ is simple: this is the expected amount that a strong bidder with type $\alpha_s(b)$ needs to pay in the resale stage if he loses against a weak bidder by a very small margin. Note, that in case of such a loss he surely buys the object in the resale stage and thus $\widetilde{r}$ becomes his effective valuation. As one can see the effective valuation of the strong bidder is equal to $\widetilde{r}$ if he lost against a weak bidder by a small margin, because then he buys the object at the resale stage for sure and pays an expected amount $\widetilde{r}$. On the other hand, his effective valuation is his use value (α_s) if he lost against a strong bidder, because then he cannot by the object, so by not buying it in the initial auction he foregoes a (gross) profit of α_s

Let us now study a small upward deviation, i.e. the case where $\widehat{b} > b$. For the same reason as in the case of a small downward deviation the deviating bidders still buys from weak bidders at the resale stage, but he starts selling to other strong bidders, which is a new component of the utility function. Let $r_s(\alpha_s(b), \widehat{b})$ denote the optimal resale offer made by the deviating bidder. As I remarked above, for a small deviation the deviating strong bidder is not able to resell to weak bidders, so when making his resale offer he ignores weak bidders, and only maximizes the utility that arises from a possible resale to other strong bidders. Setting a price above $\alpha_s(\widehat{b})$ yields no resale, while setting a reserve price below $\alpha_s(b)$ would yield a selling price below the actual valuation $\alpha_s(b)$. Therefore, the optimal resale reserve price satisfies $r_s(\alpha_s(b), \widehat{b}) \in (\alpha_s(b), \alpha_s(\widehat{b}))$. We are not interested in the exact value of this reserve price, just note that if $\widehat{b} \to b$, then $r_s(\alpha_s(b), \widehat{b}) \to \alpha_s(b)$. The deviating bidder resells the object if he beats all the weak bidders in the auction, *and* the strong bidder with the highest valuation is below $\alpha_s(\widehat{b})$ (so that the deviating bidder wins the auction), but below $r_s(\alpha_s(b), \widehat{b})$ (so that resale occurs). The probability of this event is $F_w^{n_w}(\alpha_w(\widehat{b}))[F_s^{n_s-1}(\alpha_s(\widehat{b})) - F_s^{n_s-1}(r_s(\alpha_s(b), \widehat{b}))]$. The expected revenue when resale occurs is and the reserve price is set at r_s can be written as

$$\omega(r_s) = E[v_s^2 \mid v_s^2 \in (r_s, \alpha_s(\widehat{b}))].$$

The utility of the deviating bidder can then be written as

$$U_s^{up}(\alpha_s(b), \widehat{b}) = F_s^{n_s-1}(r_s(\alpha_s(b), \widehat{b})) F_w^{n_w}(\alpha_w(\widehat{b}))[\alpha_s(b) - \widehat{b}] +$$
$$+F_s^{n_s-1}(\alpha_s(b)) \int_{\alpha_w(\widehat{b})}^{r^{-1}(\alpha_s(b))} n_w F_w^{n_w-1}(x) f_w(x) \widetilde{U}_s(\alpha_s(b), x) dx +$$
$$+F_w^{n_w}(\alpha_w(\widehat{b}))[F_s^{n_s-1}(\alpha_s(\widehat{b})) - F_s^{n_s-1}(r_s(\alpha_s(b), \widehat{b}))](\omega(r_s(\alpha_s(b), \widehat{b})) - \widehat{b}).$$

Since the reserve price $r_s(\alpha_s(b), \widehat{b})$ is chosen optimally given $b, \widehat{b}$, therefore at $r_s = r_s(\alpha_s(b), \widehat{b})$ it holds that

$$\frac{\partial}{\partial r_s}[F_s^{n_s-1}(r_s \alpha_s(b) + \{F_s^{n_s-1}(\alpha_s(\widehat{b})) - F_s^{n_s-1}(r_s)\}\omega(r_s)] = 0.$$

This implies that

$$\frac{\partial U_s^{up}}{\partial \widehat{b}} = n_w F_w^{n_w-1}(\alpha_w(\widehat{b}))f_w(\alpha_w(\widehat{b}))\alpha_w'(\widehat{b})F_s^{n_s-1}(r_s(\alpha_s(b),\widehat{b}))(\alpha_s(b)-\widehat{b})-$$

$$-n_w F_w^{n_w-1}(\alpha_w(\widehat{b}))f_w(\alpha_w(\widehat{b}))\alpha_w'(\widehat{b})F_s^{n_s-1}(\alpha_s(b)))[\alpha_s(b)-\widetilde{r}(\widehat{b})]+$$

$$+n_w F_w^{n_w-1}(\alpha_w(\widehat{b}))f_w(\alpha_w(\widehat{b}))\alpha_w'(\widehat{b})[F_s^{n_s-1}(\alpha_s(\widehat{b}))-F_s^{n_s-1}(r_s(\alpha_s(b),\widehat{b}))](\omega(r_s(\alpha_s(b),\widehat{b}))-\widehat{b})+$$

$$+F_w^{n_w}(\alpha_w(\widehat{b}))(n_s-1)F_s^{n_s-2}(\alpha_s(\widehat{b}))f_s(\alpha_s(\widehat{b}))\alpha_s'(\widehat{b})(\omega(r_s(\alpha_s(b),\widehat{b}))-\widehat{b})-$$

$$-F_s^{n_s-1}(\alpha_s(\widehat{b}))F_w^{n_w}(\alpha_w(\widehat{b})).$$

Before I describe the first order condition, note the following chain of causations

$$\widehat{b}\to b \Rightarrow \alpha_s(\widehat{b})\to\alpha_s(b)\Rightarrow r_s(\alpha_s(b),\widehat{b})\to\alpha_s(b)\Rightarrow\omega(r_s(\alpha_s(b),\widehat{b}))\to\alpha_s(b). \tag{4}$$

The first order condition of maximization implies that $\frac{\partial U_s^{up}}{\partial \widehat{b}} \mid_{\widehat{b}=b} \leq 0$ holds. Using (4), this condition becomes

$$n_w F_w^{n_w-1}(\alpha_w(b))f_w(\alpha_w(b))\alpha_w'(b)F_s^{n_s-1}(\alpha_s(b))(\widetilde{r}(b)-b)+$$

$$+F_w^{n_w}(\alpha_w(b))(n_s-1)F_s^{n_s-2}(\alpha_s(b))f_s(\alpha_s(b))\alpha_s'(b)(\alpha_s(b)-b)\geq F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b)).$$

Note, that this is identical to condition (3), just the inequality reversed. This implies that the first order condition for optimization for the strong bidder is

$$n_w F_w^{n_w-1}(\alpha_w(b))f_w(\alpha_w(b))\alpha_w'(b)F_s^{n_s-1}(\alpha_s(b))(\widetilde{r}(b)-b)+$$

$$+F_w^{n_w}(\alpha_w(b))(n_s-1)F_s^{n_s-2}(\alpha_s(b))f_s(\alpha_s(b))\alpha_s'(b)(\alpha_s(b)-b)=F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b)). \tag{5}$$

A discussion of the above derivations are useful at this point. First, the above implies that at $\widehat{b}=b$ it holds that $\frac{\partial U_s^{up}}{\partial \widehat{b}} = \frac{\partial U_s^{down}}{\partial \widehat{b}} = 0$, in other words there is no kink in the objective function at $\widehat{b}=b$. The intuition for this is that although in the second case considered the deviating buyer would resell to other strong bidders, but the gain from this is second order when $\widehat{b}$ is close to b.[12] Second, (5) can be interpreted by using the concept of effective valuations as introduced in the introduction. When a strong bidder loses against a weak bidder by a small margin, then he knows that he will buy the object at the resale stage for a price of $\widetilde{r}(b)$, so this becomes his effective valuation in this event. When he loses against another strong bidder by a small margin, then he cannot buy in the resale stage, so he loses a value exactly equal to his use value $\alpha_s(b)$, his effective valuation in this case.

Now, I turn to the analysis of the weak bidders' problem. Denote his type by $\alpha_w(b)$ and his bid $\widehat{b}$, again restricting attention to the case where $\widehat{b}-b$ is small in absolute value. First, I establish that reselling to another weak bidder is not profitable if $\widehat{b}-b$ is small in absolute value. It is clear that if $\widehat{b}<b$ then if the deviator won the auction, then all weak bidders have type $\alpha_w(\widehat{b})<\alpha_w(b)$ and thus no profitable sale can occur between the winning bidder with type $\alpha_w(b)$ and another bidder with type at most $\alpha_w(\widehat{b})$. If $\widehat{b}>b$, then the net revenue

[12] When $\widehat{b}-b$ is small (and positive), then resale takes place with a vanishing probability, moreover the expected gain *conditional on reselling* is also close to zero.

that can be gained from other weak bidders is $\max_{r_w}(F_w^{n_w-1}(\alpha_w(\widehat{b})-F_w^{n_w-1}(r_w))(r_w-\alpha_w(b))$. For this to be positive it must be that $r_w \in (\alpha_w(b), \alpha_w(\widehat{b}))$ and thus the revenue is less than $(F_w^{n_w-1}(\alpha_w(\widehat{b}) - F_w^{n_w-1}(\alpha_w(b)))(\alpha_w(\widehat{b}) - \alpha_w(b))$. But

$$\widehat{b} \to b \Rightarrow (F_w^{n_w-1}(\alpha_w(\widehat{b}) - F_w^{n_w-1}(\alpha_w(b)))(\alpha_w(\widehat{b}) - \alpha_w(b)) < \alpha_w(\widehat{b}) - \alpha_w(b) \to 0$$

by continuity of α_w, and thus for a small enough deviation it is not profitable to try to sell to other weak bidders at the resale stage.

To characterize the first order conditions we need to consider different cases again. I use the insights from the analysis for the strong bidder to simplify the details of the analysis. Specifically, I use the concept of effective valuation to derive the first order condition for the weak bidder.[13] There are three cases to consider here as opposed to the two cases for the strong bidder's problem. First, if a weak bidder loses against a strong bidder by a small margin, then he is able to sell the object to him and the expected revenue is $\widetilde{r}(b)$. Second, if he wins against another weak bidder, then he may or may not resell the object in equilibrium, depending on the highest type among the strong bidders as resale does not take place to another weak bidder. If he does not sell, then his effective valuation is his use value $\alpha_w(b)$. If he sells then his revenue is $\tau(b)$, where $\tau(b)$ is the expected price (revenue) if a weak bidder wins, beating another weak bidder with the same type $\alpha_w(b)$, but resale takes place to a lower bidder, the strong bidder with the highest type. As before, let v_s^1 and v_s^2 denote the highest and second highest types among the strong bidders. First, resale takes place if and only if $v_s^1 \in [r(\alpha_w(b)), \alpha_s(b)]$. Second, if $v_s^1 = x \in [r(\alpha_w(b)), \alpha_s(b)]$, then the expected resale price is equal to $R_b(\alpha_w(b), x) = E[\max\{v_s^2, r(\alpha_w(b))\} \mid v_s^1 = x]$, i.e. the expected value of the maximum of the reserve price and the second highest type among the strong bidders, if the highest type among the strong bidders is x. Then one can write the revenue $\tau(b)$ formally as

$$\tau(b) = E[\max\{v_s^2, r(\alpha_w(b))\} \mid v_s^1 \in [r(\alpha_w(b)), \alpha_s(b)]] =$$

$$= \frac{\int_{r(\alpha_w(b)}^{\alpha_s(b)} n_s F_s^{n_s-1}(x) f_s(x) R_b(\alpha_w(b), x) dx}{F_s^{n_s}(\alpha_s(b)) - F_s^{n_s}(r(\alpha_w(b)))}.$$

Using the above considerations one can write the first order condition for the weak bidders optimization problem as

$$\begin{gathered} n_s F_s^{n_s-1}(\alpha_s(b)) f_s(\alpha_s(b)) \alpha_s'(b) F_w^{n_w-1}(\alpha_w(b))[\widetilde{r}(b) - b] + \\ +(n_w-1) F_w^{n_w-2}(\alpha_w(b)) f_w(\alpha_w(b)) \alpha_w'(b) F_s^{n_s}(r(\alpha_w(b)))(\alpha_w(b) - b) + \\ +(n_w-1) F_w^{n_w-2}(\alpha_w(b)) f_w(\alpha_w(b)) \alpha_w'(b) [F_s^{n_s}(\alpha_s(b)) - F_s^{n_s}(r(\alpha_w(b)))](\tau(b) - b) = \\ = F_s^{n_s}(\alpha_s(b)) F_w^{n_w-1}(\alpha_w(b)). \end{gathered} \tag{6}$$

The first term in (6) corresponds to the case when the weak bidder overtakes a strong bidder on the margin, while the second and the third terms correspond to the case where a weak

[13] In effect, the analysis below substitutes from the first order condition of the weak bidder when he is choosing his optimal resale reserve price. Once such a substituiton is made, the first order condition for the bidding problem in the original auction is simplified to the one with effective valuations as it appears in (6). The details are very similar to the considerations that lead to (5), and are thus omitted.

bidder is overtaken on the margin. In the second case the strong bidder with the highest type has a type less than the reserve price set by the weak bidder $r(\alpha_w(b))$, and thus resale does not take place. In the third case resale takes place and the expected revenue (conditional on tieing) is $\tau(b)$.

The system of equations (5), (6) defines a system of ordinary differential equations, since functions $\widetilde{r}$ and τ are uniquely determined by α_w, α_s. As standard, the initial condition $\alpha_s(0) = \alpha_w(0) = 0$ cannot be used to solve our system, since the system does not satisfy the Lipschitz condition at $b = 0$. Therefore, following the rest of the literature[14] I impose an end condition $\alpha_s(\overline{b}) = \alpha_w(\overline{b}) = 1$ with an unknown value for $\overline{b}$. Then I obtain the following result:

Lemma 3 *Suppose that for some $\overline{b}$ it holds that $\alpha_s(\overline{b}) = \alpha_w(\overline{b}) = 1$ and the system of differential equations has a strictly increasing solution on $[0, \overline{b})$ such that $\alpha_s(0) = \alpha_w(0) = 0$ holds and for all $b \in (0, \overline{b})$ it holds that $\alpha_s(b) > \alpha_w(b)$. Then the solution of this differential equation (α_w, α_s) forms a pair of equilibrium inverse bid functions.*

Proof. If the above conditions hold, then one only needs to show that the bidders cannot use a large deviation in the initial auction to increase their overall utilities. This is shown in the Appendix. ■

The proof in the Appendix requires checking several additional cases, since if a bidder uses a large deviation in the initial auction, then he needs to recalculate his optimal reserve price at the resale stage. Moreover, weak bidders may become buyers and strong bidders may become sellers at the resale stage. Checking those conditions is somewhat tedious, but using that under our conditions reserve prices behave monotonically in types and initial bids provides a sufficient amount of monotonicity to preserve the second order conditions.

At this point it is also important to consider the case where $b_s(x) > b_w(x)$ for some x or where for some b it holds that $\alpha_s(b) < \alpha_w(b)$. One can show that such a case cannot occur in equilibrium. To do that formally let us consider two different subcases. First, suppose that there exists a value b^* such that $\alpha_s(b^*) = \alpha_w(b^*)$. Then by construction it holds that

$$\widetilde{r}(b^*) = \lim_{b \searrow b^*} \tau(b^*) = \alpha_s(b^*) = \alpha_w(b^*).$$

Then using (3) it holds that at $b = b^*$

$$\frac{(F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b)))'}{F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b))} = \frac{(F_s^{n_s}(\alpha_s(b))F_w^{n_w-1}(\alpha_w(b)))'}{F_s^{n_s}(\alpha_s(b))F_w^{n_w-1}(\alpha_w(b))} = \frac{1}{\alpha - b^*}.$$

This implies that at $b = b^*$ it holds that $(\frac{F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b))}{F_s^{n_s}(\alpha_s(b))F_w^{n_w-1}(\alpha_w(b))})' = 0$ or

$$(\frac{F_w(\alpha_w(b))}{F_s(\alpha_s(b))})' = 0.$$

Since by assumption $F_w(x)/F_s(x)$ is strictly decreasing in x, therefore the last equation implies that

$$(\frac{F_w(\alpha_w(b))}{F_w(\alpha_s(b))})' > 0$$

[14] See for example Lebrun (1997).

holds at $b = b^*$. The last inequality then implies that

$$b > 0, \alpha_w(b) = \alpha_s(b) \to \alpha'_w(b) > \alpha'_s(b). \tag{7}$$

Using that in equilibrium it holds that $\alpha_s(\overline{b}) = \alpha_w(\overline{b}) = 1$ implies that there exists an ε such that for all $b \in (\overline{b} - \varepsilon, \overline{b})$ it holds that $\alpha_s(b) > \alpha_w(b)$. Therefore, either for all $b \in (0, \overline{b})$ it holds that $\alpha_s(b) > \alpha_w(b)$ as conjectured or there exists a $b^* \in (0, \overline{b})$ such that $\alpha_s(b^*) = \alpha_w(b^*)$ and for all $b \in (b^*, \overline{b})$ it holds that $\alpha_s(b) > \alpha_w(b)$. However, inequality (7) implies that at such a point b^* it holds that $\alpha'_w(b^*) > \alpha'_s(b^*)$, which means that for a small enough ε it holds that $\alpha_w(b^* + \varepsilon) > \alpha_s(b^* + \varepsilon)$, which contradicts with the definition of b^*. Therefore, no such $b^* > 0$ may exist. Therefore, the only other case possible is if for all $b \in (0, \overline{b})$ it holds that $\alpha_w(b) > \alpha_s(b)$. However, following the considerations leading to (7), it must hold for a small enough ε that $\alpha_S(\overline{b} - \varepsilon) > \alpha_w(\overline{b} - \varepsilon)$, which concludes the proof that for all $b \in (0, \overline{b})$ it holds that $\alpha_s(b) > \alpha_w(b)$ or for all $x \in (0, 1)$ it holds that $b_w(x) > b_s(x)$.

The following conclusion can be drawn from this discussion:

Corollary 1 *For every pair of regular equilibrium inverse bid functions* (α_w, α_s) *for all* $b \in (0, \overline{b})$ *it holds that* $\alpha_s(b) > \alpha_w(b)$.

The above two results imply that finding a regular equilibrium is equivalent to finding an appropriate $\overline{b}$. The proof of this result is in the Appendix:

Proposition 1 *There exists a regular equilibrium of the auction game.*

The proof uses techniques from ordinary differential equations to conclude existence. However, the fundamental theorem for ordinary differential equations cannot be used without some relevant restrictions. Fortunately, one can show that Lipschitz continuity holds for the relevant cases and thus existence can be guaranteed.

4 Bid distributions

It is well known for static auctions without resale that in our setup the weak bidder bids more aggressively than the strong bidder, but produces a weaker bid distribution. Formally, let β_w, β_s be the (unique) equilibrium bid functions without resale. Then Maskin and Riley (2000) show[15] that for all $x \in (0, 1)$ it holds that $\beta_w(x) > \beta_s(x)$ and that $F_w(\beta_w(x)) < F_s(\beta_s(x))$. For the case of resale with one strong and one weak bidder the Hafalir and Krishna (2008) result implies that the bid functions are such that $b_w(x) > b_s(x)$ and that $F_w(b_w(x)) = F_s(b_s(x))$. In other words, if there is resale opportunity, the weak bidder becomes even more aggressive compared to the strong one and the weak bidder produces the same bid distribution as the strong bidder winning the object 50% of the time. The main intuition is that each bidder knows that if the weak bidder barely wins with a bid b, then there is a sure resale at a price $r(\alpha_w(b))$. Therefore, when bidding each bidder takes this r as his effective valuation. The weak bidder knows that if he barely wins he will resell the object for sure at price r, so that is how much the object is worth for him. For the same reason the strong bidder knows that if he loses by a small margin, then he will buy the object at a resale price r, which is then how much he values the object when bidding

[15] Maskin and Riley (2000) proves this result for the case where $n_s = n_w = 1$, but an extension of their results to the case of multiple bidders is routine.

for it. This logic fails when there are more than two bidders. Suppose that there are two weak bidders and one strong bidder. The strong bidder can make the same reasoning as before and thus his effective valuation is equal to the resale price at which he buys $r(\alpha_w(b))$. However, when a weak bidder wins by a small margin, then he may not be able to sell the object if the second highest bid was made by the other weak bidder. In this case his value from winning is equal to his type $\alpha_w(b)$, while in the case when he is able to sell the object his eventual utility is the resale price $r(\alpha_w)$. The expected effective valuation is then strictly between α_w and r, which is less than the effective valuation of strong buyer, which is equal to r.

Using this insight, we establish that a strong bidder wins the auction more often than a weak bidder when there are at least three bidders ($n_s + n_w \geq 3$). Let us divide equations (3) and (6) by $F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b))$ and $F_s^{n_s}(\alpha_s(b))F_w^{n_w-1}(\alpha_w(b))$ respectively. Substituting that $\widetilde{r} < \alpha_s$ implies then that if $n_s > 1$ then for all $b \in (0, \overline{b})$ it holds that

$$\frac{(F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b)))'}{F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b))} < \frac{1}{\widetilde{r}(b) - b}.$$

Using that $\widetilde{r} > \alpha_w, \tau$ implies through (6) that if $n_w > 1$ then

$$\frac{(F_s^{n_s}(\alpha_s(b))F_w^{n_w-1}(\alpha_w(b)))'}{F_s^{n_s}(\alpha_s(b))F_w^{n_w-1}(\alpha_w(b))} > \frac{1}{\widetilde{r}(b) - b}.$$

Therefore, if $n_s > 1$ or $n_w > 1$ (or both) holds, then for all $b \in (0, \overline{b})$ it holds that

$$\frac{(F_s^{n_s}(\alpha_s(b))F_w^{n_w-1}(\alpha_w(b)))'}{F_s^{n_s}(\alpha_s(b))F_w^{n_w-1}(\alpha_w(b))} > \frac{(F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b)))'}{F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b))}$$

or that

$$\left(\frac{F_s^{n_s}(\alpha_s(b))F_w^{n_w-1}(\alpha_w(b))}{F_s^{n_s-1}(\alpha_s(b))F_w^{n_w}(\alpha_w(b))}\right)' > 0.$$

But this last inequality implies that

$$\left(\frac{F_s(\alpha_s(b))}{F_w(\alpha_w(b))}\right)' > 0.$$

Noting that $F_s(\alpha_s(\overline{b})) = F_w(\alpha_w(\overline{b}))$ implies that for all $b \in (0, \overline{b})$ it holds that

$$\frac{F_s(\alpha_s(b))}{F_w(\alpha_w(b))} < 1,$$

which means that the strong bidders produce a stronger bid distribution than the weak ones and thus win more often in the initial auction if there are more than two bidders in the auction. The following theorem states the result formally:[16]

[16] In their Section 6.2, Hafalir and Krishna (2008) suggest that in the special case when there are two weak and one strong bidders bid symmetrization should hold when the winner of the initial auction makes a take it or leave it resale offer. However, our result shows that this is not the case. The intuition is that effective valuations are not equalized, because the strong bidder always buys if he lost by a small margin, while a weak bidder does not resell if he won by a small margin against the other weak bidder. In other words the sure trade property of Hafalir and Krishna (2008) fails.

Theorem 1 *Let $n_s, n_w \geq 1$ and $n_s + n_w \geq 3$. Then in a regular equilibrium it holds for any $b \in (0, \bar{b})$ that*

$$F_s(\alpha_s(b)) < F_w(\alpha_w(b))$$

and thus a strong bidder produces a more aggressive bid distribution than a weak bidder and wins the auction with a higher probability.

The logic of Theorem 1 suggests that (returning to the two group case) the asymmetry in bid distributions is reduced by the possibility of resale. Without resale the effective valuations are α_s and α_w for the strong and weak bidders respectively. With resale the effective valuation of the strong bidder belongs to the interval $[r, \alpha_s]$, while that of the weak bidder to the interval $[\alpha_w, r]$. Therefore, the asymmetry in effective valuations is reduced compared to the case without resale and thus one may expect that the bid distributions are more equal than in the case without resale.

The following result shows that our conjecture is valid for a case that can be handled formally:

Proposition 2 *Let $n_s = 1, n_w = 2$ and assume that f_s is decreasing in x and $F_w/\sqrt[4]{x}$ is increasing in x. Define $\eta(x)$ as*

$$b_s(\eta(x)) = b_w(x).$$

Let β_s, β_w denote the equilibrium bid functions of the auction without resale and let $\omega(x)$ be defined as

$$\beta_s(\omega(x)) = \beta_w(x).$$

Then it holds that for all $x \in (0, 1)$ that

$$\eta(x) > \omega(x) > x$$

and thus the bid distribution is more symmetric in the auction with resale than in the auction without resale.

The proof can be found in Appendix 2. This example shows that the bid function is more skewed in the case where resale is allowed in the sense that the weak bidders bids much more aggressively than the strong bidder if resale is allowed. But this means that a weak bidder has a higher probability to win in the case with resale compared to the no resale case, although less than the strong bidder as long as there are at least three bidders.

Although the formal analysis is not extended to the case where bidders are coming from more than two groups (i.e. not only strong and weak, but also other type distributions), using the concept of effective valuations it is possible gain intuition for that case as well. So, suppose that there are three bidders (strong, medium and weak) ordered in the sense of stochastic dominance assumed in the two-group case above. Using a similar analysis as above one can show that bid distributions are *not symmetrized*, since the effective valuations of the three bidders are different. However, obtaining any analytical result beyond that is rather difficult, and thus such an analysis is not pursued here.[17]

[17] The main difficulty is that now the effective valuation of a bidder is a weighted average of the effective valuations weighted by the probabilities of tieing with any of the two other bidders. It may be that, conditional on tieing, the medium bidder finds it much less likely to tie with the weak bidder than the strong bidder. If tieing with a strong or medium bidder leads to a much higher (conditional) effective valuation than tieing with a weak one, then this may imply that the medium bidder has higher effective valuation than a strong bidder even if the conditional effective valuation of the strong bidder is higher. This could then potentially yield a reversal of bid distributions between the strong and the medium bidders.

5 Bidding and number of buyers

It is interesting to consider some numerical results to illustrate the extent of asymmetry in bid distributions when there are more than three bidders in an auction with resale. For simplicity I consider the case with one strong bidder and several weak bidders and, $n_s = 1$, $n_w \geq 1$ and assume that $F_s(x) = x$ and $F_w(x) = \sqrt{x}$. One can then write up the first order conditions and specify conditions (6), (3) for the case at hand. Using program package Mathematica one can obtain numerical solutions for this specification.[18]

To characterize the asymmetry in bid distributions with a simple measure I use the probability of winning the auction as our starting point. Let π_ω^r, π_s^r denote the probability *under resale* that a given weak bidder wins, and the probability that a strong bidder wins, respectively. Let π_ω^n, π_s^n denote the probability *with no resale* that a given weak bidder wins, and the probability that a strong bidder wins, respectively. By construction

$$n_w \pi_w^r + \pi_s^r = n_w \pi_w^r + \pi_s^r = 1.$$

Our measure for asymmetry comparing the case with and without resale is

$$\rho(n_w) = \frac{\pi_s^r - \frac{1}{n_w+1}}{\pi_s^n - \frac{1}{n_w+1}}.$$

It is necessary to construct an indirect measure like this, since as the number of bidders becomes large, each bidder has a very small probability of winning, and thus the difference in winning probabilities $\pi_s^r - \pi_w^r$ becomes zero necessarily. To still obtain a measure of asymmetry I construct a measure that does not asymptote to zero as the number of bidders becomes large, and there is no mechanical reason for this measure to be monotone. So, if the measure turns out to be monotonic (in the number of bidders), it can be interpreted to occur because the extent of symmetrization changes as the number of bidders grows. Measure ρ satisfies these requirements, although other appropriate measures could be used as well.

Note, that this specification is a special case of Proposition 2 when $n_w = 2$ and thus it must hold that

$$0 < \rho(2) < 1,$$

because with resale the weak bidder wins more often than without resale. Also, we know it from Hafalir and Krishna (2008) that

$$\rho(1) = 0$$

and thus our conjecture is that our measure of asymmetry yields $\rho \in (0, 1)$ for any $n_w > 1$. This conjecture is valid for the case of several bidders as it is highlighted by the following results:

$$\rho(2) \approx 0.41, \rho(3) \approx 0.57, \rho(4) \approx 0.65, \rho(5) \approx 0.7, \rho(9) \approx 0.82.$$

As one can see, the asymmetry is increasing in the number of bidders and in large markets the opportunity of resale does not change winning probabilities much compared to the case of no resale where asymmetries in winning probabilities are large. The reason seems intuitive: as the number of bidders n_w becomes large, it holds that the bid functions converge to x, i.e. bid shading disappears in the limit regardless of whether there is resale or not. But then

[18] The calculations are available from the author upon request.

resale cannot take place in the limit with positive probability and thus the two allocations have to be similar in the limit.[19]

For the same example I also conduct a revenue comparison between the first and second price auctions with resale. In the two bidder case Hafalir and Krishna (2008) show that a first price auction provides a higher revenue than a second price auction when resale is introduced. Since the allocation with resale becomes more and more similar to the no resale case as the number of bidders increases, therefore one may expect that the revenue advantage is still there, but is decreasing in the number of bidders. In the Introduction I provide an alternative intuition, based on effective valuations, for this conjecture. To confirm our conjecture I provide a measure for revenue differences, and calculate the revenue difference between first and second price auction for the example above. Let R^I, R^{II}, R^M denote the revenues for the first price auction with resale, the revenues for the second price auction with (or without) resale, and the Myerson optimal auction respectively. Let

$$\mu(n_w) = \frac{R^I(n_w) - R^{II}(n_w)}{R^M(n_w) - R^{II}(n_w)}$$

represent our measure for revenue comparison. The measure μ is similar to the measure ρ in that as n_w becomes large there is no mechanical force that would determine how μ behaves. Therefore, if μ is decreasing in n_w that could be validly interpreted as a decreasing advantage of the first price auction (over the second price auction) as the number of bidder increases. Using the same example as above, we can calculate the following values for μ :

$$\mu(1) = 0.1067, \mu(2) = 0.0653, \mu(3) = 0.0475, \mu(4) = 0.0330, \mu(5) = 0.0284, \mu(9) = 0.0213. \tag{8}$$

Beyond confirming our intuition that the revenue advantage is decreasing in the number of bidders, the example also shows that the difference may be rather modest compared to how much could be gained by running a revenue maximizing auction without resale. In other words, in larger auctions the exact auction format under resale seems to matter much less, than being able to choose an optimal auction with an optimal reserve and being able to prevent resale between the bidders.

6 Conclusions

I have studied auctions with resale when there are many bidders and derived existence and characterization results under the assumption that the winner of the initial auction makes the resale offer, which takes the form of a second price auction with a reserve price. I have shown that the symmetrization result of Hafalir and Krishna (2008) does not hold when there are more than two bidders and a strong bidder is more likely to win than a weak bidder in the initial auction. I also prove that while complete symmetrization does not take place, but the bid distributions are more symmetric in the case with resale than in the case without and thus resale works toward symmetrization, even if it does not go all the way. Numerical simulations suggest that the more bidders there are the more similar the allocation to the benchmark case without resale and thus the more asymmetric the bid

[19] This argument relies on the fact that when there are many bidders and only one object, then the bid each person makes converges to his valuation. A similar insight can be gained from the case where also the number of objects goes up. In this case, the price converges to the Walrasian equilibrium price, and the allocation of the auction becomes efficient in the limit, so resale does not take place.

distributions and winning probabilities are between strong and weak bidders. We also show in an example that the revenue advantage of first price auctions over second price auctions is positive, but decreasing in the number of bidders. Future research should shed light on whether one can derive more general comparative statics results as the number of bidders change. Another open question is to what extent changing the resale mechanism would change our results.

7 Appendix

Proof of Lemma 3:

Proof. I prove that even a large deviation in the auction is not profitable for any bidder. I start with the incentive problem of the strong bidder with type $\alpha_s(b)$ when he considers bidding $\widehat{b}$. Our goal is to show that $\frac{\partial}{\partial \alpha_s(b)} \frac{\partial U}{\partial \widehat{b}} \geq 0$, which implies that the second order conditions hold globally for the strong bidder.

Case 1: Let $\widehat{b} < b$ first. Then this strong bidder buys in the resale stage with positive probability from weak buyers. This happens if a weak buyer wins and his type is less than $r^{-1}(\alpha_s(b))$, but larger than $\alpha_w(\widehat{b})$. Moreover, if he loses against a type $x \in (\alpha_w(\widehat{b}), \alpha_w(b))$ of a weak bidder, then for the strong bidder with type $\alpha_s(b)$ to be able to buy at the resale stage it must hold that the highest other strong type does not bid more than $b_w(x)$ or, in other words, that the highest other strong type is less than $\alpha_s(b_w(x))$. The utility of the strong bidder is then

$$U_s(\alpha_s(b), \widehat{b}) = F_s^{n_s-1}(\alpha_s(\widehat{b}))F_w^{n_w}(\alpha_w(\widehat{b}))[\alpha_s(b) - \widehat{b}]+$$

$$+\int_{\alpha_w(\widehat{b})}^{\alpha_w(b)} n_w F_w^{n_w-1}(x) f_w(x) F_s^{n_s-1}(\alpha_s(b_w(x)))\{\alpha_s(b) - E[\max\{r(x), v_s^2\} \mid v_s^2 \leq \alpha_s(b_w(x))]\}dx+$$

$$+F_s^{n_s-1}(\alpha_s(b)) \int_{\alpha_w(b)}^{r^{-1}(\alpha_s(b))} n_w F_w^{n_w-1}(x) f_w(x) \widetilde{U}_s(\alpha_s(b), x)dx.$$

Therefore,

$$\frac{\partial U_s}{\partial \widehat{b}} = (n_s - 1)F_s^{n_s-2}(\alpha_s(\widehat{b}))f_s(\alpha_s(\widehat{b}))\alpha_s'(\widehat{b})F_w^{n_w}(\alpha_w(\widehat{b}))[\alpha_s(b) - \widehat{b}]+$$

$$+n_w F_w^{n_w-1}(\alpha_w(\widehat{b}))f_w(\alpha_w(\widehat{b}))\alpha_w'(\widehat{b})F_s^{n_s-1}(\alpha_s(\widehat{b}))\{E[\max\{r(\alpha_w(\widehat{b})), v_s^2\} \mid v_s^2 \leq \alpha_s(\widehat{b})] - \widehat{b}\}.$$

From the last formula it follows in a straightforward manner that

$$\frac{\partial}{\partial \alpha_s(b)} \frac{\partial U_s}{\partial \widehat{b}} > 0,$$

which concludes the proof for the first case. The intuition for this result is fairly straightforward: if a strong bidder just overtakes a weak bidder (who is the high bidder) by bidding less than his equilibrium bid, then he will surely buy the object in the resale stage and pays an expected amount of $E[\max\{r(\alpha_w(\widehat{b})), v_s^2\} \mid v_s^2 \leq \alpha_s(\widehat{b})]$. This quantity is independent of the real type $\alpha_s(b)$, so all types have the same incentive to bid slightly higher. However, when overtaking another strong bidder, the effective gain is $\alpha_s(b)$, since upon losing against a strong bidder there is never any buying opportunity in the resale stage. Obviously, this value is just equal to the valuation and thus buyers with higher valuation have more incentives to increase their bids.

Case 2: Let $\widehat{b} > b$ but $r(\alpha_w(\widehat{b})) < \alpha_s(b)$. In this case the high bidder is still buying from a weak winner at the resale stage, but also starts *selling* to other strong bidders. Let $r_s(\alpha_s(b), \widehat{b})$ denote the reserve price set by a strong bidder if his type is $\alpha_s(b)$ and he bid $\widehat{b}$ in the initial auction. Then the utility of the strong bidder can be written as

$$U_s(\alpha_s(b), \widehat{b}) = F_s^{n_s-1}(r_s(\alpha_s(b), \widehat{b}))F_w^{n_w}(\alpha_w(\widehat{b}))[\alpha_s(b) - \widehat{b}]+$$

$$+F_s^{n_s-1}(\alpha_s(b)) \int_{\alpha_w(\widehat{b})}^{r^{-1}(\alpha_s(b))} n_w F_w^{n_w-1}(x) f_w(x) \widetilde{U}_s(\alpha_s(b), x)dx+$$

$$+F_w^{n_w}(\alpha_w(\widehat{b})) \int_{r_s(\alpha_s(b),\widehat{b})}^{\alpha_s(\widehat{b})} (n_s-1)F_s^{n_s-2}(x) f_s(x)(E[\max\{r_s(\alpha_s(b), \widehat{b}), v_s^3\} \mid v_s^2 = x] - \widehat{b})dx.$$

When taking a derivative with respect to $\widehat{b}$ one can use the envelope theorem by invoking that $\frac{\partial U_s}{\partial r_s} = 0$ and thus the indirect effect that enters through the dependence of r_s on $\widehat{b}$ can be neglected. Therefore,

$$\frac{\partial U_s}{\partial \widehat{b}} =$$

$$n_w F_w^{n_w-1}(\alpha_w(\widehat{b})) f_w(\alpha_w(\widehat{b}))\alpha_w'(\widehat{b})[F_s^{n_s-1}(r_s(\alpha_s(b), \widehat{b}))\alpha_s(b) - F_s^{n_s-1}(\alpha_s(b))\widetilde{U}_s(\alpha_s(b), \alpha_w(\widehat{b}))]+$$

$$+n_w F_w^{n_w-1}(\alpha_w(\widehat{b})) f_w(\alpha_w(\widehat{b}))\alpha_w'(\widehat{b})*$$

$$*\int_{r_s(\alpha_s(b),\widehat{b})}^{\alpha_s(\widehat{b})} (n_s-1)F_s^{n_s-2}(x) f_s(x) E[\max\{r_s(\alpha_s(b), \widehat{b}), v_s^3\} \mid v_s^2 = x]dx+$$

$$+F_w^{n_w}(\alpha_w(\widehat{b}))\alpha_s'(\widehat{b})(n_s-1)F_s^{n_s-2}(\alpha_s(\widehat{b})) f_s(\alpha_s(\widehat{b}))E[\max\{r_s(\alpha_s(b), \widehat{b}), v_s^3\} \mid v_s^2 = \alpha_s(\widehat{b})] -$$

$$-\frac{\partial}{\partial \widehat{b}}(F_w^{n_w}(\alpha_w(\widehat{b}))F_s^{n_s-1}(\alpha_s(\widehat{b}))).$$

To proceed, note that

$$\frac{\partial}{\partial b}\frac{\partial}{\partial \widehat{b}}(F_w^{n_w}(\alpha_w(\widehat{b}))F_s^{n_s-1}(\alpha_s(\widehat{b}))) = 0.$$

Moreover, it holds at $r_s = r_s(\alpha_s(b), \widehat{b})$ that

$$\frac{\partial}{\partial r_s}\{F_s^{n_s-1}(r_s(\alpha_s(b), \widehat{b}))\alpha_s(b)+$$

$$+\int_{r_s(\alpha_s(b),\widehat{b})}^{\alpha_s(\widehat{b})} (n_s-1)F_s^{n_s-2}(x) f_s(x) E[\max\{r_s(\alpha_s(b), \widehat{b}), v_s^3\} \mid v_s^2 = x]\} = 0,$$

because it is optimal to choose $r_s(\alpha_s(b), \widehat{b})$ in the given situation by construction. Therefore,

$$\frac{\partial}{\partial b}\frac{\partial U_s}{\partial \widehat{b}} =$$

$$n_w F_w^{n_w-1}(\alpha_w(\widehat{b})) f_w(\alpha_w(\widehat{b}))\alpha_w'(\widehat{b})[F_s^{n_s-1}(r_s(\alpha_s(b), \widehat{b}))\alpha_s'(b) - \frac{\partial}{\partial b}F_s^{n_s-1}(\alpha_s(b))\widetilde{U}_s(\alpha_s(b), \alpha_w(\widehat{b}))].$$

Finally,

$$F_s^{n_s-1}(\alpha_s(b))\widetilde{U}_s(\alpha_s(b),\alpha_w(\widehat{b})) = \int_0^{\alpha_s(b)}(n_s-1)F_s^{n_s-2}(x)f_s(x)\{\alpha_s(b)-E[\max\{r(\alpha_w(\widehat{b})),x\}]\}dx$$

and thus

$$\frac{\partial}{\partial b}F_s^{n_s-1}(\alpha_s(b))\widetilde{U}_s(\alpha_s(b),\alpha_w(\widehat{b})) = \alpha_s'(b)F_s^{n_s-1}(\alpha_s(b)).$$

Putting everything together yields that

$$\frac{\partial}{\partial b}\frac{\partial U_s}{\partial \widehat{b}} = n_w F_w^{n_w-1}(\alpha_w(\widehat{b}))f_w(\alpha_w(\widehat{b}))\alpha_w'(\widehat{b})\alpha_s'(b)[F_s^{n_s-1}(r_s(\alpha_s(b),\widehat{b}))-F_s^{n_s-1}(\alpha_s(b))] > 0,$$

which concludes the proof for Case 2.

Case 3: Let $\widehat{b} > b$ and $r(\alpha_w(\widehat{b})) > \alpha_s(b)$, but $\alpha_w(\widehat{b}) < \alpha_s(b)$. In this case the situation is simplified, our strong bidder sells to other strong bidders at the resale stage and does not have any trade with the weak bidders. Therefore, one can write down a simplified version of the Case 2 utility function (line 2 from above is now missing) and then conduct a similar analysis to above to conclude that the cross partial has the required sign. Specifically, the utility function becomes:

$$U_s(\alpha_s(b),\widehat{b}) = F_s^{n_s-1}(r_s(\alpha_s(b),\widehat{b}))F_w^{n_w}(\alpha_w(\widehat{b}))[\alpha_s(b)-\widehat{b}]+$$

$$+F_w^{n_w}(\alpha_w(\widehat{b}))\int_{r_s(\alpha_s(b),\widehat{b})}^{\alpha_s(\widehat{b})}(n_s-1)F_s^{n_s-2}(x)f_s(x)(E[\max\{r_s(\alpha_s(b),\widehat{b}),v_s^3\}\mid v_s^2=x]-\widehat{b})dx.$$

Since this is a simplified version of the case 2 utility function, one can thus perform the exact same steps as above, which leads to the exact same conclusion of the single-crossing property of U_S.

Case 4: Let $\widehat{b} > b$ and $\alpha_w(\widehat{b}) > \alpha_s(b)$. In this case our strong bidder sells with positive probability to other strong bidders at the resale stage and if $\alpha_w(\widehat{b}) - \alpha_s(b)$ is large enough, then *may sell to weak bidders* as well. He will never be able to buy in the resale stage from either the strong or the weak bidders, because $\alpha_w(\widehat{b}), \alpha_s(\widehat{b}) > \alpha_s(b)$ and thus losing with bid $\widehat{b}$ means that a bidder with a higher valuation (than $\alpha_s(b)$) won the original auction. For the rest of discussions for Case 4 I concentrate on the case when he sells with positive probability to both strong and weak bidders, otherwise (if he does not transact with the weak bidders at all) we are back to case 3 with the exact same utility function, and thus the analysis would be exactly the same.

In this case the deviating seller posts a resale reserve price $r_s(\alpha_s(b),\widehat{b}) < \alpha_s(\widehat{b}), \alpha_w(\widehat{b})$. This resale price follows from the optimization problem of a seller who faces weak bidders and strong bidders as well, and chooses optimally. The only important thing to notice about the (out of equilibrium) object r_s is that it is strictly increasing in the real type $\alpha_s(b)$.[20] Let v^1 and v^2 denote the highest and second highest types among all other (n_s+n_w-1) bidders, and let b^1 denote the highest bid among all other bidders. By bidding $\widehat{b}$, the probability of winning is $F_w^{n_w}(\alpha_w(\widehat{b}))F_s^{n_s-1}(\alpha_s(\widehat{b}))$ and thus the overall payment is $\widehat{b}F_w^{n_w}(\alpha_w(\widehat{b}))F_s^{n_s-1}(\alpha_s(\widehat{b}))$. If all other bidders have types less than $r_s(\alpha_s(b),\widehat{b})$, then the deviating strong type wins, and

[20] This follows, because a buyer with a higher valuation facing the same distribution (as determined by the bid $\widehat{b}$) of potential buyers at the resale stage will always have an incentive to set a higher reserve as straightforward calculations would show.

he will not resell the object and thus his utility (from consumption) multiplied by the probability of this event is $F_w^{n_w}(r_s(\alpha_s(b),\widehat{b}))F_s^{n_s-1}(r_s(\alpha_s(b),\widehat{b}))\alpha_s(b)$. The final component of the utility function is the revenue from reselling the object. Such a resale occurs with probability $F_w^{n_w}(\alpha_w(\widehat{b}))F_s^{n_s-1}(\alpha_s(\widehat{b}))-F_w^{n_w}(r_s(\alpha_s(b),\widehat{b}))F_s^{n_s-1}(r_s(\alpha_s(b),\widehat{b}))$. It is convenient to write this last chunk of the utility as an integral. If the highest bid of the other bidders b^1 is equal to a value $\widetilde{b}$, then the expected resale price is equal to $E[\max\{r_s(\alpha_s(b),\widehat{b}),v^2\}\mid b^1=\widetilde{b}]$. This holds, because the resale is organized as a second price auction with reserve r_s. Let $f^1(\widetilde{b})$ denote the density of the highest other bid at $\widetilde{b}$. (Since the exact value of this density does not affect our calculations, I do not calculate it.)

Putting those three components of utility function together yields that

$$U_s(\alpha_s(b),\widehat{b})=-\widehat{b}F_w^{n_w}(\alpha_w(\widehat{b}))F_s^{n_s-1}(\alpha_s(\widehat{b}))+F_w^{n_w}(r_s(\alpha_s(b),\widehat{b}))F_s^{n_s-1}(r_s(\alpha_s(b),\widehat{b}))\alpha_s(b)+$$
$$+\int_{\min\{b_s(r_s(\alpha_s(b),\widehat{b}))),b_w(r_s(\alpha_s(b),\widehat{b})))\}}^{\widehat{b}} f^1(\widetilde{b})E[\max\{r_s(\alpha_s(b),\widehat{b}),v^2\}\mid b^1=\widetilde{b}]d\widetilde{b}.$$

It is in order to explain the two bounds of the integral. Recall that the integral measures the expected value of the resale revenues. If the highest other bid b^1 was larger than $\widehat{b}$ then our strong bidder would not be able to purchase the object in the first place, and thus he would not be able to resell it either. Second, the smallest type who ever buys at the resale stage from the deviating strong bidder is equal to $r_s=r_s(\alpha_s(b),\widehat{b}))$, and thus the smallest bid any such type makes in equilibrium is equal to $\min\{b_s(r_s),b_w(r_s)\}=b_s(r_s)$. Therefore, the utility can be rewritten as[21]

$$U_s(\alpha_s(b),\widehat{b})=-\widehat{b}F_w^{n_w}(\alpha_w(\widehat{b}))F_s^{n_s-1}(\alpha_s(\widehat{b}))+F_w^{n_w}(r_s(\alpha_s(b),\widehat{b}))F_s^{n_s-1}(r_s(\alpha_s(b),\widehat{b}))\alpha_s(b)+$$
$$+\int_{b_s(r_s)}^{\widehat{b}} f^1(\widetilde{b})E[\max\{r_s(\alpha_s(b),\widehat{b}),v^2\}\mid b^1=\widetilde{b}]d\widetilde{b}.$$

To obtain the key cross partial derivative $\frac{\partial^2 U_s}{\partial b\partial\widehat{b}}$, note that $\widehat{b}$ affects U_s in two ways: indirectly through r_s and directly. The optimality condition that determines the value of $r_s(\alpha_s(b),\widehat{b})$ has a first order condition that is equivalent to $\frac{\partial U_s}{\partial r_s}=0$, and thus the indirect effect cancels[22]. Using this observation and the observation above that $\frac{\partial r_s(\alpha_s(b),\widehat{b})}{\partial b}>0$, one obtains that[23]

$$\frac{\partial^2 U_s}{\partial b\partial\widehat{b}}=f^1(\widehat{b})E[\max\{r_s(\alpha_s(b),\widehat{b}),v^2\}\mid b^1=\widehat{b}]>0,$$

which concludes the proof for Case 4, the last case for possible deviations of the strong bidder.

Now, I turn to the analysis of the weak bidders' problem. Denote his type by $\alpha_w(b)$ and his bid $\widehat{b}$, again restricting attention to the case where $\widehat{b}-b$ is small in absolute value. By

[21] Here I use the bid the others make as the dummy variable for integration, which is different from the treatment in the rest of the analysis. This allows me to treat the two cases where a strong or a weak bidder may buy at the resale stage in a unified notation, a need that did not arise in the three other cases.

[22] I assume a common support for the two groups of bidders (interval $[0,1]$), so it is always the case that the optimal resale (reserve) price is interior, since otherwise the resale profit would be zero (when r is set to high) or negative (when r is set zero). Then the first order condition for an interior optimum (using our other assumptions on differentiability) imply that $\frac{\partial U_s}{\partial r_s}=0$.

[23] The derivative with respect to $\widehat{b}$ is not well defined when the reseller is just indifferent between serving just the strong bidders or also the weak bidders. The main single crossing argument is however not affected, since when integrating such a non-existence at one point does not cause any problem.

the above argument such a bidder chooses a reserve price $r(\alpha_w(b))$ regardless of $\widehat{b}$. Then he will own the object eventually if and only if all the weak bidders have type less than $\alpha_w(\widehat{b})$ and all the strong bidders have type less than $r(\alpha_w(b))$. He will resell the object if the highest type of the strong bidders is between $\alpha_s(\widehat{b})$ and $r(\alpha_w(b))$ and all the weak bidders have types lower than $\alpha_w(\widehat{b})$. Let $R_b(\alpha_w(b), x)$ denote the expected revenue from resale if a reserve price $r(\alpha_w(b))$ is set and the highest type among the strong buyers is $x \geq r(\alpha_w(b))$ and thus resale occurs. Formally,

$$R_b(\alpha_w(b), x) = E[\max\{r(\alpha_w(b)), v_s^2\} \mid v_s^1 = x].$$

Note, that

$$R_b(\alpha_w(b), \alpha_s(b)) = \widetilde{r}(b). \tag{9}$$

Again, if a weak bidder just barely wins against a strong bidder, then his effective valuation is his expected resale price $\widetilde{r}(b)$. Let us also define the expected resale price $\tau(b)$ if a weak bidder wins, beating another weak bidder with the same type $\alpha_w(b)$, but resale takes place to a lower bidder, the strong bidder with the highest type. Formally,

$$\tau(b) = E[\max\{v_s^2, r(\alpha_w(b))\} \mid v_s^1 \in [r(\alpha_w(b)), \alpha_s(b)]] =$$

$$= \frac{\int_{r(\alpha_w(b)}^{\alpha_s(b)} n_s F_s^{n_s-1}(x) f_s(x) R_b(\alpha_w(b), x) dx}{F_s^{n_s}(\alpha_s(b)) - F_s^{n_s}(r(\alpha_w(b)))}.$$

The utility function of the weak bidder can be written as

$$U_w = F_s^{n_s}(r(\alpha_w(b))) F_w^{n_w-1}(\alpha_w(\widehat{b}))[\alpha_w(b) - \widehat{b}] +$$

$$+ F_w^{n_w-1}(\alpha_w(\widehat{b})) \int_{r(\alpha_w(b)}^{\alpha_s(\widehat{b})} n_s F_s^{n_s-1}(x) f_s(x)(R_b(\alpha_w(b), x) - \widehat{b}) dx.$$

We have then the same four cases as above:

Case 1: Let $\widehat{b} < b$ and $\alpha_s(\widehat{b}) \leq \alpha_w(b)$. In this case our weak bidder buys from other weak bidders and does not trade with the strong bidders at the resale stage.

Case 2: Let $\widehat{b} < b$, but $\alpha_s(\widehat{b}) > \alpha_w(b)$. In this case our weak bidder buys from other weak bidders and may sell to the strong bidders at the resale stage.

Case 3: Let $\widehat{b} \geq b$, but $r_w(\alpha_w(b), \widehat{b}) > \alpha_w(\widehat{b})$. In this case our weak bidder does not trade with other weak bidders and sells to the strong bidders at the resale stage.

Case 4: Let $\widehat{b} \geq b$, and $r_w(\alpha_w(b), \widehat{b}) \leq \alpha_w(\widehat{b})$. In this case our weak bidder sells to other weak bidders and sells to the strong bidders at the resale stage.

The proof of the second order condition is the same for these four cases as above with the obvious changes in notation, so it is omitted. ■

Proof of Proposition 1:

Proof. Let us start the proof by defining $\widehat{\tau}(b)$ as

$$F_s^{n_s}(r(\alpha_w))\alpha_w + (F_s^{n_s}(\alpha_s) - F_s^{n_s}(r(\alpha_w)))\tau = F_s^{n_s}(\alpha_s)\widehat{\tau}.$$

Then one can rewrite equation (6) as

$$n_s F_s^{n_s-1}(\alpha_s(b)) f_s(\alpha_s(b)) \alpha_s'(b) F_w^{n_w-1}(\alpha_w(b))[\widetilde{r}(b) - b] +$$

$$+(n_w - 1) F_w^{n_w-2}(\alpha_w(b)) f_w(\alpha_w(b)) \alpha_w'(b) F_s^{n_s}(\alpha_s)(\widehat{\tau} - b) = F_s^{n_s}(\alpha_s(b)) F_w^{n_w-1}(\alpha_w(b)).$$

After simplifications this formula is equivalent to

$$n_s f_s(\alpha_s)\alpha_s'(b)F_w(\alpha_w)[\widetilde{r}(b)-b]+(n_w-1)f_w(\alpha_w)\alpha_w'(b)F_s(\alpha_s)(\widehat{\tau}(b)-b)=F_s(\alpha_s)F_w(\alpha_w). \tag{10}$$

A similar simplification applied to formula (3) implies that

$$n_w f_w(\alpha_w)\alpha_w'(b)F_s(\alpha_s(b))[\widetilde{r}(b)-b]+(n_s-1)f_s(\alpha_s)\alpha_s'(b)F_w(\alpha_w)[\alpha_s-b]=F_s(\alpha_s)F_w(\alpha_w). \tag{11}$$

Before analyzing the above system in more details, note that for all b in the solution

$$\alpha_s \geq \widetilde{r} \geq \tau \geq \widetilde{\tau} \geq \alpha_w \tag{12}$$

with equality if and only if $\alpha_s = \alpha_w$. To see this first note that at $b = \overline{b} < 1$ it holds that $\alpha_s = \alpha_w = \widetilde{r} = \tau = \widetilde{\tau} = 1$ by definition. Then equations (10) and (11) imply that

$$f_s(1)F_w(1)\alpha_s'(\overline{b}) = f_w(1)F_s(1)\alpha_w'(\overline{b}).$$

By assumption

$$f_s(1)F_w(1) > f_w(1)F_s(1)$$

and thus

$$\alpha_s'(\overline{b}) < \alpha_w'(\overline{b}),$$

implying that for some $\varepsilon > 0$ it holds that for all $b \in (\overline{b}-\varepsilon, \overline{b})$

$$\alpha_s(b) > \alpha_w(b).$$

A similar argument implies that for any $b < \overline{b}$

$$\alpha_s(b) = \alpha_w(b) > b \Rightarrow \alpha_s'(\overline{b}) < \alpha_w'(\overline{b})$$

and thus $\alpha_s(b) \geq \alpha_w(b)$ must hold for all b in the solution of system (10), (11) as long as $\alpha_s(b) > b$ holds. From this (12) follows by the way functions $\widetilde{r}, \tau, \widetilde{\tau}$ were constructed.

Next, note that as long as

$$n_s[\widetilde{r}(b)-b] > (n_s-1)[\alpha_s-b] \tag{13}$$

and

$$n_w[\widetilde{r}(b)-b] > (n_w-1)(\widehat{\tau}(b)-b) \tag{14}$$

hold, the system satisfies the Lipschitz property and thus there is a unique solution on $[b, \overline{b}]$. It is easy to see that $\alpha_s', \alpha_w' > 0$ must hold as long as (13) and (14) hold. Moreover, (14) follows directly from (12).

Therefore, I only need to show that (13) holds for all $b > 0$ in the relevant range where $\alpha_s(b) > b$. Let $b = b^*$ be such that

$$n_s[\widetilde{r}(b)-b] = (n_s-1)[\alpha_s-b]$$

and for all $b > b^*$ condition (13) holds. Note, that (10), (11) imply that at $b = b^*$ it holds that $\alpha_w'(b^*) = 0$. Then $\alpha_w'(b^*) = 0$ and (10) imply that $\alpha_s'(b^*) > 0$ holds. Also, it must hold that

$$n_s\widetilde{r}'(b^*) \geq (n_s-1)\alpha_s'(b^*)+1, \tag{15}$$

since $n_s[\widetilde{r}(b^*) - b^*] = (n_s - 1)[\alpha_s(b^*) - b^*]$ and for all $b > b^*$ it holds that $n_s[\widetilde{r}(b) - b] > (n_s - 1)[\alpha_s - b]$.

Using the definition of $\widetilde{r}$ it holds that

$$\widetilde{r}F_s^{n_s-1}(\alpha_s) = F_s^{n_s-1}(r(\alpha_w))r(\alpha_w) + \int_{r(\alpha_w)}^{\alpha_s} (n_s - 1)F_s^{n_s-2}(x)f_s(x)xdx.$$

Using this formula and that $n_s[\widetilde{r}(b^*) - b^*] = (n_s - 1)[\alpha_s(b^*) - b^*]$ implies that

$$n_s\widetilde{r}'(b^*)F_s(\alpha_s(b^*)) = \alpha_s'(b^*)(n_s - 1)(\alpha_s(b^*) - b^*)f_s(\alpha_s(b^*)).$$

Using the assumption that $\alpha_w'(b^*) = 0$ implies through (11) that

$$\alpha_s'(b^*)(n_s - 1)(\alpha_s(b^*) - b^*)f_s(\alpha_s(b^*)) = F_s(\alpha_s),$$

and thus

$$n_s\widetilde{r}'(b^*) = 1.$$

But this last formula contradicts with (15), because $\alpha_s'(b^*) > 0$ as we established it above.

We have thus proven that the Lipschitz conditions cannot be violated as long as $\alpha_s(b) > b$ holds and that for all such b it also holds that $\alpha_s'(b), \alpha_w'(b) > 0$. Therefore, for all $\overline{b}$ two things can happen. Either one can find a value $\widetilde{b} \geq 0$ such that $\alpha_s(\widetilde{b}) = \widetilde{b}$ and for all $b > \widetilde{b}$ it holds that $\alpha_s'(b), \alpha_w'(b) > 0$ and $\alpha_s(b) > b$. If that is not possible, then it must hold that $\alpha_s(0) > 0$. If one can show that there exists a $\overline{b}$ such that the first case occurs with $\widetilde{b} = 0$, then our proof is complete.

Let $\alpha_s(b, \overline{b})$ and $\alpha_w(b, \overline{b})$ denote the solution of our system for a given value of the end condition. Let us define

$$\overline{b}^* = \inf_{\overline{b}}\{\alpha_s(\widetilde{b}, \overline{b}) = \widetilde{b} \text{ for some } \widetilde{b} > 0\}.$$

From our discussion above, it follows that $0 \leq \overline{b}^* \leq 1$.[24] The rest of the proof shows that $\alpha_s(0, \overline{b}^*) = 0$ and $\alpha_s(b, \overline{b}^*) > b$ for all $b \in (0, \overline{b}^*]$, which concludes the proof, since it shows that α_s satisfies all the relevant boundary conditions. I prove this statement by ruling out two possibilities. First, suppose that $\alpha_s(0, \overline{b}^*) > 0$ and $\alpha_s(b, \overline{b}^*) > b$ for all $b \in (0, \overline{b}^*]$. In this case the definition of $\overline{b}^*$ implies that for all $\varepsilon > 0$ it must hold that there exists $\overline{b} \in (\overline{b}^*, \overline{b}^* + \varepsilon)$ such that there exists $\widetilde{b} > 0$ such that $\alpha_s(\widetilde{b}, \overline{b}) = \widetilde{b}$. I use this statement to obtain a contradiction that rules out this first case. To do this let us construct a sequence $\{\overline{b}^i\} \searrow \overline{b}^*$, and the corresponding sequence of crossing points $\{\widetilde{b}^i\}$. By construction it holds for all i that

$$\alpha_s(\widetilde{b}^i, \overline{b}^i) = \widetilde{b}^i. \tag{16}$$

I start with some useful observations. Since function $\alpha_s(b, \overline{b}^*) - b$ is continuous in b, therefore by Weierstrass's theorem it takes its minimum, which I denote by $d > 0$. I have shown above that when $\alpha_s > b$ for all $b \geq 0$ (as it holds in this first case), then Lipschitz continuity of α_s holds, and thus for all b it holds that $\alpha_s(b, \overline{b}^*)$ is continuous in the second

[24] This follows follows, because $\alpha_s(1, 1) = 1$, and thus the property that $\{\alpha_s(b, \overline{b}) > b$ for all $b \in [0, \overline{b}]\}$ does not hold for $\overline{b} = 1$. The property holds when $\overline{b} = 0$, since then $\alpha_s(0) = 1 > 0$ and the function α_s is not even defined for any $b > 0$.

variable. Moreover, it also follows that $\alpha_s(b, \overline{b}^*)$ is uniformly continuous in the second variable, i.e. for all $\varepsilon > 0$ there exists a δ such that

$$\mid \overline{b} - \overline{b}^* \mid \leq \delta \Rightarrow \sup_{b \in [0, \max\{\overline{b}^*, \overline{b}\}]} \mid \alpha_s(b, \overline{b}^*) - \alpha_s(b, \overline{b}) \mid \leq \varepsilon.$$

This uniform convergence property implies that there exists an I, such that for all $i > I$ it holds that $\mid \alpha_s(\widetilde{b}^i, \overline{b}^*) - \alpha_s(\widetilde{b}^i, \overline{b}^i) \mid \leq d/2$ and thus (using (16))

$$\alpha_s(\widetilde{b}^i, \overline{b}^*) \leq \widetilde{b}^i + d/2. \tag{17}$$

On the other hand the definition of d implies that

$$\alpha_s(\widetilde{b}^i, \overline{b}^*) \geq \widetilde{b}^i + d,$$

which contradicts (17), and thus rules out this possibility.

To complete the proof, we need to rule out the second case where there exists some $\widetilde{b} > 0$ such that $\alpha_s(\widetilde{b}, \overline{b}^*) = \widetilde{b}$. To do this, I establish that if such a $\widetilde{b}$ exists, then there exists a small enough (positive) ε, such that for all $\overline{b} \in (\overline{b}^* - \varepsilon, \overline{b}^*)$ there exists a $b > 0$ such that $\alpha_s(b, \overline{b}) = b$, and thus the definition of $\overline{b}^*$ would be contradicted. So, let us construct a sequence $\{\overline{b}^k\} \nearrow \overline{b}^*$ and suppose that $\alpha_s(b, \overline{b}^k) > b$ for all $b > 0$ for all k. We will show that the existence of such a sequence leads to a contradiction, which completes then the proof for this second case.

Since α_s is Lipschitz continuous when $b > \widetilde{b}$ and $\overline{b} = \overline{b}^*$, and (by construction) continuous in the first variable for any $\overline{b}$, it follows that for any $\varepsilon > 0$ there exists a $K(\varepsilon)$, such that for all $k > K(\varepsilon)$ it holds that $\alpha_s(\widetilde{b}, \overline{b}^k) \in (\widetilde{b}, \widetilde{b} + \varepsilon)$.[25] To continue, let us use the system (10), (11) together with formula (12). Let us define the following variables:

$$A = n_s f_s(\alpha_s) F_w(\alpha_w)[\widetilde{r}(b) - b],$$

$$B = (n_w - 1) f_w(\alpha_w) F_s(\alpha_s)(\widehat{r}(b) - b),$$

$$C = F_s(\alpha_s) F_w(\alpha_w),$$

$$D = (n_s - 1) f_s(\alpha_s) F_w(\alpha_w)[\alpha_s - b],$$

and

$$E = n_w f_w(\alpha_w) F_s(\alpha_s(b))[\widetilde{r}(b) - b].$$

Then solving the system (10), (11) yields that

$$\alpha_s'(b, \overline{b}) = \frac{C}{A + \frac{A-D}{E-B} B}.$$

[25] To see this note that, for all $b > \widetilde{b}$ function α_s is continuous in the first variable at $(b, \overline{b}^*)$, since it solves a differential equation (in b), and is thus continuous. Therefore, there exists $\delta > 0$ such that for all $b \in (\widetilde{b}, \widetilde{b} + \delta)$ it holds that $\alpha_s(b, \overline{b}^*) \leq \alpha_s(\widetilde{b}, \overline{b}^*) + \varepsilon/3 = \widetilde{b} + \varepsilon/3$. Since for all $b > \widetilde{b}$, Lipshitz continuity holds for all $\overline{b} \leq \overline{b}^*$, therefore at such a b the function α_s is continuous in the second variable. Take any $\widehat{b} \in (\widetilde{b}, \widetilde{b} + \delta)$. Then $\exists \widehat{K}$ such that if $k \geq \widehat{K}$ then $\alpha_s(\widehat{b}, \overline{b}^k) \leq \alpha_s(\widehat{b}, \overline{b}^k) + \varepsilon/3 \leq \widetilde{b} + 2\varepsilon/3$, by the way $\widehat{b}$ was constructed. Now, suppose that an appropriate $K(\varepsilon)$ does not exist. Then $\exists k \geq \widehat{K}$ such that $\alpha_s(\widetilde{b}, \overline{b}^k) \geq \widetilde{b} + \varepsilon > \alpha_s(\widehat{b}, \overline{b}^k)$. However, α_s is increasing in b and $\widehat{b} > \widetilde{b}$, yielding a contradiction.

Using (12) implies that
$$\frac{B}{E} \leq \frac{n_w - 1}{n_w},$$
or $\frac{B}{E-B} \leq n_w - 1$. Therefore,

$$\alpha_s'(b, \overline{b}) \geq \frac{C}{A + (A - D)(n_w - 1)} \geq \frac{C/n_w}{A}. \tag{18}$$

Now, take any $\varepsilon > 0$ and let $k > K(\varepsilon)$ as defined above. Then using that $\widetilde{r} \leq \alpha_s$ implies that $0 < A(\widetilde{b}, \overline{b}^k) \leq n_s f_s(\alpha_s) F_w(\alpha_w)\varepsilon$. Thus if ε is chosen very small, then $\alpha_s'(\widetilde{b}, \overline{b}^k)$ can be made arbitrarily large, in particular much larger than a particular number greater than 1, let's say 2. By construction it also holds that $\alpha_s(\widetilde{b}, \overline{b}^k) \leq \widetilde{b} + \varepsilon$. If it holds that for all $b \in (\widetilde{b} - \varepsilon/2, \widetilde{b})$ that $\alpha_s'(\widetilde{b}, \overline{b}^k) \geq 2$ then it must be true that for some $b \in [\widetilde{b} - \varepsilon/2, \widetilde{b})$ that $\alpha_s(b, \overline{b}^k) = b$, providing the desired contradiction and concluding the proof.

For all $b \in (\widetilde{b} - \varepsilon/2, \widetilde{b})$ it holds that $C(b, \overline{b}^k) \geq F_s(\alpha_s(b, \overline{b}^k))F_w(\alpha_w(b, \overline{b}^k)) \geq F_s(b)F_w(b) \geq F_s(\widetilde{b} - \varepsilon/2)F_w(\widetilde{b} - \varepsilon/2) = \underline{C}$. Also,

$$A(b, \overline{b}^k) \leq n_s f_s(\alpha_s)F_w(\alpha_w)(\alpha_s(b, \overline{b}^k) - b) \leq (\alpha_s(b, \overline{b}^k) - b)\sup_x n_s f_s(x) = \overline{A}(\alpha_s(b, \overline{b}^k) - b) \tag{19}$$

holds.[26] This implies that
$$\alpha_s'(b, \overline{b}^k) \geq \frac{\underline{C}/n_w}{\overline{A}(\alpha_s(b, \overline{b}^k) - b)}. \tag{20}$$
If ε was chosen small enough in the first place, then
$$\frac{\underline{C}/n_w}{A(\widetilde{b}, \overline{b}^k)} \geq \frac{\underline{C}/n_w}{\overline{A}\varepsilon} > 2 \tag{21}$$

holds and thus $\alpha_s'(\widetilde{b}, \overline{b}^k) \geq 2$. Consequently, $(\alpha_s(b, \overline{b}^k) - b)$ is increasing in b at $(\widetilde{b}, \overline{b}^k)$. This implies that $\alpha_s(b, \overline{b}^k) - b) \leq \varepsilon$ if b is close enough (but less than) to $\widetilde{b}$. But as long as $\alpha_s(b, \overline{b}^k) - b) \leq \varepsilon$ holds and ε is chosen small enough, such that (21) holds, then using (20) it is still true that $\alpha_s'(b, \overline{b}^k) \geq \frac{\underline{C}/n_w}{\overline{A}(\alpha_s(b, \overline{b}^k) - b)} \geq \frac{\underline{C}/n_w}{\overline{A}\varepsilon} > 2$. Suppose that when decreasing b we leave the region where $\alpha_s'(b, \overline{b}^k) \geq 2$. By continuity at that critical value of b, denoted by b^*, it holds that $\alpha_s'(b^*, \overline{b}^k) = 2$. Since for all $b \in (b^*, \widetilde{b})$ it holds that $\alpha_s'(b, \overline{b}^k) \geq 2$ and thus $\alpha_s(b^*, \overline{b}^k) - b^* < \alpha_s(\widetilde{b}, \overline{b}^k) - \widetilde{b} \leq \varepsilon$. Therefore, (20) implies that

$$\alpha_s'(b^*, \overline{b}^k) \geq \frac{\underline{C}/n_w}{\overline{A}(\alpha_s(b^*, \overline{b}^k) - b^*)} > \frac{\underline{C}/n_w}{\overline{A}\varepsilon} > 2,$$

contradicting the starting assumption that $\alpha_s'(b^*, \overline{b}^k) = 2$. This implies that we never leave the region (as we decrease b starting from $\widetilde{b}$) where $\alpha_s'(b, \overline{b}^k) \geq 2$, which completes the proof.
■

[26] It follows from the continuity of f_s that $f_s(x)$ is bounded on interval $[0, 1]$, and thus its supremum is bounded.

8 Appendix 2

In this Appendix, I establish the result claimed in the Example of the main text, i.e. that in the three bidder auction considered in the Example, the asymmetry of bid distributions between the bidders is reduced by the presence of resale compared to the benchmark case of no resale. I assume that f_s is decreasing and that $\frac{F_w}{\sqrt[4]{x}}$ is increasing in x. Note, that these assumptions (together with the assumption that F_s/F_w is increasing) imply that $\frac{F_s(x)}{x}$ and $\frac{F_w(x)}{x}$ are decreasing in x.[27]

Evaluating (3) at $b = b_w(x)$ and using that when $n_s = 1$ then $\widetilde{r} = r$, one obtains that

$$b'_w(x) = \frac{(r(x) - b_w(x))2F_w(x)f_w(x)}{(F_w(x))^2}. \tag{22}$$

Letting $\eta = \alpha_s(b_w)$, and combining (22) with (6) (again evaluated at $b = b_w(x)$ and using that $\tau = \widetilde{r} = r$ when $n_s = 1$) imply that

$$\eta'(x) = \frac{f_w(x)}{F_w(x)f_s(\eta(x))} \frac{F_s(\eta(x))(r(x) - b_w(x)) + F_s(x)(r(x) - x)}{(r(x) - b_w(x))}. \tag{23}$$

Now, let us examine the situation with three bidders like above but with no resale. Let β_w, β_s denote the bid functions of a weak bidder and the strong, with inverse functions γ_w, γ_s and let $\omega(x) = \gamma_s(\beta_w(x))$. I show below that it must hold that $\eta(x) > \omega(x) > x$ for all $x \in (0, \overline{v})$ for all $x \in (0, \overline{v})$, which implies that there is less asymmetry in the distribution of bids produced by the two types of bidders when resale is allowed. Following the analysis of Maskin and Riley (2000) it is routine to establish that

$$\omega'(x) = \frac{f_w(x)}{F_w(x)} \frac{F_s(\omega(x))(2\omega(x) - x - \beta_w(x))}{f_s(\omega(x))(x - \beta_w(x))}, \tag{24}$$

and

$$\beta'_w(x) = \frac{(\omega(x) - \beta_w(x))2F_w(x)f_w(x)}{(F_w(x))^2}.$$

I prove at the end of this Appendix that $\eta''(1) > \omega''(1)$ and $\eta'(1) = w'(1)$. Thus, since $\eta(1) = \omega(1)$, it follows that $\eta(x) > w(x)$ for all $x \in (1 - \varepsilon, 1)$ for a low enough ε. Take the largest value $y < 1$ where $\eta(y) = \omega(y)$. By construction $\eta'(y) \geq \omega'(y)$ must hold, but I show that whenever $\eta(y) = \omega(y) = k$ holds it also holds that $\eta'(y) < \omega'(y)$, which yields contradiction and establishes the proof that no such point y exists and thus for all $x \in (0, 1)$ it holds that $\eta(x) > \omega(x)$. First, note that

$$\eta'(y) = \frac{f_w(y)}{F_w(y)f_s(k)}(F_s(k) + \frac{F_s(y)(r(y) - y)}{(r(y) - b_w(y))})$$

and

$$\omega'(y) = \frac{f_w(y)}{F_w(y)f_s(k)}(F_s(k) + \frac{F_s(k)(2k - 2y)}{(y - \beta_w(y))}).$$

Therefore, it is sufficient to prove that

$$\frac{F_s(y)(r(y) - y)}{(r(y) - b_w(y))} < \frac{F_s(k)(2k - 2y)}{(y - \beta_w(y))}. \tag{25}$$

[27] These assumptions are strong sufficient conditions, and I conjecture that they can be relaxed significantly without changing the results.

Next, note that $r(u) \in \arg\max_r r(F_s(\eta(u)) - F_s(r)) + u(F_s(r)$ and the first order condition implies that

$$F_s(\eta(u)) - F_s(r(u)) - f_s(r(u))(r(u) - u) = 0. \tag{26}$$

Since by assumption f_s is a decreasing function, therefore

$$0 = F_s(\eta(u)) - F_s(r(u)) - f_s(r(u))(r(u) - u) \leq f_s(r(u))(\eta(u) + u - 2r(u)$$

or $r(u) \leq \frac{\eta(u)+u}{2}$ holds and thus $r(y) \leq \frac{k+y}{2}$. Equation (22) implies that

$$r(y) - b_w(y) = \int_0^y (\frac{F_w(u)}{F_w(y)})^2 r'(u)du.$$

Differentiating first order condition (26) by u yields

$$f_s(\eta)\eta'(u) = f_s(r)(2r'(u) - 1) + f_s'(r)(r(u) - u).$$

Since by assumption $f_s' \leq 0$, therefore the last equation implies that $r'(u) \geq \frac{1}{2}$ for all u. Moreover, since $\frac{F_w}{x}$ is decreasing in x it holds that for all $u \leq y$

$$(\frac{F_w(u)}{F_w(y)})^2 \geq \frac{u^2}{y^2}.$$

The above then imply that

$$r(y) - b_w(y) = \int_0^y (\frac{F_w(u)}{F_w(y)})^2 r'(u)du \geq \frac{y}{6}.$$

Also, we know it from Maskin and Riley (2000) that in an auction without resale the weak bidders bid more aggressively if they face a strong bidder than if they face only weak bidders.[28] Therefore,

$$\beta_w(y) \geq \frac{1}{(F_w(y))^2} \int_0^y 2F_w(u)f_w(u)udu.$$

Using that $\frac{(F_w(x))^2}{\sqrt{x}}$ is an increasing function it follows that if $y \geq u$, then $(\frac{F_w(u)}{F_w(y)})^2 \leq \frac{\sqrt{u}}{\sqrt{y}}$ and thus a first order stochastic dominance argument yields that

$$\beta_w(y) \geq \frac{1}{(F_w(y))^2} \int_0^y 2F_w(u)f_w(u)udu \geq \frac{1}{\sqrt{y}} \int_0^y \frac{\sqrt{u}}{2} udu = \frac{y}{3}. \tag{27}$$

Putting everything together and also using that $k > y$ yields that

$$\frac{F_s(y)(r(y) - y)}{(r(y) - b_w(y))} < \frac{F_s(k)(k - y)/2}{y/6} = \frac{3F_s(k)(k - y)}{y} =$$

$$= \frac{2F_s(k)(k - y)}{2y/3} \leq \frac{2F_s(k)(k - y)}{y - \beta_w(y)},$$

which concludes the proof.

[28] They only establish it for the case of two bidders, but extending their results is routine.

Proof that $\eta''(1) > \omega''(1)$ and $\eta'(1) = w'(1)$:

The first order conditions (23) and (24) imply that at the upper end of the support of valuations ($\overline{v} = 1$) it holds that $\omega'(1) = \eta'(1) = \frac{f_w(1)}{f_s(1)} < 1$, since $\omega(1) = \eta(1) = r(1) = 1$ holds. Also, for all $x \in (0,1)$ it holds that $x < r(x) < \eta(x)$ and thus $1 > r'(1) > \eta'(1) = \omega'(1)$.[29] Now, I show that $\eta''(1) > \omega''(1)$. Using (23) and (24), it is sufficient to show that at $x = 1$ it holds that

$$(\frac{F_s(\eta(x))(r(x) - b_w(x)) + F_s(x)(r(x) - x)}{(r(x) - b_w(x))})' > (\frac{F_s(\omega(x))(2\omega(x) - x - \beta_w(x))}{(x - \beta_w(x))})'.$$

Since $\eta'(1) < 1$ holds, it follows that at $x = 1$

$$(\frac{F_s(\eta(x))(r(x) - b_w(x)) + F_s(x)(r(x) - x)}{(r(x) - b_w(x))})' > (\frac{F_s(\eta(x))(2r(x) - x - b_w(x))}{(r(x) - b_w(x))})'.$$

Therefore, it is sufficient to establish that at $x = 1$

$$(\frac{F_s(\eta(x))(2r(x) - x - b_w(x))}{(r(x) - b_w(x))})' > (\frac{F_s(\omega(x))(2\omega(x) - x - \beta_w(x))}{(x - \beta_w(x))})',$$

which is equivalent to

$$(\frac{(2r(x) - x - b_w(x))}{(r(x) - b_w(x))})' > (\frac{(2\omega(x) - x - \beta_w(x))}{(x - \beta_w(x))})',$$

since $\omega'(1) = \eta'(1)$. This last inequality is equivalent to

$$\frac{1 - r'(1)}{1 - b_w(1)} < \frac{2(1 - \omega'(1))}{1 - \beta_w(1)}. \tag{28}$$

Since $r(1) = \eta(1) = 1$ and for all $x \in (0,1)$ it holds that $r(x) < \frac{1+\eta(x)}{2}$, thus it follows that $r'(1) > \frac{1+\eta'(1)}{2} = \frac{1+\omega'(1)}{2}$ and thus

$$\frac{1 - r'(1)}{1 - b_w(1)} < \frac{1 - \omega'(1)}{2(1 - b_w(1))}.$$

Therefore, inequality (28) is satisfied if

$$4(1 - b_w(1)) > 1 - \beta_w(1).$$

Also, (27) implies that $\beta_w(1) \geq 1/3$ and thus it is sufficient to prove that $b_w(1) < \frac{5}{6}$. To prove this last inequality, consider the problem of bidder 3 with type $v_3 = 1$ and note that by bidding $b_w(1)$ he surely gets the object at a price of $b_w(1)$. Suppose now, that he deviates and bids zero in the auction stage and obtains the object at the resale stage only. In this case, resale always takes place because the winning bidder offers a resale price less than 1 with probability 1 and thus one needs to only establish that bidder 3 pays less than 5/6 for the object in expectation. To estimate the expected resale price, note that if a bidder (other than 3) wins with type x then his resale offer is $r(x) \leq \frac{x+\eta(x)}{2} < \frac{x+1}{2}$. Since $\frac{F_w(x)}{x}$ is decreasing, thus distribution F_w is stochastically dominated by distribution $G(x) = x$. Let

[29] The inequalities are strict as long as $f_w(1) > 0$, which we assumed throughout.

W be the distribution function for the type of the winner when the strong bidder bids zero in the original auction. Then

$$W(x) = F_w(x)^2 \geq x^2,$$

and thus the expected price is estimated as

$$EP = \int_0^1 r(x)dW(x) < \int_0^1 \frac{x+1}{2} * 2xdx = \frac{5}{6},$$

which concludes the proof.

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