Long-Run Risk, the Wealth-Consumption Ratio, and the Temporal Pricing of Risk

By Ralph S.J. Koijen, Hanno Lustig, Stijn Van Nieuwerburgh and Adrien Verdelhan *

Representative agent consumption-based asset pricing models have made great strides in accounting for many important features of asset returns. The long-run risk (LRR) models of Ravi Bansal and Amir Yaron (2004) is a prime example of this progress. Yet, several other representative agent models, such as the external habit model of John Y. Campbell and John H. Cochrane (1999) and the variable rare disasters model of Xavier Gabaix (2008) seem to be able to match a similar set of asset pricing moments. Additional moments would be useful to help distinguish between these models. Hanno Lustig, Stijn Van Nieuwerburgh and Adrien Verdelhan (2009) argue that the wealth-consumption ratio is such a moment. A comparison of the wealth-consumption ratio in the LRR model and in the data is favorable to the LRR model. This is no small feat because the wealth-consumption ratio is not a target in the usual calibrations of the model, and the LRR is – so far– the sole model able to reproduce both the equity premium and the wealth-consumption ratio. The LRR model matches the properties of the wealth-consumption ratio despite the fact that it implies a negative real bond risk premium. This is because it generates quite a bit of consumption cash-flow risk to offset the negative discount rate risk. This can be seen in long-horizon variance ratios for consumption. So relative to the data, the consumption cash-flow risk is too high and the discount rate (which is close to the long-horizon real bond risk premium) seems too low.

Because of a lack of data, it is hard to assess directly whether a negative real bond risk premium is counter-factual. Yet, we know that the bond risk premium at long-horizons contains crucial information about the properties of the pricing kernel. In particular, Fernando Alvarez and Urban Jermann (2005) show that the ratio of the infinite bond risk premium to the maximum risk premium is linked to the fraction of the variance of the pricing kernel that arises from the martingale component. This decomposition of the pricing kernel is model-free. Like the Lars P. Hansen and Ravi Jagannathan (1991) bound, this moment directly describes a property of the pricing kernel and links it to observable asset return characteristics. The low (nominal) bond risk premium and high equity risk premium in the data suggest that most of the shocks to the pricing kernel are shocks to the martingale component.

Since the bond market data are nominal in nature, we augment the LRR model for an inflation process and study the properties of the long-horizon nominal bond risk premium. We show that the long-run risk model, which is successful at matching the wealth-consumption ratio, high equity risk premium and the nominal yields at short maturities implies too little (much) variation in the martingale component of the nominal (real) pricing kernel. This is because the nominal bond risk premium at infinite horizon is too high, or in other words because the real

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bond risk premium at infinite horizon is too low and thus the inflation risk premium too high. We conclude that the wealth-consumption ratio, the equity risk premium, and the long horizon bond risk premium impose tight restrictions on dynamic asset pricing models.

I. Stock and Bond Risk Premia in the Long Run Risk Model

The long-run risk literature works off the class of preferences due to David Kreps and Evan L. Porteus and Larry Epstein and Stan Zin (1989). Let $U_t(C_t)$ denote the utility derived from consuming C_t. The value function of the representative agent takes the following recursive form:

$$U_t(C_t) = \left[(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta\left(E_t U_{t+1}^{1-\gamma}\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}.$$

The time discount factor is δ, the risk aversion parameter is $\gamma \geq 0$, and the inter-temporal elasticity of substitution (IES) is $\psi \geq 0$. The parameter θ is defined by $\theta \equiv (1-\gamma)/(1-\frac{1}{\psi})$. When $\psi > 1$ and $\gamma > 1$, then $\theta < 0$ and agents prefer early resolution of uncertainty.

On the technology side, we adopt the specification of Bansal and Ivan Shaliastovich (2008) for consumption growth, dividend growth, and inflation:

$$\begin{aligned}
\Delta c_{t+1} &= \mu_g + x_t + \sigma_{gt}\eta_{t+1} \\
x_{t+1} &= \rho x_t + \sigma_{xt} e_{t+1} \\
\sigma^2_{g,t+1} &= \sigma^2_g + \nu_g\left(\sigma^2_{gt} - \sigma^2_g\right) + \sigma_{gw} w_{g,t+1} \\
\sigma^2_{x,t+1} &= \sigma^2_x + \nu_x\left(\sigma^2_{xt} - \sigma^2_x\right) + \sigma_{xw} w_{x,t+1} \\
\Delta d_{t+1} &= \mu_d + \phi_x x_t + \varphi_d \sigma_{gt}\eta_{d,t+1} \\
\pi_{t+1} &= \bar{\pi}_t + \varphi_{\pi g}\sigma_{gt}\eta_{t+1} + \varphi_{\pi x}\sigma_{xt}e_{t+1} \\
&+ \sigma_\pi \xi_{t+1} \\
\bar{\pi}_{t+1} &= \mu_\pi + \alpha_\pi(\bar{\pi}_t - \mu_\pi) + \alpha_x x_t \\
&+ \varphi_{zg}\sigma_{gt}\eta_{t+1} + \varphi_{zx}\sigma_{xt}e_{t+1} \\
&+ \sigma_z \xi_{t+1}.
\end{aligned}$$

All shocks are i.i.d standard normal, except $\text{Corr}\left(\eta_{t+1}, \eta_{d,t+1}\right) \equiv \tau_{gd}$. This specification builds on Bansal and Yaron (2004) and delivers empirically plausible stock and nominal bond prices. Tim Bollerslev, George Tauchen and Hao Zhou show that heteroscedasticity is key to reproduce asset pricing moments in the LRR framework. Real consumption growth contains a persistent long-run expected growth component x_t. Shocks to (short-run) consumption growth have a stochastic volatility $\sigma^2_{g,t+1}$. As in Bansal and Shaliastovich (2008), this volatility differs from the conditional variance of the long-run component x_t, which is denoted σ^2_{xt}. The inflation process is similar to that in Jessica Wachter (2006) and Monika Piazzesi and Martin Schneider (2006).

For our numerical results, we use the calibration of Bansal and Shaliastovich (2008), repeated in Table A.1 in the appendix.[1] Table A.2 summarizes the model loadings on state variables. The model matches several key features of aggregate consumption and dividend growth, as well as inflation.

A central object in the LRR model is the log wealth-consumption ratio, $wc_t \equiv w_t - c_t$. It is the price-dividend ratio of a claim to aggregate consumption. It is affine in the state variables x_t, σ^2_{gt} and σ^2_{xt}:

$$\begin{aligned}
wc_t &= \mu_{wc} + W_x x_t + W_{gs}\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&+ W_{xs}\left(\sigma^2_{xt} - \sigma^2_x\right).
\end{aligned}$$

The appendix derives the coefficients W_x, W_{gs} and W_{xs} as functions of the structural parameters. When the IES exceeds 1, an increase in expected consumption growth and a decrease in short-run or long-run consumption volatility increase the wealth-consumption ratio. The log real stochastic discount factor (SDF) can now be written as a function of log consumption growth and the change in the log wealth-consumption ratio:

$$\begin{aligned}
sdf_{t+1} &= \left[\theta \log\delta + (\theta - 1)\kappa^c_0\right] - \gamma\Delta c_{t+1} \\
&+ (\theta - 1)\left(wc_{t+1} - \kappa^c_1 wc_t\right),
\end{aligned}$$

where κ^c_0 and κ^c_1 are linearization constants, which are a function of the long-run average log wealth-consumption ratio μ_{wc}. Note that when $\theta = 1$ ($\gamma = \frac{1}{\psi}$), the above recursive preferences collapse to the standard power utility preferences, and changes in the wealth-consumption ratio are not priced. The only priced shocks

[1] We assume that the continuation values exist. See Hansen (2009) and Borovicka et al. (2009) for more on this question.

are short-run consumption growth shocks η_{t+1}. Hence, the empirical failures of the power utility model and the successes of the LRR model must be attributable to their respective implications for the wealth-consumption ratio. Lustig, Van Nieuwerburgh and Verdelhan (2009) estimate the wealth-consumption ratio in the data, using a preference-free no-arbitrage approach. Table 1 shows that the LRR model's implications are broadly consistent with the data. In particular, the LRR model implies that the claim to aggregate consumption is not very risky, resulting in a high mean wealth-consumption ratio of 50 and a low consumption risk premium.

TABLE 1—RISK PREMIA

	Mean	*Std*	$AR(1)$
		Data	
WC	88.59	14.11	0.96
PD	27.53	7.20	0.95
ERP	6.90	9.54	0.92
$BRP^{\$}$	0.92	1.04	0.89
		Model	
WC	48.97	12.59	0.99
PD	21.71	12.17	0.99
ERP	6.25	0.49	0.99
$BRP^{\$}$	2.97	0.46	0.99

This table reports the mean, standard deviation and autocorrelation of the annualized wealth-consumption ratio (WC), price-dividend ratio (PD), equity risk premium (ERP), and the 5-year nominal bond risk premium ($BRP^{\$}$). The moments from the data are in the upper panel and are taken from Lustig et al. (2009). They pertain to the period 1953-2008. The lower panel reports the moments obtained from model simulations.

Next, we turn to stock prices. Like the wealth-consumption ratio, the price-dividend ratio of the claim to aggregate dividends is affine in the same three state variables. The bulk of the risk premium is compensation for long-run consumption risk, and short-run, and long-run consumption growth volatility risk. Table 1 shows that the model matches the properties of the price-dividend ratio and the equity risk premium well. Because the dividend claim has more exposure to long-run risk ($\phi_x > 1$), its ends up being much riskier than the consumption claim. This is reflected in a low price-dividend ratio of 22 and a high equity risk premium of 6.25 percent per year.

Finally, the log price of a n-period nominal bond is affine in the same three state variables, as well as in expected inflation $\bar{\pi}_t$. Expected inflation (short-run volatility) unambiguously increases (decreases) nominal bond yields. The effect of long-run growth (long-run volatility) on nominal yields is positive (negative) at short maturities, but negative (positive) at long maturities. These sign reversals at long maturities do not arise for real yields; they result from a negative correlation between expected inflation and long-run growth. Consistent with the findings of Bansal and Shaliastovich (2008), Table A.3 shows that the LRR model matches the one-year to five-year nominal bond yields well. The yield levels are close to the average yields in the Fama-Bliss data for 1952-2008, and the five-minus-one year yield spread of 1.18 percent is reasonably close to the historical 0.56 percent spread.

However, the same table shows that nominal yields on longer horizon bonds are very high in the model. For example, the difference between the 30-year and the 5-year bond yield is 6.44 percent per year. The same spread between constant maturity Treasury yields in the 1952-2008 data is only 0.33 percent. Hypothetical 200-year nominal yields are 20 percent per year in the model. Likewise, the nominal bond risk premium increases sharply with maturity. Table 1 shows that the five-year nominal bond risk premium is 2.97 percent, which is substantially higher than the 0.92 percent premium we estimate in the data. Table A.3 shows that the one-year risk premium on a 200-year bond is as high as 24.4 percent. In the next section, we connect the high nominal bond risk premium at very long maturities to one of the components of a decomposition of the SDF. The bottom panel of Table A.3, which is for real yields, is informative about the origins of the high nominal yields and risk premia. It shows that the real yield curve is downward sloping. Real bond risk premia are negative at all horizons and are as low as -16 percent at the 200-year maturity. Real bonds are a hedge in the LRR model because their returns are high in those states of the world where the representative agent's inter-temporal marginal rate of substitution is high (long-run growth is low or economic uncertainty is high). To generate an upward sloping nominal yield curve, inflation risk must more than offset this hedging effect. Current and future inflation are unexpectedly high exactly when long-run growth is unexpectedly low ($\varphi_{\pi,x} < 0$ and $\varphi_{z,x} < 0$), generating a capital loss on the bond in high marginal utility states of the world. When the inflation risk is calibrated to match nominal yield data for maturities of one through five years, it also implies a very high nominal bond risk premium at very long horizons.

II. Decomposing the SDF

Let the SDF be the growth rate of the pricing kernel: $SDF_{t+1} = M_{t+1}/M_t$. Following Alvarez and Jermann (2005), Hansen, John C. Heaton, and Nan Li (2008), and Hansen and Jose A. Scheinkman (2009), we study a factorization of the SDF. Under mild regularity conditions, any pricing kernel M can be decomposed in two parts: $M_t = M_t^P M_t^T$. The first component, M_t^P, is a martingale $E_t[M_{t+1}^P] = M_t^P$, and the second component M_t^T is defined as:

$$M_t^T = \lim_{\tau\to\infty} \frac{\beta^{t+\tau}}{P_t(\tau)},$$

for some number β.[2] M_t^T is the dominant pricing component for long-term bonds. We obtain expressions for both components of the SDF, as well as for their logs. We do this decomposition both for the nominal and for the real SDF, where the nominal log SDF is $sdf_{t+1}^{\$} = sdf_{t+1} - \pi_{t+1}$. We focus on the nominal decomposition here.

We define the *conditional variance ratio* ω_t as the ratio of the conditional variance of the martingale component of the nominal log SDF to the conditional variance of the entire nominal log SDF:

$$\begin{aligned}\omega_t &= \frac{\mathrm{Var}_t[sdf_{t+1}^{\$,P}]}{\mathrm{Var}_t[sdf_{t+1}^{\$}]}\\ &= 1 - \frac{\mathbb{E}_t\left[r_{t+1}^{b,\$,e}(n)\right] - \frac{1}{2}\mathrm{Var}_t\left[r_{t+1}^{b,\$,e}(n)\right]}{\max_i \mathbb{E}_t\left[r_{t+1}^{i,\$,e}(n)\right] - \frac{1}{2}\mathrm{Var}_t\left[r_{t+1}^{i,\$,e}(n)\right]}\end{aligned}$$

We show in the appendix that ω_t equals one minus the ratio of the log bond risk premium on a nominal infinite maturity bond (without Jensen adjustment) to the maximum nominal risk premium in the economy (without Jensen adjustment).

Alvarez and Jermann (2005) show that, in a model without the martingale component, the infinite horizon bond is the highest risk premium in the economy. Conversely, in a model with just the martingale component, bond risk premia of all maturities are zero and the yield curve is flat. Hence, to have realistic term structure implications, the SDF cannot have only a martingale component, but the variation of M_t^T must not be too large. In the data, long-horizon nominal bond risk premia are low compared to, say, equity risk premia. Hence, the data discipline ω_t to be close to one on average. Alvarez and Jermann (2005) argue that this conclusion holds both for nominal and real bonds. An important caveat, though, is that risk premia on bonds with infinite horizons are not precisely measured because such bonds do not exist and actual long term bonds might offer convenience yields.

Table A.4 reports moments of the SDF and its components for the benchmark LRR calibration. Unsurprisingly, the martingale component of the SDF is more volatile than the dominant pricing component, M_t^T. Our key finding is that the nominal variance ratio ω_t is very low: only 0.37 on average. The reason is that in the LRR model the long-horizon nominal bond risk premium is very high, relative to the maximum nominal risk premium in the economy. Because the real bond risk premium is highly negative, the real variance ratio is much higher than one: 1.66 on average. Hence, the LRR model fails to generate a conditional variance ratio which is close to 1. Inflation introduces too much volatility in the dominant pricing component of the nominal SDF.

[2]For related work, see Bansal and Bruce N. Lehman (1997). Hansen et al. (2008), Hansen and Scheinkman (2009), Hansen (2009), Jaroslav Borovicka et al. (2009) and Ralph S.J. Koijen, Lustig and Van Nieuwerburgh (2009) derive this decomposition for affine models. The appendix applies their insights to the LRR model, which belongs to this class. Hansen and Scheinkman (2009) and Hansen (2009) give parametric examples within the affine class in which uniqueness fails. We do not study this issue here.

This conditional variance ratio is tightly linked to the dynamics of the wealth-consumption ratio. With power utility, e.g constant relative risk-aversion preferences (CRRA), the change in the log wealth-consumption ratio is no longer a priced factor in the log SDF and the real SDF now only has the martingale component. When $\theta = 1$, the real bond risk premium is zero at all maturities. The nominal bond risk premium and maximum risk premium are very small. While the average variance ratio ω is closer to the data, the power utility model generates an equity risk premium puzzle and a nominal interest rate of 20 percent per year for the one- through five-year yields, both of which are highly counterfactual. Our analysis raises the question of whether a change in the calibration of the LRR model may solve these issues. In the appendix, we consider both changes on the real and on the nominal side of the economy. The variance ratio ω_t changes noticeably with ρ_x, α_x and α_π. However, we find it difficult to obtain a calibration that successfully matches the ratio ω_t, its components, and all the moments of consumption growth, inflation, and equity and bond returns.

III. Conclusion

Matching the wealth-consumption ratio and the ω ratio is a challenge for dynamic asset pricing models. This challenge is not unique to the LRR model, but equally applies to the habit and the rare disasters model. Future research should investigate how these models can be modified to match the variance ratio. Non-neutrality of inflation is an interesting avenue for future research.

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Mathematical Appendix

In this appendix, we first derive four risk premia: the expected excess returns on a consumption claim, on equity and on real and nominal bonds. We then obtain the Alvarez and Jermann (2005) decomposition of the SDF in the long-run risk model. Finally, we report the parameter values used in our calibration.

Wealth-Consumption Ratio and Consumption Risk Premium

We start from the aggregate budget constraint:

$$W_{t+1} = R^c_{t+1}(W_t - C_t). \tag{1}$$

The beginning-of-period (or cum-dividend) total wealth W_t that is not spent on aggregate consumption C_t earns a gross return R^c_{t+1} and leads to beginning-of-next-period total wealth W_{t+1}. The return on a claim to aggregate consumption, the *total wealth return*, can be written as

$$R^c_{t+1} = \frac{W_{t+1}}{W_t - C_t} = \frac{C_{t+1}}{C_t}\frac{WC_{t+1}}{WC_t - 1}.$$

We use the Campbell (1991) approximation of the log total wealth return $r^c_t = \log(R^c_t)$ around the long-run average log wealth-consumption ratio $\mu_{wc} \equiv E[w_t - c_t]$:

$$r^c_{t+1} = \kappa^c_0 + \Delta c_{t+1} + wc_{t+1} - \kappa^c_1 wc_t,$$

where the linearization constants κ^c_0 and κ^c_1 are non-linear functions of the unconditional mean log wealth-consumption ratio μ_{wc}:

$$\kappa^c_1 = \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1} > 1 \ \text{ and } \ \kappa^c_0 = -\log\left(e^{\mu_{wc}} - 1\right) + \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1}\mu_{wc}.$$

Throughout the paper, we use lower letters to denote logs.
The Euler equation for any asset i with lognormal return R^i implies:

$$0 = \mathbb{E}_t\left[sdf_{t+1}\right] + \mathbb{E}_t\left[r^i_{t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left[sdf_{t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left[r^i_{t+1}\right] + \mathrm{Cov}_t\left[sdf_{t+1}, r^i_{t+1}\right] \tag{2}$$

We conjecture that the wealth-consumption ratio is linear in the state variables x_t, σ^2_{gt} and σ^2_{xt}:

$$wc_t = \mu_{wc} + W_x x_t + W_{gs}\left(\sigma^2_{gt} - \sigma^2_g\right) + W_{xs}\left(\sigma^2_{xt} - \sigma^2_x\right)$$

We first compute the different components of equation 2:

$$\begin{aligned}
r^c_{t+1} &= r^c_0 + \left[1 + W_x\left(\rho - \kappa^c_1\right)\right]x_t + W_{gs}\left(\nu_g - \kappa^c_1\right)\left(\sigma^2_{gt} - \sigma^2_g\right) + W_{xs}\left(\nu_x - \kappa^c_1\right)\left(\sigma^2_{xt} - \sigma^2_x\right) \\
&\quad + \sigma_{gt}\eta_{t+1} + W_x\sigma_{xt}e_{t+1} + W_{gs}\sigma_{gw}w_{g,t+1} + W_{xs}\sigma_{xw}w_{x,t+1} \\
\mathbb{E}_t\left[r^c_{t+1}\right] &= r_0 + \left[1 + W_x\left(\rho - \kappa^c_1\right)\right]x_t + W_{gs}\left(\nu_g - \kappa^c_1\right)\left(\sigma^2_{gt} - \sigma^2_g\right) + W_{xs}\left(\nu_x - \kappa^c_1\right)\left(\sigma^2_{xt} - \sigma^2_x\right) \\
r^c_{t+1} - \mathbb{E}_t\left[r^c_{t+1}\right] &= \sigma_{gt}\eta_{t+1} + W_x\sigma_{xt}e_{t+1} + W_{gs}\sigma_{gw}w_{g,t+1} + W_{xs}\sigma_{xw}w_{x,t+1} \\
\mathbb{V}_t\left[r^c_{t+1}\right] &= \sigma^2_{gt} + W^2_x\sigma^2_{xt} + W^2_{gs}\sigma^2_{gw} + W^2_{xs}\sigma^2_{xw} \\
r^c_0 &= \kappa^c_0 + \mu_g + \left(1 - \kappa^c_1\right)\mu_{wc}
\end{aligned}$$

Epstein and Zin (1989) show that the log real stochastic discount factor is

$$\begin{aligned}
sdf_{t+1} &= \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r^c_{t+1} \\
&= \mu_s + \left\{ -\frac{\theta}{\psi} + (\theta - 1) \left[1 + W_x (\rho - \kappa^c_1)\right] \right\} x_t \\
&\quad + \{W_{gs} (\nu_g - \kappa^c_1)(\theta - 1)\} \left(\sigma^2_{gt} - \sigma^2_g\right) + \{W_{xs} (\nu_x - \kappa^c_1)(\theta - 1)\} \left(\sigma^2_{xt} - \sigma^2_x\right) \\
&\quad + \left\{ \theta \left(1 - \frac{1}{\psi}\right) - 1 \right\} \sigma_{gt} \eta_{t+1} + (\theta - 1) \{W_x \sigma_{xt} e_{t+1} + W_{gs} \sigma_{gw} w_{g,t+1} + W_{xs} \sigma_{xw} w_{x,t+1}\} \\
sdf_{t+1} - \mathbb{E}_t [sdf_{t+1}] &= \left\{ \theta \left(1 - \frac{1}{\psi}\right) - 1 \right\} \sigma_{gt} \eta_{t+1} + (\theta - 1) \{W_x \sigma_{xt} e_{t+1} + W_{gs} \sigma_{gw} w_{g,t+1} + W_{xs} \sigma_{xw} w_{x,t+1}\} \\
\mathbb{E}_t [sdf_{t+1}] &= \mu_s + \left\{ -\frac{\theta}{\psi} + (\theta - 1) \left[1 + W_x (\rho - \kappa^c_1)\right] \right\} x_t \\
&\quad + \{W_{gs} (\nu_g - \kappa^c_1)(\theta - 1)\} \left(\sigma^2_{gt} - \sigma^2_g\right) + \{W_{xs} (\nu_x - \kappa^c_1)(\theta - 1)\} \left(\sigma^2_{xt} - \sigma^2_x\right) \\
\mathbb{V}_t [sdf_{t+1}] &= \left\{ \theta \left(1 - \frac{1}{\psi}\right) - 1 \right\}^2 \sigma^2_{gt} + (\theta - 1)^2 \left\{ W^2_x \sigma^2_{xt} + W^2_{gs} \sigma^2_{gw} + W^2_{xs} \sigma^2_{xw} \right\} \\
\mu_s &= \theta \log \delta - \frac{\theta}{\psi} \mu_g + (\theta - 1) r^c_0
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}_t [r^c_{t+1}, sdf_{t+1}] &= \mathbb{E}_t [(r^c_{t+1} - \mathbb{E}_t [r^c_{t+1}])(sdf_{t+1} - \mathbb{E}_t [sdf_{t+1}])] \\
&= \left\{ \theta \left(1 - \frac{1}{\psi}\right) - 1 \right\} \sigma^2_{gt} + W^2_x (\theta - 1) \sigma^2_{xt} + W^2_{gs} (\theta - 1) \sigma^2_{gw} + W^2_{xs} (\theta - 1) \sigma^2_{xw}
\end{aligned}$$

Plugging these different components into equation (2) evaluated at $i = c$ yields:

$$0 = r^c_0 + \mu_s + \frac{\theta^2}{2} \left\{ \left(1 - \frac{1}{\psi}\right)^2 \sigma^2_g + W^2_x \sigma^2_x + W^2_{gs} \sigma^2_{gw} + W^2_{xs} \sigma^2_{xw} \right\} \tag{3}$$

$$+ \theta \left\{ -\frac{1}{\psi} + \left[1 + W_x (\rho - \kappa^c_1)\right] \right\} x_t \tag{4}$$

$$+ \frac{\theta}{2} \left\{ 2 W_{gs} (\nu_g - \kappa^c_1) + \theta \left(1 - \frac{1}{\psi}\right)^2 \right\} \left(\sigma^2_{gt} - \sigma^2_g\right) \tag{5}$$

$$+ \frac{\theta}{2} \left\{ 2 W_{xs} (\nu_x - \kappa^c_1) + \theta W^2_x \right\} \left(\sigma^2_{xt} - \sigma^2_x\right) \tag{6}$$

Then setting all coefficients equal to zero we obtain:

$$\begin{aligned}
(4) &\implies W_x = \frac{1 - \frac{1}{\psi}}{\kappa^c_1 - \rho} \\
(5) &\implies W_{gs} = \frac{\theta \left(1 - \frac{1}{\psi}\right)^2}{2 (\kappa^c_1 - \nu_g)} \\
(6) &\implies W_{xs} = \frac{\theta}{2 (\kappa^c_1 - \nu_x)} \left(1 - \frac{\frac{1}{\psi}}{\kappa^c_1 - \rho}\right)^2
\end{aligned}$$

If the IES exceeds 1, then $W_x > 0$, $W_{gs} < 0$, and $W_{xs} < 0$.
Plugging these coefficients back into equation (3) implicitly defines a nonlinear equation in one unknown (μ_{wc}), which can be solved for numerically, characterizing the average wealth-consumption ratio.
According to (2), the risk premium (expected excess real return corrected for a Jensen term) on the consumption claim is given by[3]:

$$\begin{aligned}
\mathbb{E}_t\left[r_{t+1}^{c,e}\right] &= -\mathrm{Cov}_t\left[r_{t+1}^c, sdf_{t+1}\right] \\
&= \left\{1-\theta\left(1-\frac{1}{\psi}\right)\right\}\sigma_{gt}^2 + W_x^2\left(1-\theta\right)\sigma_{xt}^2 + W_{gs}^2\left(1-\theta\right)\sigma_{gw}^2 + W_{xs}^2\left(1-\theta\right)\sigma_{xw}^2 \\
&= \lambda_\eta\sigma_{gt}^2 + W_x\lambda_e\sigma_{xt}^2 + W_{gs}\lambda_{gw}\sigma_{gw}^2 + W_{xs}\lambda_{xw}\sigma_{xw}^2
\end{aligned}$$

with the market price of risk vector $\Lambda = [\lambda_\eta, \lambda_e, \lambda_{gw}, \lambda_{xw}]$ given by:

$$\begin{aligned}
\lambda_\eta &= -\left\{\theta\left(1-\frac{1}{\psi}\right)-1\right\} = \gamma > 0 \\
\lambda_e &= \left(1-\theta\right)W_x = \frac{\gamma-\frac{1}{\psi}}{\kappa_1^c-\rho} \\
\lambda_{gw} &= \left(1-\theta\right)W_{gs} = -\frac{\left(\gamma-1\right)\left(\gamma-\frac{1}{\psi}\right)}{2\left(\kappa_1^c-\nu_g\right)} \\
\lambda_{xw} &= \left(1-\theta\right)W_{xs} = -\frac{\left(\gamma-1\right)\left(\gamma-\frac{1}{\psi}\right)}{2\left(\kappa_1^c-\nu_x\right)\left(\kappa_1^c-\rho\right)^2}
\end{aligned}$$

If the IES is sufficiently large ($\gamma > 1/\psi$), then $\lambda_e > 0$, $\lambda_{gw} < 0$, and $\lambda_{xw} < 0$.

Equity Risk Premium

We log-linearize return on portfolio: $r_{t+1} = \kappa_0 + \Delta d_{t+1} + pd_{t+1} - \kappa_1 pd_t$, and conjecture that the price-dividend ratio is linear in the state variables: $pd_t = \mu_{pd} + D_x x_t + D_{gs}\left(\sigma_{gt}^2 - \sigma_g^2\right) + D_{xs}\left(\sigma_{xt}^2 - \sigma_x^2\right)$
As we did for the return on the consumption claim, we compute innovations in the dividend claim return, and its conditional mean and variance:

$$\begin{aligned}
r_{t+1} &= r_0 + \left\{\phi_x + D_x\left(\rho-\kappa_1\right)\right\}x_t + D_{gs}\left(\nu_g-\kappa_1\right)\left(\sigma_{gt}^2-\sigma_g^2\right) + D_{xs}\left(\nu_x-\kappa_1\right)\left(\sigma_{xt}^2-\sigma_x^2\right) \\
&\quad + \varphi_d\sigma_{gt}\eta_{d,t+1} + D_x\sigma_{xt}e_{t+1} + D_{gs}\sigma_{gw}w_{g,t+1} + D_{xs}\sigma_{xw}w_{x,t+1} \\
r_{t+1} - \mathbb{E}_t\left[r_{t+1}\right] &= \varphi_d\sigma_{gt}\eta_{d,t+1} + D_x\sigma_{xt}e_{t+1} + D_{gs}\sigma_{gw}w_{g,t+1} + D_{xs}\sigma_{xw}w_{x,t+1} \\
\mathbb{E}_t\left[r_{t+1}\right] &= r_0 + \left\{\phi_x + D_x\left(\rho-\kappa_1\right)\right\}x_t + D_{gs}\left(\nu_g-\kappa_1\right)\left(\sigma_{gt}^2-\sigma_g^2\right) \\
&\quad + D_{xs}\left(\nu_x-\kappa_1\right)\left(\sigma_{xt}^2-\sigma_x^2\right) \\
\mathrm{Var}_t\left[r_{t+1}\right] &= \varphi_d^2\sigma_{gt}^2 + D_x^2\sigma_{xt}^2 + D_{gs}^2\sigma_{gw}^2 + D_{xs}^2\sigma_{xw}^2 \\
r_0 &= \kappa_0 + \mu_{pd}\left(1-\kappa_1\right) + \mu_d
\end{aligned}$$

$$\mathrm{Cov}_t\left[r_{t+1}, sdf_{t+1}\right] = \left(\theta-1\right)\left[W_{gs}D_{gs}\sigma_{gw}^2 + W_{xs}D_{xs}\sigma_{xw}^2\right] - \gamma\varphi_d\tau_{gd}\sigma_{gt}^2 + \left(\theta-1\right)W_xD_x\sigma_{xt}^2$$

[3]Recall that the log riskfree rate is $y_t(1) = -\mathbb{E}_t\left[sdf_{t+1}\right] - \frac{1}{2}\mathrm{Var}_t\left[sdf_{t+1}\right]$.

Plug these different components into equation (2):

$$
\begin{aligned}
0 &= \mu_s + r_0 + \frac{1}{2}\left[\gamma^2 - 2\gamma\varphi_d\tau_{dg} + \varphi_d^2\right]\sigma_g^2 + \frac{1}{2}\left[W_x(\theta-1)+D_x\right]^2\sigma_x^2 + \frac{1}{2}\left[W_{gs}(\theta-1)+D_{gs}\right]^2\sigma_{gw}^2 \\
(7) \quad &+\frac{1}{2}\left[W_{xs}(\theta-1)+D_{xs}\right]^2\sigma_{xw}^2 \\
(8) \quad &+\left\{-\frac{1}{\psi} + \left[\phi_x + D_x(\rho-\kappa_1)\right]\right\}x_t \\
(9) \quad &+\left\{\frac{1}{2}\left[\gamma^2 - 2\gamma\varphi_d\tau_{dg} + \varphi_d^2\right] + W_{gs}(\kappa_1^c - \nu_g)(1-\theta) + D_{gs}(\nu_g - \kappa_1)\right\}\left(\sigma_{gt}^2 - \sigma_g^2\right) \\
(10) \quad &+\left\{\frac{1}{2}\left[W_x(\theta-1)+D_x\right]^2 + W_{xs}(\kappa_1^c - \nu_x)(1-\theta) + D_{xs}(\nu_x - \kappa_1)\right\}\left(\sigma_{xt}^2 - \sigma_x^2\right)
\end{aligned}
$$

Then setting all coefficients equal to zero we get:

$$
\begin{aligned}
(8) \quad &\Longrightarrow D_x = \frac{\phi_x - \frac{1}{\psi}}{\kappa_1 - \rho} \\
(9) \quad &\Longrightarrow D_{gs} = \frac{\frac{1}{2}\left[\gamma^2 - 2\gamma\varphi_d\tau_{dg} + \varphi_d^2\right] - \frac{1}{2}\left(\gamma - \frac{1}{\psi}\right)(\gamma-1)}{\kappa_1 - \nu_g} \\
(10) \quad &\Longrightarrow D_{xs} = \frac{\frac{1}{2}\left[\frac{\phi_x - \frac{1}{\psi}}{\kappa_1 - \rho} - \frac{\gamma - \frac{1}{\psi}}{\kappa_1^c - \rho}\right]^2 - \frac{1}{2}\frac{(\gamma-1)\left(\gamma - \frac{1}{\psi}\right)}{\left(\kappa_1^c - \rho\right)^2}}{\kappa_1 - \nu_x}
\end{aligned}
$$

Plugging these into (7) implicitly defines a nonlinear equation in one unknown (i.e., μ_{pd}), which can be solved for numerically, characterizing the mean price-dividend ratio.
The D coefficients are the betas of the equity market portfolio with respect to the four fundamental consumption growth shocks.
The equity risk premium is equal to:

$$
\begin{aligned}
\mathbb{E}_t\left[r_{t+1}^e\right] &= -\text{Cov}_t\left[r_{t+1}, sdf_{t+1}\right] \\
&= (\varphi_d\tau_{gd})\lambda_\eta\sigma_{gt}^2 + D_x\lambda_e\sigma_{xt}^2 + D_{gs}\lambda_{gw}\sigma_{gw}^2 + D_{xs}\lambda_{xw}\sigma_{xw}^2 \\
&= \left[G_0 + G_{gs}\sigma_g^2 + G_{xs}\sigma_x^2\right] + G_{gs}\left(\sigma_{gt}^2 - \sigma_g^2\right) + G_{xs}\left(\sigma_{xt}^2 - \sigma_x^2\right) \\
G_0 &= D_{gs}\lambda_{gw}\sigma_{gw}^2 + D_{xs}\lambda_{xw}\sigma_{xw}^2 \\
G_{gs} &= \varphi_d\tau_{gd}\gamma \\
G_{xs} &= D_x\lambda_e
\end{aligned}
$$

Real Bond Returns and Risk Premium

We start off the expression for the real stochastic discount factor derived in the first sub-section above. Let define the following three parameters: $s_x \equiv -\frac{1}{\psi}$, $s_{gs} \equiv -\frac{1}{2}(\gamma-1)(\gamma-\frac{1}{\psi})$, and $s_{xs} \equiv -\frac{1}{2}(\gamma-1)(\gamma-\frac{1}{\psi})\frac{1}{(\kappa_1^c-\rho)^2}$. Using notation defined above and in the previous sub-sections, the real stochastic discount factor is:

$$
\begin{aligned}
sdf_{t+1} &= \mu_s + s_x x_t + s_{gs}\left(\sigma_{gt}^2 - \sigma_g^2\right) + s_{xs}\left(\sigma_{xt}^2 - \sigma_x^2\right) \\
&\quad -\lambda_\eta\sigma_{gt}\eta_{t+1} - \lambda_e\sigma_{xt}e_{t+1} - \lambda_{gw}\sigma_{gw}w_{g,t+1} - \lambda_{xw}\sigma_{xw}w_{x,t+1}
\end{aligned}
$$

Let $p_t^b(n) = \log\left(P_t^b(n)\right)$ be the log price and $y_t^b(n) = -\frac{1}{n}p_t^b(n)$ the yield of an n-period real bond. We conjecture that the log prices of real bonds are linear in the state variables: $p_t(n) = -B_0(n) - B_x(n)x_t - B_{gs}(n)\left(\sigma_{gt}^2 - \sigma_g^2\right) - B_{xs}\left(\sigma_{xt}^2 - \sigma_x^2\right)$
The coefficients are initialized at zero and satisfy the following recursions:

$$
\begin{aligned}
B_0(n) &= B_0(n-1) - \mu_s - \frac{1}{2}\left[\lambda_{gw} + B_{gs}(n-1)\right]^2 \sigma_{gw}^2 \\
&\quad -\frac{1}{2}\left\{\left[\lambda_{xw} + B_{xs}(n-1)\right]^2 \sigma_{xw}^2 + \lambda_\eta^2 \sigma_g^2\right\} \\
&\quad -\frac{1}{2}\left[\lambda_e + B_x(n-1)\right]^2 \sigma_x^2 \\
B_x(n) &= \rho B_x(n-1) + \frac{1}{\psi} \\
B_{gs}(n) &= \nu_g B_{gs}(n-1) + \frac{1}{2}(\gamma-1)(\gamma - \frac{1}{\psi}) - \frac{1}{2}\gamma^2 \\
B_{xs}(n) &= \nu_x B_{xs}(n-1) + \frac{1}{2}(\gamma-1)\frac{(\gamma-\frac{1}{\psi})}{(\kappa_1^c-\rho)^2} - \frac{1}{2}\left[\frac{\gamma-\frac{1}{\psi}}{\kappa_1^c-\rho} + B_x(n-1)\right]^2.
\end{aligned}
$$

These recursions imply the following limit values:

$$
\begin{aligned}
B_x(\infty) &= \frac{1}{\psi(1-\rho)} \\
B_{gs}(\infty) &= \frac{\frac{1}{2}(\gamma-1)(\gamma-\frac{1}{\psi}) - \frac{1}{2}\gamma^2}{1-\nu_g} \\
B_{xs}(\infty) &= \frac{\frac{1}{2}(\gamma-1)\frac{(\gamma-\frac{1}{\psi})}{(\kappa_1^c-\rho)^2} - \frac{1}{2}\left[\frac{\gamma-\frac{1}{\psi}}{\kappa_1^c-\rho} + B_x(\infty)\right]^2}{1-\nu_x}.
\end{aligned}
$$

We define $B(\infty) \equiv [B_x(\infty), B_{gs}(\infty), B_{xs}(\infty)]'$.
The real bond risk premium on monthly holding period returns is equal to:

$$
\begin{aligned}
r_{t+1}^b(n) &\equiv n y_t^b(n) - (n-1)y_{t+1}^b(n-1) \\
r_{t+1}^b(n) - \mathbb{E}_t\left[r_{t+1}^b(n)\right] &= -B_x(n-1)\sigma_{xt}e_{t+1} - B_{gs}(n-1)\sigma_{gw}w_{g,t+1} \\
&\quad -B_{xs}(n-1)\sigma_{xw}w_{x,t+1} \\
\mathbb{E}_t\left[r_{t+1}^{b,e}(n)\right] &= -\mathrm{Cov}_t\left[r_{t+1}^b, sdf_{t+1}\right] \\
&= \left[F_0(n) + F_{gs}(n)\sigma_g^2 + F_{xs}(n)\sigma_x^2\right] + F_{gs}(n)\left(\sigma_{gt}^2 - \sigma_g^2\right) + F_{xs}(n)\left(\sigma_{xt}^2 - \sigma_x^2\right) \\
F_0(n) &= -B_{gs}(n-1)\lambda_{gw}\sigma_{gw}^2 - B_{xs}(n-1)\lambda_{xw}\sigma_{xw}^2, \\
F_{gs}(n) &= 0, \\
F_{xs}(n) &= -B_x(n-1)\lambda_e.
\end{aligned}
$$

We now define some vectors and matrices to present results in a more compact way. Let the vector X_t summarize all real state variables: $X_t \equiv [x_t,\ \sigma_{gt}^2 - \sigma_g^2,\ \sigma_{xt}^2 - \sigma_x^2]'$. Let ε_{t+1} denote the corresponding gaussian, i.i.d shocks: $\varepsilon_{t+1} \equiv [e_{t+1},\ w_{g,t+1},\ w_{x,t+1}]'$. We define $\Sigma_t \equiv diag[\sigma_{xt}^2, \sigma_{gw}^2, \sigma_{xw}^2]$. The law of motion of the state vector X_t is $X_{t+1} = \Gamma X_t + \Sigma_t^{\frac{1}{2}}\varepsilon_{t+1}$, where Γ is a 3 by 3 diagonal matrix with ρ, φ_{zg}, and φ_{zx} on the diagonal. Let $B(n)$ denote all the n-period real bond parameters: $B(n) \equiv [B_x(n),\ B_{gs}(n),\ B_{xs}(n)]'$. Using this notation, we can rewrite the real bond risk premium as:

$$\mathbb{E}_t\left[r^{b,e}_{t+1}(n)\right] = -B(n-1)'\Sigma_t\Lambda.$$

Nominal Bond Returns and Risk Premium

We start off the expression for the real stochastic discount factor derived above. We use a $^{\$}$ superscript to denote nominal variables. The nominal stochastic discount factor is then:

$$\begin{aligned}
sdf^{\$}_{t+1} &\equiv sdf_{t+1} - \pi_{t+1} \\
&= \mu_s - \mu_\pi + s_x x_t + s_{gs}\left(\sigma^2_{gt} - \sigma^2_g\right) + s_{xs}\left(\sigma^2_{xt} - \sigma^2_x\right) - \left(\bar{\pi}_t - \mu_\pi\right) \\
&\quad - \left(\lambda_\eta + \varphi_{\pi g}\right)\sigma_{gt}\eta_{t+1} - \left(\lambda_e + \varphi_{\pi x}\right)\sigma_{xt}e_{t+1} - \lambda_{gw}\sigma_{gw}w_{g,t+1} - \lambda_{xw}\sigma_{xw}w_{x,t+1} - \sigma_\pi\xi_{t+1}
\end{aligned}$$

Let $p^{\$}_t(n) = \log\left(P^{\$}_t(n)\right)$ be the log price and $y^{\$}_t(n) = -\frac{1}{n}p^{\$}_t(n)$ the yield of an n-period nominal bond.

We conjecture that the log prices of nominal bonds are linear in the state variables: $p^{\$}_t(n) = -B^{\$}_0(n) - B^{\$}_x(n)x_t - B^{\$}_{gs}(n)\left(\sigma^2_{gt} - \sigma^2_g\right) - B^{\$}_{xs}\left(\sigma^2_{xt} - \sigma^2_x\right) - B^{\$}_\pi(n)\left(\bar{\pi}_t - \mu_\pi\right)$

The coefficients are initialized at zero and satisfy the following recursions:

$$\begin{aligned}
B^{\$}_0(n) &= B^{\$}_0(n-1) - \mu_s + \mu_\pi - \frac{1}{2}\left\{\left[\sigma_\pi + B^{\$}_\pi(n-1)\sigma_z\right]^2 + \left[\lambda_{gw} + B^{\$}_{gs}(n-1)\right]^2\sigma^2_{gw}\right\} \\
&\quad -\frac{1}{2}\left\{\left[\lambda_{xw} + B^{\$}_{xs}(n-1)\right]^2\sigma^2_{xw} + \left[\varphi_{\pi g} + \lambda_\eta + \varphi_{zg}B^{\$}_\pi(n-1)\right]^2\sigma^2_g\right\} \\
&\quad -\frac{1}{2}\left[\varphi_{\pi x} + \lambda_e + B^{\$}_x(n-1) + \varphi_{zx}B^{\$}_\pi(n-1)\right]^2\sigma^2_x \\
B^{\$}_x(n) &= \rho B^{\$}_x(n-1) + \alpha_x B^{\$}_\pi(n-1) - s_x \\
B^{\$}_{gs}(n) &= \nu_g B^{\$}_{gs}(n-1) - s_{gs} - \frac{1}{2}\left[\lambda_\eta + \varphi_{\pi g} + \varphi_{zg}B^{\$}_\pi(n-1)\right]^2 \\
B^{\$}_{xs}(n) &= \nu_x B^{\$}_{xs}(n-1) - s_{xs} - \frac{1}{2}\left[\lambda_e + \varphi_{\pi x} + B^{\$}_x(n-1) + \varphi_{zx}B^{\$}_\pi(n-1)\right]^2 \\
B^{\$}_\pi(n) &= \alpha_\pi B^{\$}_\pi(n-1) + 1.
\end{aligned}$$

These recursions imply the following limit values:

$$\begin{aligned}
B^{\$}_x(\infty) &= \frac{\alpha_x B^{\$}_\pi(\infty) - s_x}{1-\rho} \\
B^{\$}_{gs}(\infty) &= \frac{-s_{gs} - \frac{1}{2}\left[\lambda_\eta + \varphi_{\pi g} + \varphi_{zg}B^{\$}_\pi(\infty)\right]^2}{1-\nu_g} \\
B^{\$}_{xs}(\infty) &= \frac{-s_{xs} - \frac{1}{2}\left[\lambda_e + \varphi_{\pi x} + B^{\$}_x(\infty) + \varphi_{zx}B^{\$}_\pi(\infty)\right]^2}{1-\nu_x} \\
B^{\$}_\pi(\infty) &= \frac{1}{1-\alpha_\pi}.
\end{aligned}$$

We define $B^{\$}(\infty) \equiv [B^{\$}_x(\infty), B^{\$}_{gs}(\infty), B^{\$}_{xs}(\infty), B^{\$}_\pi(\infty)]'$.

The nominal bond risk premium on monthly holding period returns is equal to:

$$\begin{aligned}
r^{b,\$}_{t+1}(n) &\equiv ny^{\$}_t(n) - (n-1)y^{\$}_{t+1}(n-1) \\
r^{b,\$}_{t+1}(n) - \mathbb{E}_t\left[r^{b,\$}_{t+1}(n)\right] &= -\left(B^{\$}_x(n-1) + B^{\$}_\pi(n-1)\varphi_{zx}\right)\sigma_{xt}e_{t+1} - B^{\$}_{gs}(n-1)\sigma_{gw}w_{g,t+1} \\
&\quad -B^{\$}_{xs}(n-1)\sigma_{xw}w_{x,t+1} - B^{\$}_\pi(n-1)\left(\varphi_{zg}\sigma_{gt}\eta_{t+1} + \sigma_z\xi_{t+1}\right) \\
\mathbb{E}_t\left[r^{b,\$,e}_{t+1}(n)\right] &= -\mathrm{Cov}_t\left[r^{b,\$}_{t+1}, sdf^{\$}_{t+1}\right] \\
&= \left[F^{\$}_0(n) + F^{\$}_{gs}(n)\sigma^2_g + F^{\$}_{xs}(n)\sigma^2_x\right] + F^{\$}_{gs}(n)\left(\sigma^2_{gt} - \sigma^2_g\right) + F^{\$}_{xs}(n)\left(\sigma^2_{xt} - \sigma^2_x\right) \\
F^{\$}_0(n) &= -\left\{\lambda_{gw}B^{\$}_{gs}(n-1)\sigma^2_{gw} + \lambda_{xw}B^{\$}_{xs}(n-1)\sigma^2_{xw} + \sigma_\pi\sigma_z B^{\$}_\pi(n-1)\right\} \\
F^{\$}_{gs}(n) &= -\left(\lambda_\eta + \varphi_{\pi g}\right)\varphi_{zg}B^{\$}_\pi(n-1) \\
F^{\$}_{xs}(n) &= -\left(\lambda_e + \varphi_{\pi x}\right)\left(B^{\$}_x(n-1) + B^{\$}_\pi(n-1)\varphi_{zx}\right)
\end{aligned}$$

Define the following vector and matrix objects:

$$\begin{aligned}
\widehat{\Lambda}^{\$} &\equiv [\lambda_\eta + \varphi_{\pi g}, \lambda_e + \varphi_{\pi x}, \lambda_{gw}, \lambda_{xw}, \sigma_\pi], \\
\widehat{B}^{\$}(n) &\equiv [B^{\$}_\pi(n)\varphi_{zg}, B^{\$}_x(n) + B^{\$}_\pi(n)\varphi_{zx}, B^{\$}_{gs}(n), B^{\$}_{xs}(n), B^{\$}_\pi(n)\sigma_z], \\
\widehat{\Sigma}_t &\equiv diag[\sigma^2_{gt}, \sigma^2_{xt}, \sigma^2_{gw}, \sigma^2_{xw}, 1], \\
\widehat{\varepsilon}_{t+1} &\equiv [\eta_{t+1}, e_{t+1}, w_{g,t+1}, w_{x,t+1}, \xi_{t+1}]
\end{aligned}$$

Then we can write the nominal bond risk premium compactly as:

$$\mathbb{E}_t\left[r^{b,\$,e}_{t+1}(n)\right] = -\widehat{B}^{\$\prime}(n-1)\widehat{\Sigma}_t\widehat{\Lambda}^{\$}.$$

Decomposition of the Real SDF

The following proposition shows how to decompose the SDF of the long-run risk model into a martingale component and the dominant pricing component.

Proposition 1. *The stochastic discount factor of the long-run risk model can be decomposed into a martingale component and the dominant pricing component:*

$$\begin{aligned}
\frac{M^T_{t+1}}{M^T_t} &= \beta\exp\left(-B'_\infty\left(I-\Gamma\right)X_t + B'_\infty\Sigma_t^{\frac{1}{2}}\varepsilon_{t+1}\right), \\
\frac{M^P_{t+1}}{M^P_t} &= \beta^{-1}\exp\left(\mu_s + \left[S' + B'_\infty\left(I-\Gamma\right)\right]X_t - \left(\Lambda' + B'_\infty\right)\Sigma_t^{\frac{1}{2}}\varepsilon_{t+1} - \lambda_\eta\sigma_{gt}\eta_{t+1}\right).
\end{aligned}$$

To show this, we start from the definition of the dominant pricing component of the pricing kernel:

$$M^T_t = \lim_{n\to\infty}\frac{\beta^{t+n}}{P^b_t\left(n\right)},$$

Recall that log real bond prices are affine in the state vector:

$$\begin{aligned}
p^b_t(n) &= -B_0(n) - B_x(n)x_t - B_{gs}(n)\left(\sigma^2_{gt} - \sigma^2_g\right) - B_{xs}\left(\sigma^2_{xt} - \sigma^2_x\right) \\
&= -B_0(n) - B(n)'X_t.
\end{aligned}$$

We can then write the dominant pricing component of the SDF as:

$$M_t^T = \lim_{n\to\infty} \beta^{t+n} \exp\left(B_0(n) + B(n)' X_t\right).$$

The constant β is chosen in order to satisfy Assumption 1 in Alvarez and Jermann (2005):

$$0 < \lim_{n\to\infty} \frac{P_t^b(n)}{\beta^n} < \infty.$$

Recall that $B_0(n)$ is defined recursively:

$$\begin{aligned} B_0(n) &= B_0(n-1) - \mu_s - \frac{1}{2}\left\{\left[\lambda_{gw} + B_{gs}(n-1)\right]^2 \sigma_{gw}^2\right\} \\ &- \frac{1}{2}\left\{\left[\lambda_{xw} + B_{xs}(n-1)\right]^2 \sigma_{xw}^2 + \lambda_\eta^2 \sigma_g^2 + \left[\lambda_e + B_x(n-1)\right]^2 \sigma_x^2\right\} \end{aligned}$$

Because of the affine term structure of the model and the stationarity of the state vector X, the limit $\lim_{n\to\infty} B(n) = B(\infty)$ is finite. Taking limits on both sides of the equation above leads to:

$$\begin{aligned} \lim_{n\to\infty} B_0(n) - B_0(n-1) &= -\mu_s - \frac{1}{2}\left\{\left[\lambda_{gw} + B_{gs}(\infty)\right]^2 \sigma_{gw}^2\right\} \\ &\quad -\frac{1}{2}\left\{\left[\lambda_{xw} + B_{xs}(\infty)\right]^2 \sigma_{xw}^2 + \lambda_\eta^2 \sigma_g^2 + \left[\lambda_e + B_x(\infty)\right]^2 \sigma_x^2\right\} \end{aligned}$$

The limit of $B_0(n) - B_0(n-1)$ is finite, so that $B_0(n)$ grows at a linear rate in the limit. We choose the constant β to offset the growth in $B_0(n)$ as n becomes very large. Setting

$$\beta = \exp\left(\mu_s + \frac{1}{2}\left\{\left[\lambda_{gw} + B_{gs}(\infty)\right]^2 \sigma_{gw}^2 + \left[\lambda_{xw} + B_{xs}(\infty)\right]^2 \sigma_{xw}^2 + \lambda_\eta^2 \sigma_g^2 + \left[\lambda_e + B_x(\infty)\right]^2 \sigma_x^2\right\}\right)$$

guarantees that Assumption 1 in Alvarez and Jermann (2005) is satisfied.

We can now write the dominant pricing component of the SDF as:

$$\frac{M_{t+1}^T}{M_t^T} = \beta \exp\left(-B_\infty' \left(I - \Gamma\right) X_t + B_\infty' \Sigma_t^{\frac{1}{2}} \varepsilon_{t+1}\right),$$

To derive the martingale component of the SDF, let us go back to the SDF itself. Let S and Λ denote the parameters of the real SDF: $S \equiv [s_x,\ s_{gs},\ s_{xs}]'$, $\Lambda \equiv [\lambda_e,\ \lambda_{gw},\ \lambda_{xw}]'$. Then the real SDF is:

$$SDF_{t+1} = \frac{M_{t+1}}{M_t} = \exp\left(\mu_s + S' X_t - \Lambda' \Sigma_t^{\frac{1}{2}} \varepsilon_{t+1} - \lambda_\eta \sigma_{gt} \eta_{t+1}\right).$$

As a result, the martingale component of the SDF is:

$$\begin{aligned} \frac{M_{t+1}^P}{M_t^P} &= \frac{M_{t+1}}{M_t}\left(\frac{M_{t+1}^T}{M_t^T}\right)^{-1} \\ &= \beta^{-1} \exp\left(\mu_s + \left[S' + B_\infty' \left(I - \Gamma\right)\right] X_t - \left(\Lambda' + B_\infty'\right) \Sigma_t^{\frac{1}{2}} \varepsilon_{t+1} - \lambda_\eta \sigma_{gt} \eta_{t+1}\right). \end{aligned}$$

We need to verify that the martingale component is a martingale, i.e that $E_t[M_{t+1}^P / M_t^P] = 1$.

To do this, recall that the bond parameters evolve as:

$$\begin{aligned} B_x(n) &= \rho B_x(n-1) - s_x \\ B_{gs}(n) &= \nu_g B_{gs}(n-1) - s_{gs} - \frac{1}{2}\lambda_\eta^2 \\ B_{xs}(n) &= \nu_x B_{xs}(n-1) - s_{xs} - \frac{1}{2}\left[\lambda_e + B_x(n-1)\right]^2. \end{aligned}$$

Taking limits as $n \to \infty$ leads to:

$$B(\infty)'(I-\Gamma) = -S' + [0, \ -\frac{1}{2}\lambda_\eta^2, \ -\frac{1}{2}\left[\lambda_e + B_x(\infty)\right]^2]'.$$

To check the martingale condition, plug the definition of β in the following expression:

$$\mathbb{E}_t\left[\frac{M_{t+1}^P}{M_t^P}\right] = \beta^{-1}\exp\left(\mu_s + \left[S' + B'_\infty(I-\Gamma)\right]X_t + \frac{1}{2}\left(\Lambda' + B'_\infty\right)\Sigma_t\left(\Lambda + B_\infty\right) + \frac{1}{2}\lambda_\eta^2\sigma_{gt}^2\right).$$

The term in front of X_t is equal to $[0, \ -\frac{1}{2}\lambda_\eta^2, \ -\frac{1}{2}\left[\lambda_e + B_x(\infty)\right]^2]'$. Terms in σ_{gt}^2 and σ_{xt}^2 cancel out. We next plug in the expression for β and check that $E_t[\frac{M_{t+1}^P}{M_t^P}] = 1$.

We now turn to the conditional variances of the log SDF and its dominant pricing and martingale components, $\mathrm{Var}_t[sdf_{t+1}]$, $\mathrm{Var}_t[sdf_{t+1}^T]$ and $\mathrm{Var}_t[sdf_{t+1}^P]$.

$$\begin{aligned} \mathrm{Var}_t[sdf_{t+1}] &= \Lambda'\Sigma_t\Lambda + \lambda_\eta^2\sigma_{gt}^2 \\ \mathrm{Var}_t[sdf_{t+1}^T] &= B'_\infty\Sigma_t B_\infty \\ \mathrm{Var}_t[sdf_{t+1}^P] &= (\Lambda' + B'_\infty)\Sigma_t(\Lambda + B_\infty) + \lambda_\eta^2\sigma_{gt}^2. \end{aligned}$$

The conditional variance ratio $\mathrm{Var}_t[sdf_{t+1}^P]/V_t[sdf_{t+1}]$ equals

$$\frac{V_t[sdf_{t+1}^P]}{V_t[sdf_{t+1}]} = 1 - \frac{-B'_\infty\Sigma_t\Lambda - \frac{1}{2}B'_\infty\Sigma_t B_\infty}{\frac{1}{2}\Lambda'\Sigma_t\Lambda + \frac{1}{2}\lambda_\eta^2\sigma_{gt}^2}$$

The first term in the numerator corresponds to the bond risk premium $(-B'_\infty\Sigma_t\Lambda)$. It includes the Jensen term $(\frac{1}{2}B'_\infty\Sigma_t B_\infty)$. As a result, the numerator corresponds to the bond risk premium without the Jensen term. The denominator corresponds to the maximum risk premium (also without the Jensen term).

Note that the maximal Sharpe ratio in the model is:

$$\begin{aligned} MaxSR_t &= \sigma_t(\log SDF_{t+1}) \\ &= \sqrt{\lambda_e^2\sigma_{xt}^2 + \lambda_{gw}^2\sigma_{gw}^2 + \lambda_{xw}^2\sigma_{xw}^2 + \lambda_\eta^2\sigma_{gt}^2} \\ &= \left(\Lambda'\Sigma_t\Lambda + \lambda_\eta^2\sigma_{gt}^2\right)^{\frac{1}{2}} \end{aligned}$$

Decomposition of the Nominal SDF

The following proposition shows how to decompose the nominal SDF of the long-run risk model into a martingale and a dominant pricing component. To avoid confusion, we use MN to denote the nominal pricing kernel.

Proposition 2. *The stochastic discount factor of the long-run risk model can be decomposed into a* *martingale component and the dominant pricing component:*

$$\begin{aligned}
\frac{MN^T_{t+1}}{MN^T_t} &= \tilde{\beta}\exp\left(-B^{\$\prime}_\infty(I-\tilde{\Gamma})\tilde{X}_t + \widehat{B}^{\$\prime}_\infty\widehat{\Sigma}^{\frac{1}{2}}_t\widehat{\varepsilon}_{t+1}\right),\\
\frac{MN^P_{t+1}}{MN^P_t} &= \tilde{\beta}^{-1}\exp\left(\mu_s-\mu_\pi+\left[\tilde{S}'+B^{\$\prime}_\infty\left(I-\tilde{\Gamma}\right)\right]\tilde{X}_t-(\widehat{\Lambda}^\$+\widehat{B}^\$_\infty)'\widehat{\Sigma}^{\frac{1}{2}}_t\widehat{\varepsilon}_{t+1}\right).
\end{aligned}$$

To show this, we start from the definition of the dominant pricing component of the pricing kernel:

$$MN^T_t = \lim_{n\to\infty}\frac{\tilde{\beta}^{t+n}}{P^{\$b}_t(n)},$$

Recall that log real bond prices are affine in the state vector:

$$\begin{aligned}
p^{\$b}_t(n) &= -B^\$_0(n)-B^\$_x(n)x_t-B^\$_{gs}(n)\left(\sigma^2_{gt}-\sigma^2_g\right)-B^\$_{xs}\left(\sigma^2_{xt}-\sigma^2_x\right)-B^\$_\pi\left(\bar{\pi}_t-\mu_\pi\right)\\
&= -B^\$_0(n)-B^\$(n)'\tilde{X}_t,
\end{aligned}$$

where we define $\tilde{X}_t=[x_t,\sigma^2_{gt}-\sigma^2_g,\sigma^2_{xt}-\sigma^2_x,\bar{\pi}_t-\mu_\pi]$.

We can then write the dominant pricing component of the SDF as:

$$MN^T_t = \lim_{n\to\infty}\tilde{\beta}^{t+n}\exp\left(B^\$_0(n)+B^\$(n)'\tilde{X}_t\right).$$

The constant $\tilde{\beta}$ is chosen in order to satisfy Assumption 1 in Alvarez and Jermann (2005):

$$0<\lim_{n\to\infty}\frac{P^{\$b}_t(n)}{\beta^n}<\infty.$$

Recall that $B^\$_0(n)$ is defined recursively:

$$\begin{aligned}
B^\$_0(n) &= B^\$_0(n-1)-\mu_s+\mu_\pi-\frac{1}{2}\left\{\left[\sigma_\pi+B^\$_\pi(n-1)\sigma_z\right]^2+\left[\lambda_{gw}+B^\$_{gs}(n-1)\right]^2\sigma^2_{gw}\right\}\\
&\quad-\frac{1}{2}\left\{\left[\lambda_{xw}+B^\$_{xs}(n-1)\right]^2\sigma^2_{xw}+\left[\varphi_{\pi g}+\lambda_\eta+\varphi_{zg}B^\$_\pi(n-1)\right]^2\sigma^2_g\right\}\\
&\quad-\frac{1}{2}\left[\varphi_{\pi x}+\lambda_e+B^\$_x(n-1)+\varphi_{zx}B^\$_\pi(n-1)\right]^2\sigma^2_x
\end{aligned}$$

Because of the affine term structure of the model and the stationarity of the state vector $\tilde{X}$, the limit $\lim_{n\to\infty}B^\$(n)\equiv B^\$(\infty)$ is finite. Taking limits on both sides of the equation above leads to:

$$\begin{aligned}
\lim_{n\to\infty}B^\$_0(n)-B^\$_0(n-1) &= -\mu_s+\mu_\pi-\frac{1}{2}\left\{\left[\sigma_\pi+B^\$_\pi(\infty)\sigma_z\right]^2+\left[\lambda_{gw}+B^\$_{gs}(\infty)\right]^2\sigma^2_{gw}\right\}\\
&\quad-\frac{1}{2}\left\{\left[\lambda_{xw}+B^\$_{xs}(\infty)\right]^2\sigma^2_{xw}+\left[\varphi_{\pi g}+\lambda_\eta+\varphi_{zg}B^\$_\pi(\infty)\right]^2\sigma^2_g\right\}\\
&\quad-\frac{1}{2}\left[\varphi_{\pi x}+\lambda_e+B^\$_x(\infty)+\varphi_{zx}B^\$_\pi(\infty)\right]^2\sigma^2_x
\end{aligned}$$

The limit of $B^\$_0(n)-B^\$_0(n-1)$ is finite, so that $B^\$_0(n)$ grows at a linear rate in the limit. We choose the constant $\tilde{\beta}$ to offset the growth in $B_0^{\$}(n)$ as n becomes very large. Setting

$$\begin{aligned}\tilde{\beta} &= \exp\left(\mu_s - \mu_\pi + \frac{1}{2}\left\{\left[\sigma_\pi + B_\pi^{\$}(\infty)\sigma_z\right]^2 + \left[\lambda_{gw} + B_{gs}^{\$}(\infty)\right]^2 \sigma_{gw}^2\right\}\right.\\ &+\frac{1}{2}\left\{\left[\lambda_{xw} + B_{xs}^{\$}(\infty)\right]^2 \sigma_{xw}^2 + \left[\varphi_{\pi g} + \lambda_\eta + \varphi_{zg} B_\pi^{\$}(\infty)\right]^2 \sigma_g^2\right\}\\ &\left.+\frac{1}{2}\left[\varphi_{\pi x} + \lambda_e + B_x^{\$}(\infty) + \varphi_{zx} B_\pi^{\$}(\infty)\right]^2 \sigma_x^2\right)\end{aligned}$$

guarantees that Assumption 1 in Alvarez and Jermann (2005) is satisfied.

We can now write the dominant pricing component of the SDF as:

$$\begin{aligned}\frac{MN_{t+1}^T}{MN_t^T} &= \tilde{\beta}\exp\left(-B_\infty^{\$\prime}(I-\tilde{\Gamma})\tilde{X}_t + B_\pi^{\$}(\infty)\varphi_{zg}\sigma_{gt}\eta_{t+1} + B_\pi^{\$}(\infty)\sigma_z\xi_{t+1}\right.\\ &\left.+[B_x^{\$}(\infty) + B_\pi^{\$}(\infty)\varphi_{zx}]\sigma_{xt}e_{t+1} + B_{gs}^{\$}(\infty)\sigma_{gw}w_{g,t+1} + B_{xs}^{\$}(\infty)\sigma_{xw}w_{x,t+1}\right)\\ &= \tilde{\beta}\exp\left(-B_\infty^{\$\prime}(I-\tilde{\Gamma})\tilde{X}_t + \widehat{B}_\infty^{\$\prime}\widehat{\Sigma}_t^{.5}\widehat{\varepsilon}_{t+1}\right),\end{aligned}$$

where

$$\tilde{\Gamma} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \nu_g & 0 & 0 \\ 0 & 0 & \nu_x & 0 \\ \alpha_x & 0 & 0 & \alpha_\pi \end{bmatrix}$$

To derive the martingale component of the SDF, let us go back to the SDF itself. Let $\tilde{S} \equiv [s_x,\ s_{gs},\ s_{xs}, -1]'$. Then the nominal SDF is:

$$\begin{aligned}\frac{MN_{t+1}}{MN_t} &= \exp\left(\mu_s - \mu_\pi + \tilde{S}'\tilde{X}_t - (\lambda_\eta + \varphi_{\pi g})\sigma_{gt}\eta_{t+1}\right.\\ &\left.-(\lambda_e + \varphi_{\pi x})\sigma_{xt}e_{t+1} - \lambda_{gw}\sigma_{gw}w_{g,t+1} - \lambda_{xw}\sigma_{xw}w_{x,t+1} - \sigma_\pi\xi_{t+1}\right)\\ &= \exp\left(\mu_s - \mu_\pi + \tilde{S}'\tilde{X}_t - \widehat{\Lambda}^{\$\prime}\Sigma_t^{.5}\widehat{\varepsilon}_{t+1}\right)\end{aligned}$$

As a result, the martingale component of the SDF is:

$$\begin{aligned}\frac{MN_{t+1}^P}{MN_t^P} &= \frac{MN_{t+1}}{MN_t}\left(\frac{MN_{t+1}^T}{MN_t^T}\right)^{-1}\\ &= \tilde{\beta}^{-1}\exp\left(\mu_s - \mu_\pi + \left[\tilde{S}' + B_\infty^{\$\prime}\left(I-\tilde{\Gamma}\right)\right]\tilde{X}_t - [\lambda_\eta + \varphi_{\pi g} + B_\pi^{\$}(\infty)\varphi_{zg}]\sigma_{gt}\eta_{t+1}\right.\\ &-[\lambda_e + \varphi_{\pi x} + B_x^{\$}(\infty) + B_\pi^{\$}(\infty)\varphi_{zx}]\sigma_{xt}e_{t+1}\\ &\left.-[\lambda_{gw} + B_{gs}^{\$}(\infty)]\sigma_{gw}w_{g,t+1} - [\lambda_{xw} + B_{xs}^{\$}(\infty)]\sigma_{xw}w_{x,t+1} - [\sigma_\pi + B_\pi^{\$}(\infty)\sigma_z]\xi_{t+1}\right)\\ &= \tilde{\beta}^{-1}\exp\left(\mu_s - \mu_\pi + \left[\tilde{S}' + B_\infty^{\$\prime}\left(I-\tilde{\Gamma}\right)\right]\tilde{X}_t - (\widehat{\Lambda}^{\$} + \widehat{B}_\infty^{\$})'\widehat{\Sigma}_t^{.5}\widehat{\varepsilon}_{t+1}\right).\end{aligned}$$

We need to verify that the martingale component is a martingale, i.e that $E_t[M_{t+1}^P/M_t^P] = 1$. To do so, recall that the bond parameters evolve as:

$$
\begin{aligned}
B_x^{\$}(n) &= \rho B_x^{\$}(n-1) + \alpha_x B_\pi^{\$}(n-1) - s_x \\
B_{gs}^{\$}(n) &= \nu_g B_{gs}^{\$}(n-1) - s_{gs} - \frac{1}{2}\left[\lambda_\eta + \varphi_{\pi g} + \varphi_{zg} B_\pi^{\$}(n-1)\right]^2 \\
B_{xs}^{\$}(n) &= \nu_x B_{xs}^{\$}(n-1) - s_{xs} - \frac{1}{2}\left[\lambda_e + \varphi_{\pi x} + B_x^{\$}(n-1) + \varphi_{zx} B_\pi^{\$}(n-1)\right]^2 \\
B_\pi^{\$}(n) &= \alpha_\pi B_\pi^{\$}(n-1) + 1.
\end{aligned}
$$

Taking limits as $n \to \infty$ leads to:

$$
B(\infty)^{\$\prime}(I-\tilde{\Gamma})+\tilde{S}' = \left[0,\ -\frac{1}{2}\left[\lambda_\eta + \varphi_{\pi g} + \varphi_{zg} B_\pi^{\$}(\infty)\right]^2,\ -\frac{1}{2}\left[\lambda_e + \varphi_{\pi x} + B_x^{\$}(\infty) + \varphi_{zx} B_\pi^{\$}(\infty)\right]^2, 0\right]'.
$$

To check the martingale condition, we plug in the definition of $\tilde{\beta}$ in the expression for the martingale component of the nominal SDF, and use the above equation for $B(\infty)^{\$\prime}(I-\tilde{\Gamma})+\tilde{S}'$. After some algebra, we indeed find that

$$
\mathbb{E}_t\left[\frac{MN_{t+1}^P}{MN_t^P}\right] = 1.
$$

We now turn to the conditional variances of the log SDF and its dominant pricing and martingale components, $\mathrm{Var}_t[sdf_{t+1}^{\$}]$, $\mathrm{Var}_t[sdf_{t+1}^{\$,T}]$ and $\mathrm{Var}_t[sdf_{t+1}^{\$,P}]$.

$$
\begin{aligned}
\mathrm{Var}_t[sdf_{t+1}^{\$}] &= (\lambda_\eta + \varphi_{\pi g})^2 \sigma_{gt}^2 + (\lambda_e + \varphi_{\pi x})^2 \sigma_{xt}^2 + \lambda_{gw}^2 \sigma_{gw}^2 + \lambda_{xw}^2 \sigma_{xw}^2 + \sigma_\pi^2 \\
&= \widehat{\Lambda}_\infty^{\$\prime} \widehat{\Sigma}_t \widehat{\Lambda}_\infty^{\$} \\
\mathrm{Var}_t[sdf_{t+1}^{\$,T}] &= B_\pi^{\$}(\infty)^2 \varphi_{zg}^2 \sigma_{gt}^2 + [B_x^{\$}(\infty) + B_\pi^{\$}(\infty)\varphi_{zg}]^2 \sigma_{xt}^2 \\
&\quad + B_{gs}^{\$}(\infty)^2 \sigma_{gw}^2 + B_{xs}^{\$}(\infty)^2 \sigma_{xw}^2 + B_\pi^{\$}(\infty)^2 \sigma_z^2 \\
&= \widehat{B}_\infty^{\$\prime} \widehat{\Sigma}_t \widehat{B}_\infty^{\$} \\
\mathrm{Var}_t[sdf_{t+1}^{\$,P}] &= [\lambda_\eta + \varphi_{\pi g} + B_\pi^{\$}(\infty)\varphi_{zg}]^2 \sigma_{gt}^2 + [\lambda_e + \varphi_{\pi x} + B_x^{\$}(\infty) + B_\pi^{\$}(\infty)\varphi_{zx}]^2 \sigma_{xt}^2 \\
&\quad + [\lambda_{gw} + B_{gs}^{\$}(\infty)]^2 \sigma_{gw}^2 + [\lambda_{xw} + B_{xs}^{\$}(\infty)]^2 \sigma_{xw}^2 + [\sigma_\pi + B_\pi^{\$}(\infty)\sigma_z]^2 \\
&= (\widehat{\Lambda}^{\$} + \widehat{B}_\infty^{\$})' \widehat{\Sigma}_t (\widehat{\Lambda}^{\$} + \widehat{B}_\infty^{\$})
\end{aligned}
$$

The conditional variance ratio $\mathrm{Var}_t[sdf_{t+1}^P]/V_t[sdf_{t+1}]$ equals

$$
\frac{V_t[sdf_{t+1}^{\$,P}]}{V_t[sdf_{t+1}^{\$}]} = 1 - \frac{-\widehat{B}_\infty^{\$\prime}\widehat{\Sigma}_t\widehat{\Lambda}^{\$} - \frac{1}{2}\widehat{B}_\infty^{\$\prime}\widehat{\Sigma}_t\widehat{B}_\infty^{\$}}{\frac{1}{2}\widehat{\Lambda}_\infty^{\$\prime}\widehat{\Sigma}_t\widehat{\Lambda}_\infty^{\$}}
$$

The first term in the numerator corresponds to the nominal bond risk premium of an infinite horizon bond, which includes a Jensen term. The second term in the numerator is that Jensen term. As a result, the numerator corresponds to the nominal bond risk premium without the Jensen term. The denominator corresponds to the maximum nominal risk premium, also without the Jensen term.

Calibration

Table A.1 reports the model parameter values we use; they are the ones proposed in Bansal and Shaliastovich (2008). Table A.2 reports the model loadings on state variables.

The model is simulated for 60,000 months and aggregated up to quarterly frequency for comparison with our quarterly data. In the simulation, negative values for $\sigma^2_{g,t+1}$ and $\sigma^2_{x,t+1}$ are replaced by very small positive values in simulation.

Table A.4 reports the mean, standard deviation and autocorrelation of the stochastic discount factor (SDF), its martingale (SDF^P) and dominant pricing (SDF^T) components, the conditional variance ratio ω, the maximum risk premium without Jensen adjustment ($Max\ RP$) and the risk premium of an infinite maturity bond without Jensen adjustment ($BRP(\infty)$). Table A.3 reports the mean and standard deviations of the real and nominal yields and bond risk premia in the model and compare them to the same moments in the actual nominal data. Table A.5 reports moments of quarterly inflation in the model and in the data. Quarterly inflation is obtained as the sum of three consecutive monthly inflation rates.

The Bansal and Shaliastovich (2008) calibration generates an annual consumption growth rate of 2.12 percent with a standard deviation of 3.52 percent. It generates an annual inflation rate of 3.52 percent with a standard deviation of 2.49 percent.

Robustness Checks

As robustness checks, we considered both changes on the real and on the nominal side of the economy.

On the real side, we conduct two experiments. First, we find that a slight decrease in the persistence of the long-run component in consumption growth ρ_x could decrease the long-horizon consumption variance ratios and the real variance ratio significantly, and increase the long term real yield from negative to positive values. As a result, the model would need to rely less on a large inflation risk premium in order to match the nominal yield curve, thus lowering the variation of M_t^T in the nominal pricing kernel. However, if all the other parameters are maintained at their previous values, the model would then imply too much volatility of the wealth-consumption ratio and an equity risk premium that is much too low. Second, we shut down the heteroscedasticity in consumption growth by calibrating σ_{xw} and σ_{gw} to very low values. We keep all the other parameters at their previous values. In this case, the real and nominal conditional variance ratios are respectively 1.20 and 0.63 (see Table A.6). They are closer to 1, but equity and bond risk premia are constant.

On the nominal side, we first check the robustness of our results to a slightly different calibration of the inflation dynamics. First, we vary each inflation parameter independently in either direction. We report in Table A.7 the mean maximum risk premium (MRP), the mean bond risk premium BRP_J (including the Jensen term) and the mean variance ratio ω for different values of the inflation parameters. We simulate the model for a low and a high value of each parameter (25 percent above and below the benchmark value reported in Table A.1). The only exception is the parameter α_π, which we cannot increase by 25 percent without running into stationarity issues. The high value is a 10 percent increase for that parameter. We find that ω_t only changes noticeably with α_x and α_π.

To further investigate the sensitivity to these two parameters, Figure A.1 in the appendix plots ω_t (left axis) and the five-year nominal bond risk premium (right axis) against α_x (horizontal axis). As we vary α_x away from its benchmark value of -0.35, we simultaneously vary α_π to match the observed persistence of quarterly inflation. We also choose μ_π and σ_π to keep the mean and volatility of inflation at their benchmark values. The figure shows that ω_t is essentially unchanged over a wide range of values for α_x and never comes close to the desired value of one.

Next, we consider a calibration that matches the observed mean, variance, and persistence of inflation, the 5-1-year yield spread, and the persistence of the 5-year nominal bond risk premium. This calibration delivers a nominal variance ratio ω_t that is much too high.

Finally, we ask whether we can find inflation parameters that deliver a nominal variance ratio of 1. We find that we can, while matching the mean inflation, the slope of the nominal term structure, and the persistence of the nominal BRP, but inflation ends up being 2.5 times too volatile and not persistent enough.

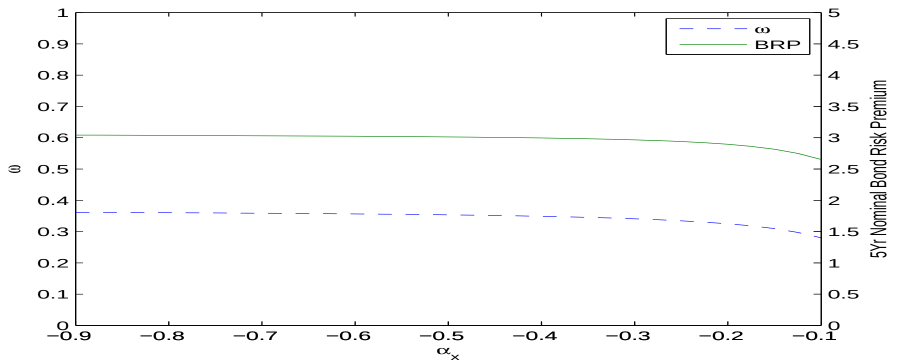

FIGURE A.1. VARIANCE RATIO AND NOMINAL BOND RISK PREMIUM: SENSITIVITY ANALYSIS

The figure plots the conditional variance ratio ω_t (against the left axis) and the five-year nominal bond risk premium (against the right axis) for different values of the parameter α_x (on the horizontal axis). As we vary α_x away from its benchmark value of -0.35, we simultaneously vary α_π to match the observed persistence of quarterly inflation. We also choose μ_π and σ_π to keep the mean and volatility of inflation at their benchmark values.

Empirical Variance Ratios

Alvarez and Jermann (2005) show that – assuming that the process X_t satisfies the same regularity conditions as above and that X_{t+1}/X_t is strictly stationary and $\lim_{k\to\infty}\frac{1}{k}\mathrm{Var}(\mathbb{E}_{t+k}[X_t]) = 0$ – then

$$\mathrm{Var}\left(\frac{X^P_{t+1}}{X^P_t}\right) = \lim_{k\to\infty}\frac{1}{k}\mathrm{Var}\left(\frac{X_{t+k}}{X_t}\right),$$

Note that the entropy measure used by Alvarez and Jermann (2005) collapses to the half-variance since all variables are conditionally normal. This result implies that long-horizon variance ratios are informative about the variance of the martingale component. We now turn to the empirical variance ratios of the two components of the SDF, e.g consumption growth and the wealth consumption ratio.

If changes in log consumption or changes in the log wealth-consumption ratio are i.i.d, then the variance of long-horizon changes in each variable should grow with the horizon. We compute variance ratios at horizon h as $VR(h) = \mathrm{Var}[\sum_{j=0}^{h}\Delta x_{t+j}]/[h\mathrm{Var}(\Delta x_t)]$, for $x = c$ and $x = wc$. We simulate the model at monthly frequency. Table A.1 in the appendix reports the model parameters. We start from the parameter values in Bansal and Shaliastovich (2008).

Figure A.2 reports these variance ratios for consumption growth, the change in the wealth-consumption ratio, and inflation. The left panel corresponds to actual data; the right panel uses simulated series. Let us first focus on actual data. The variance ratio of the wealth-consumption ratio clearly decreases with the horizon. It is below 0.6 within five years. Consumption growth exhibits a very different pattern: its variance ratio first increases for horizons up to 5 years; it then decreases, but even after 15 years, the variance ratio is still above one. As a result, there is strong evidence of persistence and mean-reversion in the wealth-consumption ratio, but not in consumption growth.

Let us now turn to simulated data. The variance ratios of the wealth-consumption ratio are in line with the data. They decrease linearly with the horizon, from 1 to approximately 0.5 at the 30-year horizon. In the data, the variance ratio decreases from 1 to 0.6. Consumption growth, however, exhibits a very different pattern. At long horizons, it displays more persistence in the model than in the data. The bottom panel shows that the inflation persistence is similar in model and data, with a slight divergence maybe at longer horizons.

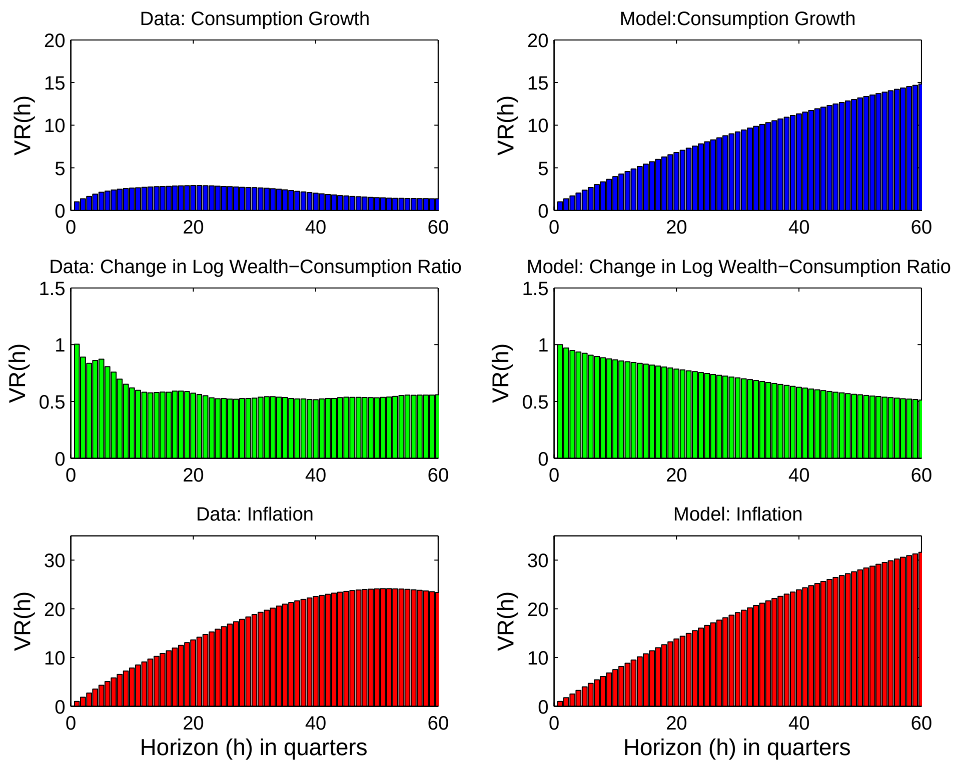

FIGURE A.2. VARIANCE RATIOS FOR CONSUMPTION GROWTH, THE CHANGE IN THE LOG WEALTH CONSUMPTION RATIO AND INFLATION IN THE DATA AND IN THE MODEL.

The variance ratio of Δx_t is equal to $VR(h) = Var[\sum_{j=0}^{h} \Delta x_{t+j}]/[hVar(\Delta x_t)]$. The left panel corresponds to actual data. The right panel corresponds to simulated data. Data are quarterly. Actual data come from Lustig et al. (2009). The sample is 1952:II-2008:IV.

Table A.1—Model Parameter Values

Parameter		$BS(2008)$
Preference Parameters:		
Subjective discount factor	δ	0.9987
Intertemporal elasticity of substitution	ψ	1.5
Risk aversion coefficient	γ	8
Consumption Growth Parameters:		
Mean of consumption growth	μ_g	0.0016
Long-run risk persistence	ρ	0.991
News volatility level	σ_g	0.004
News volatility persistence	ν_g	0.85
News volatility of volatility	σ_{gw}	$1.15e-6$
Long run-risk volatility level	σ_x	$0.004\sigma_g$
Long run-risk volatility persistence	ν_x	0.996
Long run-risk volatility of volatility	σ_{xw}	$0.06^2\sigma_{gw}$
Dividend Growth Parameters:		
Mean of dividend growth	μ_d	0.0015
Dividend leverage	ϕ_x	1.5
Dividend loading on news volatility	ϕ_{gs}	0
Dividend loading on long-run risk volatility	ϕ_{xs}	0
Volatility loading of dividend growth	φ_d	6.0
Correlation of consumption and dividend news	τ_{gd}	0.1
Inflation Parameters:		
Mean of inflation rate	μ_π	0.0032
Inflation leverage on news	$\varphi_{\pi g}$	0
Inflation leverage on long-run news	$\varphi_{\pi x}$	-2.0
Inflation shock volatility	σ_π	0.0035
Expected inflation AR coefficient	α_π	0.83
Expected inflation loading on long-run risk	α_x	-0.35
Expected inflation leverage on news	φ_{zg}	0
Expected inflation leverage on long-run news	φ_{zx}	-1.0
Expected inflation shock volatility	σ_z	$4.0e-6$

This table reports the calibrated parameters values for our simulation. We take them from Table IV and Table C.I in Bansal and Shaliastovich (2008).

TABLE A.2—MODEL LOADINGS ON STATE VARIABLES

	constant	x	$\sigma_{gt}^2 - \sigma_g^2$	$\sigma_{xt}^2 - \sigma_x^2$
wc	μ_{wc}	$W_x = \frac{1-\frac{1}{\psi}}{\kappa_1^c - \rho}$	$W_{gs} = \frac{\theta\left(\frac{1}{\psi}-1\right)^2}{2\left(\kappa_1^c - \nu_g\right)}$	$W_{xs} = \frac{\theta}{2\left(\nu_x - \kappa_1^c\right)}\left(\frac{\frac{1}{\psi}-1}{\kappa_1^c - \rho}\right)^2$
	6.4	31	-7.7	-1.8×10^6
pd	μ_{pd}	$D_x = \frac{\phi_x - \frac{1}{\psi}}{\kappa_1 - \rho}$	$D_{gs} = \frac{\frac{1}{2}\left[\gamma^2 - 2\gamma\varphi_d\tau_{dg} + \varphi_d^2\right]}{\kappa_1 - \nu_g} + \frac{\frac{1}{2}\left[\left(\gamma - \frac{1}{\psi}\right)(1-\gamma) + \phi_{gs}\right]}{\kappa_1 - \nu_g}$	$D_{xs} = \frac{\frac{1}{2}\left\{\left[\frac{\gamma - \frac{1}{\psi}}{\rho - \kappa_1^c} + \frac{\frac{1}{\psi} - \phi_x}{\rho - \kappa_1}\right]^2 + \frac{(1-\gamma)\left(\gamma - \frac{1}{\psi}\right)}{\left(\rho - \kappa_1^c\right)^2}\right\}}{\kappa_1 - \nu_x} + \frac{\phi_{xs}}{\kappa_1 - \nu_x}$
	5.6	66	1.3×10^2	-4.3×10^6
ERP	$(1-\theta)\,W_{gs}D_{gs}\sigma_{gw}^2 + (1-\theta)\,W_{xs}D_{xs}\sigma_{xw}^2 + \varphi_d\tau_{gd}\gamma\sigma_g^2 + D_x\,(\theta-1)\,W_x\sigma_x^2$		$G_{gs} = \gamma\varphi_d\tau_{gd}$	$G_{xs} = (1-\theta)\,W_x D_x$
	0.003	0	4.8	4.6×10^4
BRP (Real)	$(\theta-1)\,W_{gs}B_{gs}(n-1)\sigma_{gw}^2 + (\theta-1)\,W_{xs}B_{xs}(n-1)\sigma_{xw}^2 + (\theta-1)\,W_x B_x(n-1)\sigma_x^2$	$F_x(n) = 0$	$F_{gs}(n) = 0$	$F_{xs}(n) = (\theta-1)\,W_x B_x(n-1)$
	-0.0014	0	0	-2.1×10^4
BRP (Nominal)	$(\theta-1)\,W_{gs}B_{gs}^{\$}(n-1)\sigma_{gw}^2 + (\theta-1)\,W_{xs}B_{xs}^{\$}(n-1)\sigma_{xw}^2 + \sigma_\pi\sigma_z B_\pi^{\$}(n-1) - (\gamma+\varphi_{\pi g})\,\varphi_{zg}B_\pi^{\$}(n-1)\sigma_g^2 + ((\theta-1)\,W_x - \varphi_{\pi x})\left(B_x^{\$}(n-1) + B_\pi^{\$}(n-1)\varphi_{zx}\right)\sigma_x^2$	$F_x^{\$}(n) = 0$	$F_{gs}^{\$}(n) = -(\gamma+\varphi_{\pi g}) \times \varphi_{zg}B_\pi^{\$}(n-1)$	$F_{xs}^{\$}(n) = [(\theta-1)\,W_x - \varphi_{\pi x}] \times \left(B_x^{\$}(n-1) + B_\pi^{\$}(n-1)\varphi_{zx}\right)$
	0.0015	0	-0	4.3×10^4

This table reports the model loadings on a constant and the state variables. We consider the log wealth-consumption ratio (wc), the log price-dividend ratio (pd), the equity risk premium (ERP), the real and nominal bond risk premia (BRP) at the n-year horizon.

Table A.3—Real and Nominal Yield Curves

Maturity	1	2	3	4	5	30	200
			Nominal Bonds - Data				
Mean Yields	5.33	5.52	5.69	5.80	5.89		
Std	2.81	2.77	2.70	2.69	2.65		
			Nominal Bonds - Model				
Mean Yields	5.19	5.46	5.75	6.06	6.38	12.82	20.02
Std	2.92	2.79	2.65	2.53	2.43	1.60	0.36
Mean BRP	0.33	0.93	1.59	2.27	2.97	16.81	24.43
Std	0.07	0.18	0.28	0.38	0.46	1.13	1.18
			Real Bonds - Model				
Mean Yields	1.26	1.05	0.83	0.61	0.39	−4.71	−13.63
Std	1.39	1.35	1.32	1.30	1.29	1.10	0.25
Mean BRP	−0.39	−0.83	−1.28	−1.73	−2.19	−11.14	−16.21
Std	0.05	0.10	0.15	0.19	0.23	0.52	0.55

The top panel reports the mean and standard deviation of nominal bond yields in the Fama-Bliss data. The data are for 1952 until 2008, and only bond yields of maturities one through five years are available. The maturity is in years. The yields and returns are annualized and reported in percentage points. The middle panel does the same for nominal bond yields for a 60,000 month simulation of the LRR model. It also reports the mean and standard deviation of the nominal bond risk premia. The bottom panel reports the same model-implied moments for real bonds.

Table A.4—Conditional Variance Ratio

	Mean	*Std*	*AR*(1)
	Nominal SDF		
$SDF^{\$}$	0.99	0.23	−0.01
$SDF^{\$,P}$	1.00	0.14	−0.01
$SDF^{\$,T}$	0.98	0.10	−0.01
$\omega_t^{\$}$	**0.37**	0.06	0.98
Max RP	30.62	2.52	0.99
$BRP(\infty)$	18.72	1.04	0.99
	Real SDF		
SDF	1.00	0.23	−0.01
SDF^{P}	1.00	0.30	−0.01
SDF^{T}	1.02	0.07	−0.01
ω_t	**1.65**	0.11	0.98
Max RP	30.69	2.54	0.99
$BRP(\infty)$	−19.05	0.58	0.99

This table reports the mean, standard deviation and autocorrelation of the stochastic discount factor (SDF), its martingale (SDF^P) and dominant pricing (SDF^T) components, the conditional variance ratio ω, the maximum risk premium without Jensen adjustment (*Max RP*) and the risk premium of an infinite maturity bond without Jensen adjustment ($BRP(\infty)$). The table reports the autocorrelation of each monthly variable in logs. The top (bottom) panel focuses on the nominal (real) stochastic discount factor. The numbers are computed from a 60,000 month simulation.

Table A.5—Inflation: Model vs Data

	Data			Model		
	Mean	*Std*	*AR*(1)	*Mean*	*Std*	*AR*(1)
π_t	0.85	0.62	0.86	0.88	1.25	0.76

This table reports the mean, standard deviation and autocorrelation of the quarterly inflation rate. The left panel corresponds to actual data, from Lustig et al. (2009). The right panel corresponds to simulated data, from the model. The mean and standard deviation are in percentage.

Table A.6—Conditional Variance Ratio: No Heteroscedasticity

	Mean	*Std*	*AR*(1)
	Nominal SDF		
$SDF^{\$}$	1.00	0.12	−0.01
$SDF^{\$,P}$	1.00	0.13	−0.01
$SDF^{\$,T}$	1.00	0.01	−0.01
$\omega_t^{\$}$	1.20	0.00	1.00
$Max\ RP$	8.74	0.00	1.00
$BRP(\infty)$	−1.74	0.00	1.00
	Real SDF		
SDF	0.99	0.12	−0.01
SDF^P	1.00	0.10	−0.01
SDF^T	0.99	0.03	−0.01
ω_t	0.63	0.00	1.00
$Max\ RP$	8.70	0.00	1.00
$BRP(\infty)$	3.18	0.00	1.00

This table reports the mean, standard deviation and autocorrelation of the stochastic discount factor (SDF), its martingale (SDF^P) and dominant pricing (SDF^T) components, the conditional variance ratio ω, the maximum risk premium without Jensen adjustment ($Max\ RP$) and the risk premium of an infinite maturity bond without Jensen adjustment ($BRP(\infty)$). The table reports the autocorrelation of each monthly variable in logs. The top (bottom) panel focuses on the nominal (real) stochastic discount factor. The numbers are computed from a 60,000 month simulation.

Table A.7—Sensitivity to Inflation Specification

	Max RP		$BRP(\infty)$		ω	
	Low	*High*	*Low*	*High*	*Low*	*High*
μ_p	30.62	30.62	18.72	18.72	0.37	0.37
$\varphi_{\pi g}$	30.62	30.62	18.72	18.72	0.37	0.37
$\varphi_{\pi x}$	30.64	30.60	18.70	18.74	0.37	0.37
σ_π	30.62	30.62	18.72	18.72	0.37	0.37
α_π	30.62	30.62	5.61	26.42	0.81	0.10
α_x	30.62	30.62	14.54	21.84	0.51	0.27
φ_{zg}	30.62	30.62	18.72	18.72	0.37	0.37
φ_{zx}	30.62	30.62	18.63	18.81	0.38	0.37
σ_z	30.62	30.62	18.72	18.72	0.37	0.37

This table reports the mean maximum risk premium (*Max RP*) , the mean bond risk premium $BRP(\infty)$ (including the Jensen term) and the mean variance ratio ω. We vary one parameter at a time, and simulate the model for a low and a high value of each parameter (25 percent above and below the benchmark value reported in the first column of Table A.1). The only exception is the parameter α_π, which we cannot increase by 25 percent without running into stationarity issues. The high value is a 10 percent increase for that parameter.