Dynamic Asset Pricing in a System of Local Housing Markets

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For most people, buying a house is one of the most significant investment decisions of their lifetimes. Economists have mainly focused on the consumption aspects of this process. For example, a typical model in urban economics might frame the decision of where to live as a discrete choice over a bundle of housing and neighborhood attributes such as location, square footage, schooling options, and crime levels. The investment side of the problem has received considerably less attention, a surprising omission since housing assets comprise approximately two-thirds of the average American household's financial portfolio, serve an important role in saving for retirement and, as has become increasingly apparent, can be quite risky.

This paper views housing markets from an asset-pricing perspective, using finance theory to relate the risk premium of a housing asset (the difference between its expected return and the return for a risk-free investment) to its exposure to risk. As usual in finance, what matters for the risk premium of a housing asset is its exposure to systematic risk, not idiosyncratic risk. In our model, there are two forms of systematic risk to which housing assets are exposed: *national risk* (which is common to houses everywhere) and *local risk* (which affects all houses within a given metropolitan area, but nowhere else). Houses are said to be of the same type h if they are located in the same metropolitan area and have the same exposure to systematic risk. Our main conclusions are that (1) houses of every type face a common set of risk prices (λ^* for the national risk and λ^m for the local risk specific to metropolitan area m) that, together with appropriate measures of exposure to risk, account for the variation in risk-premiums across housing types and (2) the parameters measuring exposure to systematic risk factors can be estimated using transactions data for repeat sales of houses, data that are now readily available. We also analyze a version of the model that takes net rents rather than house values as the primitives. This special case provides some intuition regarding the impact of rent and risk premiums on the growth of house values.

I. A Model of Housing Market Risk

The setting is a collection of N single-family housing units located in M metropolitan areas. Besides metropolitan location, houses are classified into K categories. We refer to a specific pairing $h = (m, k)$ as a *housing type*, for example a large house in Los Angeles. The model is formulated as a system of stochastic differential equations (SDE's) driven by a multi-dimensional Wiener process, using as a framework the standard "multidimensional market model" in finance (see Duffie (2001) or Shreve (2004)).

We assume that our national housing market is observed over a time interval $[0, T] \subset \Re$, for example the 20-year period $[0, 20]$. The price process of house i of type $h = (m, k)$ is assumed to be the solution to the SDE

$$(1) \qquad dV_t^i = V_t^i \left[\alpha^h dt + \sigma^h dB_t^i\right]$$

Equation (1) expresses the instantaneous rate of price appreciation dV_t^i / V_t^i of house i at time t as the sum of an expected rate of price appreciation $\alpha^h dt$ and a random shock $\sigma^h dB_t^i$, where α^h (the *drift*) and σ^h (the *volatility*) are parameters and dB_t^i is the stochastic differential of a Wiener process associated with house i. The stochastic differential dB_t^i is in turn assumed to be a linear combination of three underlying *risk factors*,

$$(2) \quad dB_t^i := \frac{\sigma^{h*}}{\sigma^h} dW_t^* + \frac{\sigma^{hm}}{\sigma^h} dW_t^h + \frac{\sigma^{hh}}{\sigma^h} dW_t^i$$

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where dW_t^*, dW_t^m and dW_t^i are stochastic differentials of Wiener processes representing *national risk* W^*, *local risk* W^m specific to metropolitan area m, and *idiosyncratic risk* W^i specific to housing asset i. The parameters σ^{h*}, σ^{hm} and σ^{hh} are *covariation parameters* that measure the sensitivity of dB_t^i to the national risk factor, the local risk factor for metropolitan area m and the idiosyncratic risk factor specific to house i. The volatility parameter σ^h of equation (1) is linked to the covariation parameters of equation (2) by the following identity,

$$(\sigma^h)^2 := (\sigma^{h*})^2 + (\sigma^{hm})^2 + (\sigma^{hh})^2 \tag{3}$$

The price process of every house is assumed to be governed by a SDE of the form given by equations (1)–(3), all defined on a common filtered probability space $(\Omega, \mathfrak{F}, \mathbf{F}, P)$.

Finance imposes equilibrium restrictions on this collection of asset-price processes not by equating supply and demand for each type of asset nor by some other means of relating asset prices to fundamentals, but instead by imposing the hypothesis that in equilibrium every possible opportunity for arbitrage has been eliminated: i.e., no self-financing portfolio comprised of houses and the risk-free asset can make a positive profit with no risk of loss unless the initial investment is strictly positive a.s. (i.e., with probability one).

The *gain process* $G^i = (G_t^i)_{t\in[0,T]}$ associated with housing asset i is defined by $G_t^i = V_t^i + D_t^i$ where $D_t^i := \int_0^t \rho_t^i dt$ and ρ_t^i is the cash flow (net of expenses) received by the owner of the asset at time t. Thus $G_t^i - G_0^i$ is the sum of the *capital gain* $V_t^i - V_0^i$ and the *accumulated net cash flow* D_t^i accruing to an investor holding the asset over the interval $[0, t]$. For a landlord, ρ_t^i is simply the flow of rental income less expenses for maintenance, repairs and the like, which we will refer to as *net rental flow*. For a homeowner, ρ_t^i is the inputed net rental flow.[1]

The *Fundamental Theorem of Asset Pricing* asserts that, provided the housing market eliminates all arbitrage opportunities, there exists a pricing process $Z = (Z_t)_{t\in[0,T]}$ such that the *risk-adjusted* gain process ZG^i for every housing asset i is a martingale: i.e., for all $s, t \in [0, T]$ such that $t \geq s$

$$E(Z_t^i G_t^i \mid \mathfrak{F}_s) = Z_s^i G_s^i$$

where $\mathfrak{F}_s$ is the information set at time s for the filtered probability space $(\Omega, \mathfrak{F}, \mathbf{F}, P)$ on which all of the stochastic processes in the model are defined. When the price processes are as specified in equations (1)–(3), then the risk-pricing process Z takes a simple form. It is the stochastic process generated by the SDE

$$dZ_t = -Z_t \left[\lambda^* dW_t^* + \sum_m \lambda^m dW_t^m \right] \tag{4}$$

with $Z_0 = 1$, where the summation is over all metropolitan areas. The fact that ZG^i is a martingale implies that this process, which itself is generated by a SDE, must have zero drift. Assume the ratio ρ_t^i / V_t^i of net rent to house value is the same for all housing assets of type h and that this *net rental yield* δ_t^h remains constant over time.[2] Then

$$\alpha^h + \delta^h = \lambda^* \sigma^{h*} + \lambda^m \sigma^{hm} \tag{5}$$

for every housing type h. In the finance literature, equations (5), one for each asset type h, are called the *market-price-of-risk equations.*[3] The left-hand side of equation (5) is the *risk premium* of housing type h, the expected instantaneous total return (i.e., capital gains plus net rental yield) at time t, net of the risk-free rate. The right-hand side is the total value of risk exposure for a housing asset of type h, the sum of the price λ^* of national risk times the exposure σ^{h*} to that risk plus the price λ^m of local risk times the exposure σ^{hm} to that risk.

Rather than multiplying the gain G_t^i by Z_t, there is an equivalent way to adjust for risk by changing the probability measure. The value Z_T of the pricing process at time T is a Radon-Nikodym derivative $d\tilde{P}/dP$ that changes the true probability measure P to an equivalent

[1] Estimating ρ_t^i for homeowners is more difficult than for landlords. As we will see, no-arbitrage theory provides a way around this problem.

[2] Shortly we provide a proof that, if net rents are generated by a geometric Brownian motion, then the net rental yield must be constant.

[3] See Shreve (2004).

martingale measure (EMM). Under the EMM $\tilde{P}$ the gain process G^i itself, rather than the risk-adjusted gain process ZG^i, is a martingale: i.e., for all $s, t \in [0, T]$ such that $t \geq s$

$$\tilde{E}(G_t^i \mid \mathfrak{F}_s) = G_s^i$$

where the tilde on the expectation sign indicates that the conditional expectation is taken with respect to $\tilde{P}$. If probabilities are adjusted for risk, assets can be priced "as though" investors are risk neutral, even though they are not.

Establishing a connection between housing value and net rental flow provides a nice illustration of an arbitrage-based approach to asset pricing. As usual, it is easier to establish a link to fundamentals if we assume an infinite horizon, so for the moment we replace the time set $[0, T]$ with the time set $[0, \infty)$. We take the discounted net rental processes ρ^i as the primitives of our model, demonstrating below that this is equivalent to a model in which the value processes V^i are the primitives. Although the analysis can be generalized to handle time-varying parameters, we assume drift and volatility are constant. In a more general model, these net rental processes might depend on the value households derive from living in a particular house, including its physical features, local amenities and labor market opportunities.

When expectations are taken with respect to the EMM, house values equal the expected discounted value of future rents net of expenses. For this reason, it is easier to analyze the connection between value and net rent under the probability measure $\tilde{P}$. Letting ρ_t^i denote the flow of discounted net rent for house i at time t, suppose the process ρ^i is a geometric Brownian motion generated by the SDE

$$(6) \qquad d\rho_t^i = \rho_t^i[\tilde{\eta}^h dt + \sigma^h d\tilde{B}_t^i]$$

where $\tilde{B}_t^i$ is a Wiener process under $\tilde{P}$. The stochastic differential $d\tilde{B}_t^i$ is assumed to be a linear combination of national, local and idiosyncratic risk factors,

$$(7) \quad d\tilde{B}_t^i = \frac{\sigma^{h*}}{\sigma^h} d\tilde{W}_t^* + \frac{\sigma^{hm}}{\sigma^h} d\tilde{W}_t^m + \frac{\sigma^{hh}}{\sigma^h} d\tilde{W}_t^i$$

where $\tilde{W}_t^*$, $\tilde{W}^m$ and $\tilde{W}^i$ are Wiener processes under $\tilde{P}$ (compare equations (1) and (2) describing the SDE generating the value process V^i). In equation (6), $\tilde{\eta}^h$ is the drift of discounted net rent under $\tilde{P}$. Assume that $\tilde{\eta}^h < 0$ and define the discounted value process V^i by $V_t^i = \tilde{E}\left[\int_t^\infty \rho_u^i \, du \mid \mathfrak{F}_t\right]$ for $t \in [0, \infty)$. It follows that $V_t^i = -\rho_t^i / \tilde{\eta}^h$. Thus, under the probability measure $\tilde{P}$ the net rental yield of a house of type h is the same for all houses i of type h, and it is time invariant. Because P and $\tilde{P}$ are equivalent measures, this relationship also holds under P: i.e.,

$$(8) \qquad \delta^h := \frac{\rho_t^i}{V_t^i} = -\tilde{\eta}^h \qquad (P\text{-a.s})$$

Letting $\delta^h = -\tilde{\eta}^h$ in equation (5),

$$(9) \qquad \alpha^h = \tilde{\eta}^h + \lambda^* \sigma^{h*} + \lambda^m \sigma^{hm}$$

which offers an alternative perspective on the market-price-of-risk equation. If risk-prices are zero (so investors are in fact risk neutral) then $r + \alpha^h = r + \tilde{\eta}^h$: price appreciation on houses of type h equals the risk-free rate plus the expected rate of increase of net rent under the EMM. On the other hand, if risk prices are positive and the covariation parameters are positive, then

$$\alpha^h - \tilde{\eta}^h = \lambda^* \sigma^{h*} + \lambda^m \sigma^{hm} > 0$$

House values appreciate at a more rapid rate than $\tilde{\eta}^h$ to compensate for the risk.

What happens to the process ρ^i under the true probability measure P? Girsanov's Theorem, used to derive equation (5), implies that

$$(10) \qquad d\tilde{B}_t^i = dB_t^i + \left(\frac{\alpha^h + \delta^h}{\sigma^h}\right) dt$$

Substituting (10) into (6) and using (8) to simplify, we obtain $d\rho_t^i = \rho_t^i[\alpha^h dt + \sigma^h dB_t^i]$: under P the drift in ρ^i matches the drift in V^i. Because V^i is a scalar multiple of ρ^i, under $\tilde{P}$ $dV_t^i = V_t^i[\tilde{\eta}^h dt + \sigma^h d\tilde{B}_t^i]$. Using (10) to substitute for $d\tilde{B}_t^i$ yields equation (1), the SDE for V^i under the true probability measure P.

We conclude that, in this special case where net rent follows a geometric Brownian motion, (1) the net rent to value ratio is constant for all houses of the same type and (2) the growth rate dV_t^i/V_t^i of house value and the growth rate $d\rho_t^i/\rho_t^i$ of net rents are driven by the same process. By restricting this infinite horizon model to the interval $[0,T]$, these conclusions carry over immediately to our original finite-horizon model.

II. Hedonic Returns

In contrast to purely financial assets such as stocks or bonds, housing assets are heterogeneous and trade at very low frequency. However, data on repeat sales can be used to overcome these problems. Assume that $[0,T]$ is divided into N intervals $(t_{n-1},t_n]$, say months. Let $R^i := \log(V_t^i/V_s^i)$ denote the logarithmic return for a housing asset of type $h=(m,k)$ that sells at time s and again at time t, where the selling times $s,t \in [0,T]$ are assumed rounded to the beginning or end of a month. Define $\tau^i := t-s$, the duration of repeat sale i, and let M^i denote the set of months covered by this repeat sale. Define $\Delta t_n := t_n - t_{n-1}$, the length of month n. Similarly, let $\Delta W_{t_n}^*$ and $\Delta W_{t_n}^m$ denote the increments over month n of the Wiener processes W^* and W^m respectively.

Solving the stochastic differential equation (1) it is easy to show that

$$R^i = \gamma^h \tau^i + \sum_{n\in M^i} r_n^h + \varepsilon^i \tag{11}$$

where $\gamma^i := -(\sigma^{hh})^2/2$, $\varepsilon^i := \sigma^{hh}(W_t^i - W_s^i)$, and

$$\begin{aligned} r_n^h \quad &:= \quad [\alpha^h - \frac{(\sigma^{h*})^2+(\sigma^{hm})^2}{2}]\Delta t_n \\ &\qquad +\sigma^{h*}\Delta W_{t_n}^* + \sigma^{hm}\Delta W_{t_n}^m \end{aligned}$$

Let N^h be the set of repeat sales of houses of type $h=(m,k)$ over the time interval $[0,T]$. For $n=1,2,\ldots,N$ let $I_{\{n\in M^i\}}$ be an indicator variable that equals 1 if month n is covered by the i^{th} repeat sale and 0 otherwise. In regression form equation (11) becomes

$$R^i = \gamma^h \tau^i + \sum_{n=1}^{N} r_n^h I_{\{n\in M^i\}} + \varepsilon^i \tag{12}$$

The coefficient r_n^h in this regression is the portion of the logarithmic return for month n that is common to all housing assets of type h. We refer to the monthly time series $(r_n^h)_{n=1}^N$ generated by these regressions as *hedonic returns.*

Equation (12) bears more than a passing resemblance to the methods used by Karl Case and Robert Shiller (1989) to construct housing price indices. The differencing used to obtain the logarithmic return for the i^{th} repeat sale allows for fixed effects: the constant term V_0^i specific to repeat sale i drops out of the expression for the logarithmic return. Thus, the *level* of housing prices is allowed to be quite heterogeneous, even for houses of the same type. The homogeneity we impose only requires that log returns (the *increment* to log prices over a fixed interval of time) for houses of the same type are drawn from the same distribution. Because we see only a single realization, in this regression the realizations of W^*, W^m and W^i are fixed, but there are N^h random variables ε^i, one for each repeat sale. By definition of the Wiener processes W^i, the expectation $E\varepsilon^i = 0$ and the disturbances are independently distributed. Consequently, the parameters of equation (11) can be consistently estimated using OLS. Equation (11) highlights two effects of duration on the return. First, the variance of the disturbance term for repeat sale i is $(\sigma^{hh})^2\tau^i$. As in Case and Shiller, this heteroskedasticity is easily handled. Second, duration has a direct effect on the mean return: the regression coefficient γ^h on the duration τ^i of the i^{th} repeat sale provides an estimate of $-(\sigma^{hh})^2/2$ and hence an estimate for σ^{hh}, the volatility of the idiosyncratic risk for a housing asset i of type h. In this way, deriving equation (11) from a continuous-time structural model leads to a potentially important modification to the classic Case-Shiller specification, a mean correction for duration.

III. Estimating the Model

The market-price-of-risk equations (5) provide H linear equations (one for each house type) in $M+1$ unknowns (the price λ^* of national risk and M local risk prices λ^m). Using the monthly hedonic returns of Section II to estimate the covariation parameters σ^{h*} or σ^{hm} and the price appreciation parameters α^h is straightforward.[4] Estimating net rental yields δ^h is more difficult, especially for houses occupied by homeowners rather than renters. Fortunately, our structural model of risk pricing comes to the rescue. If we know the risk prices λ^* and λ^m, then estimates of α^h, σ^{h*} and σ^{hm} allow us to estimate δ^h. From equation (5) for house type h

$$\delta^h = \lambda^* \sigma^{h*} + \lambda^m \sigma^{hm} - \alpha^h \tag{13}$$

All we require is that risk prices be identifiable.

If $H > M$, the parameters α^h, σ^{h*} and σ^{hm} are identified. It follows from equations (5) that the risk prices are identifiable provided we can estimate net rental yields for $M+1$ housing types with at least one located in each metropolitan area. Estimating net rental yields for rental properties is relatively easy. Furthermore, if houses of type h are occupied by homeowners as well as rented, then the net rental yield imputed to homeowners must equal the net rental yield earned by landlords: all of the parameters of equation (5) except δ^h are the same, so the net rental yields must also agree. Thus, what we require for identification is $M+1$ housing types for which some houses are rented, at least one such type for each metropolitan area.

IV. Arbitrage

It is often asserted that arbitrage pricing does not apply to housing markets because the majority of transactions take place between individual owner occupants and the existence of substantial transactions and holding costs limit the ability of other investors to take advantage of arbitrage opportunities. But this view ignores the fact that a number of stake-holders (banks, landlords, developers, and land-owners) have clear financial interest in the market. The economic decisions of these stake-holders impose discipline on the market. The housing-related investments they make compete with alternative potential investments and consequently face the same risk prices. So the landlord's problem disciplines house prices in segments of the market with significant rental activity, owners of undeveloped land that might be developed discipline the returns for properties already in place, and the financial interests of banks discipline the offers buyers make to sellers.

Complex ownership structures arise in many contexts. Fischer Black and Myron Scholes (1973) and Robert Merton (1974) proposed a simple model of corporate finance in which stock is viewed as a call option giving equity-holders the right (but not the obligation) to own the firm provided they pay off the outstanding debt. The BSM model of corporate finance seems at least as relevant to financing a house. Compared to corporations, houses are traded very frequently, and repeat sales provide control for asset heterogeneity. Most housing assets, perhaps even those "owned" by landlords, are highly leveraged, and the debt is usually held by large institutions. These institutions are by far the largest stake-holders in residential real estate, they hold large portfolios of houses, and they have the incentive and the power to make sure that the assets backing this debt are correctly priced. The fact that this debt has increasingly been repackaged into mortgage-backed securities and (supposedly) hedged by credit-default swaps only serves to reinforce the view that housing markets are sophisticated asset markets. The recent market collapse suggests that our understanding of how housing markets price risk is not as good as it should be. This paper takes a step toward improving that understanding.

[4] See our working paper (2009) for details.

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