Heterogeneous Beliefs, Rare Disasters, and Asset Pricing

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Abstract

We illustrate the effects of heterogeneous beliefs about disasters on the equity premium and individual agents' trading activities. When investors disagree about the chances or severity of disasters, those optimistic investors may insure the pessimists against their disaster risk exposure. Due to the highly non-linear relationship between the consumption losses during a disaster and the risk premium, a small amount of risk sharing can significantly attenuate the effect that disasters have on the equity premium. Thus, the equity premium will remain low even when the economy is predominantly occupied by pessimistic investors, but jump up following a disaster. The effects of risk sharing become stronger when the differences in beliefs are large, or when the optimistic agents also have lower risk aversion. Other interesting predictions of the model include a non-monotonic relationship between the equity premium and the size of the disaster insurance market, as well as a negative relationship between the equity premium and the amount of disagreements about disasters.

1 Introduction

In this paper, we illustrate how heterogeneous beliefs about rare disasters affect asset prices and risk premia. Research by Rietz (1988), Longstaff and Piazzesi (2004), Barro (2006) and others show that the threat of rare economic disasters that cause severe losses in output and consumption can have large impact on the equity premium. However, the likelihoods of disasters or the size of their impact are difficult to estimate with a relatively short sample of historical data, which suggests that there are likely to be significant disagreements among market participants about the frequency and severity of disasters. We show that, with complete markets, such disagreements generate strong risk sharing motives among the investors, which can substantially lower the equity premium in the economy, even when it is predominantly populated by those who are pessimistic about disaster risks.

The setting is an exchange economy with two types of investors, whose beliefs on disasters can differ in various ways. For example, the optimists might think the probability of disasters is very low (e.g., once every 1000 years), while the pessimists believe disasters occur much more frequently (e.g., once every 60 years). Alternatively, they might disagree about the potential impact of a disaster on output and consumption. We assume that markets are complete, so that the agents can trade contingent claims and achieve optimal risk sharing. The equilibrium asset prices and risk premia depend on the beliefs and the distribution of wealth among the optimistic and pessimistic agents.

Our analysis centers around the following questions. First, how sensitive are asset prices to risk sharing in the presence of heterogeneous beliefs about disaster risks? In particular, if we introduce a small amount of optimistic agents into an economy originally occupied by pessimists, would they have any sizable impact on prices and the equity premium? Conversely, if the pessimists have the most extreme beliefs about disasters that are still "consistent with the data" (we specify the criterion for admissability later), can they have a large impact on asset prices with little wealth? Finally, how would such heterogeneity in beliefs affect the trading activities among agents in different instruments?

We show that a small amount of optimists can substantially change the level of equity premium and its dynamics in the economy. This result holds whether the disagreement is

about the intensity or impact of disasters. When we calibrate the beliefs of one agent using international data (from Barro (2006)) and the other using only consumption data from the US (where disasters have been relatively mild), raising the fraction of total wealth for the second agent from 0 to 10% lowers the equity premium from 4.4% to 2.0%. The decline in the equity premium becomes even faster when the pessimistic agent becomes more pessimistic, or when the optimistic agent also has lower risk aversion. On the flip side, if the share of total wealth the pessimistic agents own is small, even if they have the most extreme beliefs about disasters (which is bounded by the data and model), in equilibrium they might actually demand a negative premium for holding consumption claims.

The key reasons behind this result are the following: (1) the equity premium is derived almost entirely from jump (disaster) risks, (2) high prices for jump risk induce aggressive risk sharing, and (3) there is a highly nonlinear relationship between risk premium and disaster risks. First, in our economy, as is typically the case in standard power utility models, there is very little compensation for brownian risk due to the low volatility of consumption and moderate levels of risk aversion. Consequently, the equity premium derives primarily from disaster risk, and the compensation for bearing disaster risk must be high. For example, if there is a single type of disaster resulting in a 40% loss to the consumption claim and the equity premium due to disaster risk is 4%, then the annual premium for a disaster insurance contract which pays $1 when disaster strikes must be at least 10 cents.

Second, the high premium for disaster risk and high prices for disaster insurance provide a strong motivation for risk sharing when agents have different beliefs about disasters. In a benchmark case of our model, the pessimists may be willing to pay up to 13 cents per $1 of coverage, even though the payoff probability is only 1.7% under their own beliefs, or 0.1% under the beliefs of the optimistic agents. Such high prices induce the optimists to underwrite insurance contracts with notional value up to 40% of their total wealth, despite the risk of losing 70% of their consumption if a disaster strikes.

Third, the disaster risk premium is highly non-linear in the size of disasters, so that even small amounts of risk sharing may have significant effects on risk premia. Since disasters are rare, in order for them to have significant impact on the risk premium ex ante, marginal utility in the disaster states needs to rise substantially as the size of the consumption drop

increases. As a result, the equity premium is sensitive to changes in the size of individual consumption losses during a disaster. For example, if the pessimistic agents manage to reduce their consumption loss in a disaster by 10% (in logs), the corresponding reduction in risk premium is $1 - e^{-.1\gamma}$, or 33% when $\gamma = 4$. Thus, a small number of optimistic agents aggressively selling disaster insurances can reduce the disaster risk exposure of the pessimistic agents enough to significantly lower the equity premium they demand.

Our model thus explains why the market risk premium can remain low even though the majority of market participants are becoming more concerned with the risks of major disasters. Before a disaster strikes, the optimistic investors gain more wealth by selling disaster insurance, which gradually drives down the equity premium. However, when a disaster occurs, these optimists will lose a large fraction of their wealth, and their risk sharing capacity will be greatly reduced. As a result, the equity premium will jump up significantly.

A number of other interesting results arise from our analysis. First, we show that historical consumption data combined with simple economic restrictions can provide meaningful bounds on agents' beliefs about disasters. The data is quite informative about the mean of consumption growth, but less so about the frequencies of disasters, which suggests that disagreements about rare disasters should be an important source of heterogeneity in beliefs in asset pricing. Second, changes in the distribution of wealth among agents with different beliefs drive the fluctuations in the equity premium, volatility of stock returns, and riskfree rates in our model, and that heterogeneous beliefs about disasters can generate sizable variations in equity premium. Interestingly, our model predicts that the equity premium is not necessarily increasing in the weighted average probability of disasters under the beliefs of market participants, and that the equity premium can be higher when there is less disagreement among market participants. Third, similar to Longstaff and Wang (2008), who establish a link between asset prices and the size of the market of riskless money market funds, our model predicts a non-monotonic relationship between the equity premium and the size of the disaster insurance market.

The amount of risk sharing in our complete markets model, as measured by the disaster insurance trading, can be extreme at times. In practice, implementing such trades can be difficult due to moral hazard. Even exchange trading and daily mark-to-market will not

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eliminate the counterparty risks associated with these contracts, because disasters will lead to sudden large changes in prices. If we were to limit the ability for agents to share risk (e.g., by imposing collateral constraints for selling catastrophe bonds or shorting the stock), the equilibrium consumption of the pessimistic agent will become more risky, which raises the equity premium.

Another possible way to reduce the effects of heterogeneous beliefs is through ambiguity aversion. As Hansen (2007) and Hansen and Sargent (2009) show, if investors are ambiguity averse, they deal with model/parameter uncertainty by slanting their beliefs pessimistically. In the case with disaster risks, confronting the agents with the same model uncertainty facing econometricians could lead to agents behave as if they believe the disaster probabilities are high, even though their actual priors might suggest otherwise. This could reduce the heterogeneity of the distorted beliefs among agents, thus limiting the effects of risk sharing. We leave these implications to future research.

This paper contributes to the disaster risk literature, which goes back to the work of Rietz (1988). Barro (2006, 2009) has reinvigorated this literature by providing international evidence that disasters have been frequent and severe enough to have a large impact on the equity premium. A series of recent studies demonstrate that disaster risks can also help match a wide range of facts in financial markets, including asset volatility, return predictability, corporate bond spreads, option pricing, exchange rates, etc. Among these studies are Liu, Pan, and Wang (2005), Gabaix (2009), Wachter (2009), Farhi and Gabaix (2009), and others. The majority of these studies adopt a representative-agent framework. The few exceptions include Dieckmann and Gallmeyer (2005), Bates (2008), and Dieckmann (2009). The paper closest to ours is Dieckmann (2009), who also study a model of heterogenous beliefs about disasters under both complete markets and incomplete markets. He only considers the case of log utility and constant disaster risks, where risk sharing has limited effects on the equity premium.

Our paper builds on the literature of heterogeneous beliefs models. See Basak (2005) for a survey. Recent developments on heterogeneous beliefs and asset pricing include Kogan, Ross, Wang, and Westerfield (2006), Buraschi and Jiltsov (2006), Yan (2008), David (2008), Dumas, Kurshev, and Uppal (2009), Xiong and Yan (2009), among others. Our main finding

is related to the results of Kogan, Ross, Wang, and Westerfield (2006), who show that irrational traders can still have large price impact when their wealth becomes negligible. Our affine heterogeneous beliefs model provides a tractable yet flexible framework, through the generalized transform results of Chen and Joslin (2009), to study the implications of very general forms of heterogeneity in beliefs about disasters. In the special case with constant disaster probability, we derive closed form solutions for prices, risk premia, and portfolio positions for the cases where relative risk aversion $\gamma > 1$. We also provide explicit parameter restrictions for asset prices to be finite.

Finally, we also compare our results to models of heterogeneous preferences. Among the works on this topic are Dumas (1989), Wang (1996), Chan and Kogan (2002), and more recently Longstaff and Wang (2008). When the risk aversions are sufficiently different, we show that the effects on the equity premium are similar to the case with heterogeneous beliefs. Moreover, combing lower risk aversion with optimistic beliefs can make the effects of risk sharing on the equity premium particularly strong.

The rest of the paper is organized as follows. Section 2 presents our basic model of heterogeneous agents and disasters, beginning first with a review of a benchmark single agent economy. Section 3 introduces bounds on beliefs consistent both the historical data and asset prices. Section 4 analyzes the effect of heterogeneity in beliefs. Section 5 compares the results with the model of heterogeneous risk aversion. Section 6 concludes.

2 Model Setup

We first present the results of the general model where agents have both heterogeneous beliefs and preferences, and the disaster risk is time-varying.[1] Then we review the results for a special case with homogeneous agents and constant disaster risks.

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[1] This model is a special case of the affine heterogenous beliefs model in Chen and Joslin (2009).

2.1 Disasters and Heterogenous Agents

We consider a continuous-time, pure exchange economy. There are two agents (A, B), each being the representative of her own class. Agent A believes that the aggregate endowment is $C_t = e^{c_t^c + c_t^d}$, where c_t^c is the diffusion component of log aggregate endowment, which follows

$$dc_t^c = \bar{g}dt + \sigma_c dB_t^c, \tag{1}$$

and c_t^d is a pure jump process whose jumps arrive with stochastic intensity λ_t,

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t}dB_t^\lambda, \tag{2}$$

and the jump size has distribution ν^A. We summarize Agent A's beliefs with the probability measure $\mathbb{P}_A$.

Agent B believes that the probability measure is $\mathbb{P}_B$, which we shall suppose are equivalent to that of Agent A. She may disagree about the growth rate of consumption, the likelihood of disasters or the distribution of the severity of disasters when they occur. We assume that the two agents are aware of each others' beliefs, but nonetheless "agree to disagree".[2]

Specifically, Agent B's beliefs are characterized by the Radon-Nikodym derivative $\eta_t \equiv (d\mathbb{P}_B/d\mathbb{P}_A)_t$, where

$$\eta_t = e^{a_t + bc_t^c - I_t}, \tag{3}$$

$$I_t = \int_0^t \left(b\bar{g} + \frac{1}{2}b^2\sigma_c^2 + \lambda_t \left(\frac{\bar{\lambda}^B}{\bar{\lambda}^A} - 1 \right) \right) dt, \tag{4}$$

and a_t is a pure jump process whose jumps are coincident with the jumps in c_t^d and have size

$$\Delta a_t = \log \left(\frac{\bar{\lambda}^B}{\bar{\lambda}^A} \frac{d\nu^B}{d\nu^A} \right). \tag{5}$$

Here, $\frac{d\nu^B}{d\nu^A}$ will be a function of the disaster and will reflect the disagreement about the distribution of disaster; $\frac{d\nu^B}{d\nu^A}$ will be large (small) for the disasters that agent B thinks are

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[2]We do not explicitly model learning about disasters. Given the nature of disasters, such learning will likely be quite slow, and the main source of disagreements will be the priors.

relatively more (less) likely than agent A.

The variable η_t expresses Agent B's differences in beliefs in that when η_t is high, Agent B believes an event is more likely than Agent A thinks. Thus a, b reflect Agent B's pessimism or optimism regarding the likelihood of disasters and the growth rate of consumption. Under agent B's beliefs, the expected growth rate of consumption without jumps is $\bar{g} + \sigma_c^2 b$, and a disaster occur with intensity $\lambda_t \times \frac{\bar{\lambda}^B}{\lambda^A}$. Agent B believes that the disasters have distribution ν^B (which is equivalent to ν^A). Whenever disasters occur, the Radon-Nikodym derivative jumps by the ratio of the likelihood of the particular disaster under the two agents beliefs. Thus this specification of beliefs about disasters is quite flexible. Agent B not only can disagree with Agent A on the average frequency of disasters, but also the likelihoods for disasters of different magnitude, which effectively makes her disagree on on the distribution of jump size conditional on a disaster. This setting remains within the affine family as $X_t = (c_t^c, c_t^d, \log \eta_t, \lambda_t)$ follow a jointly affine process where the moment generating function of the jumps in X_t can be computed using the moment generating function of the disasters and (5).

We assume that the agents are infinitely lived and have constant relative-risk aversion utility over life time consumption:

$$U^i(C^i) = E_0^i \left[\int_0^\infty e^{-\rho_i t} \frac{(C_t^i)^{1-\gamma_i}}{1 - \gamma_i} dt \right], \tag{6}$$

for $i = A, B$. We suppose also that markets are complete and agents are endowed with some fixed share of aggregate consumption $(\theta_A, \theta_B = 1 - \theta_A)$.

The equilibrium allocations can be characterized as the solution of the following planner's problem, specified under the probability measure $\mathbb{P}_A$,

$$\max_{C_t^A, C_t^B} E_0^A \left[\int_0^\infty e^{-\rho_A t} \frac{(C_t^A)^{1-\gamma_A}}{1 - \gamma_A} + \tilde{\zeta}_t e^{-\rho_B t} \frac{(C_t^B)^{1-\gamma_B}}{1 - \gamma_B} dt \right], \tag{7}$$

$$s.t. \quad C_t^A + C_t^B = C_t, \tag{8}$$

where $\tilde{\zeta}_t \equiv \zeta \eta_t$ is the stochastic Pareto weight for Agent B. The first order conditions imply

$$e^{-\rho_A t}(C_t^A)^{-\gamma_A} = \tilde{\zeta}_t e^{-\rho_B t}(C_t^B)^{-\gamma_B}, \tag{9}$$

which together with the market clearing condition (8) gives the equilibrium consumption allocations as implicit functions

$$C_t^A = f^A(\hat{\zeta}_t)C_t , \tag{10}$$

$$C_t^B = (1 - f^A(\hat{\zeta}_t))C_t . \tag{11}$$

where $\hat{\zeta}_t = e^{(\rho_A - \rho_B)t}C_t^{\gamma_A - \gamma_B}\tilde{\zeta}_t$.

The stochastic discount factor under Agent A's beliefs, M_t, is given by

$$M_t = e^{-\rho_A t}(C_t^A)^{-\gamma_A} = e^{-\rho_A t}f^A(\hat{\zeta}_t)^{-\gamma_A}C_t^{-\gamma_A}. \tag{12}$$

Then, we can solve for ζ through the life-time budget constraint for one of the agents (see Cox and Huang (1989)), which is linked to the initial allocation of endowment.

Since our emphasis is on heterogeneous beliefs about disasters, for the remainder of this section we focus the case where there is no disagreement about the distribution of Brownian shocks, and the two agents have the same preferences.[3] In this case, $b = 0$, $\gamma_A = \gamma_B = \gamma$, $\rho_A = \rho_B = \rho$. The equilibrium consumption share simplifies to

$$f^A(\tilde{\zeta}_t) = \frac{1}{1 + \tilde{\zeta}_t^{\frac{1}{\gamma}}} . \tag{13}$$

From the stochastic discount factor, we can also compute the riskfree rate

$$r_t = \rho + \gamma\bar{g} - \frac{1}{2}\gamma^2\sigma_c^2 - \lambda_t \left(E_t \left[e^{\gamma\Delta c_t^d}\frac{f^A\left(\tilde{\zeta}_t e^{\Delta a_t}\right)}{f^A(\tilde{\zeta}_t)}\right] - 1 \right), \tag{14}$$

where a_t is given in (5) and $f^A(\tilde{\zeta}_t)$ is given in (13).

We see that as a disaster of size d occurs, $\tilde{\zeta}_t$ is multiplied by $\frac{\bar{\lambda}^A}{\bar{\lambda}^B}\frac{d\nu^B}{d\nu^A}(d)$. If the agent is

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[3]In Section 5, we investigate the case with heterogeneous preferences.

pessimistic in the sense that $\bar{\lambda}^B > \bar{\lambda}^A$ and ν^A stochastically dominates ν^B, the pessimistic agent will have a higher weight in the planner's problem when severe disasters occur so that their (relative) consumption increases. These equilibrium allocations are realized through competitive trading by the agents. We consider the following three types of securities: (1) a risk-free money market account, (2) a claim to aggregate consumption, and (3) a series of disaster insurance contracts with 1 year maturity, which pay \$1 on the maturity date if a disaster of size d occurs within the year.

The price of the aggregate endowment claim is

$$P_t = \int_0^\infty E_t \left[e^{-\rho\tau} \frac{M_{t+\tau}}{M_t} C_{t+\tau} \right] d\tau \ , \tag{15}$$

which can be viewed as a portfolio of zero coupon consumption claims

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\begin{aligned}
M_t P_t^{t+T} &= E_t[e^{-\rho T} M_{t+T} C_{t+T}] \\
&= e^{(1-\gamma)c_t - T(\rho + \bar{g}(1-\gamma) - \frac{1}{2}\sigma_c^2(1-\gamma)^2)} \times \\
&\quad E_t \left[e^{(1-\gamma)c_{t+T}^d} \left(1 + (\zeta_0 e^{\log \eta_{t+T}})^{\frac{1}{\gamma}} \right)^\gamma \right].
\end{aligned}
\tag{16}
$$

Under our assumption of integral γ, the final term will take the form as a sum of expectations of the form

$$e^{\alpha_i T} E_t[e^{(1-\gamma)c_{t+T}^d + \beta_i \log \eta_{t+T}}] = e^{A_i(T) + (1-\gamma)c_t^d + \beta_i \log \eta_t + B_i(T)\lambda_t} \ , \tag{17}$$

where (A_i, B_i) satisfy a simplified version of the familiar Riccati differential equations

$$\dot{B}_i = -\frac{\bar{\lambda}^B}{\bar{\lambda}^A}\beta_i - \kappa B_i + \frac{\sigma_\lambda^2}{2}B_i^2 + (\phi(\langle 1 - \gamma, \beta_i \rangle) - 1) \ , \quad B_0(0) = 0 \ . \tag{18}$$

$$\dot{A}_i = \kappa\theta B_i + \alpha_i \ , \quad A_i(0) = 0 \ , \tag{19}$$

where ϕ is the moment generating function of jumps in $\langle c_t^d, a_t \rangle$. It follows that price/consumption ratios vary only with the stochastic weight $\tilde{\zeta}_t$ and the disaster intensity:

$$P_t^{t+T} = C_t h^T(\lambda_t, \tilde{\zeta}_t) \ . \tag{20}$$

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Since the stochastic weight, $\tilde{\zeta}_t$, does not depend on the brownian shock (due to our assumption that agents only disagree about disasters), the price-consumption ratio is independent of these shocks as well. In the case that λ_t is constant, the price of the consumption claim further reduces to closed form solutions. Similar equations are also found for the price of the equilibrium consumption of Agent A and Agent B. See Appendix for details.

With stochastic disaster risk, in order for prices to be finite in the heterogeneous-agent economy, it is necessary and sufficient that prices are finite under each agent's beliefs in a single-agent economy (see Appendix B for a proof). The conditions can be found by considering the attracting fixed point of (18). Finite prices require, for the parameters of both agents perceived consumption processes, that the following two inequalities hold:

$$0 < \kappa^2 - 2\sigma_\lambda^2(\phi_d(1-\gamma) - 1), \tag{21}$$

$$0 > \kappa\theta\frac{\kappa - \sqrt{\kappa^2 + 2\sigma_\lambda^2(1 - \phi^i(1-\gamma))}}{\sigma_\lambda^2} - \rho + (1-\gamma)\bar{g} + \frac{1}{2}(1-\gamma)^2\sigma_c^2, \tag{22}$$

where ϕ_d is the moment generating function of jumps in c_t^d. The first inequality reflects the fact that the volatility of the disaster intensity cannot be too large relative to the rate of mean reversion or the convexity effect induced by the possibility of large values of the intensity will generate zero coupon equity prices that increase faster than than any possible exponential discounting. The second inequality reflects that they must be enough discounting to counteract the discount rate effects that growth and disasters induces in zero coupon equity prices.

The disaster insurance security is priced similarly through the stochastic discount factor. For the simple case of a single type of disaster, we can compute the price of disaster insurance by considering the counting processes, N_t, which counts the number of disasters that have occurred:

$$M_t P_t^{DI} = E_t[M_{t+1}1_{\{N_{t+1}>N_t\}}]$$
$$= E_t[M_{t+1}] - E_t[M_{t+1}|N_{t+1} = N_t]\Pr(N_{t+1} = N_t). \tag{23}$$

In the case the λ_t is constant, this reduces to a particularly simple expression. For the general

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case, we use the transform analysis of Duffie, Pan, and Singleton (2000). See Appendix for details.

Finally, we can compute the expected excess return of the various securities under the probability measure $\mathbb{P}_i$ ($i = A, B$). In each case, we make use of the risk-neutral dynamics to compute the expected excess return for a price process that has the form $P_t^o = h(c_t^c, c_t^d, \tilde{\zeta}_t)$. In the case of single disaster type, the expected excess return is of the form

$$ep_t = \sigma_c^2 \partial_{c^c} h + \lambda^i r^{\mathbb{P}_i} - \lambda^{\mathbb{Q}} r^{\mathbb{Q}}, \tag{24}$$

where r^m is the return, conditional on a disaster occurring, under the measure m. The risk neutral intensity, $\lambda^{\mathbb{Q}}$, is simply $E_{t-}^i[M_t/M_{t-}]\lambda_t^i$. In the more general case, we simply average this formula over the conditional disaster distribution ν^i, taking into account, where again the conditional disaster distribution under the risk-neutral measure, $\nu^{\mathbb{Q}}$, is computed directly from the stochastic discount factor.

In the remainder of the paper, we report the equity premium relative to the probability measure of the pessimist, $\mathbb{P}^A$. One interpretation for picking $\mathbb{P}^A$ as the reference probability measure is that the pessimist has the correct beliefs, and we are studying the impact of the incorrect beliefs of an optimist on asset prices.

2.2 Homogeneous agents and constant disaster risk

When agents have the same preferences and beliefs about disasters, and that the disaster intensity λ_t is constant, we recover the basic version of the representative agent rare disaster model. We first review this case before presenting the results of the heterogeneous-agent model.

The aggregate endowment process is a special case of the process in Section 2, where c_t^d is now a pure jump process with constant intensity λ and moment generating function (MGF) φ for the jump size distribution. The stochastic discount factor, M_t, is given by $M_t = e^{-\rho t} C_t^{-\gamma}$. From the stochastic discount factor we can compute the constant riskfree rate

$$r_t = -\frac{\mathcal{D}M}{M} = \rho + \gamma \bar{g} - \frac{1}{2}\gamma^2 \sigma_c^2 + \lambda(\varphi(-\gamma) - 1). \tag{25}$$

Additionally, the stochastic discount factor allows us to easily compute the risk neutral dynamics, which facilitates the computation and interpretation of excess returns. Under $\mathbb{Q}$, the risk-neutral measure,

$$dc_t = (\bar{g} - \sigma_c^2 \gamma)dt + \sigma_c dB_t^{\mathbb{Q}} + dJ_t^{\mathbb{Q}}, \tag{26}$$

where disasters arrive with intensity $\lambda^{\mathbb{Q}} = \varphi(-\gamma)\lambda$ under this measure and have distribution with moment generating function $\varphi^{\mathbb{Q}}(s) = \varphi(s-\gamma)/\varphi(-\gamma)$. When the riskfree rate and disaster intensity are close to zero, the risk-neutral disaster intensity $\lambda^{\mathbb{Q}}$ can be approximately viewed as the value of a one-year disaster insurance contract that pays \$1 when a disaster occurs within a year.[4]

The risk adjustments for the jumps are quite intuitive. If consumption drops during a disaster, then $\varphi(-\gamma) > 1$ for $\gamma > 0$, so that disasters occur more frequently under the risk-neutral measure. Moreover, the risk-adjusted distribution of jump size conditional on a disaster is $dv^{\mathbb{Q}}/dv = e^{-\gamma \Delta c}/\varphi(-\gamma)$, which slants the probabilities towards large jumps, making severe disasters more likely.

The price of the claim to aggregate dividends is

$$P_t = E_t \left[\int_0^\infty e^{-\rho\tau} \frac{M_{t+\tau}}{M_t} C_{t+\tau} d\tau \right] = \frac{C_t}{\theta},$$

where

$$\theta = \rho - (1-\gamma)\bar{g} - \frac{1}{2}(1-\gamma)^2\sigma_c^2 - \lambda(\varphi(1-\gamma) - 1), \tag{27}$$

and prices are finite if and only if $\theta > 0$.

Finally, the instantaneous expected excess return of the aggregate dividend claim is

$$E_t \left[\frac{dP_t}{P_t} + \frac{C_t}{P_t}dt \right] /dt - r_t = \gamma\sigma_c^2 - \lambda E_\nu \left[(e^{-\gamma \Delta c} - 1)(e^{\Delta c} - 1) \right]$$

$$= \gamma\sigma_c^2 - \left[\lambda^{\mathbb{Q}}(\varphi^{\mathbb{Q}}(1) - 1) - \lambda(\varphi(1) - 1) \right]. \tag{28}$$

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[4]The value of the disaster insurance is $D_t^1 = E_t \left[\int_t^{t+1} \lambda^{\mathbb{Q}} e^{-(r+\lambda^{\mathbb{Q}})(s-t)} ds \right]$. When r and $\lambda^{\mathbb{Q}}$ are close to 0, $D_t^1 \approx \lambda^{\mathbb{Q}}$.

Thus, the expected excess return arises from (1) exposure to brownian risk and (2) exposure to jump risk. The risk premium for exposure to jumps reflects both the increased likelihood of disasters under $\mathbb{Q}$, $\lambda^{\mathbb{Q}}$ (relative to λ), and also the increased severity of losses for a given disaster under $\mathbb{Q}$, $\varphi^{\mathbb{Q}}(1) - 1$ (relative to $\varphi(1) - 1$). Importantly, the risk premium rises exponentially with the size of the consumption drops. Thus, a small reduction in the consumption exposure to disasters (especially the most severe ones) could have large impact on the risk premium. This mechanism is key to the results of the heterogeneous agents model.

3   Bounds for Extreme Beliefs

While the beliefs of individual agents about consumption growth and disasters are not directly observable, historical consumption data as well as simple economic restrictions can provide guidance on how extreme these beliefs could be, which helps with the calibration of our model. We define an admissible belief as satisfying the following conditions: (i) the price of consumption claim under the belief is finite, and (ii) the belief cannot be rejected by the data at a given significance level α.

We consider whether an agent with a given null hypothesis about either the growth rate of consumption or the likelihood of disasters would be able to reject this hypothesis using historical data. Figure 1 plots the p-value associated with different beliefs about the expected growth rate $\bar{g}$ in a Gaussian model (no disasters) and beliefs about λ in a constant disaster risk model. The p-value in the Gaussian case, given in the left axis, reflects the highest significance level at which one could not reject the null that the true growth rate is below the observed historical value by an amount specified in the top axis, based on 120 years of data. The diagonal line plots the effect of such pessimism on the equity premium (right axis) for base parameters of $\gamma = 4$ and $\sigma_c = 3\%$. The p-value falls rapidly, reaching 1% when the expected growth rate is just 0.8% below the historical mean. We see that such tiny difference in beliefs has little impact on asset prices: even if the pessimistic agents own all the wealth, the corresponding equity premium under the optimistic agents' beliefs is only 1.3%. In order to get a large premium (say 4%), the beliefs of the pessimistic agents have to be so extreme that it is difficult to reconcile with the data.
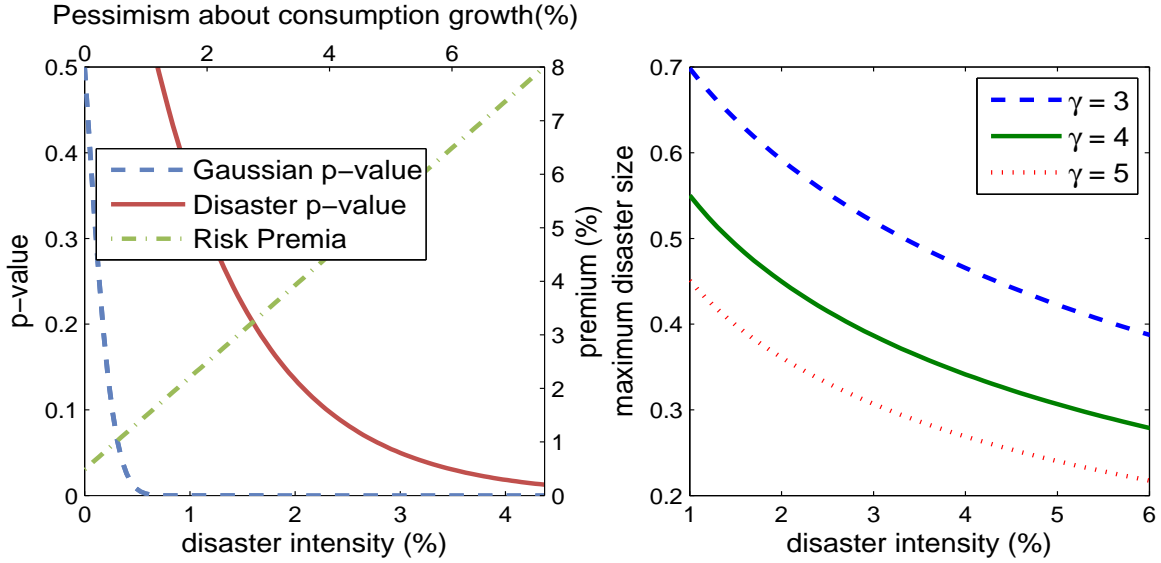
Figure 1: **Bounds for extreme beliefs.** The left panel plots the p-value for various disaster intensities (bottom axis) and mean growth rates of consumption (top axis) based on 120 years of data. The right panel plots the maximum disaster size that gives finite prices for various disaster intensities.

One might argue that there can be more disagreement about the growth rates of dividend than consumption, which can generate high risk premium for claims on dividends as opposed to consumption. However, this assumption implies that at least for some agents the long run growth rate of consumption will differ from that of dividends, which means that consumption and dividends can not be cointegrated, a reasonable restriction that is supported empirically.

In contrast, the model of heterogeneous beliefs about disasters shows more promise. The disaster we consider later in the paper are in fact significantly more severe than those observed in US history, where consumption has never declined more than 10% in a given year. If we make an extreme assumption that no disasters have occurred over the last 120 years, then the p-value is the probability of observing no disasters assuming the true intensity is consistent with the agents' beliefs (bottom axis). Figure 1 shows that $\lambda = 4\%$ roughly corresponds to a p-value of 1%, which provides an upper bound on how frequent disaster can be within the set of admissible beliefs.

While consumption data has no information about the size of disasters that have not

occurred in the past, we can bound the jump size using the parameter restrictions given in Section 2. For example, in the case with deterministic disaster size, with $\gamma = 4$ and $\lambda = 4\%$, the largest admissible drop in consumption during a disaster is 35% ($\bar{d} = 0.43$).

Compared with the Gaussian model, models of heterogeneous beliefs about disasters have a better chance in matching the equity premium. Again assuming that the pessimistic agent owns all the endowment and have the most extreme version of admissible beliefs ($\lambda = 4\%$ and $\bar{d} = 0.5$), the equity premium under the optimistic agents' beliefs is 8%. Next, we investigate what happens to asset prices when pessimistic agents no longer have all the wealth.

4 Heterogeneous Beliefs: Constant Disaster Risk

In this section, we analyze the effects of heterogeneous beliefs for rare disasters on asset prices and consumption allocations. To cleanly demonstrate the effects of heterogeneous beliefs and the risk-sharing mechanism, we first keep the risk of rare disasters constant, i.e., $\lambda_t = \bar{\lambda}$. We consider two natural examples of disagreements about rare disasters, one where agents disagree about the frequencies of disasters, the other where they disagree about the size of disasters. After analyzing these two cases, we then extend the model by calibrating the beliefs to the US and international data on disasters.

4.1 Example I: Disagreement about the Frequency of Disasters

In the first example, we assume that the disaster size is deterministic, $d_t = \bar{d}$, and the two agents disagree about the frequencies of disasters (λ). We set $\bar{d} = -0.51$ so that the MGF $\varphi(-\gamma)$ in this model matches the calibration of Barro (2006) for $\gamma = 4$. This $\bar{d}$ is admissible according to Figure 1. It implies that aggregate consumption falls by 40% when a disaster occurs.[5] Agent A (pessimist) believes that disasters occur with intensity $\lambda^A = 1.7\%$, which is also taken from Barro (2006). The remaining parameters are $\bar{g} = 2.5\%$, $\sigma_c = 2\%$, and $\rho = 3\%$. Agent B (optimist) believes that disasters are much less likely, $\lambda^B = 0.1\%$, but she agrees with Agent A on the size of jump in aggregate consumption given a disaster as well

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[5]This value is higher than the average disaster size in Barro (2006) due to the fact that larger but more rare jumps can have big impact on the MGF, especially when γ is large.
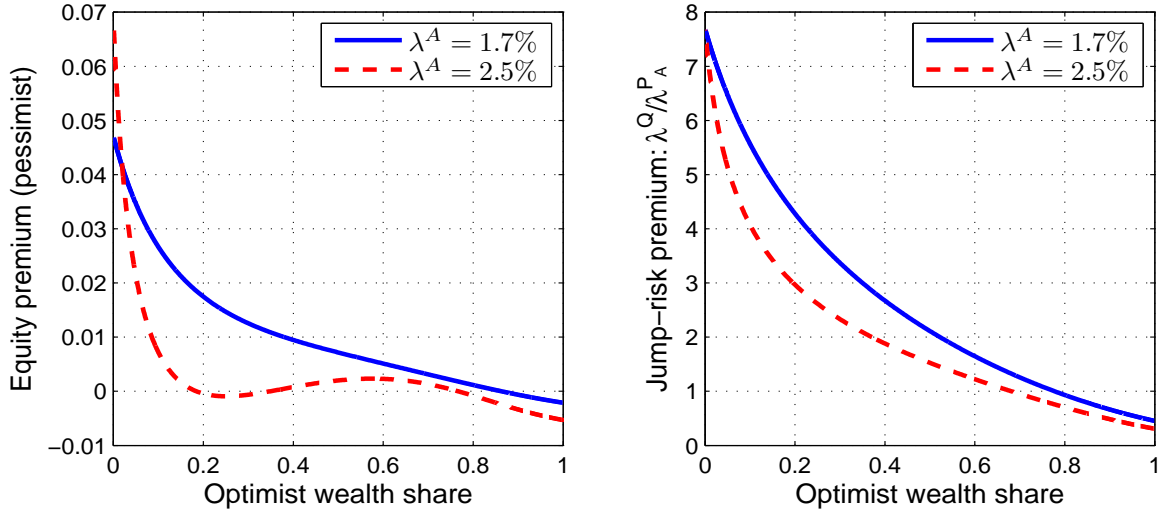
Figure 2: **Disagreement about the frequency of disasters.** The left panel plots the equity premium under the pessimist's beliefs as a function of the wealth share of the optimist. The right panel plots the jump-risk premium for the pessimist. We consider two sets of beliefs for the pessimist: $\lambda^A = 1.7\%$ and $\lambda^A = 2.5\%$.

as the Brownian risks in consumption. She also has the same preferences as Agent A.

Figure 2 shows the conditional equity premium under Agent A's beliefs, as well as the jump-risk premium for Agent A. If all the wealth is owned by the pessimistic agent, the equity premium is 4.7%, and the riskfree rate is 1.3%. The left panel shows that the equity premium falls as we allocate more wealth in this economy to the more optimistic agent, who views the chances of disasters as negligible (once every thousand years).

Qualitatively, this fact is as expected. Without much disaster risk, Agent B demands a very low equity premium. When she has all the wealth, this premium is only 0.43% under her own beliefs, or -0.21% under the pessimist's beliefs. While we expect the premium to fall as we endow Agent B with more wealth, the speed at which the premium falls is impressive. When the optimistic agent owns 10% of the total wealth, the equity premium has fallen from 4.7% to 2.7%. When the wealth of the optimist reaches 20%, the equity premium falls to just 1.7%.

To understand the behavior of the equity premium, we can decompose the premium:

$$ep_t = \gamma\sigma_c^2 + \lambda^{\mathbb{P}_A}\left(\frac{\lambda_t^{\mathbb{Q}}}{\lambda^{\mathbb{P}_A}} - 1\right)\left(\frac{h(\tilde{\zeta}_t e^{a\bar{d}})e^{\bar{d}}}{h(\tilde{\zeta}_t)} - 1\right). \tag{29}$$

The first term $\gamma\sigma_c^2$ is the standard compensation for bearing brownian risk. Heterogeneity has no effect on this term since the agents agree about the brownian risk. Given the value of risk aversion and consumption volatility we consider, this term has negligible effect on the premium. The second term reflects the compensation for disaster risks. It can be further decomposed into three factors. The first factor, $\lambda^{\mathbb{P}_A}$, is independent of the wealth distribution. The second, $\frac{\lambda_t^{\mathbb{Q}}}{\lambda^{\mathbb{P}_A}} - 1$, gives the relative jump of the stochastic discount factor at the time of a disaster, where $\lambda^{\mathbb{Q}}$ is the risk-neutral disaster intensity (the same for both agents with complete markets). The ratio $\lambda_t^{\mathbb{Q}}/\lambda^{\mathbb{P}_A}$ is often referred to as the jump-risk premium. The third factor is the return of the consumption claim in a disaster, which is due to changes in consumption as well as the price-consumption ratio.

How does the wealth distribution affect the jump-risk premium? In this example,

$$\lambda_t^{\mathbb{Q}}/\lambda_t^{\mathbb{P}_A} = e^{-\gamma\Delta c_t^A},$$

where Δc_t^A is the jump size of the equilibrium log consumption for Agent A in a disaster, which could be very different from the jump size in aggregate endowment due to trading. The jump-risk premium is large when the economy is entirely occupied by the pessimistic agent. Without trading, the jump in the equilibrium log consumption for Agent A will be $\bar{d}$, which generates a jump-risk premium of 7.7. As we show earlier, $\lambda^{\mathbb{Q}}$ is approximately the premium of a one-year disaster insurance with notional value \$1. Thus, $\lambda^{\mathbb{Q}}/\lambda^{\mathbb{P}_A} = 7.7$ corresponds to an annual premium of 13 cents for \$1 of protection against a disaster event that occurs with probability 1.7%. This premium falls rapidly as the optimistic agent's wealth increases. When the optimist owns 20% of total wealth, the jump-risk premium drops to 4.2. According to equation (29), such a drop in the jump-risk premium alone will cause the equity premium to fall by more than half to 2.2%, which accounts for the majority of the change in the premium (from 4.7% to 1.7%).

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The additional change in the premium is due to the return of the consumption claim in a disaster becoming less sensitive to disasters as the optimist's wealth increases. With CRRA utility, the effect of the fall in riskfree rate can exceed that of the rise in the risk premium following a disaster, which raises the price-consumption ratio. In the region where the optimist has low wealth, this effect gets stronger as her wealth increases, thus offsetting more of the losses in consumption and resulting in a less negative return in the consumption claim. Wachter (2009) finds similar results in a representative agent rare disaster model with time-varying disaster probabilities and CRRA utility. However, our decomposition above shows that the reduction of the jump-risk premium is the main reason behind the fall in premium.

As the formula of the jump-risk premium clearly shows, reduced compensation for disaster risk for the pessimist comes from reduced consumption exposure to disasters, which is the result of optimal risk sharing. Since the optimist views disasters as very unlikely events, she is willing to trade away their claims in the future disaster states in exchange for higher consumption in normal times. For example, she will find selling an $1 disaster insurance to the pessimist and collecting a 13 cents premium a lucrative trade. However, her capacity for underwriting such insurance is limited by her wealth, as she needs to ensure that her consumption/wealth is positive in all future states, in particular when a disaster occurs (no matter how unlikely such an event is). In fact, she stays away from this limit imposed by the wealth constraint because the more disaster insurance she sells, the more her consumption falls in the disaster states, which makes her less willing to take on additional disaster risks. The more wealth the optimistic agent has, the more disaster insurance she is able to sell without making her consumption too risky when a disaster strikes. Such a mechanism can substantially reduce the disaster risk exposure of the pessimistic agent in equilibrium.

If the disaster risk premium from a "mild" version of pessimism about disaster risks can be offset by an optimistic agent with limited wealth, can we improve the model performance by making the pessimist more pessimistic? The dash-lines in Figure 2 plot the results when Agent A believes that $\lambda = 2.5\%$ (everything else equal), which according to Figure 1 is still "reasonable" (with p-value of 8%). The results are striking. While the equity premium becomes significantly higher (6.8%) when the pessimistic agent owns all the wealth in the

economy, it falls to 4.1% with just 2% of total wealth allocated to the optimistic agent (already lower than the previous case with $\lambda^A = 1.7\%$), and is below 1% when the wealth of the optimistic agent exceeds 8.5%. As the wealth of the optimistic agent grows higher, the premium can even become negative. While part of the sharp fall in equity premium (especially near the left boundary) is due to the price-consumption ratio effect that is specific to CRRA utility, the lower jump-risk premium is also important. For example, when the optimist has 10% of total wealth, the jump-risk premium falls to 4.0, which will drive the premium down to 3.1% (60% of the total fall).

Another interesting implication of this comparative static exercise is that, holding the average belief constant (weighted by wealth share), increasing the disagreement between the two agents can drive the equity premium lower. This is because the amount of risk sharing becomes larger as the beliefs of the two agents become more different, which can dominate the effect of more pessimism for the pessimist.

To better examine the risk sharing mechanism between agents, we compute their portfolio positions in the aggregate consumption claim, disaster insurance, and the money market account. The first thing to notice is that each agent will hold a constant proportion of the consumption claim. Intuitively, this is because they agree on the brownian risk and share it proportionally. Disagreement over disaster risk is resolved through trading in the disaster insurance market, which is financed by the money market account.

Figure 3 Panel A plots the notional value of the disaster insurance sold by the optimistic agent as her share of total wealth. This shows the degree to which the optimist insures the pessimist against the disaster event. The dash-line plots the maximum amount of disaster insurance (as a fraction of her wealth) the optimist can sell subject to her budget constraint. When the optimist has very little wealth, the notional value of the disaster insurance she sells is about 35% of her wealth. This value initially rises and then falls as the optimist gains more wealth. The reason is that when the optimist has little wealth, the pessimist has great demand for disaster insurance and is willing to pay a high premium, which induces the optimist to sell more insurance relative to her wealth. As the optimist gets more wealth, risk sharing improves, and the premium on the disaster insurance falls, so that the optimist is no longer as aggressive (relative to her wealth) in underwriting the insurance.
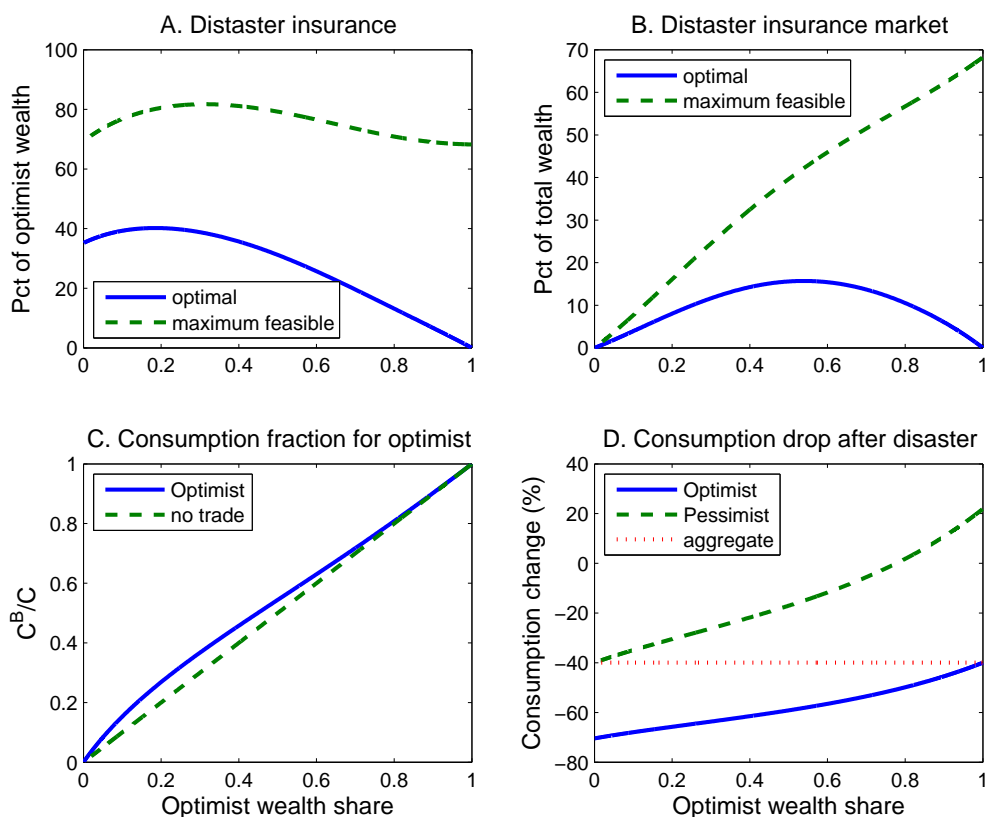
Figure 3: **Risk sharing:** $\lambda^A = 1.7\%$. Panel A and B plot the total notional value of disaster insurance relative to the wealth of the optimist and total wealth in the economy. Panel C plots the consumption share for the optimist in equilibrium. Panel D compares the two agents' consumption drops in a disaster with that of the aggregate endowment.

This graph also helps us judge whether the risk sharing in equilibrium is too "extreme". At its peak, the amount of disaster insurance sold by the optimist is about half of the maximum amount that she can underwrite while still keeping her wealth positive with probability 1, which might appear quite reasonable. However, in reality, underwriters of disaster insurance will likely be required to collateralize their promises to pay in the disaster states. According to the model, all the wealth is from the claim on future endowment income, which might be not be used as collateral (just as labor income cannot be used as collateral). We will have more discussion of the collateral constraint later.

Panel B plots the size of the disaster insurance market (the total notional value normalized by total wealth). Naturally, the size of this market is zero when either agent has all the

wealth, and the market is the biggest when wealth is close to be evenly distributed. At its peak, the notional value of the disaster insurance market is about 16% of the total wealth of the economy. Notice that the model generates a non-monotonic relation between the size of the disaster insurance market and the equity premium. The premium is high when there is a lot of demand for disaster insurance but little supply, and is low when the opposite is true. In either case, the size of the disaster insurance market will be small.

Panel C plots the equilibrium consumption shares of the optimistic agent for different wealth distributions. The 45-degree line corresponds to the case of no trading. The optimist's consumption share is above the 45-degree line, especially when her wealth is small, suggesting that she is consuming a larger share of total consumption than her endowment in the non-disaster states. However, the price for getting more to consume in normal times is more exposure to the fall in consumption when disaster strikes.

Panel D shows the impact of a disaster on the equilibrium consumption of the two agents. To see how aggressive the optimist is in betting on disaster risks, consider that, when she has little wealth, she will suffer a 70% loss in consumption in the event of a disaster (compared to 40% drop in aggregate consumption). The optimist is willing to face this catastrophic downside risk because she thinks disasters are rare and the price of disaster insurance is very high. As for the pessimistic agent, the less wealth she possesses, the more disaster insurance she buys relative to her wealth. This will gradually lower her disaster risk exposure, and can eventually turn the disaster insurance into a speculative position, which actually makes her consumption jump up (as high as 20%) during a disaster. This "over-insurance" explains why the equity premium under the pessimist's beliefs can turn negative when the optimist has most of the wealth.

Panel D also helps explain why the equity premium falls so rapidly as the optimist gets more wealth. While the pessimist's consumption downfall in a diaster gets reduced gradually, the jump-risk premium falls at an exponential rate, which then causes the equity premium to fall at an exponential rate as well. This effect will be stronger when the agents are more risk averse.

When Agent A has more pessimistic beliefs, trading between the two agents also becomes
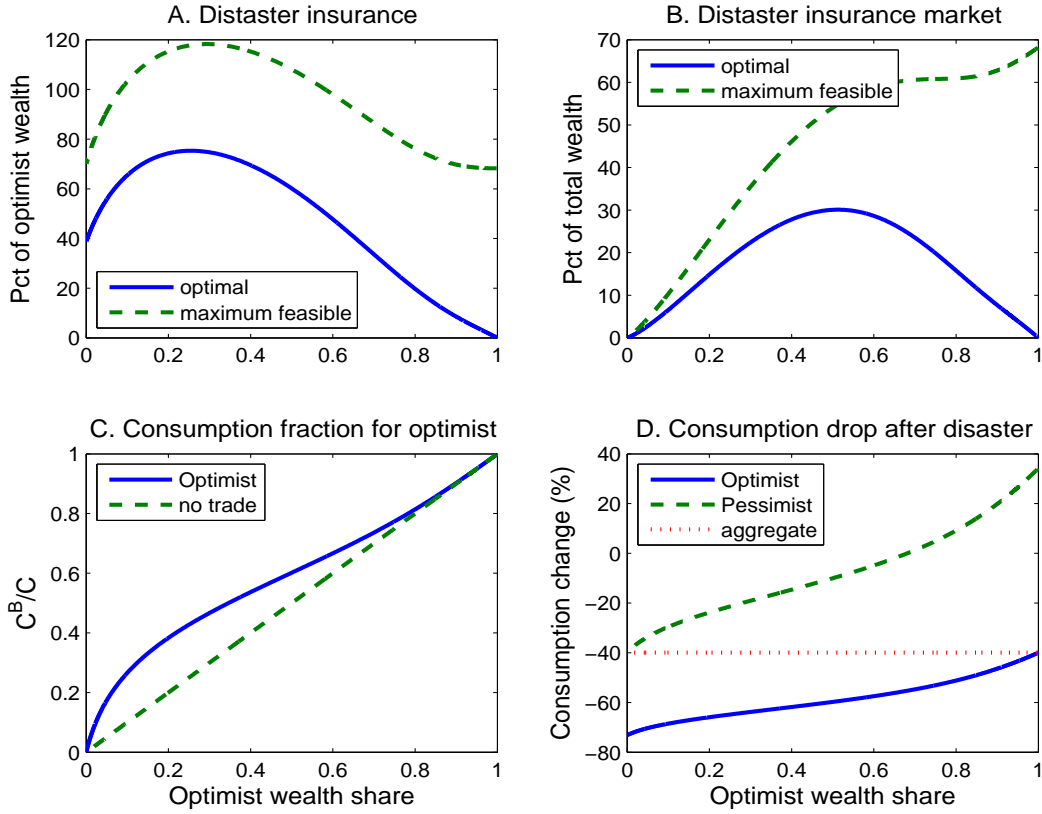
Figure 4: **Risk sharing:** $\lambda^A = 2.5\%$**.** Panel A and B plot the total notional value of disaster insurance relative to the wealth of the optimist and total wealth in the economy. Panel C plots the consumption share for the optimist in equilibrium. Panel D compares the two agents' consumption drops in a disaster with that of the aggregate endowment.

more active. Figure 4 shows the results. The price of disaster insurance rises significantly compared to the case of mild pessimism. Naturally, the amount of disaster insurance sold (both relative to the wealth of the optimistic agent and to total wealth in the economy) also becomes higher than the case of milder pessimism. The equilibrium consumption shares is now significantly more nonlinear, especially in the region where the wealth share of the optimist is low. Moreover, a disaster now has an even bigger impact on the consumption of the optimistic agent. On the other hand, the pessimistic agent will buy insurance more aggressively, which reduces her exposure to diaster risks at a faster pace. This is evident in the plot of the consumption drop in a disaster. Compared to Figure 3, the reduction in consumption drop is considerably faster near the left boundary. When the pessimist has

little wealth, trading can result in an upward jump of up to 35% in her consumption when a disaster arrives.

Thus, having an agent who is more pessimistic about disasters does not necessarily drive up the equity premium in the economy. The pessimist will pay more for disaster insurance, which presents a better trading opportunity for the optimistic agent and increases the amount of insurance she provides. More risk sharing can then lead to safer consumption stream for the pessimistic agent and make the equity premium lower.

A final question for this example is whether the effect of risk sharing on the equity premium becomes stronger or weaker as the size of disaster increases. On the one hand, for larger disasters, the equity premium becomes more sensitive to changes in the size of consumption drops, which means the premium will decline more for the same amount of risk sharing between the agents. On the other hand, the optimist will be increasingly reluctant to take on extra losses in the disaster state because her marginal utility rises exponentially in the (log) size of consumption losses. To study the net effects, we increase $\bar{d}$, yet keep the risk premium for the pessimist in the single-agent case as well as the relative difference in beliefs unchanged (by lowering λ^A and keeping λ^B/λ^A fixed). Our results show that the second effect dominates. The decline in equity premium becomes closer to linear as $\bar{d}$ gets larger, and the amount of risk sharing becomes smaller.

4.2   Example II: Disagreement about the Size of Disasters

Having examined disagreement about the likelihoods of disasters, we next study how disagreement about the distribution of disaster sizes affects asset pricing. For simplicity, we assume that when a disaster occurs, the fall in aggregate consumption follows a binomial distribution, with the possible drop being 10% and 40%. Both agents agree on the intensity of a disaster ($\lambda = 1.7\%$). Agent A (pessimist) assigns a 99% probability to a 40% drop in aggregate consumption, thus having essentially the same beliefs as in the previous example. On the contrary, Agent B (optimist) believes that disasters are much less severe. She assigns only a 1% probability to a 40% drop, but 99% probability to a 10% drop. The rest of the parameter values are the same as in Example I.
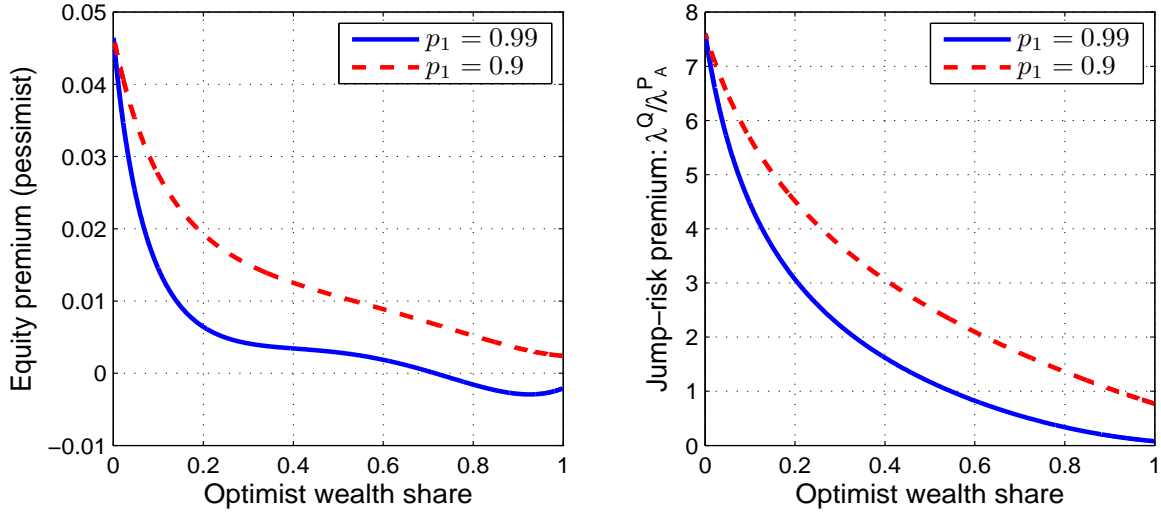
Figure 5: **Disagreement about the size of disasters.** The left panel plots the equity premium under the pessimist's beliefs. The right panel plots the jump-risk premium for the pessimist. The values of p_1 in the two cases specify the probability that Agent B assigns to the smaller disaster (with size -10%) conditional on a disaster occurring.

Figure 5 plots the conditional equity premium and jump-risk premium in this case, both of which are again under the pessimist's probability measure $\mathbb{P}^A$. When the pessimist has all the wealth, the equity premium is 4.6% (almost the same as in Example I). Again, the equity premium falls rapidly as we starts to shift wealth to the optimist. The premium falls by almost half to 2.4% when the optimist owns just 5% of total wealth, and becomes 1.4% when the optimist's share of total wealth grows to 10%. Similarly, the jump-risk premium falls from 7.6 to 4.4 with the optimist's wealth share reaching 10%, which by itself will lower the premium to 2.4%. Finally, the effects of risk sharing become smaller as the disagreement on the jump size distribution is reduced (we do so by increasing the probability that Agent B assigns to a 40% drop).

This example shows that in terms of asset pricing, introducing an agent who disagrees about the severity of disasters is quite similar to introducing one who disagrees about the frequency of disasters. In either case, the optimistic agent will aggressively insure the pessimistic agent against the severe diasters, which drives down the equity premium exponentially.

4.3   Calibrating Disagreement: Is the US Special?

The two examples we have presented so far are quite stylized in the way disagreements are modelled. In this section, we extend the analysis to a more realistic model of beliefs on disasters. The way we calibrate the beliefs of the two types of agents is as follows. Agent A believes that the US is no different from the rest of the world in its disaster risk exposure. Hence her beliefs are calibrated using cross-country consumption data. Agent B, on the other hand, believes that the US is special. She forms her beliefs on disaster risk using only the US consumption data.

An important contribution of Barro (2006) is to provide detailed accounts of the major consumption declines cross 35 countries in the twentieth century. Rather than directly using the empirical distribution from Barro (2006), we estimate a truncated Gamma distribution for the log jump size[6] from Barro's data using maximum likelihood (MLE). Our estimation is based on the assumption that all the disasters in the sample were independent, and that the consumption declines occurred instantly.[7] We also bound the jump size between 5% and 75%. In comparison, the smallest and largest declines in per capital GDP in Barro's sample are 15% and 64%, respectively. The disaster intensity is still $\lambda^A = 1.7\%$. The remaining parameters are the mean growth rate and volatility of consumption without a diaster, $\bar{g} = 2.5\%$ and $\sigma_c = 2\%$, which are consistent with the US consumption data post WWII.

As for Agent B, we assume that she agrees with the values of $\bar{g}$ and σ_c, but we estimate the truncated Gamma distribution of diaster size using MLE from annual per-capita consumption data in the US 1890-2008.[8] Over the sample of 119 years, there are three years where consumption falls by over 5%. Thus, we set $\lambda^B = 3/119 = 2.5\%$. Alternatively, we can estimate λ^B jointly with the jump size distribution, which will make jumps more frequent and have smaller sizes.

Panel A of Figure 6 plots the probability density functions of the log jump size distri-

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[6]The truncated Gamma distribution has PDF $f(d; \alpha, \beta | \underline{d}, \overline{d}) = f(d; \alpha, \beta) / \left(F(\overline{d}; \alpha, \beta) - F(\underline{d}; \alpha, \beta) \right)$, where $f(x; \alpha, \beta)$ and $F(x; \alpha, \beta)$ are the PDF and CDF of the standard Gamma distribution with shape parameter α and scale parameter β.

[7]These assumptions are debatable. For example, many of the major declines cross European countries are in WWI and WWII. Moreover, many of the declines spanned several years. See Donaldson and Mehra (2008) for more discussions on the issue of observation frequency.

[8]The data is taken from Robert Shiller's web site http://www.econ.yale.edu/~shiller/data.htm
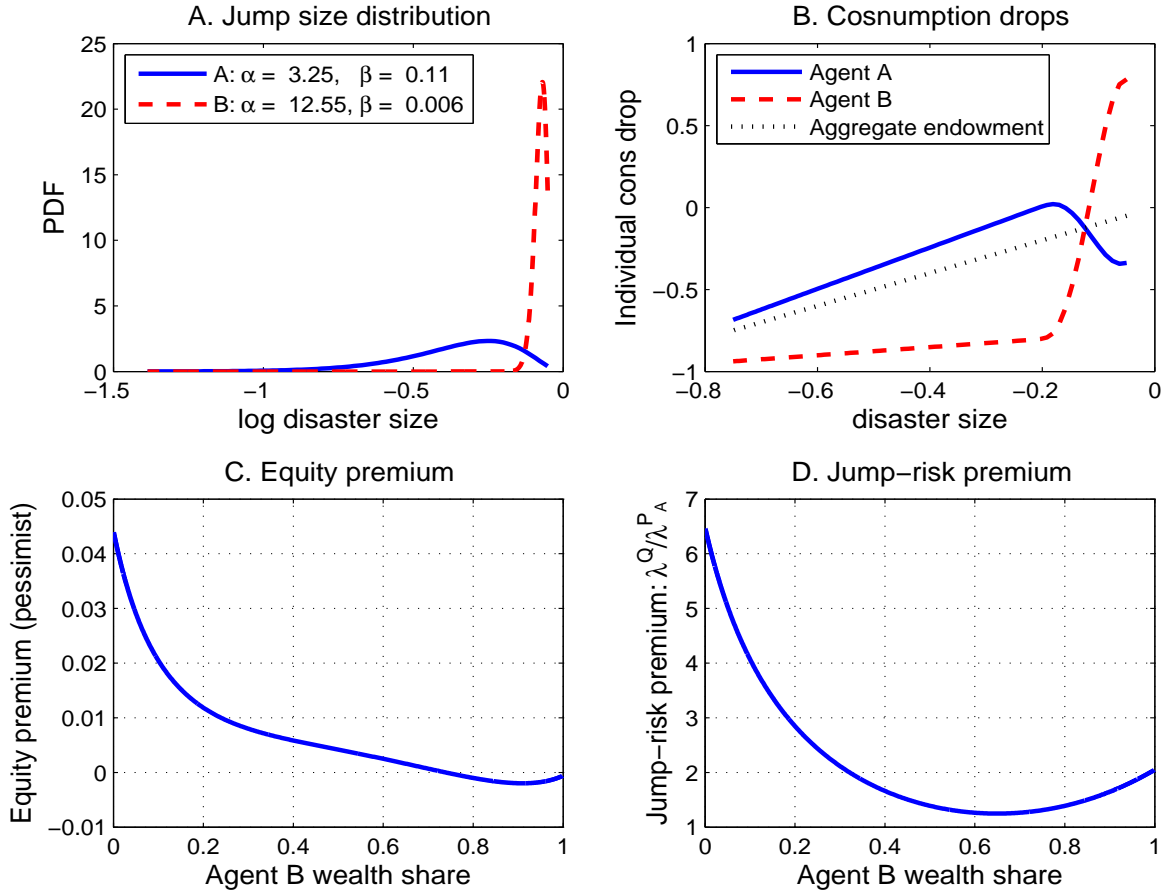
Figure 6: **Calibrated Disagreements: International vs US Experiences.** Panel A plots the truncated Gamma distribution $\Gamma(\alpha, \beta | \underline{d}, \overline{d})$ for the two agents. We assume $\underline{d} = 5\%$, $\overline{d} = 75\%$. Panel B plots the equilibrium consumption drops for the two agents given the size of the disaster. Panel C and D plot the equity premium and jump-risk premium under Agent A's beliefs.

butions for the two agents, which are very different from each other. The solid line is the distribution fitted to the international data on disasters. The average log drop is 0.36, which is equivalent to 30% drop in the level of consumption. In the US data, the average drop in log consumption is only 0.075, or 7.3% in level. In addition, Agent A's distribution has much fatter left tailers than Agent B's. Thus, while Agent A assigns significantly higher probabilities than B to large disasters (where consumption drops by 15% or more), Agent B assigns more probabilities to small disasters, especially those ranging $5\% - 12\%$.

Such differences in beliefs lead the two agents to insure each other against the types

of disasters they fear more about. Such trading can be implemented with a continuum of disaster insurance contracts with coverage specific to the disaster size. Panel B plots drops in the equilibrium consumption (level) for the two agents when disasters of different sizes occur, assuming that Agent B (the optimist) owns 10% of total wealth. The graph shows that through disaster insurances, Agent A is able to reduce her consumption downfall in large disasters (comparing the solid line to the dotted line). For example, her own consumption will only fall by 24% in a disaster where aggregate consumption falls by 40%, a sizable reduction especially considering the small amount of wealth that Agent B has. At the same time, she also provides insurances to Agent B on smaller disasters, which increases her consumption losses when such disasters strike. This trading strategy is somewhat analogous to a "bear put spread" in option trading. Agent B's consumption drops are close to a mirror image of Agent A's. However, the changes are magnified both for large and small disasters due to her small wealth share.

Panel C shows the by-now familiar exponential drop in the equity premium as the wealth share of the optimist increases. The equity premium is 4.4% for a population entirely consisted of agents who form their beliefs about disasters based on international data, but drops to 2.0% when just 10% of total wealth is allocated to the agents who form their priors using only the US data. The main reason for the lower equity premium is due to the decrease of the jump-risk premium (plotted in Panel D), which falls from 6.5 to 4.0. This effect alone drives the equity premium to 2.4%. Finally, it is interesting to notice that the jump-risk premium is no longer monotonic in the wealth share of Agent B. This is due to the fact that when Agent A has little wealth, she would be selling insurances for small disasters so aggressively that the effect of bigger losses in small disasters can dominate the effect of smaller losses in bigger disasters.

In summary, the results from this calibrated model of heterogeneous beliefs about disasters demonstrate that our main findings on how risk sharing quickly reduces the equity premium are robust to general specifications of disagreements.
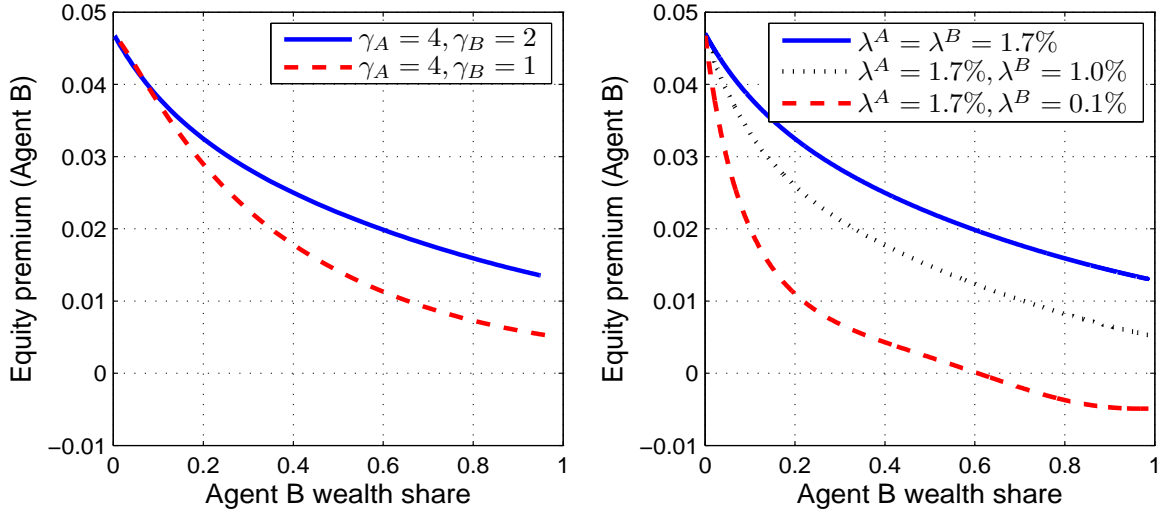
Figure 7: **The effects of heterogeneous risk aversion.** The left panel plots the equity premium when the agents have the same beliefs but different risk aversion. The right panel plots the joint effects of heterogeneous beliefs and heterogeneous risk aversion. In the three scenarios the two agents have risk aversion $\gamma_A = 4, \gamma_B = 2$.

5 Heterogeneous Risk Aversion

Intuitively, besides heterogeneous beliefs, heterogeneity in risk aversion should also be able to induce risk sharing among agents and reduce the equity premium in the equilibrium. Recall that in the case when disaster size is constant, the jump-risk premium is $\lambda_t^{\mathbb{Q}}/\lambda_t^{\mathbb{P}_A} = e^{-\gamma \Delta c}$, which is just as sensitive to changes in disaster size Δc as it is to changes in the relative risk aversion γ. Thus, we expect that heterogeneous risk aversion can have similar effects on the equity premium as heterogeneous beliefs on disasters.

To check this intuition, we consider the following special case of the model. Agent A is the same as in Example I with constant disaster size: $\lambda^A = 1.7\%$, $\gamma_A = 4$. Agent B has identical beliefs about disasters but is less risk averse: $\lambda^B = 1.7\%$, $\gamma_B < \gamma_A$. The left panel of Figure 7 plots the equity premium as a function of Agent B's wealth share for two levels of γ_B. The equity premium does decline as Agent B's wealth share rises. However, the decline is slow and closer to being linear. When $\gamma_B = 2$, in order for the equity premium to fall by 2%, the wealth share of the less risk-averse agent needs to rise from 0 to 30%. The decline in the equity premium becomes faster as we further reduce the risk aversion of Agent B (see the

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dash line), but the non-linearity is still less pronounced than in the cases with heterogeneous beliefs.

Combining heterogenous beliefs about disasters and different risk aversion can amplify risk sharing and accelerate the decline in the equity premium. As shown in the right panel, if Agent B believes disasters are less likely than does Agent A, and she happens to be less risk averse, the equity premium falls even faster. Consider the case where Agent B believes disasters only occur once every hundred years (dotted line). With 20% of total wealth, she drives the equity premium down by almost a half to 2.5%.

6 Concluding Remarks

We illustrate the equilibrium effects of reasonable disagreement about disasters on risk premia and trading activities of agents. When agents disagree about disaster risk, optimist may insure pessimist agents against their disaster risk. Because of the highly non-linear effect of disaster on risk premia, disagreement and the related trading activities greatly attenuate the effect of disasters on the equity premium. Our analysis suggests a potentially important role in such models for market incompleteness where agents may not be able to effectively hedge such risks. While we model disagreement through a fixed prior, additional channels such as parameter uncertainty, learning, and ambiguity aversion may also have important consequence in the model of heterogeneous beliefs about disasters.

Appendix

A Securities' prices and portfolio positions

In this appendix we present the expressions of first agent's consumption claim price, catastrophe bond price, and the equilibrium portfolio positions. Closed-form expressions are obtained in the following simple setting where agents have same relative risk aversion *gamma*, and constant-size disasters.

Price of Agent A's consumption claim

$$P_t^A = \int_t^\infty E_t \left[\frac{M_{t+T}}{M_t} C_{t+T}^A \right] dT$$

where by expanding the binomial

$$
\begin{aligned}
E_t \left[M_{t+T} C_{t+T}^A \right] &= E_t \left[e^{-\rho(t+T)} (C_{t+T}^A)^{1-\gamma} \right] = e^{-\rho(t+T)} E_t \left[\left(1 + (\tilde{\zeta}_{t+T})^{1/\gamma} \right)^{1-\gamma} C_{t+T}^{1-\gamma} \right] \\
&= e^{-\rho(t+T)} C_t^{1-\gamma} \sum_{k=0}^{\gamma-1} \left(\begin{array}{c} \gamma - 1 \\ k \end{array} \right) E_t \left[\frac{(\tilde{\zeta}_{t+T})^{k/\gamma} C_{t+T}^{1-\gamma}}{C_t^{1-\gamma}} \right]
\end{aligned}
$$

Plugging in the explicit expressions for aggregate consumption C_t and state price density M_t, we obtain Price of Agent A's consumption claim

$$P_t^A = C_t \sum_{k=0}^{\gamma-1} \frac{\alpha_{k,t}^A}{\beta_k^A}$$

with

$$
\begin{aligned}
\alpha_{k,t}^A &\equiv \left(\begin{array}{c} \gamma - 1 \\ k \end{array} \right) \frac{(\tilde{\zeta}_t)^{k/\gamma}}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} \\
\beta_k^A &\equiv \rho + (\gamma-1)\bar{g} - \frac{1}{2}\sigma_c^2(\gamma-1)^2 - \bar{\lambda}(e^{(\gamma-1)\bar{d}+\frac{ka}{\gamma}} - 1) + \frac{\bar{\lambda}k}{\gamma}(e^a - 1) \quad (30)
\end{aligned}
$$

The restriction $\beta_k^A > 0$ is needed to ensure finite value for $P_{A,t}$.

Price of catastrophe bond

Let $P_{t,t+T}^{DI}$ denotes the price of disaster insurance (or catastrophe bond) which pays 1\$ at maturity time $t+T$ if there was at least one disaster taking place in the time interval $(t, t+T)$. In the main text we consider disaster insurance P_t^{DI} of maturity $T = 1$ in particular.

$$P_{t,t+T}^{DI} = E_t \left[\frac{M_{t+T}}{M_t} \mathbf{1}_{(N_{t+T} > N_t)} \right] = \frac{e^{-\rho T}}{(C_t^A)^{-\gamma}} E_t \left[(C_T^A)^{-\gamma} \mathbf{1}_{(N_{t+T} > N_t)} \right] \quad (31)$$

$$= \frac{e^{(-\rho - \gamma \bar{g} + \frac{1}{2}\gamma^2 \sigma_c^2)T}}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} E_t [e^{\gamma \bar{d} \Delta N_T} (1 + (\tilde{\zeta}_{t+T})^{1/\gamma} e^{(a\Delta N_T - \bar{\lambda}T(e^a - 1))/\gamma})^\gamma \mathbf{1}_{(\Delta N_T > 0)}]$$

$$= \frac{e^{(-\rho - \gamma \bar{g} + \frac{1}{2}\gamma^2 \sigma_c^2)T}}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} \{ E_t [e^{\gamma \bar{d} \Delta N_T} (1 + (\tilde{\zeta}_{t+T})^{1/\gamma} e^{(a\Delta N_T - \bar{\lambda}T(e^a - 1))/\gamma})^\gamma]$$

$$- (1 + (\tilde{\zeta}_t)^{1/\gamma} e^{-\bar{\lambda}T(e^a - 1)/\gamma})^\gamma \mathrm{Prob}(\Delta N_T = 0) \}$$

where $\Delta N_T \equiv N_{t+T} - N_t$ is number of disasters taking place in $[t, t + T]$, and $\mathrm{Prob}(\Delta N_T = 0) = e^{-\bar{\lambda}T}$ is the probability that no such disaster did happen. Again by expanding the binomial $(1 + (\tilde{\zeta}_{t+T})^{1/\gamma} e^{(a\Delta N_T - \bar{\lambda}T(e^a - 1))/\gamma})^\gamma$, and then computing the expectation of each resulting term, we obtain

$$P_{t,t+T}^{DI} = \frac{a_T}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} \{ [\sum_{k=0}^{\gamma} b_{k,T} (\tilde{\zeta}_t)^{k/\gamma}] - e^{-\bar{\lambda}T} (1 + (\tilde{\zeta}_t)^{1/\gamma} e^{-\bar{\lambda}T(e^a - 1)/\gamma})^\gamma \}$$

where

$$a_T = e^{(-\rho - \gamma \bar{g} + \frac{1}{2}\gamma^2 \sigma_c^2)T}$$

$$b_{k,T} = \begin{pmatrix} \gamma \\ k \end{pmatrix} e^{-\bar{\lambda}kT(e^a - 1)/\gamma} e^{\bar{\lambda}T[e^{(\gamma d + \frac{ak}{\gamma})} - 1]}$$

Equilibrium portfolio positions

In the current case of constant jump size with two dimensions of uncertainties (Brownian motion and disaster jump), the market is complete when agents are allowed to trade contingent claims on aggregate consumption (stock) P_t, money market account RFB_t and disaster insurance P_t^{DI}. We can use generalized Ito lemma on jump-diffusion price processes P_t^A, P_t, P_t^{DI} to write generically (where indexes b and j respectively correspond to Brownian and

jump shocks)

$$dP_t^A = \sigma^{P_A,b}dB_t + \sigma^{P_A,j} + \mathcal{O}(dt) \tag{32}$$

$$\text{where } \sigma^{P_A,b} = P^A\sigma_c; \quad \sigma^{P_A,j} = P_{t+}^A - P_{t-}^A$$

$$dP_t = \sigma^{P,b}dB_t + \sigma^{P,j} + \mathcal{O}(dt) \tag{33}$$

$$\text{where } \sigma^{P,b} = P\sigma_c; \quad \sigma^{P,j} = P_{t+} - P_{t-}$$

$$dP_t^{DI} = \sigma^{DI,b}dB_t + \sigma^{DI,j} + \mathcal{O}(dt) \tag{34}$$

$$\text{where } \sigma^{DI,b} = 0; \quad \sigma^{DI,j} = RFB_{t,t+T} - P_t^{DI}$$

where the sensitivity of catastrophe bond with respect to the jump is derived from the fact that, immediately after the jump, the catastrophe bond will surely pays 1\$ at maturity, so its post-jump price is equal to that of a riskfree bond $RFB_{t,t+T}$ of the same maturity.

From another perspective, the self-financing property of agent A's portfolio $\{\theta_{P,t}^A, \theta_{DI,t}^A, \theta_{RFB,t}^A\}$ (these are agent A's positions in stock, disaster insurance and instantaneously risk-free bond respectively):

$$
\begin{aligned}
dP^A &= \theta_t^{A,P}dS_t + \theta_t^{A,DI}dP_t^{DI} + \theta_t^{A,RFB}dRFB_t + \mathcal{O}(dt) \\
&= \#dt + (\theta_t^{A,P} \ \theta_t^{A,DI})\begin{pmatrix} \sigma^{P,b} & \sigma^{P,j} \\ \sigma^{DI,b} & \sigma^{DI,j} \end{pmatrix}\begin{pmatrix} dB_t \\ \Delta N_t \end{pmatrix}
\end{aligned} \tag{35}
$$

By identifying the diffusion and jump parts of dP^A in (32), (35) we have

$$(\theta_t^{A,P} \ \theta_t^{A,DI})\begin{pmatrix} \sigma^{P,b} & \sigma^{P,j} \\ \sigma^{DI,b} & \sigma^{DI,j} \end{pmatrix} = (\sigma^{P_A,b} \ \sigma^{P_A,j}) \Rightarrow \begin{pmatrix} \theta_t^{1,P} \\ \theta_t^{1,DI} \end{pmatrix} = \begin{pmatrix} \sigma^{P,b} & \sigma^{DI,b} \\ \sigma^{P,j} & \sigma^{DI,j} \end{pmatrix}^{-1}\begin{pmatrix} \sigma^{P_A,b} \\ \sigma^{P_A,j} \end{pmatrix} \tag{36}$$

We need the "sensitivities" $\sigma^{P,b}, \sigma^{DI,b}, \sigma^{P,j}, \sigma^{DI,j}, \sigma^{P_A,b}, \sigma^{P_A,j}$ in (32), (33), (34) to determine

the portfolio positions.

$$
\begin{pmatrix} \theta_t^{A,P} \\ \theta_t^{A,DI} \end{pmatrix} = \begin{pmatrix} \sigma^{P,b} & 0 \\ \sigma^{P,b} & \sigma^{DI,j} \end{pmatrix}^{-1} \begin{pmatrix} \sigma^{P_A,b} \\ \sigma^{P_A,j} \end{pmatrix} = \begin{pmatrix} \frac{\sigma^{P_A,b}}{\sigma^{P,b}} \\ -\frac{\sigma^{P,j}\sigma^{P_A,b}}{\sigma^{P,b}\sigma^{DI,j}} + \frac{\sigma^{P_A,j}}{\sigma^{DI,j}} \end{pmatrix}
$$

And agent A's position in money market account

$$
\theta_t^{A,RFB} = P_t^A - \theta_t^{A,P} P_t - \theta_t^{A,DI} P_t^{DI} \tag{37}
$$

We note in particular, from (32), (33) we have $\sigma^{P_A,b} = P_1\sigma_c$, $\sigma^{P,b} = P\sigma_c$, so stock position is $\theta_t^{1,P} = \frac{\sigma^{P_A,b}}{\sigma^{P,b}} = \frac{P_t^A}{P_t}$, or value fraction invested in stock of agent 1 is always one

$$
\frac{\theta_t^{A,P} P_t}{P_t^A} = 1
$$

Thus from (37), agent 1's position is riskless bond is $\theta_t^{1,RFB} = -\theta_t^{1,DI} P_t^{DI}$. Agent B's portfolio positions can be found from market clearing condition: $\theta^{B,P} = 1 - \theta^{A,P}$; $\theta^{B,DI} = -\theta^{A,DI}$; $\theta^{B,RFB} = -\theta^{A,RFB}$.

B Boundedness of prices

This appendix discusses the boundedness of securities prices in general heterogeneous-agent economy. As claimed in the main text, as long as agents have different but equivalent beliefs, necessary and sufficient condition for finite price of a security in heterogeneous-agent economy is that this price be finite under each agent's beliefs in a single-agent economy. The proof proceeds as follows.

Suppose that the security pays dividend stream D_t (which can be either continuous or discrete in time). Let us denote S, S^A, S^B its prices in heterogeneous-agent, and single-agent

economies respectively.

$$
\begin{aligned}
S_t^A &= E_t\left[\int_0^\infty \frac{\tilde{\zeta}_{t+\tau}^A C_{t+\tau}^{-\gamma^A}}{\tilde{\zeta}_t^A C_t^{-\gamma^A}} D_{t+\tau} d\tau\right] \\
S_t^B &= E_t\left[\int_0^\infty \frac{\tilde{\zeta}_{t+\tau}^B C_{t+\tau}^{-\gamma^B}}{\tilde{\zeta}_t^B C_t^{-\gamma^B}} D_{t+\tau} d\tau\right] \\
S_t &= E_t\left[\int_0^\infty \frac{\tilde{\zeta}_{t+\tau}^A (C_{t+\tau}^A)^{-\gamma^A}}{\tilde{\zeta}_t^A (C_t^A)^{-\gamma^A}} D_{t+\tau} d\tau\right] = E_t\left[\int_0^\infty \frac{\tilde{\zeta}_{t+\tau}^B (C_{t+\tau}^B)^{-\gamma^B}}{\tilde{\zeta}_t^B (C_t^B)^{-\gamma^B}} D_{t+\tau} d\tau\right]
\end{aligned}
$$

where the last equality is a consequence of the FOC in heterogenous-agent economy.

Necessary condition $S < \infty \Rightarrow S^A, S^B < \infty$: This is immediate by noting that since individual consumptions are always non-negative $0 \leq C_{t+\tau}^A, C_{t+\tau}^B \leq C_{t+\tau} \ \forall \tau$, we have

$$
\tilde{\zeta}_{t+\tau}^A (C_{t+\tau}^A)^{-\gamma^A} \leq \tilde{\zeta}_{t+\tau}^A C_{t+\tau}^{-\gamma^A},
$$

and thus S_t^A is finite whenever S_t is finite. By identical reason, S_t^B is finite whenever S_t is finite.

Sufficient condition $S^A, S^B < \infty \Rightarrow S < \infty$: This is straightforward by noting that, for *any* fixed number $k \in (0, 1)$ (without loss of generality, we can e.g. fix $k = 0.5$ to visualize this):

$$
\frac{C^A}{C} < k \Leftrightarrow \frac{(C^A)^{-\gamma^A}}{C^{-\gamma^A}} > k^{-\gamma^A} \Rightarrow \frac{C^B}{C} > 1 - k \Leftrightarrow \frac{(C^B)^{-\gamma^B}}{C^{-\gamma^B}} < (1-k)^{-\gamma^B}
$$

and vice versa. That is, at any moment $t + \tau$, the integrand of price S is *always* bounded (up to a finite factor) by either the integrand of S^A or S^B. Now as both S^A, S^B are finite, S is also finite.[9]

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[9]The technical point that sum of possibly infinite numbers of same-direction inequalities remain an inequality of same direction is assured simply by the boundedness of both S^A, S^B.

References

Barro, R. J., 2006, "Rare disasters and asset markets in the twentieth century," *The Quarter Journal of Economics*, 121, 823–866.

Basak, S., 2005, "Asset Prices with Heterogenous Beliefs," *Journal of Banking & Finance*, 29, 2849–2881.

Bates, D. S., 2008, "The Market for Crash Risk," *Journal of Economic Dynamics and Control*, 32, 22912321.

Buraschi, A., and A. Jiltsov, 2006, "Model Uncertainty and Option Markets with Heterogeneous Beliefs," *Journal of Finance*, 61, 2841–2897.

Chan, Y. L., and L. Kogan, 2002, "Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices," *Journal of Political Economy*, 110, 1255–1285.

Chen, H., and S. Joslin, 2009, "Generalized Transform Analysis of Affine Processes And Asset Pricing Applications," Working Paper, MIT.

Cox, J. C., and C. Huang, 1989, "Optimum Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, 33–83.

David, A., 2008, "Heterogeneous Beliefs, Speculation, and the Equity Premium," *Journal of Finance*, 63, 41–83.

Dieckmann, S., 2009, "Rare Event Risk and Heterogeneous Beliefs: The Case of Incomplete Markets," Working Paper.

Dieckmann, S., and M. Gallmeyer, 2005, "The equilibrium allocation of diffusive and jump risks with heterogeneous agents," *Journal of Economic Dynamics and Control*, 29, 1547–1576.

Donaldson, J., and R. Mehra, 2008, "Risk Based Explanations of the Equity Premium," in R. Mehra (ed.), *Handbook of the Equity Risk Premium*, Elsevier, Amsterdam, Netherlands.

Duffie, J. D., J. Pan, and K. Singleton, 2000, "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, 68, 1343–1376.

Dumas, B., 1989, "Two-Person Dynamic Equilibrium in the Capital Market," *Review of Financial Studies*, 2, 157–188.

Dumas, B., A. Kurshev, and R. Uppal, 2009, "Equilibrium Portfolio Strategies in the Presence of Sentiment Risk and Excess Volatility," *Journal of Finance*, 64, 579–629.

Farhi, E., and X. Gabaix, 2009, "Rare Disasters and Exchange Rates," Working papers, National Bureau of Economic Research, Inc.

Gabaix, X., 2009, "Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance," Working paper, National Bureau of Economic Research, Inc.

Hansen, L., and T. J. Sargent, 2009, "Fragile Beliefs and the price of uncertainty," University of Chicago, working paper.

Hansen, L. P., 2007, "Beliefs, Doubts and Learning: Valuing Macroeconomic Risk," *American Economic Review*, 97, 1–30.

Kogan, L., S. A. Ross, J. Wang, and M. M. Westerfield, 2006, "The Price Impact and Survival of Irrational Traders," *Journal of Finance*, 61, 195–229.

Liu, J., J. Pan, and T. Wang, 2005, "An Equilibrium Model of Rare-Event Premia and Its Implication for Option Smirks," *Review of Financial Studies*, 18, 131–164.

Longstaff, F., and J. Wang, 2008, "Asset Pricing and the Credit Market," Working Paper, UCLA and MIT.

Longstaff, F. A., and M. Piazzesi, 2004, "Corporate earnings and the equity premium," *Journal of Financial Economics*, 74, 401–421.

Rietz, T. A., 1988, "The Equity Premium: A Solution," *Journal of Monetary Economics*, 22, 117–131.

Wachter, J., 2009, "Can time-varying risk of rare disasters explain aggregate stock market volatility?," University of Pennsylvania, working paper.

Wang, J., 1996, "The Term Structure of Interest Rates in a Pure Exchange Economy with Heterogenous Investors," *Journal of Financial Economics*, 41, 75–110.

Xiong, W., and H. Yan, 2009, "Heterogeneous Expectations and Bond Markets," *Review of Financial Studies,* forthcoming.

Yan, H., 2008, "Natural Selection in Financial Markets: Does It Work?," *Management Science*, 54, 1935–1950.