An Equilibrium Term Structure Model with Recursive Preferences

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Equilibrium, affine asset pricing models with L. Epstein and S. Zin (1989)'s preferences typically generate time-variation in risk premiums through time variation in the *quantity* of risks, with the market *prices* of risks (*MPR*) held constant. This is true of models with built in long-run consumption risks (*LRR*) (e.g., R. Bansal and A. Yaron (2004), R. Bansal, D. Kiku and A. Yaron (2009)), as well as of the broader formulations in B. Eraker and I. Shaliastovich (2008). For pricing bonds[1] such formulations may be overly constrained as reduced-form models suggest that it is time variation in the *MPR*'s, more than stochastic yield volatilities, that resolves the expectations puzzles in bond markets.[2]

Constant *MPR*s are not an inherent feature of equilibrium pricing models with recursive preferences, but rather they arise as a consequence of the linearizations underlying the affine approximations to these models that have been explored empirically. The essential ingredients of these econometric formulations are (P1) recursive (Epstein-Zin) preferences, (P2) risk-neutral ($\mathbb{Q}$), affine pricing, and (P3) the assumption that the state of the economy is described by an affine process under the historical ($\mathbb{P}$) distribution. Key to achieving property (P2), given P1 and P3, is the assumption that the valuation ratio (the log "price/consumption" ratio) associated with the claim that pays aggregate consumption is an affine function of the state.

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[1] Examples of term structure models with *LRR*'s that presume constant MPR's include R. Bansal and I. Shaliastovich (2009) and T. Doh (2008).

[2] E.g., G. Duffee (2002) and Q. Dai and K. Singleton (2002).

We develop a dynamic term structure model with recursive preferences that preserves properties P1 and P2, but that relaxes the assumption that the price/consumption ratio be linear in the state. Preserving P2 ensures our model inherits the tractable pricing of models in which the state process is affine under $\mathbb{Q}$. Equally importantly, allowing the price/consumption ratio to depend nonlinearly on the state– a quadratic function in our case– leads to an equilibrium model with time-varying *MPR*s in addition to state-dependent volatilities (quantities of risk). While we necessarily give up P3– the state follows a nonlinear (non-affine) process under $\mathbb{P}$– we show that the model-implied likelihood function is known in closed form.

Key to obtaining these properties is a new modeling scheme, closely related to that of A. Le, K. Singleton and Q. Dai (2009). The state is assumed to follow an affine process under $\mathbb{Q}$, which is central to delivering analytical expressions for bond prices (P2). We then derive the data-generating process for consumption, inflation, and bond yields from this $\mathbb{Q}$ process using the change-of-measure associated with Epstein and Zin (1989) preferences. In deriving the discrete-time Radon-Nykodym derivative, we adopt a linearization scheme that gives rise to state-dependent *MPR*s whose time variation is *endogenously* determined by investors' preferences. The nonlinear *MPR*s, when combined with a $\mathbb{Q}$-affine state process, result in nonlinear physical dynamics. Nevertheless, the conditional $\mathbb{P}$-density of the state is known in closed form and, hence, so is the likelihood function of the data. We also provide sufficient conditions under which the state is geometrically $\mathbb{P}$-ergodic.

Preserving properties P1 and P2 of the extant literature while relaxing P3 gives us considerably more latitude in modeling the historical joint distribution of consumption, inflation, and bond yields. Within our equilibrium model with recursive preferences, a non-affine structure to the $\mathbb{P}$ distribution of the state arises directly as a consequence of state-dependent *MPR*s. An interesting question for future research is whether the data calls for time-varying *MPR*s and, thereby, for richer models of the data-generating process for consumption, inflation, and yields than has heretofore been explored in the literature on *LRR*s.

I. A Pricing Kernel with Time-Varying MPRs

Following Bansal and Yaron (2004) and others, we assume that agents rank consumption profiles according to the Epstein and Zin (1989)'s recursive utility:

$$U_t = \left[(1-\delta^{\Delta t})(C_t\Delta t)^{\frac{1-\gamma}{\theta}} + \delta^{\Delta t}(E_t U_{t+1}^{1-\gamma})^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}$$

where Δt is the (small) time interval,[3] C_t denotes time-t real consumption rate, δ denotes the (annualized) time discount factor; ψ denotes the inter-temporal elasticity of substitution (IES) while $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. As shown by Epstein and Zin (1989), this recursive utility leads to the following nominal pricing kernel, in log form:

$$(1) \quad m_{t+1} = \theta\, log\,\delta\,\Delta t - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)(r_{c,t+1}-\pi_{t+1}) - \pi_{t+1}$$

where c_{t+1}, $r_{c,t+1}$ and π_{t+1} denote log consumption rate, nominal return on the consumption series and realized inflation, respectively.

We assume that the log price-consumption ratio, $z_t = log(P_t/C_t)$, is quadratic in the vector of state variables, $\mathbf{x}_t$:

$$(2) \qquad z_t = \lambda_0 + \lambda_x'\mathbf{x}_t + \mathbf{x}_t'\Omega\mathbf{x}_t.$$

The presence of the quadratic term is a key differentiating feature of our formulation.[4]

The nominal return on the consumption series can be written as:

$$r_{c,t+1} = \Delta c_{t+1} + log(\Delta t + e^{z_{t+1}}) - z_t + \pi_{t+1}.$$

Applying a standard log-linear approximation, we can write:

$$r_{c,t+1} = \Delta c_{t+1} + \pi_{t+1} + (\kappa_0 + \kappa_1 z_t)\Delta t + \kappa_2\Delta z_{t+1},$$

where κ_0, κ_1, and κ_2 are dependent on the steady state value of z_t, $\bar{z}$, and Δt, with non-trivial continuous-time limits.[5]

To make $r_{c,t+1}$ conditionally affine in $\mathbf{x}_{t+1}$, which will prove convenient in subsequent derivations, we linearize its quadratic part around the lagged value of the states, $\mathbf{x}_t$:[6]

$$(3) \quad \mathbf{x}_{t+1}'\Omega\mathbf{x}_{t+1} = \mathbf{x}_t'\Omega\mathbf{x}_t + \mathbf{x}_t'(\Omega+\Omega')\Delta\mathbf{x}_{t+1}.$$

Substituting for $\mathbf{x}_{t+1}'\Omega\mathbf{x}_{t+1}$, the return on the consumption series and the stochastic discount factor become conditionally affine in $\mathbf{x}_{t+1}$:

$$\begin{aligned} r_{c,t+1} = &\quad \Delta c_{t+1} + \pi_{t+1} \\ &\quad + r_0(\mathbf{x}_t)\Delta t + r_x(\mathbf{x}_t)'\Delta\mathbf{x}_{t+1} \\ -m_{t+1} = &\quad \gamma\Delta c_{t+1} + \pi_{t+1} \\ &\quad + m_0(\mathbf{x}_t)\Delta t + m_x(\mathbf{x}_t)'\Delta\mathbf{x}_{t+1}, \end{aligned}$$

where

$$\begin{aligned} r_0(\mathbf{x}_t) = &\quad \kappa_0 + \kappa_1 z_t, \\ r_x(\mathbf{x}_t) = &\quad \kappa_2(\lambda_x + (\Omega+\Omega')\mathbf{x}_t), \\ m_0(\mathbf{x}_t) = &\quad -\theta\, log(\delta) - (\theta-1)(\kappa_0+\kappa_1 z_t), \\ m_x(\mathbf{x}_t) = &\quad -(\theta-1)\kappa_2(\lambda_x + (\Omega+\Omega')\mathbf{x}_t). \end{aligned}$$

The weight $m_x(\mathbf{x}_t)$ on $\mathbf{x}_{t+1}$ in the pricing kernel m_{t+1} varies linearly in the current states $\mathbf{x}_t$ as long as Ω is non-zero. Consequently, under this setup, the *MPR* is time-varying– excess returns are predictable– even if the quantity of

[3] By making the time interval explicit, we will be able to assess the magnitude of subsequent approximation errors as the time interval approaches zero.

[4] L. Hansen, J. Heaton and N. Li (2008) obtain a quadratic pricing kernel by linearizing their economy around the case $\psi = 1$.

[5] $\kappa_2 = \frac{e^{\bar{z}}}{\Delta t + e^{\bar{z}}}$, $\kappa_0 = \frac{log(\Delta t + e^{\bar{z}}) - \kappa_2\bar{z}}{\Delta t}$, $\kappa_1 = -\frac{1}{\Delta t + e^{\bar{z}}}$

[6] We choose this approach for parsimony but note that the approximation can be improved by including a second-order term in the spirit of the Ito's lemma, thereby reducing the error to order $O(\Delta t^{3/2})$. However, this second-order term will not change the resulting *MPR* which is our current focus.

risks (conditional variance of $\Delta \mathbf{x}_{t+1}$) is constant. By way of contrast, expected excess returns are constant in most models with *LRR* when the quantity of risk is constant.

II. Risk Neutral Dynamics

We assume that $\mathbf{x}_t$ follows an affine process under the risk-neutral measure, so its conditional Laplace transform is exponentially affine:

$$E_t^Q[e^{\mathbf{u}'\Delta \mathbf{x}_{t+1}}] = e^{\left(a(\mathbf{u})+\mathbf{b}(\mathbf{u})'\mathbf{x}_t\right)\Delta t}, \tag{4}$$

with known one-step ahead density $f^Q(\mathbf{x}_{t+1}|\mathbf{x}_t)$.

In addition, we assume that the nominal short rate is affine in $\mathbf{x}_t$:

$$r_t = \left(\delta_0 + \delta_x' \mathbf{x}_t\right)\Delta t. \tag{5}$$

It follows that nominal bond prices are exponentially affine: $P_{n,t} = e^{-A_n - \mathbf{B}_n' \mathbf{x}_t}$, with n being the number of periods until maturity and A_n and B_n being determined through standard recursions (D. Duffie and R. Kan (1996)).

Real consumption growth is assumed to follow the process

$$\Delta c_{t+1} = g_0(\mathbf{x}_t)\Delta t + g_x(\mathbf{x}_t)\epsilon^Q_{c,t+1}\sqrt{\Delta t}, \tag{6}$$

where $\epsilon^Q_{c,t+1}$ is an i.i.d. standard normal random variable under Q. We capture possible conditional correlation between Δc_{t+1} and $\mathbf{x}_{t+1}$ through a vector $\sigma_{\mathbf{x},g}$ that satisfies:

$$\sigma_{\mathbf{x},g} = \frac{\partial \mathbf{x}_{t+1}}{\partial c_{t+1}}. \tag{7}$$

Introducing a component $\mathbf{x}^*_{t+1}$ that is conditionally independent of $\epsilon^Q_{c,t+1}$, we assume that:

$$\mathbf{x}_{t+1} = \mathbf{x}^*_{t+1} + \sigma_{\mathbf{x},g} g_x(\mathbf{x}_t)\epsilon^{\mathbb{Q}}_{c,t+1}\sqrt{\Delta t}. \tag{8}$$

Observed inflation follows the process

$$\pi_{t+1} = \pi_0(\mathbf{x}_t)\Delta t + \pi_x'\Delta \mathbf{x}_{t+1} + \sigma_{\pi,c} g_x(\mathbf{x}_t)\epsilon^Q_{c,t+1}\sqrt{\Delta t}.$$

Conditional correlation between Δc_{t+1} and π_{t+1} may arise through nonzero $Corr(\Delta x_{t+1}, \epsilon^Q_{c,t+1})$ and $\sigma_{\pi,c}$. The choices of $g_0(.), g_x(.)$, and $\pi_0(.)$ are discussed below.

III. The Implied Physical Dynamics

Le, Singleton and Dai (2009) shows that, in a discrete-time setting, the physical density of the states together with observable Δc_{t+1} and π_{t+1} can be computed as:

$$\begin{aligned} & f^P(\mathbf{x}_{t+1}, \Delta c_{t+1}, \pi_{t+1}|\mathbf{x}_t) = \\ & \quad \frac{e^{-m_{t+1}}}{E_t^Q[e^{-m_{t+1}}]} f^Q(\mathbf{x}_{t+1}, \Delta c_{t+1}, \pi_{t+1}|\mathbf{x}_t). \end{aligned} \tag{9}$$

Since f^Q is, by assumption, known analytically, (9) gives f^P in closed form. Up to regularity conditions that guarantee stationarity of $\mathbf{x}_t$, the combination of a known physical density and affine bond pricing renders *ML* estimation computationally tractable.

An approximate expression for the first moment of $y_{t+1} = \{\Delta \mathbf{x}_{t+1}, \Delta c_{t+1}, \pi_{t+1}\}$ under P[7] is obtained by assuming that y_{t+1} is conditionally Gaussian and utilizing Stein's lemma:

$$E_t^P[y_{t+1}] = E_t^Q[y_{t+1}] - var_t^Q[y_{t+1}]\frac{\partial m_{t+1}}{\partial y_{t+1}}, \tag{10}$$

where $E_t^Q[y_{t+1}]$, $var_t^Q[y_{t+1}]$, and $\frac{\partial m_{t+1}}{\partial y_{t+1}}$ are known from the relevant Laplace transforms.

Applying (10), it can be shown that:

$$\begin{aligned} \frac{E_t^P[\Delta \mathbf{x}_{t+1}]}{\Delta t} &= \frac{\partial a}{\partial u}(0) + \frac{\partial b}{\partial u}(0)'\mathbf{x}_t \\ &+ \left(\frac{\partial^2 a}{\partial u \partial u'}(0) + \sum \frac{\partial^2 b_i}{\partial u \partial u'}(0)'\mathbf{x}_{i,t}\right) Dx_t, \end{aligned} \tag{11}$$

where $Dx_t = (\gamma + \sigma_{\pi,c})./\sigma_{\mathbf{x},g} + \pi_x + m_x(\mathbf{x}_t)$ and ./ denotes element by element division. From (11), and the fact that Dx_t is linear in $\mathbf{x}_t$, it follows that geometric ergodicity of the state can be imposed by constraining the magnitudes of the relevant elements of the matrix Ω so that $\mathbf{x}_t$ is sufficiently mean-reverting.[8]

We define the steady state value of $\mathbf{x}_t$ as the vector $\bar{\mathbf{x}}$ that sets the right-hand side of (11) to zero: $E_t^P[\Delta \mathbf{x}_{t+1}|\mathbf{x}_t = \bar{\mathbf{x}}] = 0$. Internal model consistency requires that $\bar{\mathbf{x}}$ and $\bar{z}$ be related ac-

[7] This approximation becomes more accurate as Δt gets smaller. Note that we do not need this assumption in evaluating the physical density.

[8] See Le, Singleton and Dai (2009) for a more in depth discussion of ergodicity.

cording to (2) as follows:

$$\bar{z} = \lambda_0 + \lambda_x' \bar{\mathbf{x}} + \bar{\mathbf{x}}' \Omega \bar{\mathbf{x}}. \tag{12}$$

The corresponding first moments of consumption growth and realized inflation are:

$$\frac{E^P[\Delta c_{t+1}]}{\Delta t} = g_0(\mathbf{x}_t) + g_x(\mathbf{x}_t)^2 Dc_t, \tag{13}$$

where $Dc_t = \gamma + \sigma_{\pi,c} + (\pi_x + m_x(\mathbf{x}_t))' \sigma_{\mathbf{x},g}$, and

$$\frac{E^P[\pi_{t+1}]}{\Delta t} = \pi_0(\mathbf{x}_t) - \sigma_{\pi,c} g_0(\mathbf{x}_t) + \pi_x' \frac{E_t^P[\Delta x_{t+1}]}{\Delta t} + \sigma_{\pi,c} \frac{E_t^P[\Delta c_{t+1}]}{\Delta t}. \tag{14}$$

IV. Equilibrium Restrictions

So far we have left unspecified the dimension of $\mathbf{x}_t$ and the functional forms of $g_0(.)$, $g_x(.)$, and $\pi_0(.)$. The choice of the conditional volatility of consumption growth, $g_x(.)$, is simple: a constant, σ_g, in case of constant volatility, or $\sigma_t \sigma_g$ in case of stochastic volatility (where σ_t^2 is a non-negative element of $\mathbf{x}_t$).

Given $g_x(.)$, equilibrium pricing determines the functional forms of $g_0(.)$ and $\pi_0(.)$. Specifically, r_t determines the mean of the pricing kernel, and m_{t+1} must price the return on the consumption claim:

$$E_t^Q[e^{r_{c,t+1}}] = e^{r_t}, \tag{15}$$

$$E_t^Q[e^{-m_{t+1}}] = e^{r_t}. \tag{16}$$

It can be shown that (15) is equivalent to:

$$\begin{aligned} g_0(\mathbf{x}_t) + \pi_0(\mathbf{x}_t) = \delta_0 + \delta_x' \mathbf{x}_t \\ -r_0(\mathbf{x}_t) - g_x(\mathbf{x}_t)^2 Dr_t - a(\pi_x + r_x(\mathbf{x}_t)) \\ -b(\pi_x + r_x(\mathbf{x}_t))' \mathbf{x}_t, \end{aligned}$$

where

$$Dr_t = \frac{1}{2}(1+\sigma_{\pi,c})^2 + (1+\sigma_{\pi,c})(\pi_x + r_x(\mathbf{x}_t))' \sigma_{\mathbf{x},g}.$$

Similarly, (16) is equivalent to:

$$\begin{aligned} \gamma g_0(\mathbf{x}_t) + \pi_0(\mathbf{x}_t) = \delta_0 + \delta_x' \mathbf{x}_t \\ -m_0(\mathbf{x}_t) - g_x(\mathbf{x}_t)^2 Dm_t - a(\pi_x + m_x(\mathbf{x}_t)) \\ -b(\pi_x + m_x(\mathbf{x}_t))' \mathbf{x}_t, \end{aligned}$$

where

$$Dm_t = \frac{1}{2}(\gamma+\sigma_{\pi,c})^2 + (\gamma+\sigma_{\pi,c})(\pi_x + m_x(\mathbf{x}_t))' \sigma_{\mathbf{x},g}.$$

Assuming γ is different from one, (15) and (16) can be solved for the $g_0()$ and $\pi_0()$ that are consistent with our economy.

V. Discussion

Typically, a three-dimensional state vector captures most of the variation in bond yields (R. Litterman and J. Scheinkman (1991), Q. Dai and K. Singleton (2000)). In models with stochastic, conditional consumption volatility, the volatility-related state variables can be modeled within our affine setting as autoregressive-gamma processes (C. Gourieroux and J. Jasiak (2006), Le, Singleton and Dai (2009)).

Importantly, by specifying the conditional distribution of the state $\mathbf{x}_t$ under the risk-neutral measure as a primitive of our model, we are free to adopt any identified, canonical form for $f^Q(\mathbf{x}_{t+1}|\mathbf{x}_t)$. S. Joslin (2007) and S. Joslin, K. Singleton and H. Zhu (2009) develop normalizations that, we anticipate, will offer significant computational advantages in estimating equilibrium term structures models with our flexible affine structure under Q.

The physical dynamics of $(\Delta c_{t+1}, \pi_{t+1})$ implied by our model will be nonlinear (e.g., have nonlinear conditional means), as long as Ω, a free matrix of parameters in our setup, is nonzero. This nonlinearity enters through the equilibrium functional forms of $g_0()$ and $\pi_0()$, and it remains in the continuous time limit of our discrete-time economy. On the other hand, if $\Omega = \mathbf{0}$, our model is affine under both P and Q (satisfies both P2 and P3), and the *MPR*s are time-invariant. In this sense, our setup nests many prior studies that adopt an affine representation of the price-consumption ratio.

Because we start from the Q distribution of $\mathbf{x}_t$, and then derive the P distribution that is consistent with Epstein-Zin preferences, the parameters that govern the price-consumption ratio (λ_0, λ_x, and Ω) and the short rate (δ_0 and δ_x) are not tied down by other fundamental parameters that describe the physical cash flows. Instead, λ_0, λ_x, Ω, δ_0, and δ_x are the fundamental parameters of our model. What is derived in our setup are (not just parameters but rather) functionals that regulate the dynamics of the cash flows. By not requiring the physical dynamics of cash flows to fit any pre-specified form, we gain considerable flexibility in modeling the price consumption ratio - an important component of the pricing kernel - as well as the short rate. The flexibility in modeling these two components translates into flexibility in modeling the entire term structure of interest rates.

The "cost" of our modeling strategy is that we cannot assign specific economic roles to elements of the state $\mathbf{x}_t$ (other than that a subset might govern the stochastic volatility of $(\Delta c_{t+1}, \pi_{t+1})$). In contrast, it is standard in the *LRR* literature to assume that the physical mean of consumption growth is driven by an element of $\mathbf{x}_t$. For some special cases of our model it appears possible to enforce such an interpretation. Adding this requirement means that λ_0, λ_x, and Ω are no longer (entirely) free parameters.

At this juncture, proceeding with the flexibility of a general affine representation of the state (up to the choice of the numbers of factors and drivers of stochastic volatility) seems advantageous, in that it gives our equilibrium setting maximal flexibility in fitting the term structure with *both time-varying market prices and quantities of risks.* Of interest will be whether, with this flexibility, the model gives rise to a *LRR*-like structure to the drift of consumption growth. From (13) it is seen this will depend on the estimated functional forms of $g_0(\mathbf{x}_t)$ and $m_x(\mathbf{x}_t)$.

REFERENCES

Bansal, R., and A. Yaron. 2004. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *Journal of Finance*, 59: 1481–1509.

Bansal, R., and I. Shaliastovich. 2009. "Risk and Return in Bond, Currency and Equity Markets." Duke University.

Bansal, R., D. Kiku, and A. Yaron. 2009. "An Empirical Evaluation of the Long-Run Risks Model for Asset Price." Duke University.

Dai, Q., and K. Singleton. 2000. "Specification Analysis of Affine Term Structure Models." *Journal of Finance*, 55: 1943–1978.

Dai, Q., and K. Singleton. 2002. "Expectations Puzzles, Time-Varying Risk Premia, and Affine Models of the Term Structure." *Journal of Financial Economics*, 63: 415–441.

Doh, T. 2008. "Long Run Risks in the Term Structure of Interest Rates: Estimation." Federal Reserve Bank of Kansas City, RWP 08-11.

Duffee, G. 2002. "Term Premia and Interest Rate Forecasts in Affine Models." *Journal of Finance*, 57: 405–443.

Duffie, D., and R. Kan. 1996. "A Yield-Factor Model of Interest Rates." *Mathematical Finance*, 6: 379–406.

Epstein, L., and S. Zin. 1989. "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Theoretical Framework." *Econometrica*, 57: 937–969.

Eraker, B., and I. Shaliastovich. 2008. "An Equilibrium Guide to Designing Affine Asset Pricing Models." *Mathematical Finance*, 18: 519–543.

Gourieroux, C., and J. Jasiak. 2006. "Autoregressive Gamma Processes." *forthcoming, Journal of Forecasting.*

Hansen, L., J. Heaton, and N. Li. 2008. "Consumption Strikes Back? Measuring Long-Run Risk." *Journal of Political Economy*, 116: 260–302.

Joslin, S. 2007. "Pricing and Hedging Volatility in Fixed Income Markets." Working Paper, MIT.

Joslin, S., K. Singleton, and H. Zhu. 2009. "A New Perspective on Gaussian DTSMs." Stanford University.

Le, A., K. Singleton, and Q. Dai. 2009. "Discrete-time Affine^Q Term Structure Models with Generalized Market Prices of Risk." *Review of Financial Studies*, Forthcoming.

Litterman, R., and J. Scheinkman. 1991. "Common Factors Affecting Bond Returns." *Journal of Fixed Income*, 1: 54–61.