The delegated Lucas-tree*

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Abstract

We introduce delegation into a standard Lucas exchange economy, where trading in financial assets is delegated to funds, but the endowment process is owned by their clients. Flow-performance incentive functions describe how much capital investors provide to funds at each date, as a function of past performance. We consider a rich set of flow-performance functions including examples with both convex and concave regions, and derive implications for asset prices and trading patterns of various incentive schemes. Delegation affects the Sharpe ratio through two channels: discount rate and capital flow. The two work in opposite directions leaving the aggregate effect ambiguous, in general. For some flow-performance functions even if all investors are identical funds trade among themselves and returns are dispersed in the cross-section. In contrast, when the flow-performance relationship is convex for some funds and concave for others they might not trade at all. In this case, delegation does not effect the Sharpe ratio. Also, the direction of lending and borrowing between funds with different incentives can depend on the sign of the skew of the endowment process.

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1 Introduction

While it is undisputed that financial intermediaries play a central role in financial markets, our understanding of the impact of these intermediaries on asset prices is fairly limited.[1] Specifically, we know fairly little on how do the incentives of delegated portfolio managers impact asset prices when different types of delegated portfolio managers co-exist.

The importance of improving our understanding of the link between the incentives of these intermediaries and asset prices was highlighted in the recent financial crisis. Some have suggested that the incentives of financial institutions played an important role in the emergence of the crisis, as well as in amplifying it once it began. To analyze the link between the incentives of these institutions and asset prices, we introduce delegation into a standard Lucas exchange economy, by introducing financial intermediaries (funds) that make dynamic investment decisions on behalf of their clients, who own the endowment process. A general insight from our analysis is that the implications of convexities in incentives are richer than the popular view that they induce risk taking. For example fund managers with the most convex incentives might not leverage up and their presence might not decrease the Sharpe-ratio. Also, even if agents are identical, convexities in the flow-performance relationship induce heterogeneous strategies, trade, preference for positive skewness and excessively volatile prices.

Similar to a standard Lucas exchange economy there is a Lucas tree paying a stochastic dividend each period, there is a stock which is a a claim on the endowment process, and a riskless bond which is in zero net supply. Financial markets are populated by fund managers. These managers are allowed to set their fees freely every period, which they need to consume. Each period they invest the capital they have under management in a portfolio of the two financial assets. The endowment process is owned by funds' clients, who cannot invest directly in financial assets. Each period clients allocate capital to different funds depending on each fund's past performance relative to the market. The relationship between last period's return compared to the market and new capital flow is described by each manager's *incentive function*. We interpret the incentive function as a short-cut for an unmodelled learning process by clients on managers'. Its empirical counterpart is the flow-performance relationship. We are agnostic as to whether the learning process is rational or not. Instead of deriving the incentive function from first principles, we allow for a rich set of possible exogenous specifications including examples with both convex and concave intervals. We allow different types of funds to co-exist in the same economy by considering settings where

[1] See, for example, the presidential address of Allen (2001) for an elaborate discussion on the importance of the role of financial intermediaries.

different funds can have different incentive functions.

To obtain an analytically tractable model we combine a few simplifying assumptions. First, managers have log utility. Second, managers are forced to consume their fees, and can not trade on their own account. Third, incentive functions are composites of constant elasticity functions. That is, we divide the set of possible relative returns into an arbitrary number of segments. Within each segment the incentive function has constant elasticity. However, the elasticity can freely vary across segments allowing for flat, linear, concave or convex segments within the same incentive function.[2] We are particularly interested in increasing elasticity incentive functions, i.e. where higher segments imply larger elasticity. The combination of log utility and piece-wise constant elasticity functions is the key methodological contribution that allows us to derive simple analytical formulas for the trading pattern and asset prices under various incentive functions, even when managers with different incentive functions co-exist.

We present the main insights of the model through a sequence of three examples. In the first, funds are heterogeneous with differing constant elasticity incentive functions. We refer to funds with constant elasticity incentives as mutual funds. In the second, funds are identical with an increasing elasticity incentive function. Specifically, the incentive function has two segments where the response in capital flows has a larger elasticity in the higher segment, when the return of the fund exceeds the market return by a prespecified proportion. We refer to funds with such incentives as hedge funds. In the third, mutual funds and hedge funds co-exist so that one group of funds has a constant elasticity incentive function and the other has an increasing elasticity incentive function.

When funds are identical with an increasing elasticity incentive function there is trade among ex ante identical agents, dispersion in realized returns and excess volatility in prices. In particular, even if hedge funds are identical ex ante, a subset of them sell bonds to the rest and invest in a leveraged portfolio. In contrast, buyers of the bond have a smaller then unity (possibly negative) exposure to the market. If hedge funds were to follow an identical strategy their return would equal to the market return in all states and they would never be evaluated at the higher segment of their incentive function. This implies gains from trade. With asymmetric strategies, a different subset of hedge funds outperforms the market in each state, getting extra capital flows in that state. The benefit of the added flows outweighs the cost induced by distorting the portfolio. As the performance of the two groups varies across the two states, so does the total delegated endowment. As managers bid up prices in the state when their total capital is higher, the price-dividend ratio also varies across states. We interpret this as excess volatility, since in the standard no-delegation Lucas-economy

[2] With sufficient number of intervals we can basically approximate a wide set of incentive functions.

the price-dividend ratio is constant. While the price-dividend ratio varies across states, state prices and the Sharpe ratio remains constant. However, they do get impacted by the presence of delegation, where the aggregate effect is driven by two effects. First, an extra unit of return is appreciated more when the total capital of the fund industry is lower. This is the standard discount rate effect. Second, an extra unit of return is appreciated more when it increases capital flows through the incentive function. We call this the capital-flow effect. Generally, the two effects work in the opposite direction, because the average hedge fund is richer when she is evaluated at the higher-elasticity segment of her incentive function. In aggregate, increasing elasticity incentives can both increase and decrease the Sharpe ratio.

To contrast the argument that incentives with increasing elasticity can create trade among identical agents, we show that when incentive functions all have constant elasticities that may differ across funds there is no trade, all funds hold the market portfolio, and the Sharpe ratio is the same as it would be without delegation. In particular, we consider two groups of mutual funds, where each of the groups can have a convex, concave, flat or linear constant elasticity incentive function. Constant elasticities change marginal utilities by the same proportion in both states. Therefore, even if the marginal utility of a unit return of agents differ across groups, their marginal rate of substitution do not, and there are no gains from trade. The capital flows and endowment share of mutual funds are independent from the dividend growth shock because each fund performs exactly like the market in each state. This implies no discount rate effect and no capital flow effect of delegation, so the Sharpe ratio remains constant.

In our last example, mutual funds and hedge funds co-exist. In this case, the relative capital of the two groups influences both portfolios allocations and prices. When hedge funds own all the capital, then the equilibrium is as in the example with only hedge funds: there is an asymmetric equilibrium where ex ante identical hedge funds trade among each other. As the capital share of hedge funds decrease, the dispersion in hedge fund strategies and returns decreases as well. When the capital share of hedge funds is sufficiently low all hedge funds follow the same strategy. That is, hedge funds do not trade among themselves, but trade with the mutual funds. Interestingly, when all hedge funds follow the same strategy, they borrow from mutual funds and leverage up only when a boom is less likely than a recession. Otherwise they lend to mutual funds and leverage down. For any underlying skewness of the endowment process this implies a positively skewed relative return distribution for hedge funds. Thus, in equilibrium increasing elasticity incentive functions imply the opposite of the popular observation that hedge funds "pick up nickels in front of a steam roller".

Our paper is related to several branches of the literature. First, papers that study the effects of delegated portfolio management on asset prices (e.g. Shleifer-Vishny 1997, Vayanos,

2003, Cuoco-Kaniel, 2007, Dasgupta-Prat, 2006, 2008, Guerrieri-Kondor, 2009,) Most of these papers use models that are fairly different than the Lucas-tree exchange economy, making it hard to compare the results to standard consumption based asset pricing models. Apart from our work, to our knowledge the only other exception is He and Krishnamurthy (2008) who also studies the effect of delegation in a standard Lucas-economy. The main difference is that in He and Krishnamurthy (2008) managers are not directly interested in managing larger funds. In their model, fund flows effect the decision of managers only indirectly, through the effect of flows on equilibrium prices. Therefore, they cannot asses the effect of convexities in the flow-performance relationship on asset prices through managers' incentives. In contrast, this is our main focus.

Second, the literature on consumption based asset pricing with heterogeneous risk aversion (e.g. Dumas (1989), Wang (1996), Chan-Kogan (2002), Bhamra-Uppal, (2007), Longstaff-Wang (2008)). Models in these papers have similar structure to ours but very different implications. The closest to our work is Chan and Kogan (2002) who assumes that agents value their consumption relative to others. In flavor, this is similar to our assumption that fund flows are a function of relative performance. However, the two structures have very different implications. This demonstrate well that heterogeneous incentives and heterogeneous risk aversion have different asset pricing implications. For example, unlike in our structure, in Chen and Kogan trade among identical traders is never required in equilibrium. Also, in Chan and Kogan relative consumption evaluation implies a stationary distribution of consumption shares. This is not the case in our structure. In our basic model only one of the intermediaries survive in the long term. Finally, just like in any other paper with heterogeneous risk-aversion, in Chan and Kogan less risk averse agents always lend to more risk averse agents. In our paper, hedge funds typically borrow from mutual funds when a high shock is unlikely but lend to mutual funds when a high shock is likely.

additional related literature to be added here....

The structure of the paper is as follows. In the next section we present the general model. We discuss the general set up, our equilibrium concept and the main properties of the equilibrium. In Section 3, we derive the main insights of our model by presenting three examples. Readers more interested in insights then techniques might start directly with Section 3. In Section 4 we present two extensions (allowing funds to endogenously decide whether to be a mutual fund or a hedge fund, and allowing funds to endogenously choose between two markets with differing dividend growth processes), while in Section 5 we discuss the main assumptions of the model. Finally, we conclude.

2 The general model

In this section we introduce the general framework, define our equilibrium concept and present sufficient conditions for the existence of such an equilibrium and its basic properties. This gives a useful toolbox which we will use in the subsequent sections to analyze various examples.

2.1 The Economy

We consider a discrete-time, infinite-horizon exchange economy with complete financial markets and a single perishable consumption good. There is only one source of uncertainty and participants trade in financial securities to share risk.

Our model has two non-standard elements. First, the endowment process is owned by the clients of fund managers who cannot trade financial assets directly. Clients invest a proportion of the endowment through each manager depending on the past relative performance of the manager and described by their *incentive function*. The empirical counterpart of the incentive function is the flow-performance relationship.[3] Instead of deriving the incentive function from first principles, we take it exogenously in the spirit of Shleifer and Vishny (1997). However, we allow for a rich set of possible exogenous specifications including examples with kinks and convex intervals. Second, we allow for two groups of fund managers with differing incentive functions to coexist.

Our objective is to analyze the effect of various versions of convexities in incentives on trading patterns and asset prices, as well as to understand the equilibrium interactions between different types of delegated portfolio managers.

Securities. The aggregate endowment process is described by the binomial tree

$$\delta_{t+1} = y_t \delta_t$$

where the growth process y_t has two i.i.d. states: $s_t = H, L$. The dividend growth is either high y_H or low y_L, with $y_H > y_L$. The probability of the high and the low states are p and $(1-p)$ respectively.

Investment opportunities are represented by a one period riskless bond and a risky stock.

[3] There is a large empirical literature exploring the relationship between past performance and future fund flows. This literature shows that there is a positive relationship between performance and flows for most type of financial intermediaries but the shape of this relationship is effected by the type of the fund. See Chevalier and Ellison (1997), Sirri and Tufano (1998), and Chen et al. (2003)) for evidence on mutual funds, Bares, Gibson, and Gyger (2002), Brown, Goetzman, and Ibbotson (1999), Edwards and Cagalyan (2001), and Kat and Menexe (2002) for evidence on hedge funds and Kaplan and Schoar (2004) for evidence on private equity partnerships.

The riskless bond is in zero net supply. The stock is a claim to the dividend stream δ_t and is in unit supply. The price of the stock and the interest rate on the bond are q_t and R_t respectively.

Fund Managers. The economy is populated by two groups of fund managers $i = 1, 2$. We assume each group is comprised of a continuum of managers with a total mass of one. All managers derive utility from inter-temporal consumption, and have *log* utility. In period, t, each manager determines the fraction ψ_t^i of beginning of period assets under management w_t^i she will receive as a fee. We assume the manager must consume her fee $\psi_t^i w_t^i$.[4] She then invests the remaining $(1-\psi_t^i)w_t^i$ in a portfolio with α_t^i share in the stock and $(1-\alpha_t^i)$ share in the bond. The return on the fund's portfolio is

$$\rho_{t+1}\left(\alpha_t^i, s_{t+1}\right) \equiv \alpha_t^i \left(\frac{q_{t+1}(s_{t+1}) + \delta_{t+1}(s_{t+1})}{q_t} - R_t\right) + R_t,$$

The representative manager in group i solves the problem

$$\max_{\{\psi_t^i, \alpha_t^i\}} E\left[\sum_t \beta^t \ln \psi_t^i w_t^i\right] \qquad s.t.\ \ w_{t+1}^i = g^i\left(Y_{t+1}^i\right) w_{t+1,-}^i \tag{1}$$

where $g^i(\cdot)$ is the incentive function,

$$\begin{aligned} w_{t+1,-}^i &\equiv \rho_{t+1}\left(\alpha_t^i, s_{t+1}\right)\left(1-\psi_t^i\right) w_t^i \\ Y_{t+1}^i(\alpha_t^i, s_{t+1}) &\equiv \frac{\rho_{t+1}\left(\alpha_t^i, s_{t+1}\right)}{\left(\frac{q_{t+1}(s_{t+1})+\delta_{t+1}(s_{t+1})}{q_t}\right)} \end{aligned}$$

are the the assets under management at the end of the previous period (i.e., after the time $t+1$ return has been realized, but before investors decide how much to allocate to the fund to manage between $t+1$ and $t+2$) and the return on her portfolio relative to the market.

Clients. To close the model, we have to exogenously specify the decision of the owners of the capital as their consumption-saving decision is given by the incentive function. In particular, the owners of capital consume

$$\delta_t + q_t - \left(g^1\left(Y_{t+1}^1\right) w_{t,-}^1 + g^2\left(Y_{t+1}^2\right) w_{t,-}^2\right).$$

[4] The assumption that managers cannot invest their fees is a major simplification allowing us not to keep track the private wealth of fund managers. Note also that on one hand, we are allowing ψ_t to be conditional on any variable in the managers' information set in t. That is, we do not constrain our attention to proportional fees ex ante. On the other hand, our assumptions imply that fees are proportional in equilibrium, managers effectively maximize capital under management and fees do not play any role in the portfolio decision.

Incentive Functions. A fund's assets under management at a beginning of a period are proportional to assets under management at the end of the previous period. The proportion $g^i(\cdot)$ depends on a fund's return relative to the market portfolio, and can be a nonlinear and non-concave function of this relative return.

In particular, incentive functions belong to the following class:

$$w_{t+1}^i = \ A_m^i \left(Y_{t+1}^i\right)^{n_m^i - 1} w_{t+1,-}^i \quad if \quad Y_{t+1}^i \in \kappa_m^i \tag{2}$$

where $n_m^i \geq 1$, $A_m^i > 0$ and we pick $M^i - 1$ positive and increasing "kinks", $k_m^i, m = 1, ...M^i - 1$. The kinks divide the positive segment of the real line into M^i segments, $\kappa_1^i, ...\kappa_M^i$ defined as

$$\begin{aligned} \kappa_1^i &\in [0, k_1^i) \\ \kappa_m^i &= [k_{m-1}^i, k_m^i) \\ \kappa_M^i &= [k_{M-1}^i, \infty). \end{aligned}$$

This specification allows for varying shapes across segments. For example, $n_m^i = 1$ implies in segment m the incentive function is flat. Similarly, $n_m^i = 2$, $1 < n_m^i < 2$, $n_m^i > 2$ imply a linear, concave, and convex segment respectively. We can think of the relative size of the n_m^i coefficients across segments of i's incentive function as the "log-log convexity" of the incentive function in the following sense. In each segment m

$$\frac{\partial \ln g^i \left(Y_{t+1}^i\right)}{\partial \ln Y_{t+1}^i} = n_m^i,$$

therefore, if n_m^i is increasing in m, $\ln g^i(\cdot)$ is convex in the log of the relative return. In this sense, a specification with increasing elasticity can be interpreted as convexity in incentives. Although our general approach would allow for discontinuities, throughout the analysis we assume that the incentive function is continuous by imposing the restriction

$$A_{m+1}^i = A_m^i \left(k_m^i\right)^{n_m^i - n_{m+1}^i}.$$

Note that with the choice of no kink and $A^i = n^i = 1$ our structure nests the standard case of an individual log-investor.

After characterizing the equilibrium in general, we will analyze possible equilibria in the three following examples of increasing complexity.

Example 1 (no kink) *For* $i = 1, 2$

$$w_{t+1}^{i} = A^{i} \left(Y_{t+1}^{i}\right)^{n^{i}-1} w_{t+1,-}^{i} \tag{3}$$

The incentive functions of the two groups can differ, but neither of the two has a kink. We refer to managers with such constant elasticity incentive functions as mutual funds.

Example 2 (only hedge funds) *Assume* $k \geq 1$, *and* $n_2 \geq n_1$. *For* $i = 1, 2$

$$w_{t+1}^{i} = \begin{cases} A_1 \left(Y_{t+1}^{i}\right)^{n_1-1} w_{t+1,-}^{i} & if Y_{t+1}^{i} < k \\ A_1 k^{n_1-n_2} \left(Y_{t+1}^{i}\right)^{n_2-1} w_{t+1,-}^{i} & if Y_{t+1}^{i} \geq k. \end{cases} \tag{4}$$

Since the two groups have the same incentive functions, effectively there is a single group. The assumption that $n_2 \geq n_1$ implies that the elasticity of the incentive function is higher above the kind than below, implying an increasing elasticity incentive function.

We refer to a manager with such incentive function as a hedge fund.[5] Intuitively, hedge fund managers represent the group among which the financial press is constant search for "stars". When the relative return of one of the funds reaches a particular level, her performance hits the headlines and the elasticity of her flow-performance relationship increases. Accordingly, we will see that the significance of the kink in the results is to allow for increasing elasticity as opposed to introducing non-differentiability.

Example 3 (hedge funds and mutual funds) *Assume* $k \geq 1$, *and* $n_2 \geq n_1$. *For* $i = 1, 2$

$$w_{t+1}^{1} = \begin{cases} A_1 \left(Y_{t+1}^{1}\right)^{n_1-1} w_{t+1,-}^{1} & if Y_{t+1}^{1} < k \\ A_1 k^{n_1-n_2} \left(Y_{t+1}^{1}\right)^{n_2-1} w_{t+1,-}^{1} & if Y_{t+1}^{1} \geq k. \end{cases} \tag{5}$$

$$w_{t+1}^{2} = A \left(Y_{t+1}^{2}\right)^{n-1} w_{t+1,-}^{2}. \tag{6}$$

Managers of type 1 are hedge funds, as in Example 2, while managers of type are mutual funds, as in Example 1. Note that there are two alternative interpretation of the structure in Example 2. We can see it as an example with two type of financial intermediaries, where only one of them (the hedge fund) is subject to the pressure to show stellar performance for the extra capital flow. A draw back of this interpretation is that here aggregate investment is driven purely by the relative performance of intermediaries as opposed to any standard

[5] Our labeling of hedge funds and mutual funds is purely for presentational purposes. Pairing types of financial institutions with the characteristics of their flow-performance functions is outside of the scope of this paper.

consumption smoothing arguments. However, an alternative interpretation comes if we set $A = 1, n = 2$. This implies that mutual funds are in fact individual traders. They are not subject to any in- or outflow of capital. Thus, instead of two financial intermediaries, we have only one (hedge funds), but we have two types of individuals. There is an active group who do not invest in the hedge fund, but trade directly and optimally on the financial market. And there is an inactive group (clients) who invest only through hedge fund managers based on their relative performance. The initial relative size of the two groups is arbitrary. This second interpretation can be also applied to the general model. None of the qualitative results change.

2.2 The Asymmetric Interior Equilibrium

In this section, we define the equilibrium, derive sufficient conditions for its existence and provide some general characterizations. In the following sections we use these characterization results to analyze in more detail Examples 1-3.

In our proposed equilibrium, there is a single state variable: the relative share of wealth share of group 1:

$$\omega_t \equiv \frac{w_t^1}{w_t^2 + w_t^1}.$$

We use the time script only when it is necessary to avoid confusion. Otherwise, variables with no subscript refer to period t and we denote variables referring to $t+1$ period by prime.

We use the following definitions in our equilibrium concept.

Definition 1 *An* ***lh*−portfolio** *is a portfolio, α, for which a fund's return relative the the market Y_{t+1} is in the $l-$th segment of the incentive function following a low shock $s_{t+1} = L$ and in h-th segment following a high shock $s_{t+1} = H$.*

If α is an ***lh*-portfolio**, then

$$\begin{aligned} l &= \sum_{m=1}^{M^i} 1_{\{Y_{t+1}(\alpha,L)\in\kappa_m^i\}}\, m \\ h &= \sum_{m=1}^{M^i} 1_{\{Y_{t+1}(\alpha,H)\in\kappa_m^i\}} m. \end{aligned}$$

In an asymmetric equilibrium managers in a given group i follow heterogeneous strategies. However, we conjecture that in a given state ω there is a unique $lh-$portfolio which is optimal for managers in group i. We refer to this locally optimal portfolio as $\alpha_{lh}^i(\omega)$. Therefore, to

describe asymmetric strategy profiles it is sufficient to specify measures $\mu_{lh}^{i}(\omega)$ of managers in group i which choose corresponding $\alpha_{lh}^{i}(\omega)$ portfolios.[6]

Definition 2 *A* **strategy profile** $\mathcal{P}^{i}(\omega)$ *is a triplet* $(\mathcal{H}^{i}(\omega), \mathcal{M}^{i}(\omega), \mathcal{A}^{i}(\omega))$ *where for any* ω*:*

- $\mathcal{H}^{i}(\omega)$ *is a set of* lh *index-pairs,*
- $\mathcal{A}^{i}(\omega)$ *is a set of corresponding* $\alpha_{lh}^{i}(\omega)$*-portfolios,*
- $\mathcal{M}^{i}(\omega)$ *is a set of* $\mu_{lh}^{i}(\omega) : [0,1] \times \mathcal{H}^{i}(\omega) \to (0,1]$ *functions such that* $\sum_{lh \in \mathcal{H}^{i}(\omega)} \mu_{lh}^{i}(\omega) = 1$.

The above definition nests also the case where all managers of group i follow the same strategy. In that case $\mathcal{H}^{i}(\omega)$ and $\mathcal{A}^{i}(\omega)$ are singletons, and $\mu_{lh}^{i}(\omega) = 1$ for $lh \in \mathcal{H}^{i}(\omega)$.

The next definition defines the asymmetric interior equilibrium.

Definition 3 *An* **asymmetric interior equilibrium** *is a price process* $q_{s'}(\omega)$ *for the stock and* $R(\omega)$ *for the bond, a law of motion* $\omega' = \Omega_{s'}(\omega)$ *for the relative wealth share of group 1, and strategy profiles* $\mathcal{P}^{i}(\omega)$ *for* $i = 1, 2$ *such that*

1. *consumption choices* $\psi^{i}(\omega)$ *and trading strategies* $\alpha_{lh}^{i}(\omega)$ *are optimal for managers given the equilibrium prices and the law of motion for the relative wealth share of group 1,*
2. *prices* $q_{s'}(\omega)$*, and* $R(\omega)$ *clear both good and asset markets,*
3. *the law of motion for the relative share of group 1 is consistent with managers:*

$$\omega' = \Omega_{s'}(\omega).$$

2.2.1 Characterization and Existence

To characterize the equilibrium, it is useful to define the *individual shape-adjusted probability of a high state*

$$\xi_{lh}^{i} \equiv \frac{pn_{h}^{i}}{pn_{h}^{i} + (1-p)\, n_{l}^{i}}, \tag{7}$$

[6] Searching for asymmetric equilibria is necessary as there are very simple and intuitive cases where there is no symmetric equilibrium. We will demonstrate this by Example 2 and 3.

and the *the aggregate shape-adjusted probability of a high state*

$$\tilde{\xi}(\omega) \equiv \omega \sum_{lh \in \mathcal{H}^1(\omega)} \mu^1_{lh}(\omega)\, \xi^1_{lh} + (1-\omega) \sum_{lh \in \mathcal{H}^2(\omega)} \mu^2_{lh}(\omega)\, \xi^2_{lh}. \tag{8}$$

ξ^i_{lh} is simply the probability of a high state adjusted to the relative elasticity of the incentive function in the high state, assuming that investor i forms an $lh-$portfolio. $\tilde{\xi}(\omega)$ is a weighted average of the individual shape adjusted probabilities. When the incentive function elasticity is constant $n^i_h = n^i_l$ and $\tilde{\xi}(\omega) = \xi^i_{lh} = p$.

We also define the end of period share of wealth of a given group i after a given shock s' relative to the total, cum-dividend value as

$$W^i_{s'}(\omega) = \frac{\sum_{lh \in \mathcal{H}^i(\omega)} \mu^i_{lh}(\omega)\, A_{m_{lh}(s')} \left(Y^i_{t+1}(\alpha^i_{lh}(\omega), s')\right)^{n_{m_{lh}(s')}-1} w^i_{t+1,-}(\alpha^i_{lh}(\omega), s')}{q_{s'} + \delta'}$$

and the aggregate version is

$$\tilde{W}_{s'}(\omega) = W^1_{s'}(\omega) + W^2_{s'}(\omega).$$

The representative manager in group i solves the problem

$$\begin{aligned} V\left(w^i_t, \omega_{t-1}, s_{t-1}\right) &= \max_{\psi^i_t, \alpha^i_t} \ln \psi^i_t w^i_t + \beta E\left(V\left(w^i_{t+1}, \omega_t, s_t\right)\right) \\ & \quad s.t. \\ w^i_{t+1} &= g^i\left(Y^i_{t+1}\right) w^i_{t+1,-} \end{aligned} \tag{9}$$

In the sequel instead of tracking the stock price $q(\omega)$, and the stock price next period $q_{s'}(\omega)$, it is more convenient to track the price-dividend ratio

$$\pi(\omega) = \frac{q(\omega)}{\delta}.$$

and the price dividend ratio next period

$$\pi_{s'}(\omega) = \frac{q_{s'}(\omega)}{\delta'}.$$

The following proposition characterizes the equilibrium

Proposition 1 *In an asymmetric interior equilibrium,*

1. *the optimal consumption rule of agent i is*

$$\psi^i(\omega) = (1-\beta), \tag{10}$$

2. *her optimal trading strategy is*

$$\alpha^i_{lh}(\omega) = \frac{1-\xi^i_{lh}}{1-\frac{y_H(1+\pi_H(\omega))}{R(\omega)\pi(\omega)}} + \frac{\xi^i_{lh}}{1-\frac{y_L(1+\pi_L(\omega))}{R(\omega)\pi(\omega)}} \tag{11}$$

for some $lh \in \mathcal{H}^i(\omega)$

3. *the value function is*

$$V^i\left(w^i_t, \omega_{t-1}, s_{t-1}\right) = \frac{1}{1-\beta}\ln w^i_t + \Lambda^i\left(\omega_{t-1}, s_{t-1}\right) \tag{12}$$

in any period $t \geq 1$ and

$$V^i_0\left(w^i_0, \omega_0, \tilde{W}_0\right) = \frac{1}{1-\beta}\ln w^i_0 + \Lambda^i_0\left(\omega_0, \tilde{W}_0\right) \tag{13}$$

in $t=0$, where $\tilde{W}_0$ is the initial total share of endowment of all managers.

4. *the total share of capital of group i is*

$$W^1_H(\omega) = \omega \sum_{lh\in\mathcal{H}^1(\omega)} \mu^1_{lh}(\omega) A^1_h \left(\frac{\xi^1_{lh}}{\tilde{\xi}(\omega)}\right)^{n^1_h}. \tag{14}$$

$$W^1_L(\omega) = \omega \sum_{lh\in\mathcal{H}^1(\omega)} \mu^1_{lh}(\omega) A^1_l \left(\frac{1-\xi^1_{lh}}{1-\tilde{\xi}(\omega)}\right)^{n^1_l} \tag{15}$$

$$W^2_H(\omega) = (1-\omega) \sum_{lh\in\mathcal{H}^2(\omega)} \mu^2_{lh}(\omega) A^2_h \left(\frac{\xi^2_{lh}}{\tilde{\xi}(\omega)}\right)^{n^2_h} \tag{16}$$

$$W^2_L(\omega) = (1-\omega) \sum_{lh\in\mathcal{H}^2(\omega)} \mu^2_{lh}(\omega) A^2_l \left(\frac{1-\xi^2_{lh}}{1-\tilde{\xi}(\omega)}\right)^{n^2_l}, \tag{17}$$

5. *the price dividend ratio is*

$$\pi_{s'}(\omega) = \frac{\beta\tilde{W}_{s'}(\omega)}{1-\beta\tilde{W}_{s'}(\omega)} \tag{18}$$

where $\tilde{W}_{s'}(\omega) = W^1_{s'}(\omega) + W^2_{s'}(\omega)$ is the total share of endowment of all managers,

6. *the interest rate is*

$$R(\omega) = \frac{\theta(\omega)}{\pi(\omega)}$$

where $\theta(\omega)$ *solves*

$$\tilde{\xi}(\omega) \frac{1}{y_H(1+\pi_H(\omega))} + \left(1-\tilde{\xi}(\omega)\right) \frac{1}{y_L(1+\pi_L(\omega))} = \frac{1}{\theta(\omega)}, \tag{19}$$

7. *the law of motion is*

$$\Omega_{s'}(\omega) = \frac{W^1_{s'}(\omega)}{\tilde{W}_{s'}(\omega)}. \tag{20}$$

The following proposition describes sufficient conditions for existence of an asymmetric interior equilibrium.

Proposition 2 *The strategy profiles* $\mathcal{P}^i(\omega)$, *prices* $\pi_{s'}(\omega), R(\omega)$ *and law of motion* $\omega' = \Omega_{s'}(\omega)$ *characterized by (12)-(20) form an asymmetric interior equilibrium, if for any* $\omega \in [0,1]$, *and* $i = \{1,2\}$

1. *for any* $l'h' \in \mathcal{H}^i(\omega)$ *and* $l''h''$

$$p \ln \frac{A^i_{h''}\left(\frac{\xi^i_{l''h''}}{\tilde{\xi}(\omega)}\right)^{n^i_{h''}}}{A^i_{h'}\left(\frac{\xi^i_{l'h'}}{\tilde{\xi}(\omega)}\right)^{n^i_{h}}} + (1-p)\ln \frac{A^i_{l''}\left(\frac{1-\xi^i_{l''h''}}{1-\tilde{\xi}(\omega)}\right)^{n^i_{l''}}}{A^i_{l'}\left(\frac{1-\xi^i_{l'h'}}{1-\tilde{\xi}(\omega)}\right)^{n^i_{l'}}} \leq 0 \tag{21}$$

holds with equality whenever $l''h'' \in \mathcal{H}^i(\omega)$, *and holds with strict inequality whenever* $l''h'' \notin \mathcal{H}^i(\omega)$.

2. *for any* $lh \in \mathcal{H}^i(\omega)$, $\alpha^i_{lh}(\omega)$, *defined in (11), is an* $lh-$*portfolio, that is,*

$$h = \sum_{m=1}^{M^i} 1_{\{\frac{\xi^i_{lh}}{\tilde{\xi}(\omega)} \in \kappa^i_m\}} m, \text{ and } l = \sum_{m=1}^{M^i} 1_{\{\frac{1-\xi^i_{lh}}{1-\tilde{\xi}(\omega)} \in \kappa^i_m\}} m \tag{22}$$

3. *owners of capital consume a positive amount,*

$$\omega \sum_{lh\in\mathcal{H}^1(\omega)} \mu^1_{lh}(\omega) A^1_l \left(\frac{1-\xi^1_{lh}}{1-\tilde{\xi}(\omega)}\right)^{n^1_l} + (1-\omega) \sum_{lh\in\mathcal{H}^2(\omega)} \mu^2_{lh}(\omega) A^2_l \left(\frac{\xi^2_{lh}}{\tilde{\xi}(\omega)}\right)^{n^2_l} \leq 1 \tag{23}$$

$$\omega \sum_{lh\in\mathcal{H}^1(\omega)} \mu^1_{lh}(\omega) A^1_h \left(\frac{\xi^1_{lh}}{\tilde{\xi}(\omega)}\right)^{n^1_h} + (1-\omega) \sum_{lh\in\mathcal{H}^2(\omega)} \mu^2_{lh}(\omega) A^2_h \left(\frac{\xi^2_{lh}}{\tilde{\xi}(\omega)}\right)^{n^2_h} \leq 1 \tag{24}$$

We prove these propositions in the Appendix. Although the proof is relatively long, its logic is simple. The difficulty of finding the equilibrium comes from the possible convexities in the incentive function. In particular, as the problem might not be concave in the portfolio choice, α, the first order condition might not be sufficient to find the equilibrium choice. However, because of the interaction of log-utility and our piece-wise constant elasticity specification of the incentive function, there would not be such convexity issues, if the manager should not have to consider the various segments of her incentive function. For example, suppose we modify the incentive function of a manager in a way that regardless of her portfolio she is compensated according to the parameters A_l^i, n_l^i (A_h^i, n_h^i) after a low (high) shock, where l (h) is a given segment of the original incentive function. Then the manager's problem has a well behaving first order condition. In fact, for given prices $\pi_{s'}(\omega), R(\omega)$, her optimal portfolio is $\alpha_{lh}^i(\omega)$ defined in (11).

Now consider the following modified economy. Suppose that we modify the incentive function of each manager in a similar way by choosing an lh index pair. This index pair might be different across managers even within the same group. Think of the sets $\mathcal{M}^i(\omega)$ as the distribution of lh index pairs across managers. Then all managers choose the portfolio (11) for the given lh and prices $\pi_{s'}(\omega), R(\omega)$, consume according to (10), and their value function indeed has the form of (12). Aggregating across their first order conditions and imposing that their total holding of the stock has to sum up to 1, implies that prices satisfy (19). The choice (11) and expression (19) imply that the relative return Y_{t+1}^i of a manager with guess lh is

$$\frac{\xi_{lh}^i}{\tilde{\xi}(\omega)}, \ \frac{1-\xi_{lh}^i}{1-\tilde{\xi}(\omega)} \tag{25}$$

after a high shock and low shock, respectively. This in turn, gives expressions (14)-(17) and the law of motion (20). Market clearing in the goods market implies the price-dividend ratio (18). Finally, conditions (23)-(24) ensure that the consumption of the owners of capital is non-negative. Thus, we have just established that expressions (12)-(20) is an equilibrium of the modified economy.

Now consider the original economy. The equilibrium of the modified economy might not be an equilibrium of the original economy because of two potential problems. First, some of the $\alpha_{lh}^i(\omega)-$ portfolios with a measure $\mu_{lh}^i(\omega) > 0$ in $\mathcal{M}^i(\omega)$ might result in relative returns which are not in the given lh segments of the original incentive function. That is, $\alpha_{lh}^i(\omega)$ might not be an $lh-$portfolio under the equilibrium prices. Condition (22) makes sure that this is not the case. Second, given that in the original economy the choice of the portfolio influences at which segment of her incentive function the manager would be compensated, the manager might prefer to deviate to another lh segment. Condition (21) ensures that

such deviation is not profitable.

Given Proposition 2, finding the equilibrium amounts to guessing and verifying on which segments managers are likely to be evaluated for a given ω. That is guessing and verifying a particular choice of $\mathcal{H}^i(\omega), \mathcal{M}^i(\omega)$. In Section 3, we illustrate this process by considering Examples 1-3. This process is trivial in Example 1, because incentive functions have a single segment. In Example 2, there is a limited number of combinations to consider as $\omega \equiv 1$ and there are only two segments. We also find the unique asymmetric interior equilibrium in the more complex Example 3.

2.2.2 Distribution of returns and the Sharpe-ratio

Throughout the analysis, we are especially interested in the effect of incentives to the distribution of returns, to the relative state-prices and to the resulting Sharpe-ratio. Expressions derived in the following Proposition will help to discover these effects.

Proposition 3 *In any given asymmetric interior−equilibrium*

1. *the distribution of relative returns is characterized by the distribution of lh−strategies $\mathcal{M}^i(\omega)$ and the corresponding relative returns. In particular, the one-period ahead variance of relative return of a given manager i following an lh-strategy is*

$$C_{lh}^i(\omega) \equiv p(1-p)\left(\frac{\xi_{lh}^i}{\tilde{\xi}(\omega)} - \frac{1-\xi_{lh}^i}{1-\tilde{\xi}(\omega)}\right)^2.$$

The cross-sectional dispersion can be characterized by the width of support of returns in a given state

$$\begin{aligned} D_H(\omega) &\equiv \max_i \frac{\xi_{lh}^i}{\tilde{\xi}(\omega)} - \min_i \frac{\xi_{lh}^i}{\tilde{\xi}(\omega)} \\ D_L(\omega) &\equiv \max_i \frac{1-\xi_{lh}^i}{1-\tilde{\xi}(\omega)} - \min_i \frac{1-\xi_{lh}^i}{1-\tilde{\xi}(\omega)} \end{aligned}$$

in the high state and low state respectively, and the mass following the particular strategy, $\mathcal{M}^i(\omega)$.

2. *the state price of the low state relative to the high state is*

$$\frac{y_H}{y_L} X(\omega)$$

where

$$X(\omega) \equiv \frac{\frac{\left(1-\tilde{\xi}(\omega)\right)}{1-p}}{\frac{\tilde{\xi}(\omega)}{p}} \frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)}.$$

3. *the Sharpe-ratio is*

$$S(\omega) = \frac{p^{\frac{1}{2}}(1-p)^{\frac{1}{2}} \left\| y_H X(\omega) - y_L \right\|}{p y_L + (1-p) y_H X(\omega)}. \tag{26}$$

Although, we will gain most of the intuition about an asymmetric interior−equilibrium through the examples, it is useful to note some properties already at this stage. First, substituting (19) into (11) gives

$$\alpha_{lh}^i(\omega) = \frac{\frac{\xi_{lh}^i}{\tilde{\xi}(\omega)} - \frac{\theta(\omega)}{y_H(1+\pi_H(\omega))}}{1 - \frac{\theta(\omega)}{y_H(1+\pi_H(\omega))}}. \tag{27}$$

The over or under exposure to the market risk depends on the size of the shape adjusted probability ξ_{lh}^i relative to its aggregate counterpart $\tilde{\xi}(\omega)$. For example, if the sensitivity to high states relative to low state is the same for all managers than $\xi_{lh}^i = \tilde{\xi}(\omega)$ and each manager holds the market. In general, managers with low relative sensitivity to high states are underexposed to market risk and lend to managers with high sensitivity to the high state. Importantly, ξ_{lh}^i depends jointly on the shape of the incentive function and the chosen strategy. Second, the distribution of relative returns depends exclusively on the distribution of relative size of ξ_{lh}^i and $\tilde{\xi}(\omega)$. Managers with low relative sensitivity to high states will do relatively well in low states and badly in high states. Third, the price dividend ratio, (18), is a monotonically increasing function of the share of the endowment the owners delegate to managers in a given state, $\tilde{W}_{s'}(\omega)$. If this share were 1 in all state, we would be back to the standard Lucas-tree model with a constant price-dividend ratio of $\frac{\beta}{1-\beta}$. Finally, both the deviation of relative state prices and the Sharpe ratio from the standard Lucas-tree case is driven by the term $X(\omega)$. Delegation increases the Sharpe-ratio and relative state prices if and only if $X(\omega)$ is larger than 1. The term $X(\omega)$ is determined by the relative size of two effects. We interpret the term

$$\frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)}$$

as the discount factor-effect. It is large when the average manager gets a higher share of the endowment in the high state. In this case, she will appreciate a dollar less in the high state

which pushes the relative state price and the Sharpe ratio up. We interpret the term

$$\frac{\frac{\left(1-\tilde{\xi}(\omega)\right)}{1-p}}{\frac{\tilde{\xi}(\omega)}{p}} \tag{28}$$

as the capital-flow effect. The capital-flow effect is similar to the classic cash-flow effect in asset pricing. Because of the incentive function, a dollar return in a given state might attract more or less future capital flows. The term (28) shows the relative capital-flow generating ability of a dollar in the low state versus the high state for the average manager. When the incentive function of the average manager is relatively more sensitive in the high state, then a dollar is more valuable in that state which pushes the relative state price and the Sharpe ratio down. As the average manager tends to be richer in the state when a marginal dollar attracts more capital, the capital-flow effect and the discount factor effect tend to drive state prices in the opposite directions. As we will see, the direction of the aggregate effect can go either way.

3 Three examples

In this section, we present three examples. With the help of the first example, we show that if all agents' have constant elasticity incentive functions, i.e., they are all mutual funds, then agents do not trade and delegation does not affect the Sharpe ratio, even if incentive functions differ in their convexity. By the second example, we show that if all agents are identical, but each has an increasing elasticity incentive function, i.e., they are hedge funds, then there is trade among identical agents leading to excess volatility in prices and dispersion in returns. By the third example, we demonstrate how strategies and prices vary with the relative wealth share of agents, if hedge funds trade with mutual funds. We also connect the characteristics of returns to the skewness of the underlying dividend growth process.

3.1 Example 1: no kink

In this example incentive functions of the two type of fund managers differ, but both feature constant elasticity throughout the positive segment of the real line as it is specified in (3). Hence, the incentive function of any agent can be convex or concave, but it is never "log-log convex". With this example, we demonstrate the importance of log-log convexity, that is, increasing elasticity, versus convexity in our structure.

First note that with only a single segment in the incentive functions, conditions (21) and (22) are automatically satisfied. Also, this specification implies that both the individual and

the aggregate shape adjusted probability is $p, \xi^i_{lh} = \tilde{\xi}(\omega) = p$. Thus, (11) implies the same strategy for all managers. By market clearing, this strategy must be that each manager holds the market:

$$\alpha^i = 1.$$

Therefore, relative returns are always 1. Thus, $A^1, A^2 \leq 1$ is sufficient to satisfy conditions (23) and (24). The next corollary follows.

Corollary 1 *If both incentive functions are described by a single segment, $M = 1$, and $A^1, A^2 \leq 1$, then an asymmetric interior equilibrium exists, where each agent holds the market, $\alpha^i = 1$.*

Note also that the law of motion and prices are deterministic and they depend only on the share of endowment the owners delegate when each manager perform as the market, A^1, A^2:

$$\begin{aligned}
\Omega_H(\omega) &= \Omega_L(\omega) = \frac{A^1\omega}{A^1\omega + A^2(1-\omega)} \\
\pi_L(\omega) &= \pi_H(\omega) = \beta\frac{A^1\omega + A^2(1-\omega)}{1-\beta(A^1\omega + A^2(1-\omega))} \\
\theta(\omega) &= \frac{(1+\pi(\omega))}{\frac{p}{y_H} + \frac{(1-p)}{y_L}} = \frac{1}{1-\beta(A^1\omega + A^2(1-\omega))}\frac{y_H y_L}{y_L p + y_H(1-p)} \\
R(\omega) &= \frac{1 - \frac{(A_1(1-\omega)+\omega A_2)}{\beta A_1 A_2}}{\beta(A_2(1-\omega)+\omega A_1) - 1}\frac{y_H y_L}{y_L p + y_H(1-p)}.
\end{aligned}$$

The relative wealth share, ω, and prices, $\pi_H(\omega), \pi_L(\omega), R(\omega)$ are time varying only to the extent that $A^1 \neq A^2$. When $A^1 = A^2 = A$, there is a single group of agents and the price dividend ratio and the normalized interest rate simplify to

$$\pi = \beta\frac{A}{1-\beta A},\ \theta = \frac{1}{1-\beta A}\frac{y_H y_L}{y_L p + y_H(1-p)},$$

which implies the constant interest rate

$$R = \frac{\theta}{\pi} = \frac{1}{\beta A}\frac{y_H y_L}{p y_L + (1-p) y_H}.$$

In contrast, if $A^1 > A^2$, then owners deterministically increase the wealth share of type 1 managers and as they do so, they increase the total endowment share of all managers. This increases the price dividend ratio and decreases the interest rate R.

Interestingly, the relative state price and the Sharpe ratio are not influenced by delegation, because

$$X(\omega) = \frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)} = \frac{\frac{\left(1-\tilde{\xi}(\omega)\right)}{1-p}}{\frac{\tilde{\xi}(\omega)}{p}} = 1.$$

There is neither a discount rate effect nor a capital-flow effect for any A^1, A^2, n^1, n^2. Relatedly, the sensitivity parameters, n^i, do not have any effect on the equilibrium. Convexity does not matter in itself.[7] This is because of the interaction of log utility and constant elasticity incentive functions. The marginal utility from a dollar linearly increases in n^i. Given that n^i is the same across states, the marginal rate of substitution is not affected by n^i. Thus, the marginal rate of substitution is the same for both agents. Hence, there are no gains from trade.

3.2 Example 2: only hedge funds

In this example, we consider an economy with a single type of managers. These managers have an increasing elasticity incentive function with a single kink as specified in (4), that is, they are hedge funds.

3.2.1 Characterization and Existence

Given that there is only one type of managers, the wealth share is constant: $\omega = 1$ and we can omit ω and the reference to the type of manager, i, from every object. Because we have a single kink, it makes sense to index the two segments of the incentive function as A, B for the cases when the manager is compensated above and below the kink, respectively. For example, a $BA-$strategy for given prices is an lh-strategy for which the manager is compensated above the kink in the high state and below the kink in the low state. We show that depending on the parameters, there is always a unique asymmetric interior equilibrium. This equilibrium can be one of four different types.

In a BB equilibrium each manager invests all her wealth in the risky asset, $\alpha_{BB} = 1$, and the equilibrium is very similar to the one of Example 1. There are also four types of $l'h' - l''h''$ equilibria where $l'h' - l''h'' = AB - BB, BA - BB, AB - BA, BA - AB$. In an $l'h' - l''h''$ equilibrium, $\mathcal{H} = \{l'h', l''h''\}$ and $\mathcal{M} = \{\mu_{l'h'}, \mu_{l''h''}\}$. This implies, after simplify the notation and writing μ instead of $\mu_{l''h''}$, that μ proportion of managers follow an $l''h''$ strategy while $1-\mu$ fraction of them follow an $l'h'$ strategy in a $l'h' - l''h''$ equilibrium. As

[7] The importance of considering the utility function and the incentives was also pointed out by Ross (2006).

$\omega = 1$, the aggregate shape adjusted probability, $\tilde{\xi}(\omega)$ is a constant $\bar{\xi}$ By (8),

$$\bar{\xi} = \mu \xi_{l'h'} + (1-\mu)\, \xi_{l''h''}$$

which we can rewrite as

$$\mu = \frac{\bar{\xi} - \xi_{l''h''}}{\xi_{l'h'} - \xi_{l''h''}} \tag{29}$$

where $\xi^1_{l'h'}, \xi^1_{l''h''}$ are defined in (7) and $\bar{\xi}$ is determined by the condition that managers have to be indifferent between the two strategies. As it is stated by (21) in Proposition 2, this indifference condition is equivalent to

$$p \ln \frac{A_{h''} \left(\frac{\xi_{l''h''}}{\bar{\xi}}\right)^{n_{h''}}}{A_{h'} \left(\frac{\xi_{l'h'}}{\bar{\xi}}\right)^{n_h}} + (1-p) \ln \frac{A_{l''} \left(\frac{1-\xi_{l''h''}}{1-\bar{\xi}}\right)^{n_{l''}}}{A_{l'} \left(\frac{1-\xi_{l'h'}}{1-\bar{\xi}}\right)^{n_{l'}}} \equiv 0. \tag{30}$$

By substituting $\tilde{\xi}(\omega) = \bar{\xi}$, expression (27) describes the equilibrium portfolios of hedge funds following $lh = l'h', l''h''$ strategies.

In the next proposition we show how the parameters determine the type of asymmetric interior equilibrium.

Proposition 4 *There are critical values* $\hat{p}_{BA-BB}(k, A^1, n_1, n_2)$, $\hat{p}_{BA-AB}(k, A^1, n_1, n_2) \in \left(0, \frac{1}{2}\right)$ *and* $\hat{k}_{high}(n_2, n_1), \hat{k}_{low}(n_2, n_1)$ *that*

1. *if* $k > \hat{k}_{high}$, *there is a unique asymmetric interior equilibrium and it is a* BB *equilibrium* ,

2. *if* $\hat{k}_{low} < k < \hat{k}_{high}$, *there is a unique asymmetric interior equilibrium and its type depends on* p *as follows:*

$p \in (0, \hat{p}_{BA-BB})$	$p \in (\hat{p}_{BA-BB}, 1 - \hat{p}_{BA-BB})$	$p \in (\hat{p}_{BA-BB}, 1)$
$BA - BB$ *equilibrium*	BB *equilibrium*	$AB - BB$ *equilibrium*

.

3. *if* $k < \hat{k}_{low}$,*there is a unique asymmetric interior equilibrium and its type depends on* p *as follows:*

$p \in (0, \hat{p}_{BA-AB})$	$p \in \left(\hat{p}_{BA-AB}, \frac{1}{2}\right)$	$p \in \left(\frac{1}{2}, 1 - \hat{p}_{BA-AB}\right)$	$p \in (1 - \hat{p}_{BA-AB}, 1)$
$BA - BB$ *equilibrium*	$BA - AB$ *equilibrium*	$AB - BA$ *equilibrium*	$AB - BB$ *equilibrium*

.

Consider the parameters for which the equilibrium is of the $l'h' - l''h''$ type. Our result shows that increasing elasticity incentive functions create trading across ex-ante identical

agents. The idea is that having opposite positions help the managers to being evaluated above the kink at least in one of the states. If they were all to follow the same strategy, their relative performance would be 1 and they would not get the extra capital flow in any states. So in equilibrium, managers form two subgroups. One subgroup leverages up by selling bonds to the other group. The first group is above the market after a high shock. The other group has less exposure to the market risk, therefore they will be above the market after a low shock. The excess return adjusts in a way that each manager is indifferent across following any of the strategies.

It is useful to disentangle the role of various elements in hedge funds' incentives in this result. The incentive structure of hedge fund managers is a combination of the log-utility and the flow-performance relationship described by the incentive function. The argument of the incentive function is relative performance and it has a kink and increasing elasticity. Which of these elements is critical? Which type of utility function would imply the same result?

We argue that the critical property is that hedge funds' portfolio decision problem is not globally concave in α. Consider the following thought experiment. Suppose that the economy is populated by individual traders living for two periods. In period 2, the economy is in a high or a low state with equal probability. In period 1, each trader can place a bet of size ρ on which state will realize, in period 2 each trader consumes her wealth. Trader's utility function is $u(\cdot)$. Consider the following inequalities

$$\begin{aligned} \frac{1}{2}\left(u\left(w_1+\rho\right)+u\left(w_1-\rho\right)\right) &> u\left(w_1\right) \\ \frac{1}{2}\left(u\left(w_1-\rho\right)+u\left(w_1+\rho\right)\right) &> u\left(w_1\right). \end{aligned}$$

If both inequalities hold, then traders starting with wealth w_1 are better off splitting up and bet than not betting at all. Note that the inequalities do not hold for any ρ if $u(\cdot)$ is concave, but holds for any ρ if it is convex. Note also, that if $u(\cdot)$ is convex than the left hand side of both inequalities are increasing in ρ. The parallel with Example 2 is apparent. There are gains from trade among hedge funds, because their increasing elasticity incentive function imply a convex segment in their portfolio problem. However, if their incentives would be globally convex, they would prefer to increase bets without bounds and there were no equilibrium. Intuitively, the increasing elasticity incentive function introduces enough convexity for trade, but logarithmic utility introduces enough concavity for the existence of an equilibrium.

3.2.2 Distribution of returns and the Sharpe-ratio

To understand better the effect of incentives on the distribution of returns, we analyze comparative statics on kink, k. Note that a marginal increase on k hurts the manager in two ways. First, it requires the manager to achieve higher relative return to reach the high-elasticity segment. Second, it decreases her flows in the high-elasticity segment by decreasing A_2 as $A_2 = A_1 k^{n_1 - n_2}$. The following Proposition states the results.

Proposition 5 *Consider a marginal increase in k.*

1. *(volatility of relative returns) In any $l'h' - l''h''$ equilibrium, the variance of relative returns, C_{lh}, increases for the manager with larger exposure to the market, and decreases for the manager with the smaller exposure to the market.*

2. *(cross-sectional dispersion of relative returns) The support of relative returns D_H, D_L narrows in the smaller probability state and widens in the larger probability state. The mass of hedge funds which over perform the market decreases in the smaller probability state and increases in the larger probability state.*

Turning to the price dividend ratio, the wealth share of hedge funds is given by (14), (15) as

$$\begin{aligned} W_H &= \tilde{W}_H = \mu A_{h'} \left(\frac{\xi_{l'h'}}{\bar{\xi}} \right)^{n_h} + (1-\mu) A_{h''} \left(\frac{\xi_{l''h''}}{\bar{\xi}} \right)^{n_{h''}} \\ W_L &= \tilde{W}_L = \mu A_{l'} \left(\frac{1-\xi_{l'h'}}{1-\bar{\xi}} \right)^{n_{l'}} + (1-\mu) A_{l''} \left(\frac{1-\xi_{l''h''}}{1-\bar{\xi}} \right)^{n_{l''}} . \end{aligned}$$

Trivially, in this example the wealth share of hedge funds and the endowment share of all funds, $\tilde{W}_{s'}$, is the same. As the wealth share depends on the shock s', this is also true for the equilibrium price dividend ratio

$$\pi_{s'} = \frac{\beta \tilde{W}_{s'}}{1 - \beta \tilde{W}_{s'}} \tag{31}$$

for $s' = L, H$. The normalized interest rate is given by (19). Expression (31) show that the increasing elasticity incentive function also creates return volatility. Asymmetric strategies imply that the distribution of relative returns differ after a low shock and a high shock. Therefore, both the share of the endowment delegated to each manager and the total share, $\tilde{W}_{s'}$, varies with the shock. Thus, the price dividend ratio, (31), also varies with the shock.

3.3 Example 3: the hedge fund and the mutual fund

In this example, we consider an economy with two types of fund managers. Managers in the first group have increasing elasticity incentive functions as it is specified in (4). This type of managers, whom we refer to as hedge funds, were the only participants in asset markets in Example 2. Managers in the second group have constant elasticity incentive functions as specified in (6). We refer to them as mutual funds. This example nests Example 2 when all the capital is managed by hedge funds, $\omega = 1$, and it is similar to Example 1, when all the capital is managed by mutual funds, $\omega = 0$.

3.3.1 Characterization and Existence

There is always a unique asymmetric interior equilibrium in this example. The types of equilibria are similar to Example 2. In a BB equilibrium each manager invests all her wealth in the risky asset,$\alpha^1_{BB} = \alpha_{BB} = 1$ regardless of the wealth share of hedge funds, ω.

There are also four types of $l'h' - l''h''$ equilibria where $l'h' - l''h'' = AB - BB, BA - BB, AB - BA, BA - AB$. In an $l'h' - l''h''$ equilibrium hedge funds potentially follow asymmetric strategies. That is, there is a $\mu_{l'h'}(\omega)$ and $\mu_{l''h''}(\omega)$ that $\mathcal{H}^1(\omega) = \{l'h', l''h''\}$ and $\mathcal{M}^1(\omega) = \{\mu_{l'h'}(\omega), \mu_{l''h''}(\omega)\}$. Similarly to Example 2, we simply notation by referring to $\mu_{l''h''}(\omega)$ as $\mu(\omega)$ and to $\mu_{l'h'}(\omega)$ as $1 - \mu(\omega)$. Contrast to Example 2, the set of hedge funds following a particular strategy varies with the wealth share of hedge funds. In particular, in each $l'h' - l''h''$ equilibrium there is a threshold $\hat{\omega} \in (0, 1)$ that

$$\mu(\omega) = \left\{ \begin{array}{ccc} 1 & if & \omega \leq \hat{\omega} \\ \hat{\mu}(\omega) & if & \omega > \hat{\omega} \end{array} \right\}. \tag{32}$$

where $\hat{\mu}(\omega)$ is a monotonically decreasing function. That is, each hedge fund follow the same $l'h'$ strategy as long as their wealth share, ω, is relatively small, but an increasing measure of them follows the $l''h''$ strategy as their wealth share increases. In any of the equilibria, all mutual fund managers follow symmetric strategies.

The set of managers following the two equilibrium strategies determined similarly than in Example 2. By (8)

$$\tilde{\xi}(\omega) = \omega \left(\mu(\omega) \xi^1_{l'h'} + (1 - \mu(\omega)) \xi^1_{l''h''} \right) + (1 - \omega) p.$$

Given that managers has to be indifferent between the two equilibrium strategies for any $\omega > \hat{\omega}$, just as in Example 2, for any $\omega > \hat{\omega}$, $\tilde{\xi}(\omega)$ is the constant $\bar{\xi}$ determined by the

indifference condition (30). Thus,by (8) and (32),

$$\hat{\mu}(\omega) = \frac{\bar{\xi} - \omega \xi^1_{l''h''} - (1-\omega)\, p}{\omega \left(\xi^1_{l'h'} - \xi^1_{l''h''}\right)}.$$

For any, $\omega < \hat{\omega}$, by (8) and (32),

$$\tilde{\xi}(\omega) = \omega \xi^1_{l'h'} + (1-\omega)\, p.$$

Finally, $\tilde{\xi}(\hat{\omega}) = \bar{\xi}$ gives

$$\hat{\omega} \equiv \frac{\bar{\xi} - p}{\xi^1_{l'h'} - p}.$$

The following proposition states that the type of equilibria depends on the parameters the same way as in Example 2.

Proposition 6 *If managers in group 1 have incentives of the form (4), and managers in group 2 have incentives of the form (6), then there is a unique asymmetric interior equilibrium for any set of parameters. The type of the equilibrium depends on the parameters exactly as it is stated in Proposition 4.*

Consistently to Proposition 1, the portfolios of agents are given by (27) where $lh = l'h', l''h''$ for hedge funds, while prices and low of motion of ω are given by (18)-(20) with the straightforward substitution of the equilibrium distribution of strategies of hedge funds and mutual funds.

To see the intuition behind the equilibrium consider first the case when the wealth share of hedge funds has a measure of zero, $\omega = 0$. In this case all mutual funds hold the market just as in Example 1 and their preferences determine equilibrium prices. Consider the decision of the first hedge fund who enters the market. Because the hedge fund manager's increasing elasticity incentive function, her relative evaluation of a unit of marginal utility in the two states differ from the relative evaluation of the mutual fund. Namely, she prefers a unit more in the state when it generates more capital flow. Interestingly, she can decide which state this should be. The hedge fund can sell bonds and leverage up, aiming for high relative return and the corresponding extra capital flow after a high shock. Alternatively, she can hold bonds aiming for the extra capital flow after a the low shock. The presence of the kink implies that the problem (9) is non-concave in α. Thus, instead of comparing marginal deviations, we have to compare two locally optimal portfolios. Proposition 6 shows that at $\omega = 0$, the hedge fund prefers to get the extra capital flow always in the lower probability state. As ω increases, hedge funds affect prices more and more and make their preferred

strategy less and less attractive up to the point of $\hat{\omega}$. From that point on hedge funds are indifferent between the optimal $l'h'$-strategy and the optimal $l''h''$-strategy and follow an asymmetric strategy.

3.3.2 Distribution of returns and the Sharpe-ratio

Consider again the result in Proposition 6 that as long as hedge funds follow a symmetric strategy, they prefer to get the extra capital flow in the lower probability state. This implies that the relative return of the hedge fund is always positively skewed: it is higher in the lower probability state.We summarize this in the following corollary.

Corollary 2 *In any $lh = l'h', l''h''$ equilibrium, whenever $\omega < \hat{\omega}$ the relative return of all hedge funds is positively skewed.*

3.3.3 Sharpe ratio

By the general analysis of an asymmetric interior equilibrium, the Sharpe ratio with delegation is larger (smaller) than the Sharpe ratio without delegation if and only if

$$X(\omega) = \frac{\frac{1-\tilde{\xi}(\omega)}{1-p}}{\frac{\tilde{\xi}(\omega)}{p}} \frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)} > (<) 1$$

in the given $l'h' - l''h''$ equilibrium. Consider first the capital flow effect

$$\frac{\frac{1-\tilde{\xi}(\omega)}{1-p}}{\frac{\tilde{\xi}(\omega)}{p}}.$$

For any $\omega < \hat{\omega}$, this term is

$$\frac{\omega\frac{n_{l'}}{pn_{h'}+(1-p)n_{l'}} + (1-\omega)}{\omega\frac{n_{h'}}{pn_{h'}+(1-p)n_{l'}} + (1-\omega)}.$$

It is 1 at $\omega = 0$, and decreasing (increasing) if $n_{h'} > n_{l'}$. For example, for all sets of parameters which result in $l'h' = BA$, this term tends to decrease the Sharpe-ratio while if $l'h' = AB$, it tends to increase the Sharpe-ratio. For any $\omega > \hat{\omega}$ the term is constant at the level

$$\frac{\frac{1-\hat{\xi}}{1-p}}{\frac{\hat{\xi}}{p}}.$$

Consider now the discount rate effect

$$\frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)}.$$

The discount rate effect is 1 at $\omega = 0$. For any $\omega < \hat{\omega}$, hedge funds trade against mutual funds, therefore their endowment shares, $W^1_{s'}(\omega)$ and $W^2_{s'}(\omega)$ moves into opposite directions. Given that hedge fund incentives are log-convex while mutual fund incentives are log-linear, typically (but not always) the change of the endowment share of hedge funds dominates that of mutual funds in the aggregate term, $\tilde{W}_{s'}(\omega)$. Therefore, whenever $l'h' = BA$ ($l'h' = AB$) the discount rate effect is increasing (decreasing) at $\omega = 0$ and it is larger (smaller) than 1 for any $\omega < \hat{\omega}$. As the following Lemma states, for $\omega > \hat{\omega}$, the discount rate effect is always monotonic in ω, but the sign of the derivative depends on the parameters.

Lemma 1 *If, $\omega > \hat{\omega}$, then the discount rate effect is monotonic in ω. In particular,*

$$sgn\left(\frac{\partial\left(\frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)}\right)}{\partial\omega}\right) = sgn\left((a_H - a_L) + (b_L - b_H) - \beta\left(a_H b_L - b_H a_L\right)\right)$$

where

$$\begin{aligned}
a_H &= \varepsilon A_{h''}\left(\frac{\xi^1_{l''h''}}{\bar{\xi}}\right)^{n_{h''}} + (1-\varepsilon)A_{h'}\left(\frac{\xi^1_{l'h'}}{\bar{\xi}}\right)^{n_{h'}} \\
a_L &= \varepsilon A_{l''}\left(\frac{1-\xi^1_{l''h''}}{1-\bar{\xi}}\right)^{n_{l''}} + (1-\varepsilon)A_{l'}\left(\frac{1-\xi^1_{l'h'}}{1-\bar{\xi}}\right)^{n_{l'}} \\
b_H &= A\left(\frac{p}{\bar{\xi}}\right)^{n},\ b_L = A\left(\frac{1-p}{\bar{\xi}}\right)^{n} \\
\varepsilon &= \frac{\xi^1_{l'h'} - p}{\xi^1_{l'h'} - \xi^1_{l''h''}}.
\end{aligned}$$

Note that the above Lemma also determines whether the Sharpe ratio is pro-cyclical or counter-cyclical in the region $\omega \in [\hat{\omega}, 1]$. In that region the capital flow effect is constant in ω, so the Sharpe ratio changes in line with the change of the discount factor effect. In an equilibrium where $l'h' = BA$ ($l'h' = AB$), a high shock always increases (decreases) the wealth share of hedge funds, ω. Therefore, the Sharpe-ratio is countercyclical whenever $l'h' = BA$ and $\frac{\partial\left(\frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)}\right)}{\partial\omega} < 0$ or when $l'h' = AB$ and $\frac{\partial\left(\frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)}\right)}{\partial\omega} > 0$. It is procyclical in all other cases.

In terms of the general effect of delegation on the Sharpe-ratio, the above results implies that whether delegation increases or decreases Sharpe ratio depends on two, generally

opposing, forces. For example, in an $l'h' = BA$ equilibrium, a dollar in the high state generates more fund flows, because in this case managers are above the kink. This decreases the Sharpe-ratio for any ω because the asset is more attractive with delegation than without delegation for any ω. On the other hand, the average manager tends to can keep a larger share in the high state, thus the asset is less attractive in terms of consumption smoothing. This increases the Sharpe-ratio. Numerical simulations show that the net effect can go either way.

4 Extensions

In this section we introduce two extensions. For both, our starting point is our last example where a group of hedge funds and a group of mutual funds trade on the financial market. In the first extension, we keep the incentive functions of mutual funds and hedge funds given, but we let managers to pick their type. We are interested in stability conditions which make both type of institutions coexist in equilibrium. In the second extension, we keep the fraction of hedge funds and mutual funds as given but we introduce a second market where the dividend growth process follows a different Bernoulli distribution. We assume that managers have to pick one of the two markets at period 0 because of some fixed initial capital investment. We are interested in the effect of incentives on managers' preference across markets.

4.1 Hedge funds or mutual funds?

In this part, we allow managers to pick whether they would like to operate as hedge fund managers or as mutual fund managers. We allow them to switch types at period 0 taking everyone else's decision as given. This amounts to comparing the value functions of the two type of fund managers at period 0. From (13), we define the difference in value functions as

$$\Delta_\Lambda(\omega_0) \equiv V_0^1\left(w_0^1, \omega_0, \tilde{W}_0\right) - V_0^2\left(w_0^2, \omega_0, \tilde{W}_0\right).$$

We show in the Appendix that the difference is indeed only a function of the initial share of capital, ω_0. There are three ways that an equilibrium can arise in this extended problem. If

$$\Delta_\Lambda(0) < 0,$$

or

$$\Delta_\Lambda(1) > 0$$

then we have an equilibrium with only hedge funds or only mutual funds respectively. However, if a ω_0^* exists for which

$$\Delta_\Lambda(\omega_0^*) = 0,$$

then we have an equilibrium where both type of agents coexists. We can also check whether

$$\frac{\partial \Delta_\Lambda(\omega_0)}{\partial \omega_0}|_{\omega_0=\omega_0^*} < 0,$$

holds, which we can interpret as a stability criterion. Intuitively, a small increase in ω_0 from ω_0^* would generate incentives for hedge fund managers to switch their type to mutual funds and a small decrease in ω_0 would generate opposite incentives. Thus, a perturbation on ω_0^* would generate incentives to push back the equilibrium to ω_0^*.

The next proposition states simple and intuitive conditions on the incentive functions for the existence of an equilibrium where both mutual funds and hedge funds coexists and the stability criterion holds.

Proposition 7 *For any other parameters there are finite thresholds* $\left(\hat{A}_{low}, \hat{A}_{high}\right)$ *that if*

$$\ln \frac{A_{h'}^{(1-p)} A_{l'}^{p}}{A} \in \left(\hat{A}_{low}, \hat{A}_{high}\right) \tag{33}$$

then an equilibrium of the extended problem exists where mutual funds and hedge funds coexist, i.e. $\omega_0^* \in (0,1)$ *If* $n_1 \leq n < n_2$ *also holds, then there is a* $\omega_0^* \in (0,1)$ *that*

$$\frac{\partial \Delta_\Lambda(\omega_0)}{\partial \omega_0}|_{\omega_0=\omega_0^*} < 0.$$

Condition (33) simply states that for the coexistence of mutual funds and hedge funds, the incentive functions cannot provide much better fund flows to one of the types in average. For example, if A is very large compared to the geometric average of A_1 and A_2, then being a mutual funds is much more attractive.

4.2 Negatively skewed or positively skewed market?

In this extension, we consider the choice of a given manager over the particular market she wants to enter. To keep the exercise simple, we consider two markets which are fully segmented after period 0. Think of each market as a country. Each country has its own dividend process owned by a different set of clients. The type of each manager is given, but at period 0, each can choose which market to enter. What we have in mind is that entering a

particular market might require market specific fixed investment in terms of human capital or in terms of collecting proprietary data and building market specific models. Therefore, each manager prefer to specializes to a single country. Before, we consider the choice of the manager, we analyze the two economies of opposite skewness in isolation.

4.2.1 Economies with opposite skewness

We compare two otherwise similar economies with positively skewed dividend growth process in one of them, and negatively skewed dividend process in the other. In particular, we pick a $\underline{p} > \frac{1}{2}$ and assume that the probability of the high state is $p = 1 - \underline{p}$ in the first economy and $p = \underline{p}$ in the second economy. In the first economy, dividend growth is still described by y_H, y_L, but we allow for a different dividend growth process in the second economy characterized by $\underline{y}_H, \underline{y}_L$. This allows for various comparisons. For example, if $y_H - y_L = \underline{y}_H - \underline{y}_L$ then the variance of dividend growth is $\underline{p}\left(1-\underline{p}\right)(y_H - y_L)^2$ in both economies. One can also require that the mean growth rate also the same by satisfying $(1-\bar{p})\, y_H + \bar{p} y_L = \bar{p}\underline{y}_H + (1-\bar{p})\, \underline{y}_L$. Alternatively, one can require that in a case with no delegation ($A_1 = A_2 = A = n = n_1 = n_2 = 1$), each manager would be indifferent between the two economies. We will show that this amounts to the parameter restriction

$$\left(1-\underline{p}\right) \ln \frac{y_H}{\underline{y}_L} + \underline{p} \ln \frac{y_L}{\underline{y}_H} = 0.$$

In any case, the first market has positive skewness, while the second market has negative skewness.

In the next Proposition we show that our framework admits a large degree of symmetry between our two economies. In particular, when a manager would follow an BA strategy in the first economy, then her optimal strategy is to follow AB in the second economy and vice-versa. Also, what happens with the distribution of realized returns, law of motion of ω, and the price dividend ratio in the low state in one economy, happens in the high state in the other economy.

Proposition 8 *There are critical values* $\hat{p}_{BA-BB}(k, A^1, n_1, n_2)$, $\hat{p}_{BA-AB}(k, A^1, n_1, n_2) \in \left(0, \frac{1}{2}\right)$ *and* $\hat{k}_{high}(n_2, n_1), \hat{k}_{low}(n_2, n_1)$ *that*

1. *if* $k > \hat{k}_{high}$, *there is a unique asymmetric interior equilibrium in both economies and it is a* BB *equilibrium* ,

2. *if* $\hat{k}_{low} < k < \hat{k}_{high}$, *there is a unique asymmetric interior equilibrium in both economies*

and its type depends on p as follows:

	$\underline{p} \in \left(\frac{1}{2}, 1-\hat{p}_{BA-BB},\right)$	$\underline{p} \in (\hat{p}_{BA-BB}, 1)$
$p = \underline{p}$	BB-*equilibrium*	$AB - BB$ *equilibrium*
$p = 1 - \underline{p}$	BB-*equilibrium*	$BA - BB$ *equilibrium*

3. *if $k < \hat{k}_{low}$,there is a unique asymmetric interior equilibrium in both economies and its type depends on p as follows:*

	$\underline{p} \in \left(\frac{1}{2}, \hat{p}_{BA-AB}\right)$	$\underline{p} \in (\hat{p}_{BA-AB}, 1)$
$p = \underline{p}$	$AB - BA$ *equilibrium*	$AB - BB$ *equilibrium*
$p = 1 - \underline{p}$	$BA - BA$ *equilibrium*	$BA - BB$ *equilibrium*

.

Furthermore, denoting the variables and functions corresponding the second economy by underline,

$$\begin{aligned} \xi_{AB} &= 1 - \underline{\xi}_{BA} \\ \tilde{\xi}_{BA-BB}(\omega) &= 1 - \underline{\tilde{\xi}}_{AB-BB}(\omega), \ \tilde{\xi}_{BA-AB}(\omega) = 1 - \underline{\tilde{\xi}}_{AB-BA}(\omega) \end{aligned}$$

and

$$\begin{aligned} \tilde{W}_H(\omega) &= \underline{\tilde{W}}_L(\omega) \\ \pi_H(\omega) &= \underline{\pi}_L(\omega) \\ \Omega_H(\omega) &= \underline{\Omega}_L(\omega). \end{aligned}$$

Let us turn to the Sharpe-ratio. Observe that without delegation the two economies have generally different Sharpe ratio as

$$\frac{\underline{p}\left(1-\underline{p}\right)^{\frac{1}{2}}(y_H - y_L)}{\left(1-\underline{p}\right)\underline{y}_L + \bar{p}\underline{y}_H} \neq \frac{\underline{p}\left(1-\underline{p}\right)^{\frac{1}{2}}\left(\underline{y}_H - \underline{y}_L\right)}{\bar{p}y_L + \left(1-\underline{p}\right)y_H},$$

For the case with delegation, recall from (26) that the Sharpe ratio is

$$\frac{p^{\frac{1}{2}}(1-p)^{\frac{1}{2}}\|y_H X - y_L\|}{py_L + (1-p)\,y_H X}.$$

Observe that by the previous proposition,

$$X|_{p=1-\bar{p}} = \frac{\frac{1-\tilde{\xi}_{BA-l''h''}(\omega)}{\bar{p}}}{\frac{\tilde{\xi}_{BA-l''h''}(\omega)}{1-\bar{p}}} \frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)} = \frac{1}{\underline{X}} = \frac{\frac{\tilde{\xi}_{AB-l''h''}(\omega)}{\bar{p}}}{\frac{1-\tilde{\xi}_{AB-l''h''}(\omega)}{1-\bar{p}}} \frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)}.$$

This implies the following corollary.

Corollary 3 *For any $\underline{p} \neq \frac{1}{2}$ and $\underline{y}_L, \underline{y}_H, y_H, y_L$ and ω, whenever delegation increases the Sharpe-ratio in a given economy, it also decreases the Sharpe ratio in the other economy.*

In the next part, we turn to the choice of managers between the two economies.

4.2.2 The choice of managers

We compare the value functions on the two countries of a given hedge fund keeping the relative capital, ω_0 and the initial share of endowment, $\tilde{W}_0$ the same across countries. This implies the following proposition.

Proposition 9 *If in both countries the relative capital, ω_0 and the initial share of endowment, $\tilde{W}_0$, is the same, then*

$$V_0^i\left(w_0^i, \tilde{W}_0, \omega_0\right) - \underline{V}_0^i\left(w_0^i, \tilde{W}_0, \omega_0\right) = \frac{\beta}{(1-\beta)^2}\left(\left(1-\underline{p}\right)\ln\frac{y_H}{\underline{y}_L} + \underline{p}\ln\frac{y_L}{\underline{y}_H}\right).$$

Note that the difference in value functions across the two countries is independent of the parameters of delegation, A_1, A, k, n_2, n_1, n. This shows that an individual trader would pick the same country as any type of a mutual fund or hedge fund. In equilibrium, the preference on markets with different skewness is independent of the shape of the incentive function.

5 Conclusion

In this paper, we introduce delegation into a standard Lucas exchange economy. In our model, all financial assets are traded by professional investors, but the endowment process is owned by their clients. Fluctuations in capital under management are driven by the exogenously specified incentive functions. We consider a rich set of possible shape of incentive functions including examples with both convex and concave intervals. We allow for up to two types of professional investors with managers with different incentives. We derive most of the insights of the model by presenting three examples. We focus on two basic types of financial institutions. Managers with an incentive function with increasing elasticity are

referred to as hedge funds, while managers with incentive functions of constant elasticity are referred to as mutual funds.

In our first example, we show that when only mutual funds populate the market, delegation does not effect trading strategies or the Sharpe-ratio. This is the case, even if one group of mutual fund managers have convex incentives, while the other group has concave incentives. In our second example, we show that when the market is populated by only identical hedge funds, hedge funds trade among each other, their returns are dispersed and price-dividend ratios are excessively volatile. In our third example, we show that when hedge funds and mutual funds trade with each other, hedge funds typically lends from mutual funds if a recession is more likely than a boom, and borrows in the opposite case. Relative returns of hedge funds are positively skewed. Finally, in general, delegation effects the Sharpe ratio by two opposing channels leaving the aggregate effect ambiguous.

Our paper is a first step to understand the effects of delegation in standard asset pricing models. There are a number of natural extensions. It would be useful to build a more sophisticated model of endogenous formation of hedge funds vs. mutual funds. Second, by deriving the decision of clients from a maximization problem, one could analyze the welfare effects of the different type of financial intermediaries.

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6 Appendix

A Proof of Propositions 1 and 2

The logic of the proof is at is described in the main text. First we show that Proposition 1 described an equilibrium of a modified problem where managers are evaluated at a given l segment of their incentive function after a low shock and after a given h segment after a high shock. Then we show that under conditions described in Proposition 2, the equilibrium of the modified problem is the equilibrium in the original problem. Finally, we show that the

A.1 Equlibrium in the modified problem

Fix the sets $\mathcal{M}^i(\omega), \mathcal{H}^i(\omega)$. Modify the incentive functions of each manager in a way that for any ω, for each $lh \in \mathcal{H}^i(\omega)$, $\mu^i_{lh}(\omega)$ measure of managers face the incentive function

$$w^i_{t+1} = \left\{ \begin{array}{lll} A^i_h \left(Y^i_{t+1}\right)^{n^i_h - 1} w^i_{t+1,-} & if & s_{t+1} = H \\ A^i_l \left(Y^i_{t+1}\right)^{n^i_l - 1} w^i_{t+1,-} & if & s_{t+1} = L \end{array} \right\}.$$

We show by a series of lemmas, that in this modified economy, expressions in Proposition 1characterize an equilibrium. We will use the observation that the market clearing on the bond market implies

$$\begin{aligned} \alpha^1 \left(\bar{w}^1 - \bar{c}^1\right) + \alpha^2 \left(\bar{w}^2 - \bar{c}^2\right) &= q \\ \omega\alpha^1 + (1-\omega)\alpha^2 &= 1. \end{aligned} \tag{34}$$

Lemma 2 *Strategies $(\mathcal{M}^i(\omega), \mathcal{H}^i(\omega), \mathcal{A}^i(\omega))$ and Euler-equation (19) imply that the total share of capital of each group after each shock is given by (14)-(17), law of motion of ω is given by (20) and the price dividend ratio is given by.(18). Furthermore, relative returns are given by (25).*

Proof. By (11) and the market clearing conditions

$$\begin{aligned} w^1 - c^1 + w^2 - c^2 &= q \\ w^1 - c^1 &= \frac{(w^1 - c^1)\, q}{w^1 - c^1 + w^2 - c^2} = \omega q \\ w^2 - c^2 &= \beta (1-\omega)\, q. \end{aligned} \tag{35}$$

imply that the return of a manager in group 1 after a low shock at the end of the period is

$$\left(\alpha^1\left(\frac{\delta_{t+1}+q_{t+1}}{q_t}-R_t\right)+R_t\right)=$$
$$=R_t\left(\alpha^1\left(\frac{\delta_{t+1}+q_{t+1}}{q_tR_t}-1\right)+1\right)$$

If $s_{t+1}=L$, then this equal to

$$\begin{aligned}
R_t\left(\alpha^1(\omega)\left(\frac{y_L(1+\pi_L(\omega))}{\theta(\omega)}-1\right)+1\right) &= R_t\left(\left(1-\xi^i\right)\frac{\left(\frac{y_L(1+\pi_L(\omega))}{\theta(\omega)}-1\right)}{1-\frac{y_H(1+\pi_H(\omega))}{\theta(\omega)}}+\left(1-\xi^i\right)\right)= \\
&= R_t\left(\left(1-\xi^i\right)\left(\frac{\frac{y_L(1+\pi_L(\omega))}{\theta(\omega)}-\frac{y_H(1+\pi_H(\omega))}{\theta(\omega)}}{1-\frac{y_H(1+\pi_H(\omega))}{\theta(\omega)}}\right)\right)= \\
&= R_t\left(\left(1-\xi^i\right)\left(\frac{y_H(1+\pi_H(\omega))-y_L(1+\pi_L(\omega))}{y_H(1+\pi_H(\omega))-\theta(\omega)}\right)\right)
\end{aligned}$$

Using that from (19), we can rewrite this as

$$\frac{(1-\xi^1_{lh})}{1-\tilde{\xi}(\omega)}\frac{\delta_t}{q_t}y_L(1+\pi_L(\omega)). \tag{36}$$

similarly, after a high shock it is

$$\frac{\xi^1_{lh}}{\tilde{\xi}(\omega)}\frac{\delta_t}{q_t}y_H(1+\pi_H(\omega)) \tag{37}$$

and for the manager in the second group it is

$$\frac{(1-\xi^2_{lh})}{1-\tilde{\xi}(\omega)}\frac{\delta_t}{q_t}y_L(1+\pi_L(\omega)) \tag{38}$$

$$\frac{\xi^2_{lh}}{\tilde{\xi}(\omega)}\frac{\delta_t}{q_t}y_H(1+\pi_H(\omega)) \tag{39}$$

after a low shock and a high shock respectively. Thus, after a high shock

$$A\left(\frac{\rho_{t+1}(\alpha^1_t)}{\frac{q_{t+1}+\delta_{t+1}}{q_t}}\right)^{n-1}\rho_{t+1}(\alpha^1_t)\left(w^i_t-c^i_t\right)=A\left(\frac{\xi^1_{lh}}{\tilde{\xi}(\omega)}\right)^{n-1}\frac{\xi^1_{lh}}{\tilde{\xi}(\omega)}\frac{\delta_t}{q_t}y_H(1+\pi_H(\omega))\left(w^i_t-c^i_t\right)$$

integrating over all managers in group i it implies (14)-(17) and (20). For example,

$$\Omega_H(\omega) =$$

$$= \frac{\sum_{lh\in\mathcal{H}^1(\omega)} \mu^1_{lh}(\omega) A^1_h \left(\frac{\xi^1_{lh}}{\tilde{\xi}(\omega)}\right)^{n^1_h} \delta_t y_H (1+\pi_H(\omega))\omega}{\sum_{lh\in\mathcal{H}^1(\omega)} \mu^1_{lh}(\omega) A^1_h \left(\frac{\xi^1_{lh}}{\tilde{\xi}(\omega)}\right)^{n^1_h} \delta_t y_H (1+\pi_H(\omega))\omega + \sum_{lh\in\mathcal{H}^2(\omega)} \mu^2_{lh}(\omega) A^2_h \left(\frac{\xi^2_{lh}}{\tilde{\xi}(\omega)}\right)^{n^2_h} (1-\omega)(\delta_t y_H (1+\pi_H(\omega)))\beta} =$$

$$= \frac{W^1_H(\omega)}{\tilde{W}_H(\omega)}.$$

Note, that the market clearing condition for the good market is

$$\delta_{t+1} = (\delta_{t+1}+q_{t+1})\left(\left[1-\tilde{W}_{s'}(\omega)\right] + (1-\beta)\tilde{W}_{s'}(\omega)\right) = (\delta_{t+1}+q_{t+1})\left(1-\beta\tilde{W}_{s'}(\omega)\right)$$

which implies

$$\pi_{s'}(\omega) = \frac{q_{t+1}}{\delta_{t+1}} = \frac{\beta\tilde{W}_{s'}(\omega)}{1-\beta\tilde{W}_{s'}(\omega)}$$

Also, (36)-(39) imply the formula for the relative returns. For example, after a high shock we get

$$\frac{\alpha^i\left(\frac{\delta_{t+1}+q_{t+1}}{q_t} - R_t\right) + R_t}{\frac{\delta_{t+1}+q_{t+1}}{q_t}} = \frac{\frac{\xi^i_{lh}}{\tilde{\xi}(\omega)}\frac{\delta_t}{q_t} y_H(1+\pi_H(\omega))}{\frac{\delta_{t+1}+q_{t+1}}{q_t}} = \frac{\xi^i_{lh}}{\tilde{\xi}(\omega)}.$$

■

Lemma 3 *Strategies* $(\mathcal{M}^i(\omega), \mathcal{H}^i(\omega), \mathcal{A}^i(\omega))$ *and the market clearing condition (34) implies (19).*

Proof. By simple substitution

$$\frac{\omega\sum_{lh\in\mathcal{H}^1(\omega)} \mu^1_{lh}(\omega)\xi^1_{lh} + (1-\omega)\sum_{lh\in\mathcal{H}^2(\omega)} \mu^2_{lh}(\omega)\xi^2_{lh}}{1-\frac{y_L(1+\pi_L(\omega))}{\theta(\omega)}} +$$

$$+ \frac{\omega\sum_{lh\in\mathcal{H}^1(\omega)} \mu^1_{lh}(\omega)\left(1-\xi^1_{lh}\right) + (1-\omega)\sum_{lh\in\mathcal{H}^2(\omega)} \mu^2_{lh}(\omega)\left(1-\xi^2_{lh}\right)}{1-\frac{y_H(1+\pi_H(\omega))}{\theta(\omega)}} =$$

$$= \tilde{\xi}(\omega)\frac{1}{1-\frac{y_L(1+\pi_L(\omega))}{\theta(\omega)}} + \left(1-\tilde{\xi}(\omega)\right)\frac{1}{1-\frac{y_H(1+\pi_H(\omega))}{\theta(\omega)}} = 1$$

which gives (19). ■

Lemma 4 *In the modified economy, prices given by (18) and (19) imply that any manager*

has a value function (12) and her consumption and portfolio choices are described by (10) and (11).

Proof. For any $t \geq 1$, conjecture that the value function has the form of

$$V^i\left(w^i, \omega_{t-1}, s_{t-1}\right) = \frac{1}{1-\beta}\ln w^i + \Lambda^i\left(\omega_{t-1}, s_{t-1}\right).$$

Under our conjecture we can write problem as

$$\begin{aligned} V\left(w^i, \omega_{t-1}, s_{t-1}\right) &= \max_{\alpha_t^i, \psi^i} \ln \psi^i w^i + \frac{\beta}{1-\beta} E\left(\ln A_{m_{lh}^i(s')}\left(Y_{t+1}^i\right)^{n_{m_{lh}^i(s')}-1} w_{t+1,-}^i\right) \\ &+ \beta E\left(\Lambda\left(\omega_t, s_{t+1}\right)\right) \\ \omega_t &= \Omega_{s_t}\left(\omega_{t-1}, s_{t-1}\right) \end{aligned}$$

for the given lh. Let us fix an arbitrary α_t^i. The first order condition in c^i has the form of

$$\frac{1}{\psi^i} = \frac{\beta}{1-\beta}\frac{1}{1-\psi^i}$$

which gives

$$1 - \psi^i = \beta$$

We rewrite the problem as

$$\begin{aligned} V\left(w^i, \omega, s_t\right) = &\max_{\alpha^i} \ln\left(1-\beta\right) w^i + \\ &+ \frac{\beta}{1-\beta} p \ln A_h \left(\frac{\rho_{t+1}\left(\alpha_t^1, H\right)}{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}}\right)^{n_h-1} \rho_{t+1}\left(\alpha_t^1, H\right)\beta w^i + \\ &+ \frac{\beta}{1-\beta}\left(1-p\right)\ln A_l \left(\frac{\rho_{t+1}\left(\alpha_t^1, L\right)}{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}\right)^{n_l-1} \rho_{t+1}\left(\alpha_t^1, L\right)\beta w^i \\ &+ \beta\left(p\Lambda\left(\omega_t, H\right) + \left(1-p\right)\Lambda\left(\omega_t, L\right)\right) \end{aligned}$$

Note that this problem is strictly concave in α. The first order condition is

$$pn_h \frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t} - R_t}{\alpha^i\left(\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t} - R_t\right) + R_t} + \\ \left(1-p\right) n_l \frac{\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t} - R_t\right)}{\alpha^i\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t} - R_t\right) + R_t} = 0$$

which is equivalent to

$$\xi_{lh}^{i}\frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}-R_t}{\alpha^i\left(\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}-R_t\right)+R_t}+ \tag{40}$$
$$+\left(1-\xi_{lh}^{i}\right)\frac{\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}-R_t\right)}{\alpha^i\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}-R_t\right)+R_t}=0.$$

Solving for α^i gives $\alpha_{lh}^{i}(\omega)$.

Define $\widehat{\mathcal{H}^i}(\omega)$ which, for every ω, contains a single index pair lh, the one which is assigned to the given manager in the modified economy.Substituting back α^i and ψ^i into the value function implies that our conjecture is correct with the choice of function $\Lambda(\omega_{t-1}, s_{t-1})$ solving

$$\begin{aligned}\Lambda(\omega_{t-1}, s_{t-1}) &= \ln(1-\beta)+ \qquad (41)\\ &+\beta\frac{1}{1-\beta}p\ln\sum_{lh\in\mathcal{H}^i(\omega_t)}1_{lh\in\widehat{\mathcal{H}^i}(\omega_t)}A_h\left(\frac{\xi_{lh}^{1}}{\tilde{\xi}(\omega_t)}\right)^{n_h}\frac{1}{\pi_{s_t}(\omega_{t-1})}y_H(1+\pi_H(\omega_t))\beta+\\ &+\beta\frac{1}{1-\beta}(1-p)\ln\sum_{lh\in\mathcal{H}^i(\omega_t)}1_{lh\in\widehat{\mathcal{H}^i}(\omega_t)}A_l\left(\frac{1-\xi_{lh}^{1}}{1-\tilde{\xi}(\omega_t)}\right)^{n_l}\frac{1}{\pi_{s_t}(\omega_{t-1})}y_L(1+\pi_L(\omega_t))\beta\\ &+\beta(p\Lambda(\omega_t,H)+(1-p)\Lambda(\omega_t,L))\\ \omega_t &= \Omega_{s_t}(\omega_{t-1}, s_{t-1})\end{aligned}$$

which has the conjectured form.

For the value function in period 0, suppose that we start the system at relative wealth ω_0 and and total wealth share $\tilde{W}_0$. Thus, we can write the price dividend ratio in period 0 as

$$\pi_0(\omega_{-1})=\frac{\tilde{W}_0}{1-\beta\tilde{W}_0}.$$

Then we can run the same argument as in the case of $t \geq t$, with the only change that we from (41), we rewrite $\Lambda^i(\omega_{t-1}, s_{t-1})$ for period 0 as

$$\begin{aligned}
\Lambda_0^i\left(\tilde{W}_0, \omega_0\right) &= \ln(1-\beta) + \qquad (42)\\
&+ \beta\frac{1}{1-\beta}p\ln\sum_{lh\in\mathcal{H}^i(\omega_0)} 1_{lh\in\widehat{\mathcal{H}}^i(\omega_0)} A_h^i\left(\frac{\xi_{lh}^i}{\tilde{\xi}(\omega_0)}\right)^{n_h^i}\frac{\tilde{W}_0}{1-\beta\tilde{W}_0}y_H\frac{1}{1-\beta\tilde{W}_1(H)}\beta+\\
&+ \beta\frac{1}{1-\beta}(1-p)\ln\sum_{lh\in\mathcal{H}^i(\omega_0)} 1_{lh\in\widehat{\mathcal{H}}^i(\omega_0)} A_l^i\left(\frac{1-\xi_{lh}^i}{1-\tilde{\xi}(\omega_0)}\right)^{n_l^i}\frac{\tilde{W}_0}{1-\beta\tilde{W}_0}y_L\frac{1}{1-\beta\tilde{W}_1(L)}\beta\\
&+ \beta\left(p\Lambda_0^i\left(\Omega_H(\omega_0), \tilde{W}_H(\omega_0)\right) + (1-p)\Lambda_0^i\left(\Omega_L(\omega_0), \tilde{W}_L(\omega_0)\right)\right)
\end{aligned}$$

which gives the result. ■

A.2 Original problem

Here we show that the equilibrium of the modified problem is an equilibrium of the original problem if conditions in Proposition 2 are satisfied.

For consistency, we need that the portfolio described by (11) is indeed an lh-portfolio. That is, (22) has to be satisfied. Also, the consumption of clients has to be positive, which implies the conditions (23) and (24). Finally, from (12) and (41), the payoff-difference from a deviation from the assigned $lh-$portfolio, $\alpha_{l'h'}^i(\omega)$ to another locally optimal $\alpha_{l''h''}^i(\omega)$ is given by the left hand side of (21). Thus, condition (21) ensures that a deviation from the assigned strategy is suboptimal.

A.3 Proof of Proposition 3

The first part of the proposition is a trivial consequence of (25) and the structure of equilibrium strategies. For the Sharpe ratio and relative state prices, observe that reading (19) as $E(\phi_s) = \frac{1}{R}$ where ϕ_s is the state price, one can see that

$$\begin{aligned}
\phi_H &= \frac{\tilde{\xi}(\omega)}{p}\frac{1}{\frac{1}{\pi_{s_t}(\omega_{t-1})}y_H\left(1+\pi_H(\omega_t)\right)} = \frac{\tilde{\xi}(\omega)}{p}\frac{1-\beta W_H}{\frac{1}{\pi_{s_t}(\omega_{t-1})}y_H}\\
\phi_L &= \frac{\left(1-\tilde{\xi}(\omega)\right)}{1-p}\frac{1}{\frac{1}{\pi_{s_t}(\omega_{t-1})}y_L\left(1+\pi_L(\omega_t)\right)} = \frac{\tilde{\xi}(\omega)}{p}\frac{1-\beta W_L}{\frac{1}{\pi_{s_t}(\omega_{t-1})}y_L}.
\end{aligned}$$

Writing $X(\omega) = \frac{\phi_L}{\phi_H}$.gives the Sharpe-ratio by

$$\begin{aligned} S(\omega) &= \frac{\sqrt{Var(\phi_s)}}{E(\phi_s)} = \frac{p^{\frac{1}{2}}(1-p)^{\frac{1}{2}}\left\|\frac{\left(1-\tilde{\xi}\omega\right)}{1-p}y_H(1+\pi_H\omega) - \frac{\tilde{\xi}(\omega)}{p}y_L(1+\pi_L(\omega))\right\|}{\tilde{\xi}(\omega)\,y_L(1+\pi_L(\omega)) + \left(1-\tilde{\xi}(\omega)\right)y_H(1+\pi_H(\omega))} \\ &= \frac{p^{\frac{1}{2}}(1-p)^{\frac{1}{2}}\left\|y_H\frac{\frac{(1-\tilde{\xi}(\omega))}{1-p}}{\frac{\tilde{\xi}(\omega)}{p}}\frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)} - y_L\right\|}{py_L + (1-p)\,y_H\frac{\frac{(1-\tilde{\xi}(\omega))}{1-p}}{\frac{\tilde{\xi}(\omega)}{p}}\frac{1-\beta\tilde{W}_L(\omega)}{1-\beta\tilde{W}_H(\omega)}} = \frac{p^{\frac{1}{2}}(1-p)^{\frac{1}{2}}\left\|y_H X(\omega) - y_L\right\|}{py_L + (1-p)\,y_H X(\omega)}. \end{aligned}$$

B Proof of Propositions 4, 6 and 8

We show that for a given set of parameters, the sufficient conditions in Proposition 2 hold in the given equilibrium described by Proposition 4. In the first part, we introduce the analytical formulas for deviations from the prescribed equilibrium strategies. In the second part, we show that condition (21) holds for $\omega = 0$. In the third part, we show that condition (21) holds for $\omega > 0$. In the last part, we show that condition (22) holds for any ω. We will show that

$$\begin{aligned} \hat{k}_{high} &\equiv \exp\left(\frac{\ln\frac{n_2}{n_1}}{\left(1-\frac{n_1}{n_2}\right)} + 1\right) \\ \hat{k}_{low} &= \exp\left(\frac{n_1\ln n_1 + n_2\ln n_2 - (n_2+n_1)\ln\frac{n_1+n_2}{2}}{n_2-n_1}\right) \end{aligned}$$

and $\hat{p}_{BA-BB}$ and $\hat{p}_{BA-AB}$ are given by the unique solution in $\left[0, \frac{1}{2}\right]$ of

$$\begin{aligned} \Delta^{BA-BB}(\hat{p}_{BA-BB}) &\equiv 0. \\ \hat{p}_{BA-AB}\exp\left(\frac{\Delta^{BA-BB}(\hat{p}_{BA-AB})}{\hat{p}_{BA-AB}(n_2-n_1)}\right) + (1-\hat{p}_{BA-AB})\exp\left(\frac{\Delta^{AB-BB}(\hat{p}_{BA-AB})}{(n_2-n_1)(1-\hat{p}_{BA-AB})}\right) &\equiv 1, \end{aligned}$$

respectively where

$$\Delta^{l_1h_1-l_2h_2}(p) \equiv p\ln\frac{A_{h_1}}{A_{h_2}}\frac{\left(\frac{\xi^1_{l_1h_1}}{p}\right)^{n_{h_1}}}{\left(\frac{\xi^1_{l_2h_2}}{p}\right)^{n_{h_2}}} + (1-p)\ln\frac{A_{l_1}}{A_{l_2}}\frac{\left(\frac{1-\xi^1_{l_1h_1}}{1-p}\right)^{n_{l_1}}}{\left(\frac{1-\xi^1_{l_2h_2}}{1-p}\right)^{n_{l_2}}}.$$

Example 3 nests Example 2 by the choice of $\omega = 1$, while the proof of Proposition 8 comes by simple substitution of the different values of p. Therefore we provide only a single proof for the three propositions.

B.1 Differences in the value functions

Note that

$$\frac{1-\beta}{\beta}\left(V^{l_1h_1}_{l'h'-l''h''}(\omega)-V^{l_2h_2}_{l'h'-l''h''}(\omega)\right)=$$

$$p\ln\frac{A_{h_1}}{A_{h_2}}\frac{\left(\frac{\xi^1_{l_1h_1}}{\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{h_1}}}{\left(\frac{\xi^1_{l_2h_2}}{\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{h_2}}}+(1-p)\ln\frac{A_{l_1}}{A_{l_2}}\frac{\left(\frac{1-\xi^1_{l_1h_1}}{1-\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{l_1}}}{\left(\frac{1-\xi^1_{l_2h_2}}{1-\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{l_2}}}=$$

$$=\Delta^{l_1h_1-l_2h_2}(p)+p\ln\frac{\left(\frac{p}{\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{h_1}}}{\left(\frac{p}{\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{h_2}}}+(1-p)\ln\frac{\left(\frac{1-p}{1-\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{l_1}}}{\left(\frac{1-p}{1-\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{l_2}}}.$$

which implies (for example)

$$V^{BA}_{BA-BB}(\omega)-V^{BB}_{BA-BB}(\omega)=$$

$$=\left\{\begin{array}{cc}\Delta^{BA-BB}(p)-p(n_2-n_1)\ln\left(\omega\frac{n_2}{pn_2+(1-p)n_1}+(1-\omega)\right) & for\quad \omega<\hat{\omega}_{BA-BB}\\ \Delta^{BA-BB}(p)-p(n_2-n_1)\ln\left(\hat{\omega}_{BA-BB}\frac{n_2}{pn_2+(1-p)n_1}+(1-\hat{\omega}_{BA-BB})\right) & otherwise\end{array}\right\}$$

where $\hat{\omega}_{BA-BB}$ is defined as

$$\Delta^{BA-BB}(p)\equiv p(n_2-n_1)\ln\left(\hat{\omega}\frac{n_2}{pn_2+(1-p)n_1}+(1-\hat{\omega})\right)$$

Similarly,

$$V^{BA}_{BA-AB}(\omega)-V^{AB}_{BA-AB}(\omega)=$$

$$=\left\{\begin{array}{cc}\begin{array}{c}\Delta^{BA-AB}(p)-(n_2-n_1)p\ln\left(\omega\frac{n_2}{pn_2+(1-p)n_1}+(1-\omega)\right)-\\(n_2-n_1)(1-p)\ln\left(\omega\frac{n_1}{pn_2+(1-p)n_1}+(1-\omega)\right)\end{array} & for\quad \omega<\hat{\omega}_{BA-AB}\\ \begin{array}{c}\Delta^{BA-AB}(p)-(n_2-n_1)p\ln\left(\hat{\omega}_{BA-AB}\frac{n_2}{pn_2+(1-p)n_1}+(1-\hat{\omega}_{BA-AB})\right)-\\-(n_2-n_1)(1-p)\ln\left(\hat{\omega}_{BA-AB}\frac{n_1}{pn_2+(1-p)n_1}+(1-\hat{\omega}_{BA-AB})\right)\end{array} & otherwise\end{array}\right\}$$

where $\hat{\omega}_{BA-AB}$ is determined by

$$\begin{aligned}\Delta^{BA-AB}(p)\quad\equiv\quad&(n_2-n_1)p\ln\left(\omega\frac{n_2}{pn_2+(1-p)n_1}+(1-\omega)\right)-\\&-(n_2-n_1)(1-p)\ln\left(\omega\frac{n_1}{pn_2+(1-p)n_1}+(1-\omega)\right).\end{aligned}$$

B.2 $\Delta(p)$ functions

In the following Lemmas we show that under the classification in Proposition 4, at least when $\omega = 0$, deviations to other locally optimal lh strategies are globally suboptimal.

Lemma 5 *If*

$$-\left(1-\frac{n_1}{n_2}\right)(\ln k+1)-\ln\frac{n_1}{n_2}<0$$

then $V_x^{BA}(0)-V_x^{BB}(0)<0$ *for all* p. *If*

$$-\left(1-\frac{n_1}{n_2}\right)(\ln k+1)-\ln\frac{n_1}{n_2}>0$$

then there is a $\hat{p}_{BA-BB}$ *that* $V_x^{BA}(0)-V_x^{BB}(0)>0$ *for all* $p<\hat{p}_{BA-BB}$ *and* $V_x^{BA}(0)-V_x^{BB}(0)<0$ *for all* $p>\hat{p}_{BA-BB}$. *Furthermore,* $\hat{p}_{BA-BB}<(>)\frac{1}{2}$ *iff*

$$\left(\frac{1}{2}(n_1\ln n_1+n_2\ln n_2)-\frac{(n_2+n_1)}{2}\ln\frac{n_1+n_2}{2}\right)<(n_2-n_1)\frac{1}{2}\ln k$$

Proof. Note that

$$\Delta^{BA-BB}(p)=-(n_2-n_1)p\ln k-n_2p\ln\frac{((1-p)n_1+pn_2)}{n_2}-(1-p)n_1\ln\frac{((1-p)n_1+pn_2)}{n_1}.$$

Observe that

$$\begin{aligned}
\Delta^{BA-BB}(0) &= 0\\
\Delta^{BA-BB}(1) &= -(n_2-n_1)\ln k\\
\frac{\partial\Delta^{BA-BB}(p)}{\partial p} &= -(n_2-n_1)\ln k-(n_2-n_1)\ln((1-p)n_1+pn_2)+n_2\ln n_2-n_1\ln n_1-(n_2-n_1)\\
\frac{\partial^2\Delta^{BA-BB}(p)}{\partial^2 p} &= \frac{-(n_2-n_1)^2}{((1-p)n_1+pn_2)}<0
\end{aligned}$$

Given that the second derivative is negative, if

$$\frac{\partial\Delta^{BA-BB}(p)}{\partial p}|_{p=0}<0$$

then $\Delta^{BA-BB}(p)$ is decreasing everywhere, while if

$$\frac{\partial\Delta^{BA-BB}(p)}{\partial p}|_{p=0}>0$$

then $\Delta^{BA-BB}(p)$ is either increasing everywhere, or increasing until a given point and then decreasing. As $n_1 < n_2$ implies $\Delta^{BA-BB}(1) > 0$, this implies that $\Delta^{BA-BB}(p) > 0$ for all p. As $n_1 > n_2$ implies $\Delta^{BA-BB}(1) < 0$,

$$\frac{\partial \Delta^{BA-BB}(p)}{\partial p}|_{p=0} < 0$$

implies $\Delta^{BA-BB}(p) < 0$ for all p and

$$\frac{\partial \Delta^{BA-BB}(p)}{\partial p}|_{p=0} > 0$$

implies the existence of $\hat{p}$ of the Lemma. The last part of the Lemma comes from the observation that

$$\Delta^{BA-BB}\left(\frac{1}{2}\right) = -(n_2 - n_1)\frac{1}{2}\ln k + \left[\frac{1}{2}(n_1 \ln n_1 + n_2 \ln n_2) - \frac{(n_2 + n_1)}{2}\ln\frac{n_1 + n_2}{2}\right]$$

where the term in the bracket is positive if $n_2 > n_1$, as the function $x \ln x$ is convex . ■

Lemma 6 *If $n_1 > n_2$ then $\Delta^{AB-BB}(p) > 0$ for all p. If $n_2 > n_1$ and*

$$\left(1 - \frac{n_1}{n_2}\right)(\ln k + 1) + \ln\frac{n_1}{n_2} > 0$$

then $V_x^{AB}(0) - V_x^{BB}(0) < 0$ for all p. If $n_2 > n_1$ and

$$\left(1 - \frac{n_1}{n_2}\right)(\ln k + 1) + \ln\frac{n_1}{n_2} < 0$$

then there is a $\hat{p}_{AB-BB}$ that $V_x^{AB}(0) - V_x^{BB}(0) > (<) 0$ for $p > (<) \hat{p}_{AB-BB}$. Furthermore $\hat{p}_{AB-BB} > (<) \frac{1}{2}$, iff

$$\frac{1}{2}(n_1 \ln n_1 + n_2 \ln n_2) - \frac{(n_1 + n_2)}{2}\ln\frac{n_1 + n_2}{2} < (>) \frac{1}{2}(n_2 - n_1)\ln k$$

Proof. Note that

$$\Delta^{AB-BB}(p) \equiv -(1-p)(n_2 - n_1)\ln k - pn_1 \ln\frac{pn_1 + (1-p)n_2}{n_1} - (1-p)n_2 \ln\frac{(pn_1 + (1-p)n_2)}{n_2}.$$

The statement comes from simple analysis of the cases $n_2 > n_1$ and $n_2 < n_1$ observing that

$$\begin{aligned}
\Delta^{AB-BB}(0) &= -(n_2 - n_1)\ln k \\
\Delta^{AB-BB}(1) &= 0 \\
\frac{\partial \Delta^{AB-BB}(p)}{\partial p} &= (n_2 - n_1)(\ln k + 1) + (n_2 - n_1)\ln(pn_1 + (1-p)n_2) + n_1 \ln n_1 - n_2 \ln n_2 \\
\frac{\partial \Delta^{AB-BB}(p)}{\partial p}|_{p=1} &= (n_2 - n_1)(\ln k + 1) + n_2 \ln \frac{n_1}{n_2} \\
\frac{\partial^2 \Delta^{AB-BB}(p)}{\partial^2 p} &= -\frac{(n_1 - n_2)^2}{(pn_1 + (1-p)n_2)} < 0
\end{aligned}$$

■

and that

$$\Delta^{AB-BB}\left(\frac{1}{2}\right) = -\frac{1}{2}(n_2 - n_1)\ln k + \left[\frac{1}{2}(n_1 \ln n_1 + n_2 \ln n_2) - \frac{(n_1 + n_2)}{2}\ln\frac{n_1 + n_2}{2}\right]$$

where the term in the bracket is positive if $n_2 > n_1$, as $x \ln x$ is a convex function.

Lemma 7 *If* $(n_2 - n_1)\left(\frac{1}{2}p - 1\right) > (<) 0$ *then* $\Delta^{BA-AB}(p) < (>) 0$.

Proof. Consider that

$$\Delta^{BA-AB}(p) \equiv (n_2 - n_1)((1-p) - p)\ln k + \ln \frac{\left(\frac{n_1}{((1-p)n_1 + pn_2)}\right)^{(1-p)n_1}\left(\frac{n_2}{((1-p)n_1 + pn_2)}\right)^{pn_2}}{\left(\frac{n_2}{((1-p)n_2 + pn_1)}\right)^{(1-p)n_2}\left(\frac{n_1}{((1-p)n_2 + pn_1)}\right)^{pn_1}}.$$

We need $\Delta(p) > 0$ for a BA equilibrium and $\Delta(p) < 0$ for an AB equilibrium. Observe that

$$\begin{aligned}
\Delta^{BA-AB}(1) &= -(n_2 - n_1)\ln k \\
\Delta^{BA-AB}(0) &= (n_2 - n_1)\ln k \\
\Delta^{BA-AB}\left(\frac{1}{2}\right) &= 0.
\end{aligned}$$

Also

$$\begin{aligned}
\frac{\partial \Delta(p)}{\partial p} &= [-2(n_2 - n_1)\ln k - 2n_1 \ln n_1 + 2n_2 \ln n_2 - 2(n_2 - n_1)] \\
&\quad - (n_2 - n_1)[\ln((1-p)n_1 + pn_2)((1-p)n_2 + pn_1)].
\end{aligned}$$

As $n_2 > n_1$ then $\Delta(p)$ is positive at $p = 0$ and negative at $p = 1$. Given that it is a continuous function, it's derivative cannot be positive for all $p \in [0,1]$. As the term in the second bracket is maximal for $p = \frac{1}{2}$, monotonically increasing for $p < \frac{1}{2}$ and monotonically decreasing for $p > \frac{1}{2}$ and the term in the first bracket is constant in p, there cannot be a minimum in $p \in \left(0, \frac{1}{2}\right)$ or a maximum at $p \in \left(\frac{1}{2}, 0\right)$. Thus, $p > \frac{1}{2}$ implies $\Delta(p) < 0$ and $p < \frac{1}{2}$ implies $\Delta(p) > 0$. ■

B.3 comparing thresholds $\hat{\omega}$

Now we proceed for $\omega > 0$. In this part, we show that in a given $l'h' - l''h''$ equilibrium, there is no $lh \neq l''h''$ that

$$V^{lh}_{l'h'-l''h''}(\omega) - V^{l'h'}_{l'h'-l''h''}(\omega) > 0$$

for any $\omega > 0$. We already know that this is true for $\omega = 0$. Given the monotonicity in ω of any

$$V^{lh}_{l'h'-l''h''}(\omega) - V^{l'h'}_{l'h'-l''h''}(\omega)$$

functions, and given that for $\omega > \hat{\omega}, \tilde{\xi}(\omega)$ is constant, thus

$$V^{lh}_{l'h'-l''h''}(\omega)$$

is also constant for any lh, we only have to show that in a $l'h' - l''h''$ equilibrium,

$$\hat{\omega}_{l'h'-l''h''} < \hat{\omega}_{l'h'-lh}$$

for any $lh \neq l''h'', l'h'$ where $\hat{\omega}_{l'h'-lh}$ is the $\hat{\omega}$ in a given $l'h' - lh$ equilibrium defined as

$$V^{lh}_{l'h'-l''h''}(\hat{\omega}_{l'h'-lh}) = V^{l'h'}_{l'h'-l''h''}(\hat{\omega}_{l'h'-lh}).$$

This amounts to a comparison between $\hat{\omega}_{BA-BB}(p)$ and $\hat{\omega}_{BA-AB}(p)$ defined implicitly by the functions

$$\Delta^{BA-BB}(p) \equiv p(n_2 - n_1)\ln\left(\hat{\omega}_{BA-BB}\frac{n_2}{pn_2 + (1-p)n_1} + (1 - \hat{\omega}_{BA-BB})\right) \tag{43}$$

$$\begin{aligned}\Delta^{BA-AB}(p) \equiv\ & (n_2 - n_1)p\ln\left(\hat{\omega}_{BA-AB}\frac{n_2}{pn_2 + (1-p)n_1} + (1-\omega)\right) - \\ & - (n_2 - n_1)(1-p)\ln\left(\hat{\omega}_{BA-AB}\frac{n_1}{pn_2 + (1-p)n_1} + (1 - \hat{\omega}_{BA-AB})\right)\end{aligned} \tag{44}$$

Lemma 8 *Suppose $n_2 > n_1$.If*

$$\frac{n_1 \ln n_1 + n_2 \ln n_2 - (n_2 + n_1) \ln \frac{n_1+n_2}{2}}{n_2 - n_1} < \ln k \tag{45}$$

then $\hat{\omega}_{BA-BB}(p) < \hat{\omega}_{BA-AB}(p)$ whenever the functions exist. If

$$\frac{n_1 \ln n_1 + n_2 \ln n_2 - (n_2 + n_1) \ln \frac{n_1+n_2}{2}}{n_2 - n_1} > \ln k \tag{46}$$

then there exist a $\hat{p} < \frac{1}{2}$ that $\hat{\omega}_{BA-BB}(p) < \hat{\omega}_{BA-AB}(p)$ for all $p < \hat{p}$ and $\hat{\omega}_{BA-BB}(p) > \hat{\omega}_{BA-AB}(p)$ for all $p > \hat{p}$

Proof. First, I show that the system

$$\begin{aligned} \Delta^{BA-BB}(p) &\equiv p(n_2 - n_1) \ln\left(\omega \frac{n_2}{pn_2 + (1-p)n_1} + (1-\omega)\right) \\ \Delta^{BA-BB}(p) - \Delta^{BA-AB}(p) &= \Delta^{AB-BB}(p) = (n_2 - n_1)(1-p) \ln\left(\omega \frac{n_1}{pn_2 + (1-p)n_1} + (1-\omega)\right) \end{aligned} \tag{47}$$

has no solution if (45) holds and a single solution $(\hat{p}, \hat{\omega})$ where $\hat{\omega} = \hat{\omega}_{BA-BB}(\hat{p}) = \hat{\omega}_{BA-AB}(\hat{p})$ if (46) holds. For this, note that the system is equivalent to

$$\begin{aligned} \exp\left(\frac{\Delta^{BA-BB}(p)}{p(n_2 - n_1)}\right) &\equiv \left(\omega \frac{n_2}{pn_2 + (1-p)n_1} + (1-\omega)\right) \\ \exp\left(\frac{\Delta^{AB-BB}(p)}{(n_2 - n_1)(1-p)}\right) &= \left(\omega \frac{n_1}{pn_2 + (1-p)n_1} + (1-\omega)\right), \end{aligned}$$

hence, any solution of the system has to satisfy

$$p \exp\left(\frac{\Delta^{BA-BB}(p)}{p(n_2 - n_1)}\right) + (1-p) \exp\left(\frac{\Delta^{AB-BB}(p)}{(n_2 - n_1)(1-p)}\right) \equiv 1.$$

From

$$\begin{aligned} \frac{\Delta_{BA-BB}(p)}{p(n_2 - n_1)} &= -\ln k - \frac{n_2}{n_2 - n_1} \ln \frac{((1-p)n_1 + pn_2)}{n_2} - \frac{(1-p)}{p} \frac{n_1}{n_2 - n_1} \ln \frac{((1-p)n_1 + pn_2)}{n_1} \\ \frac{\Delta_{AB-BB}(p)}{(1-p)(n_2 - n_1)} &= -\ln k - \frac{n_2}{n_2 - n_1} \ln \frac{(pn_1 + (1-p)n_2)}{n_2} - \frac{p}{1-p} \frac{n_1}{n_2 - n_1} \ln \frac{pn_1 + (1-p)n_2}{n_1} \end{aligned}$$

observe that this function is symmetric in the sense that if

$$\Pi(p) \equiv p \exp\left(\frac{\Delta^{BA-BB}(p)}{p(n_2 - n_1)}\right)$$

then

$$\tilde{\Pi}(p) \equiv \Pi(p) + \Pi(1-p) = p \exp\left(\frac{\Delta^{BA-BB}(p)}{p(n_2-n_1)}\right) + (1-p)\exp\left(\frac{\Delta^{AB-BB}(p)}{(n_2-n_1)(1-p)}\right).$$

Also

$$\begin{aligned}\frac{\partial \Pi(p)}{\partial p} &= e^{\frac{\Delta_{BA-BB}(p)}{p(n_2-n_1)}}\left(1 + p\frac{\partial\left(\frac{\Delta_{BA-BB}(p)}{p(n_2-n_1)}\right)}{\partial p}\right) = \\ &= e^{\frac{\Delta_{BA-BB}(p)}{p(n_2-n_1)}}\frac{1}{p}\frac{n_1}{n_2-n_1}\ln\frac{((1-p)n_1+pn_2)}{n_1} > 0.\end{aligned}$$

and

$$\Pi(0) = \tilde{\Pi}(0) = \tilde{\Pi}(1) = \frac{1}{k} < 1.$$

Thus, $\tilde{\Pi}(p)$ is increasing for $p < \frac{1}{2}$ and decreasing for $p > \frac{1}{2}$ and its maximum is at $p = \frac{1}{2}$. If (45) holds, then

$$\tilde{\Pi}\left(\frac{1}{2}\right) = \Pi\left(\frac{1}{2}\right) < 1,$$

which implies that $\tilde{\Pi}(p) = 1$ does not have a solution. However, if (46) holds, then $\tilde{\Pi}(p) = 1$ has two solutions. If we denote the first by $\hat{p} < \frac{1}{2}$ then the second one is $(1-\hat{p})$. However, $\hat{\omega}_{BA-BB}(p)$ exists for a given p, only if $\Delta_{BA-BB}(p) > 0$. It is easy to $\Delta_{BA-BB}(\hat{p}) > 0$, but $\Delta_{BA-BB}(1-\hat{p}) < 0$. This concludes the proof. ■

B.4 Condition (22)

In this part we show that if in a given equilibrium, a given strategy characterized by (11) for a given lh, is preferred to the BB strategy, then this implies that (11) is an lh portfolio.

1. $V_{BA}^{BA}(\omega) - V_{BA}^{BB}(\omega) > 0$ implies $\frac{\frac{pn_2}{pn_2+(1-p)n_1}}{\xi_{BA}(\omega)} = \frac{\frac{n_2}{pn_2+(1-p)n_1}}{\omega\frac{n_2}{pn_2+(1-p)n_1}+(1-\omega)} > k$

$$\begin{aligned}0 &< V_{BA}^{BA}(\omega) - V_{BA}^{BB}(\omega) = V_{BA}^{BA}(0) - V_{BA}^{BB}(0) - p(n_2-n_1)\ln\left(\omega\frac{pn_2}{pn_2+(1-p)n_1} + p(1-\omega)\right) = \\ &= (n_2-n_1)p\ln\frac{\frac{n_2}{((1-p)n_1+pn_2)}}{k} + n_1\ln\frac{n_2^p n_1^{(1-p)}}{((1-p)n_1+pn_2)} - p(n_2-n_1)\ln\left(\omega\frac{n_2}{pn_2+(1-p)n_1} + (1-\omega)\right) \\ &= (n_2-n_1)p\ln\frac{\frac{n_2}{((1-p)n_1+pn_2)}}{k\left(\omega\frac{n_2}{pn_2+(1-p)n_1}+(1-\omega)\right)} + n_1\ln\frac{n_2^p n_1^{(1-p)}}{((1-p)n_1+pn_2)}\end{aligned}$$

As $\frac{n_2^p n_1^{(1-p)}}{((1-p)n_1+pn_2)} < 1$ because of the inequality of arithmetic and geometric means, $\frac{\frac{n_2}{((1-p)n_1+pn_2)}}{\left(\omega\frac{n_2}{((1-p)n_1+pn_2)}+(1-\omega)\right)} >$

k must hold

2. $V_{AB}^{AB}(\omega) - V_{AB}^{BB}(\omega) > 0$ implies $\frac{\frac{(1-p)n_2}{(1-p)n_2+pn_1}}{1-\xi_{AB}(\omega)} = \frac{\frac{n_2}{(1-p)n_2+pn_1}}{\omega\frac{n_2}{(1-p)n_2+pn_1}+(1-\omega)} > k$

$$0 < V_{AB}^{AB}(\omega) - V_{AB}^{BB}(\omega) = V_{AB}^{AB}(0) - V_{AB}^{BB}(0) - (1-p)(n_2-n_1)\ln\left(1-\tilde{\xi}_{AB}(\hat{\omega})\right) =$$
$$= -(1-p)(n_2-n_1)\ln k - pn_1\ln\frac{pn_1+(1-p)n_2}{n_1} - (1-p)n_2\ln\frac{(pn_1+(1-p)n_2)}{n_2} -$$
$$-(1-p)(n_2-n_1)\ln\left(\omega\frac{n_2}{((1-p)n_1+pn_2)}+(1-\omega)\right) =$$
$$= -(1-p)(n_2-n_1)\ln k - pn_1\ln\frac{pn_1+(1-p)n_2}{n_1} - (1-p)(n_2-n_1)\ln\frac{(pn_1+(1-p)n_2)}{n_2}$$
$$-(1-p)n_1\ln\frac{(pn_1+(1-p)n_2)}{n_2} - (1-p)(n_2-n_1)\ln\left(\omega\frac{n_2}{((1-p)n_1+pn_2)}+(1-\omega)\right) =$$
$$= (1-p)(n_2-n_1)\ln\frac{\frac{n_2}{(pn_1+(1-p)n_2)}}{k\left(\omega\frac{n_2}{((1-p)n_1+pn_2)}+(1-\omega)\right)} + n_1\ln\frac{n_1^p n_2^{1-p}}{(pn_1+(1-p)n_2)}$$

the second part is negative, so $\frac{\frac{n_2}{(pn_1+(1-p)n_2)}}{\left(\omega\frac{n_2}{pn_2+(1-p)n_1}+(1-\omega)\right)} > k$ must hold.

3.$V_{BA-\mu}^{AB}(\omega) - V_{BA-\mu}^{BB}(\omega) > 0$ implies $\frac{\frac{(1-p)n_2}{(1-p)n_2+pn_1}}{1-\xi_{BA}(\hat{\omega})} = \frac{\frac{n_2}{(1-p)n_2+pn_1}}{\hat{\omega}\frac{n_1}{(1-p)n_2+pn_1}+(1-\hat{\omega})} > k$ for all $\omega > \hat{\omega}$

$$0 < V_{AB}^{AB}(\omega) - V_{AB}^{BB}(\omega) = V_{AB}^{AB}(0) - V_{AB}^{BB}(0) - (1-p)(n_2-n_1)\ln\left(1-\tilde{\xi}_{BA}(\hat{\omega})\right) =$$
$$= -(1-p)(n_2-n_1)\ln k - pn_1\ln\frac{pn_1+(1-p)n_2}{n_1} - (1-p)n_2\ln\frac{(pn_1+(1-p)n_2)}{n_2} -$$
$$-(1-p)(n_2-n_1)\ln\left(\hat{\omega}\frac{n_1}{((1-p)n_1+pn_2)}+(1-\hat{\omega})\right) =$$
$$= -(1-p)(n_2-n_1)\ln k - pn_1\ln\frac{pn_1+(1-p)n_2}{n_1} - (1-p)(n_2-n_1)\ln\frac{(pn_1+(1-p)n_2)}{n_2}$$
$$-(1-p)n_1\ln\frac{(pn_1+(1-p)n_2)}{n_2} - (1-p)(n_2-n_1)\ln\left(\hat{\omega}\frac{n_1}{((1-p)n_1+pn_2)}+(1-\hat{\omega})\right) =$$
$$= (1-p)(n_2-n_1)\ln\frac{\frac{n_2}{(pn_1+(1-p)n_2)}}{k\left(\hat{\omega}\frac{n_1}{((1-p)n_1+pn_2)}+(1-\hat{\omega})\right)} + n_1\ln\frac{n_1^p n_2^{1-p}}{(pn_1+(1-p)n_2)}$$

the second part is negative so $\frac{\frac{n_2}{(pn_1+(1-p)n_2)}}{k\left(\hat{\omega}\frac{n_1}{((1-p)n_1+pn_2)}+(1-\hat{\omega})\right)} > 1$

4. $V_{AB}^{BA}(\omega) - V_{AB}^{BB}(\omega) > 0$ implies $\frac{\frac{pn_2}{pn_2+(1-p)n_1}}{\xi_{AB}(\hat{\omega})} = \frac{\frac{n_2}{pn_2+(1-p)n_1}}{\hat{\omega}\frac{n_1}{pn_2+(1-p)n_1}+(1-\hat{\omega})} > k$

$$
\begin{aligned}
0 &< V_{AB}^{BA}(\omega) - V_{AB}^{BB}(\omega) = V_{BA}^{BA}(0) - V_{BA}^{BB}(0) - p(n_2 - n_1)\ln\left(\hat{\omega}\frac{n_1}{pn_2 + (1-p)n_1} + (1-\hat{\omega})\right) = \\
&= (n_2 - n_1)\, p \ln \frac{\frac{n_2}{((1-p)n_1 + pn_2)}}{k} + n_1 \ln \frac{n_2^p n_1^{(1-p)}}{((1-p)n_1 + pn_2)} - p(n_2 - n_1)\ln\left(\hat{\omega}\frac{n_1}{pn_2 + (1-p)n_1} + (1-\hat{\omega})\right) \\
&= (n_2 - n_1)\, p \ln \frac{\frac{n_2}{((1-p)n_1 + pn_2)}}{k\left(\hat{\omega}\frac{n_1}{pn_2 + (1-p)n_1} + (1-\hat{\omega})\right)} + n_1 \ln \frac{n_2^p n_1^{(1-p)}}{((1-p)n_1 + pn_2)}
\end{aligned}
$$

As $\frac{n_2^p n_1^{(1-p)}}{((1-p)n_1 + pn_2)} < 1$ because of the inequality of arithmetic and geometric means, $\frac{\frac{n_2}{((1-p)n_1 + pn_2)}}{\left(\hat{\omega}\frac{n_1}{pn_2 + (1-p)n_1} + (1-\hat{\omega})\right)} > k$ must hold

C Other proofs

C.1 Proof of Proposition 5

The following Lemma and the definitions in Proposition 3 give the result.

Lemma 9 *$\frac{\partial \bar{\xi}}{\partial k}$ is negative for $BA - BB$ and $BA - AB$ equilibria and positive for $AB - BB$ and $AB - BA$ equilibria. $\frac{\partial \mu}{\partial k} < 0$ for any equilibria.*

Proof.

BA-BB Solving (30) explicitly yields

$$
\bar{\xi} = \frac{p}{k}\left((n_2)^{n_2 p}(n_1)^{n_1(1-p)}\left(\frac{1}{n_1(1-p) + n_2 p}\right)^{n_2 p + n_1(1-p)}\right)^{\frac{1}{(n_2 - n_1)p}}
$$

AB-BB Solving (30) explicitly yields

$$
\bar{\xi} = 1 - \frac{1-p}{k}\left((n_1)^{n_1 p}(n_2)^{n_2(1-p)}\left(\frac{1}{n_2(1-p) + n_1 p}\right)^{n_1 p + n_2(1-p)}\right)^{\frac{1}{(-n_1 + n_2)(1-p)}}
$$

BA-AB Taking a derivative of (30) with respect to k yields

$$
(((n_2 - n_1)(1 - 2p))/k) > 0,
$$

since $p < \frac{1}{2}$ in a $BAAB$ equilibrium.

A derivative of (30) with respect to $\bar{\xi}$ yield s

$$-((n_1 - n_2)(-\bar{\xi} + p(-1 + 2\bar{\xi})))/((-1 + \bar{\xi})\bar{\xi}),$$

which has the same sign as

$$-(-\bar{\xi} + p(-1 + 2\bar{\xi})).$$

Since in a $BA - AB$ equilibrium $p < \frac{1}{2}$ $(-\bar{\xi}_{l'h'-l''h''} + p(-1 + 2\bar{\xi}_{l'h'-l''h''}))$ is decreasing in $\bar{\xi}_{l'h'-l''h''}$. Furthermore, it equals zero at $\frac{p}{(-1+2p)} < 0$. Therefore, the derivative of (30) with respect to $\bar{\xi}_{l'h'-l''h''}$ is positive.

The results then follows from the implicit function theorem.

AB-BA Taking a derivative of (30) with respect to k yields

$$(((n_2 - n_1)(-1 + 2p))/k) > 0,$$

since $p > \frac{1}{2}$ in a $ABBA$ equilibrium. A derivative of (30) with respect to $\bar{\xi}_{l'h'-l''h''}$ yields

$$((n_1 - n_2)(-\bar{\xi} + p(-1 + 2\bar{\xi})))/((-1 + \bar{\xi})\bar{\xi}),$$

which has the same sign as

$$(-\bar{\xi} + p(-1 + 2\bar{\xi}_{l'h'-l''h''})).$$

Specifically, the sign of the derivative is the same as the sign of

$$(-\bar{\xi} + p(-1 + 2\bar{\xi})).$$

Since $p > \frac{1}{2}$ in an $AB - BA$ equilibrium $(-\bar{\xi} + p(-1 + 2\bar{\xi}))$ is increasing in $\bar{\xi}$. Furthermore, the lowest value of $\frac{p}{(-1+2p)}$ is attained at $p = 1$, implying that the derivative with of (30) with respect to $\bar{\xi}$ is negative.

The results then follows from the implicit function theorem.

Finally, by inspection of (29) the result for μ follows by combining the results for $\bar{\xi}$ with the fact that $\xi^1_{l'h'} < \xi^1_{l''h''}$ in $AB - BB$ and $AB - BA$ equilibria and $\xi^1_{l'h'} > \xi^1_{l''h''}$ in $BA - BB$ and $BA - AB$ equilibria. ■

C.2 Proof of Proposition 7

From (42) define

$$\Delta_\Lambda(\omega_0) \equiv \Lambda_0^1\left(\tilde{W}_0, \omega_0\right) - \Lambda_0^2\left(\tilde{W}_0, \omega_0\right).$$

Then the function $\Delta_\Lambda(\omega_0)$ in the general case is defined by the recursion

$$\begin{aligned}
\Delta_\Lambda(\omega_0) &= \beta\frac{1}{1-\beta}p\ln\frac{\sum_{lh\in\mathcal{H}^1(\omega_0)}1_{lh\in\widehat{\mathcal{H}^1}(\omega_0)}A_h^1\left(\frac{\xi_{lh}^1}{\tilde{\xi}(\omega_0)}\right)^{n_h^1}}{\sum_{lh\in\mathcal{H}^2(\omega_0)}1_{lh\in\widehat{\mathcal{H}^2}(\omega_0)}A_h^2\left(\frac{\xi_{lh}^2}{\tilde{\xi}(\omega_0)}\right)^{n_h^2}} + \\
&\quad +\beta\frac{1}{1-\beta}(1-p)\ln\frac{\sum_{lh\in\mathcal{H}^1(\omega_0)}1_{lh\in\widehat{\mathcal{H}^1}(\omega_0)}A_l^1\left(\frac{1-\xi_{lh}^1}{1-\tilde{\xi}(\omega_0)}\right)^{n_l^1}}{\sum_{lh\in\mathcal{H}^2(\omega_0)}1_{lh\in\widehat{\mathcal{H}^2}(\omega_0)}A_l^2\left(\frac{1-\xi_{lh}^2}{1-\tilde{\xi}(\omega_0)}\right)^{n_l^2}} + \\
&\quad +\beta\left(p\Delta_\Lambda(\Omega_H(\omega_0)) + (1-p)\Delta_\Lambda(\Omega_L(\omega_0))\right).
\end{aligned} \tag{48}$$

In the case of Example 3, in an $l'h' - l''h''$ equilibrium it is

$$\begin{aligned}
\Delta_\Lambda(\omega_0) &= \beta\frac{1}{1-\beta}p\ln\frac{A_{h'}\left(\frac{\xi_{l'h'}^1}{\tilde{\xi}_{l'h'-l''h''}(\omega_0)}\right)^{n_{h'}}}{A\left(\frac{p}{\tilde{\xi}_{l'h'-l''h''}(\omega_0)}\right)^{n}} + \beta\frac{1}{1-\beta}(1-p)\ln\frac{A_{l'}\left(\frac{1-\xi_{l'h'}^1}{1-\tilde{\xi}_{l'h'-l''h''}(\omega_0)}\right)^{n_{l'}}}{A\left(\frac{1-p}{1-\tilde{\xi}_{l'h'-l''h''}(\omega_0)}\right)^{n}} + \\
&\quad +\beta\left(p\Delta_\Lambda(\Omega_H(\omega_0)) + (1-p)\Delta_\Lambda(\Omega_L(\omega_0))\right).
\end{aligned}$$

We rewrite this as

$$\begin{aligned}
\Delta_\Lambda(\omega_0) &= \frac{\beta}{(1-\beta)^2}\left(p\ln A_{h'} + (1-p)\ln A_{l'} - \ln A\right) + \frac{\beta}{(1-\beta)^2}\left(p\ln\frac{\left(\xi_{l'h'}^1\right)^{n_{h'}}}{(p)^n} + (1-p)\ln\frac{\left(1-\xi_{l'h'}^1\right)^{n_{l'}}}{(1-p)^n}\right) \\
&\quad +\frac{\beta}{1-\beta}p\left(n-n_{h'}\right)\ln\tilde{\xi}_{l'h'-l''h''}(\omega_0) + \beta\frac{1}{1-\beta}(1-p)\left(n-n_{l'}\right)\ln\left(1-\tilde{\xi}_{l'h'-l''h''}(\omega_0)\right) + \\
&\quad +\beta\left(p\hat{\Delta}_\Lambda(\Omega_H(\omega_0)) + (1-p)\hat{\Delta}_\Lambda(\Omega_L(\omega_0))\right)
\end{aligned}$$

where

$$\begin{aligned}
\hat{\Delta}_\Lambda(\omega_0) &= \frac{\beta}{1-\beta}\left(p\left(n-n_{h'}\right)\ln\tilde{\xi}_{l'h'-l''h''}(\omega_0) + (1-p)\left(n-n_{l'}\right)\ln\left(1-\tilde{\xi}_{l'h'-l''h''}(\omega_0)\right)\right) \\
&\quad +\frac{\beta}{1-\beta}\beta\left(p\hat{\Delta}_\Lambda(\Omega_H(\omega_0)) + (1-p)\hat{\Delta}_\Lambda(\Omega_L(\omega_0))\right).
\end{aligned} \tag{49}$$

Consider the next Lemma first

Lemma 10 *1.*

2. *If*

$$\bar{\Delta}_\Lambda + \left(p\left(n-n_{h'}\right)\ln\tilde{\xi}_{l'h'-l''h''}\left(1\right)+\left(1-p\right)\left(n-n_{l'}\right)\ln\left(1-\tilde{\xi}_{l'h'-l''h''}\left(1\right)\right)\right)$$

and

$$\bar{\Delta}_\Lambda + \left(p\left(n-n_{h'}\right)\ln p+\left(1-p\right)\left(n-n_{l'}\right)\ln\left(1-p\right)\right)$$

have opposite signs, where

$$\bar{\Delta}_\Lambda = \ln\frac{A_{h'}^{(1-p)}A_{l'}^{p}}{A}+p\ln\frac{\left(\xi_{l'h'}^{1}\right)^{n_{h'}}}{\left(p\right)^{n}}+\left(1-p\right)\ln\frac{\left(1-\xi_{l'h'}^{1}\right)^{n_{l'}}}{\left(1-p\right)^{n}} \tag{50}$$

then there is a $\omega_0^* \in (0,1)$.

3. *if* $n_1 \leq n \leq n_2$ *then* $\Delta_\Lambda(1) < \Delta_\Lambda(0)$.

Proof. Observe that $\Omega_H(1) = \Omega_L(1) = 1$, $\Omega_H(0) = \Omega_L(0) = 0$ and (**??**) implies that

$$\begin{aligned}
\hat{\Delta}_\Lambda(1) &= \frac{\beta}{\left(1-\beta\right)^2}\left(p\left(n-n_{h'}\right)\ln\tilde{\xi}_{l'h'-l''h''}\left(1\right)+\left(1-p\right)\left(n-n_{l'}\right)\ln\left(1-\tilde{\xi}_{l'h'-l''h''}\left(1\right)\right)\right)\\
\hat{\Delta}_\Lambda(0) &= \frac{\beta}{\left(1-\beta\right)^2}\left(p\left(n-n_{h'}\right)\ln p+\left(1-p\right)\left(n-n_{l'}\right)\ln\left(1-p\right)\right)
\end{aligned}$$

Thus, :

$$\begin{aligned}
\Delta_\Lambda(1) &= \frac{\beta}{\left(1-\beta\right)^2}\left(\bar{\Delta}_\Lambda+\left(p\left(n-n_{h'}\right)\ln\tilde{\xi}_{l'h'-l''h''}\left(1\right)+\left(1-p\right)\left(n-n_{l'}\right)\ln\left(1-\tilde{\xi}_{l'h'-l''h''}\left(1\right)\right)\right)\right)\\
\Delta_\Lambda(0) &= \frac{\beta}{\left(1-\beta\right)^2}\left(\bar{\Delta}_\Lambda+\left(p\left(n-n_{h'}\right)\ln p+\left(1-p\right)\left(n-n_{l'}\right)\ln\left(1-p\right)\right)\right).
\end{aligned}$$

As $\Delta_\Lambda(\omega_0)$ is continuous in ω_0, if $\Delta_\Lambda(1)\,\Delta_\Lambda(0) < 0$, there must be a ω_0^*.

As

$$\Delta_\Lambda(1)-\Delta_\Lambda(0) = \frac{\beta}{\left(1-\beta\right)^2}\left(p\left(n-n_{h'}\right)\ln\frac{\bar{\xi}}{p}+\left(1-p\right)\left(n-n_{l'}\right)\ln\frac{1-\bar{\xi}}{1-p}\right) =$$

$$= \frac{\beta}{\left(1-\beta\right)^2}\left(p\left(n-n_{h'}\right)\ln\frac{\hat{\omega}\xi_{l'h'}+\left(1-\hat{\omega}\right)p}{p}+\left(1-p\right)\left(n-n_{l'}\right)\ln\frac{1-\left(\hat{\omega}\xi_{l'h'}+\left(1-\hat{\omega}\right)p\right)}{1-p}\right),$$

if $n_2 \geq n \geq n_1$ then $l'h' = BA$, implies $\frac{\hat{\omega}\xi_{l'h'}+(1-\hat{\omega})p}{p} > 1 > \frac{1-(\hat{\omega}\xi_{l'h'}+(1-\hat{\omega})p)}{1-p}$ and $n_{h'} = n_2$ and $n_l = n_1$, so,

$$\frac{\beta}{(1-\beta)^2}\left(p\left(n-n_{h'}\right)\ln\frac{\hat{\omega}\xi_{l'h'}+(1-\hat{\omega})\,p}{p}+(1-p)\left(n-n_{l'}\right)\ln\frac{1-\left(\hat{\omega}\xi_{l'h'}+(1-\hat{\omega})\,p\right)}{1-p}\right) < 0$$
$$\Delta_\Lambda(1) < \Delta_\Lambda(0).$$

Also $l'h' = AB$, then $\frac{\hat{\omega}\xi_{l'h'}+(1-\hat{\omega})p}{p} < 1 < \frac{1-(\hat{\omega}\xi_{l'h'}+(1-\hat{\omega})p)}{1-p}$ and $n_{h'} = n_1$ and $n_l = n_2$, so

$$\frac{\beta}{(1-\beta)^2}\left(p\left(n-n_{h'}\right)\ln\frac{\hat{\omega}\xi_{l'h'}+(1-\hat{\omega})\,p}{p}+(1-p)\left(n-n_{l'}\right)\ln\frac{1-\left(\hat{\omega}\xi_{l'h'}+(1-\hat{\omega})\,p\right)}{1-p}\right) < 0$$
$$\Delta_\Lambda(1) < \Delta_\Lambda(0).$$

This implies that there is at least one ω_0^* for which the stability criterion holds. ■

This lemma implies the proposition if we notice from the definition of (50) that for any other parameters, we can pick a $\ln\frac{A_{h'}^{(1-p)}A_{l'}^{p}}{A}$ to satisfy the conditions of the lemma.

C.3 Proof of Proposition 9

Consider the case of hedge funds. Consider the term $\Lambda_0\left(\tilde{W}_0,\omega_0\right)$ in (13). Note that this term in the first market where $p = 1-\underline{p}$ is

$$\begin{aligned}\Lambda_0^1\left(\tilde{W}_0,\omega_0\right) &= \ln(1-\beta)+ \\ &+\beta\frac{1}{1-\beta}\left(1-\underline{p}\right)\ln A_2\left(\frac{\xi_{BA}^i}{\tilde{\xi}(\omega_0)}\right)^{n_2}\frac{\tilde{W}_0}{1-\beta\tilde{W}_0}y_H\frac{1}{1-\beta\tilde{W}_H(\omega_0)}\beta+ \\ &+\beta\frac{1}{1-\beta}\underline{p}\ln A_1\left(\frac{1-\xi_{BA}^i}{1-\tilde{\xi}(\omega_0)}\right)^{n_1}\frac{\tilde{W}_0}{1-\beta\tilde{W}_0}\frac{1}{1-\beta\tilde{W}_L(\omega_0)}\beta \\ &+\beta\left(\left(1-\underline{p}\right)\Lambda\left(\Omega_H(\omega_0),\tilde{W}_H(\omega_0)\right)+\underline{p}\Lambda\left(\Omega_L(\omega_0),\tilde{W}_L(\omega_0)\right)\right)\end{aligned} \tag{51}$$

while on the second market where $p = \underline{p}$, it is

$$\begin{aligned}\underline{\Lambda}_0^1\left(\tilde{W}_0,\omega_0\right) &= \ln(1-\beta)+ \\ &+\beta\frac{1}{1-\beta}\underline{p}\ln A_1\left(\frac{1-\xi_{BA}^i}{1-\tilde{\xi}(\omega_0)}\right)^{n_1}\frac{\tilde{W}_0}{1-\beta\tilde{W}_0}\underline{y}_H\frac{1}{1-\beta\tilde{W}_L(\omega_0)}\beta+ \\ &+\beta\frac{1}{1-\beta}\left(1-\underline{p}\right)\ln A_2\left(\frac{\xi_{BA}^i}{\tilde{\xi}(\omega_0)}\right)^{n_2}\frac{\tilde{W}_0}{1-\beta\tilde{W}_0}\underline{y}_L\frac{1}{1-\beta\tilde{W}_H(\omega_0)}\beta \\ &+\beta\left(\underline{p}\underline{\Lambda}\left(\Omega_L(\omega_0),\tilde{W}_L(\omega_0)\right)+\left(1-\underline{p}\right)\underline{\Lambda}\left(\Omega_H(\omega_0),\tilde{W}_H(\omega_0)\right)\right)\end{aligned} \tag{52}$$

Subtracting one from the other gives the result for hedge funds. For mutual funds the proof is analogous with the substitution of $\xi_{lh} = p$, $A_1 = A_2 = A$ and $n_1 = n_2 = n$.