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MATURITY, INDEBTEDNESS, AND DEFAULT RISK

Satyajit Chatterjee
Federal Reserve Bank of Philadelphia

Burcu Eyigungor
Koç University

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Satyajit Chatterjee

Federal Reserve Bank of Philadelphia

Burcu Eyigungor

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Abstract

We present a novel and tractable model of long-term sovereign debt. We make two sets of contributions. First, on the substantive side, using Argentina as a test case we show that unlike one-period debt models, our model of long-term sovereign debt is capable of accounting for the average spread, the average default frequency, and the average debt-to-output ratio of Argentina over the 1991-2001 period without any deterioration in the model's ability to account for Argentina's cyclical facts. Using our calibrated model we determine what Argentina's debt, default frequency and welfare would have been if Argentina had issued only short-term debt. Second, on the methodological side, we advance the theory of sovereign debt begun in Eaton and Gersovitz (1981) by establishing the existence of an equilibrium pricing function for long-term sovereign debt and by providing a fairly complete set of characterization results regarding equilibrium default and borrowing behavior. In addition, we identify and solve a computational problem associated with pricing long-term unsecured debt that stems from nonconvexities introduced by the possibility of default.

1   Introduction

We study an equilibrium model of unsecured debt and default in which borrowers issue long-term debt. The existing literature on this subject – both the consumer debt and sovereign debt parts – has mostly considered one-period debt. In reality, both consumers and countries can and do borrow long term. The maturity structure we introduce in this paper brings equilibrium models of unsecured debt and default closer to the maturity structures observed in the real world.

Our motivation for this extension derives from a deficiency of one-period unsecured debt models, a deficiency most clearly evident in the extant quantitative sovereign-debt literature. This literature has achieved notable success in accounting for the key cyclical patterns in output, consumption, trade balance, and interest rates for emerging market economies that borrow in international credit markets. As demonstrated in Aguiar and Gopinath (2006) and Arellano (2008), a model of unsecured sovereign debt of the type developed in Eaton and Gersovitz (1981) can explain why such emerging market economies have consumption volatility almost as high as output volatility and why their trade balance moves countercyclically – two facts that appear anomalous relative to the experience of developed economies (Neumeyer and Perri (2005)). However, these efforts also have some notable misses. Although they succeed in accounting for second moments (cyclical patterns), they do not account for first moments of the key variables; in particular, they do not account for the generally high level of indebtedness of emerging market economies or the high spreads on their sovereign debt.

We make two contributions in this paper, one substantive and the other methodological. The substantive contribution is to show, using Argentina as a test case, that incorporating long-term bonds of median maturity observed in the data allows the model to match the average debt and spread levels for Argentina (over the 1991-2001 period). Furthermore, although no attempt is made to target cyclical properties of the Argentine data, the model has the same level of success in accounting for the cyclical facts as Arellano (2008) and Aguiar and Gopinath (2006). We show that it is the lengthened maturity that makes this improvement in model performance possible: specifically, a one-period debt model cannot match average spreads and debt levels without generating counterfactually high volatility of consumption and trade balance.

There is a simple intuition for our findings. With the long-maturity bond, fluctuations in the default spread caused by fluctuations in output do not strongly constrain the ability of the sovereign to service debt. Even if the debt level is very high (and it is about 100 percent of quarterly output in the case of Argentina in the 1990s), not all of this debt is due for repayment in any given period. Thus, the amount of new debt that needs to be issued at potentially high interest rates is relatively small. In contrast, when spreads rise on one-period debt either very large amounts of new debt at high interest rates have to be issued – implying low level of consumption next period – or consumption needs to be drastically reduced in the current period. In this situation the sovereign is willing to acquire the observed high level of debt only if it is very impatient and faces a very high cost of default. However, the implied equilibrium volatility of consumption is almost twice as high, and that of the trade balance is almost four times as high, compared to the data.

In contrast, long-term debt gives the sovereign both the opportunity and the incentive to acquire substantial amounts of debt. Because all debt does not have to be refinanced each period, the incentive to default on additional unit of long-term debt is much lower relative to short-term debt and this reduces the elasticity of spreads with respect to debt. Thus, with long-term debt, the sovereign has the opportunity to borrow additional amounts at interest rates that rise relatively slowly with debt. In addition, the sovereign has the incentive to acquire more debt as well because the sovereign does not internalize the decline in the value of outstanding debt caused by additional borrowing. This is the well-known debt-dilution effect. Once debt has been issued, the decline in its value from additional borrowing in the future is of no concern to the sovereign. In contrast, with one-period debt, the sovereign recognizes that additional debt lowers the price (raises the interest rate) on all debt issued. Both factors – the low elasticity of spreads as well as the debt-dilution effect – contributes the sovereign's acquiring a large amount of debt. With one-period debt, these contributing factors are absent.

Our quantitative results also shed light on an important welfare issue. Given the ease of rolling over (or servicing) long-term debt, a common intuition in the literature is that borrowing long-term is beneficial to a sovereign facing default risk. But when we compare the welfare implications of long-term versus short-term debt for parameter values that account for average Argentine debt and spreads, we find that Argentina is better off with short-term debt. The main reason is that,

keeping all model parameters constant (other than maturity), the sovereign borrows much more and defaults much more frequently with long-term bonds. This is due to the two reasons noted above: the low elasticity of spreads with respect to debt and the incentive to acquire debt because of the debt-dilution effect. Although we might expect this decrease in welfare to be counterbalanced by the ease of rolling over long-term debt, this effect does not turn out to be dominant with our calibrated parameters, mainly because, with short-term debt, the country never ends up borrowing high enough (and have high spreads) to make this repayment flexibility afforded by long-term debt a valuable option. We suspect that long-term debt might look attractive if we take into account the considerable transactions costs of participating in the international capital market or where there are issues of multiple equilibria and long-term debt might prevent the "run" equilibrium where each small foreign lender refuses to issue new debt because each expects the other lenders to refuse as well (as in Broner, Lorenzoni and Schmukler (2007) and Cole and Kehoe (2000)).

Turning next to our methodological contributions, there are two. Long-term debt introduces new theoretical and computational issues, which we address. On the theoretical side, the pricing of long-term debt depends not only on the probability of default in the following period but also on the borrowing behavior of the sovereign in the event of repayment. This is different from the case of one-period debt where in the event of repayment the pay-off is certain and given. Thus it is not apparent that the price of debt is decreasing in the amount of debt issued. We establish that the equilibrium pricing function for long-term debt must have this property. In the process, we characterize the sovereign's default and borrowing behavior with respect to the level of debt. These characterization results provide intuition on how a model with long-term debt works. We also establish the existence of an equilibrium pricing function, extending the existence result for one-period debt in Chatterjee et al. (2007) to the case of long-term debt with a constant risk-free rate.[1]

On the computational side there are new issues that also stem from the fact that the price of debt depends on the sovereign's borrowing behavior in the event of repayment. Although our way of modeling long-term debt keeps the state space low, the nonconvexities introduced by the possibility

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[1]Consumer debt is not a focus of this paper. It is worth noting, however, that the model of unsecured consumer debt introduced in Athreya (2002) and extended and analyzed further in Chatterjee et al. (2007), Livshits, MacGee and Tertilt (2007), and others, bears a strong resemblance to models of sovereign debt *a la* Eaton and Gersovitz.

of default can result in cycles in the sovereign's borrowing behavior and the model solution need not converge with standard grid-based methods. This problem is solved in a manner explained in the computation section of the paper. The solution makes use of some additional characterization results provided in the theoretical section. Also, we describe a novel algorithm for computing the optimal borrowing decision rule in the presence of nonconvexities.

There is a related literature on sovereign debt that attempts to go beyond one-period debt. Hatchondo and Martinez (2008) introduce long-duration bonds into an Arellano-style sovereign debt model. Their motivation is to improve upon the volatility of spreads predicted by Arellano, which is arguably too low relative to the data. In contrast, the quantitative focus of our paper is on first moments of debt and spreads. There is also a difference in methodology. As we explain later in the paper, lengthening the maturity of bonds in a standard way leads to a computationally intractable model. Therefore, some "trick" is needed to analyze long-term bonds. The strategy relies on making the long-term bond "memoryless" so that it is not necessary to keep track of the date-of-issue of the bond. Our strategy is more general and easier to apply to the data than the one applied in Hatchondo and Martinez. Bi (2006) focuses on maturity choice in a model of one- and two-period debt. Arellano and Ramnarayan (2008) also focus on maturity choice but use the long-duration model proposed in Hatchondo and Martinez.

The paper is organized as follows. In section 2 we briefly discuss why incorporating long-term debt in the standard way into models of unsecured debt can lead to computationally intractable models and then describe our strategy for circumventing this problem. In section 3 we introduce the sovereign debt environment we analyze. In section 4 we present characterization results for the equilibrium pricing function and the default and debt decision rules. These results provide intuition on the working of the model and aid in its computation. Section 5 discusses computational issues. As mentioned above, grid-based algorithms for computing equilibrium models of default encounter convergence problems. These difficulties, and the way they are addressed in this paper, are explained in section 5. Section 6 presents the results of incorporating long-term debt for Argentina and explains how incorporating long-term debt helps improves the ability of this class of models to explain the emerging market facts. Section 7 concludes.

2 Modeling Long-Term Debt

A natural way to introduce long-term debt is to assume that debt issued in period t is due for repayment in period $t + T$. Since new debt can be issued each period, this means that the issuer's state vector contains the vector $(b_0, b_1, b_2, \ldots, b_{T-1})$ where b_τ is the quantity of bonds due for repayment τ periods in the future. Then, the probability of default on a bond due for repayment in period τ is the sum of the probability of default in the current period plus the probability of repayment in the current period but default in the next period plus the probability of repayment in the next two periods followed by default in the third period and so on all the way to period τ in the future. Even for modest values of T (such as 3 or 4), these calculations can become quite demanding because we will have at least T state variables, each of which can potentially take many, many values.

Our approach is to simplify the maturity structure of debt in a way that calculation of default probabilities from the individual's decision problem becomes easier. We analyze long-term debt contracts that mature probabilistically. Specifically, each unit of outstanding debt matures next period with probability λ. If the unit does not mature – which happens with probability $1 - \lambda$ – it gives out a coupon payment z. Observe that if $\lambda = 1$, then the bond is a one-period discount bond, and if $z > 0$ and $\lambda = 0$ then the bond is a consol promising to pay z units each period. For intermediate values of λ, we have a bond that matures, on average, in $1/\lambda$ periods.

Why is this probabilistic maturity structure easier to analyze? The benefit comes from the "memory-less" nature of the bond. Going forward, a unit bond of type (z, λ) issued $k \geq 1$ periods in the past has exactly the same payoff structure as another (z, λ) unit bond issued $k' > k$ periods in the past. This means we can aggregate all outstanding (z, λ) unit bonds regardless of the date of issue and thereby cut down on the number of state variables relevant to the individual's decision problem. This reduction in turn reduces the computational burden of computing default probabilities.

In what follows we will assume that unit bonds are infinitesimally small – meaning that if b unit bonds of type (z, λ) are outstanding at the start of next period, the issuer's coupon obligations next period will be $z \cdot (1 - \lambda)b$ for sure and her payment-of-principal obligations will be λb for sure.

And if no new bonds are issued or no outstanding bonds redeemed next period, $(1-\lambda)b$ unit bonds will be outstanding for sure at the start of the following period.

Hatchondo and Martinez (2008) use a similar trick of rendering outstanding obligations "memory-less" in order to analyze sovereign debt that lasts more than one (model) period. In their setup all bonds last forever (consols) but each pays a geometrically declining sequence of coupon payments. Thus, a bond issued in the current period promises to pay the sequence $\{1, \delta, \delta^2, \delta^3, \ldots\}$. One period later, the promised sequence of payments is $\{\delta, \delta^2, \delta^3, \delta^4, \ldots\}$, which is no different than the promised sequence on δ units of the bond issued last period. It is as if $(1-\delta)$ fraction of outstanding bonds mature each period and there is a coupon payment of 1 on every outstanding bond, including the ones that mature.

3 Environment

3.1 Preferences and Endowments

Time is discrete and denoted $t \in \{0, 1, 2, \ldots\}$. The sovereign receives a strictly positive endowment y_t each period. The stochastic evolution of y_t is governed by a finite-state Markov chain with state space $Y \subset \mathbb{R}_{++}$ and transition law $\Pr\{y_{t+1} = y' | y_t = y\} = F(y, y')$, y and $y' \in Y$.

The sovereign maximizes expected utility over consumption sequences, where the utility from any given sequence c_t is given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t - m_t),\ \beta < 0 \tag{1}$$

The momentary utility function $u(\cdot) : [0, \infty) \to \mathbb{R}$ is continuous, strictly increasing, and strictly concave. The term $m_t \in M = [0, \bar{m}]$ is a minimum consumption requirement drawn independently each period from a probability distribution with continuous cdf $G(m)$.

The presence of the minimum consumption shock deserves some comment. The shock is included to make robust computation of the model possible. It is important that the shock be drawn from a continuous distribution and that it be i.i.d. The role played by these two assumptions in the

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computation of the equilibrium is discussed later. In the quantitative application, volatility of m is taken as low as possible so that it does not affect quantitative results significantly, but convergence of the model solution is ensured. It is worth pointing out that the minimum consumption requirement setup is isomorphic to a setup where there is no minimum consumption requirement but there are temporary i.i.d. shocks to endowments.[2]

3.2   Option to Default and the Market Arrangement

The sovereign can borrow in the international credit market and has the option to default on a loan. If the sovereign defaults, it cannot borrow in the period of default and, in the future, it is excluded from the international credit market for a random length of time. Specifically, upon default the sovereign is permitted to borrow in the future with probability $0 < \xi < 1$ and once it is permitted to borrow it can borrow in all subsequent periods until it defaults again. During the periods in which the sovereign is excluded from borrowing – including the period of default – it loses some amount $\phi(y) > 0$ of its output y. In addition, in the period of default, the sovereign's minimum consumption requirement rises to its maximum value $\bar{m}$. We will assume that $y - \phi(y) - \bar{m} > 0$ for all y, which ensures that discretionary output (total output less minimum consumption) is always strictly positive under all circumstances.

There is a single type of bond of type (z, λ) available in this economy. We will assume that lenders are risk-neutral and the market for sovereign debt is competitive. The unit price of a bond of size b is given by $q(y, b)$. Note that the unit price does not depend on the transitory shock m because knowledge of current period m does not help predict either m or y in the future and, therefore, does not inform the likelihood of future default. We will assume that the sovereign can choose the size of her bond issues from a finite set $B = \{b_I, b_{I-1}, \ldots b_2, b_1, 0\}$, where $b_I < b_{I-1} < \ldots < b_2 < b_1 < 0$.[3] As is customary in this literature, we will view borrowing as negative assets. Thus when we refer to the level of bonds or debt, b, it should be understood that b is a negative number.

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[2]This is the environment analyzed in Chatterjee et al. (2007); so the theoretical results on long-term debt reported later in the paper apply to that (consumer debt) environment as well.

[3]For simplicity, we do not allow the sovereign to save. This restriction is not important for the theory, and in the application, the no-savings constraint is never binding.

3.3 Decision Problem

Consider the decision problem of a sovereign with $b \in B$ of type (z, λ) bonds outstanding, endowment y, and minimum consumption requirement m. Because the sovereign is indebted, it has the option to default. Denote the sovereign's lifetime utility conditional on repayment, i.e. maintaining access to international credit markets, by the function $V(y, m, b) : Y \times M \times B \to \mathbb{R}$ and its lifetime utility conditional on being excluded from international credit markets by the function $W(y, m) : Y \times M \to \mathbb{R}$.

Then:

$$W(y, m) = u(c - m) + \beta\{[1 - \xi]E_{(y' m')|y}W(y', m') + \xi E_{(y' m')|y}V(y', m', 0)\} \qquad (2)$$

s.t.

$$0 \le c \le y - \phi(y)$$

The sovereign's lifetime utility under exclusion reflects the possibility that it may be let back into the credit market with probability ξ. If it is not let back, its situation next period will be the same as it is in the current period under exclusion – it loses $\phi(y)$ of its output and can expect to be let back into the credit market next period with probability ξ. In the period of default, the sovereign's lifetime utility is $W(y, \bar{m})$. Since $u(.)$ is strictly increasing, it is optimal for a sovereign who defaults or is excluded from international markets due to a prior default to simply consume all its available output.

And

$$V(y, m, b) = \max_{b'} u(c - m) + \beta E_{(y' m')|y} \max\{V(y', m', b'), W(y', \bar{m})\} \qquad (3)$$

s.t.

$$0 \le c \le y + [\lambda + [1 - \lambda]z]b - q(y, b')[b' - [1 - \lambda]b]$$

The above implicitly assumes that the budget set under repayment is nonempty, meaning there is at least one choice of b' that leads to nonnegative discretionary consumption. But it is possible that (y, b, m) is such that all choices of b' lead to negative discretionary consumption. In this case,

repayment is simply not an option and the sovereign must default. But in order to ensure that $V(.)$ is defined over the entire domain, we will assume that if the budget set is empty, then

$$V(y, m, b) = u(0) + \beta\{[1 - \xi]E_{(y' \, m')|y}W(y', m') + \xi E_{(y' \, m')|y}V(y', m', 0)\}$$

Observe that in this definition the future looks exactly the same as it does under default. Thus, with this definition it is strictly optimal for the sovereign to choose default over "repayment" when the budget set under repayment is empty. This is so because the sovereign can get strictly positive consumption by declaring default – recall that by assumption $y - \phi(y) - \bar{m} > 0$.

We will assume that if the sovereign is indifferent between repayment and default, it repays. Hence, the country will default on debt b if and only if $W(y, \bar{m}) > V(y, m, b)$.

This decision problem implies a default decision rule $d(y, m, b)$, where $d = 1$ implies default and $d = 0$ implies repayment, and, conditional on repayment, a debt choice rule $a(y, m, b)$.

3.4   Equilibrium

The world one-period risk-free rate r_f is taken as exogenous. Given a competitive market in sovereign debt, the unit price of a bond of size b – $q(y, b')$ – must be consistent with zero profits adjusting for the probability of default. That is:

$$q(y, b') = E_{(y' \, m')|y}\left[[1 - d(y', m', b')]\frac{\lambda + [1 - \lambda][z + q(y', a(y', m', b'))]}{1 + r_f}\right] \tag{4}$$

Observe that in the future states in which the sovereign defaults, the creditors get nothing. In the absence of default, the creditors get λ, which is the fraction of the unit bond that matures next period, and on the remaining fraction, the creditors get the coupon payment z. In addition, the fraction that does not mature will have some value next period that depends on the sovereign's endowment and borrowing next period.

4   Characterization of Equilibrium

In this section we characterize and prove the existence of an equilibrium. The characterization results provide intuition on the nature of the model and aid in the computation of the equilibrium. They are all in the nature of monotonicity results.

Proposition 1: There exist unique continuous and bounded functions $W(y, m)$ and $V(y, m, b)$ that solve the functional equations (2)-(4). In the region of the domain where repayment is feasible, $V(y, m, b)$ is strictly increasing in b and strictly decreasing in m.

Proof: The existence of unique, continuous and bounded solutions to the functional equations follow from standard contraction mapping arguments that will not be repeated here.

With regard to the monotonicity of V, observe that if $m^0 < m^1$, then every b' that is feasible under (y, m^1, b) is also feasible under (y, m^0, b) and yields strictly higher discretionary consumption. Since the value of m does not affect the probability distribution of (y', m'), it follows that $V(y, m^0, b) > V(y, m^1, b)$.

Next, observe that if $b^0 < b^1$, then for every $b' \in B$ and every $y \in Y$ we have $[\lambda + [1 - \lambda]z]b^0 + q(y, b')[1 - \lambda]b^0 < [\lambda + [1 - \lambda]z]b^1 + q(y, b')[1 - \lambda]b^1$. This follows because $[\lambda + [1 - \lambda]z] > 0$ and $q(y, b') \geq 0$. Hence, any b' that is feasible under (y, m, b^0) is also feasible under (y, m, b^1) and affords strictly greater discretionary consumption. Therefore, $V(y, m, b^0) < V(y, m, b^1)$.

$\square$

The next proposition establishes that default is at least as likely under a higher debt level as under a lower debt level.

Proposition 2: If $b_0 < b_1$, then $d(y, m, b^0) \geq d(y, m, b^1)$.

Proof: Suppose, to get a contradiction, that $d(y, m, b^0) < d(y, m, b^1)$. Then it must be the case that $d(y, m, b^0) = 0$ and $d(y, m, b^1) = 1$. The former implies that $V(y, m, b^0) \geq W(y, \bar{m})$ and the latter implies $W(y, \bar{m}) > V(y, m, b^1)$. Then, we must have $V(y, m, b^0) > V(y, m, b^1)$. But this contradicts Proposition 1. Hence, $d(y, m, b^0) \geq d(y, m, b^1)$.

□

One would expect Proposition 2 to imply that the equilibrium pricing function, $q(y, b')$, is increasing in b' (or, equivalently, that the equilibrium default premium is increasing in the level of indebtedness). In the case of one-period bonds this is indeed true. If $z = 0$ and $\lambda = 1$, the equilibrium pricing equation (4) reduces to $q_{0,1}(y, b') = E_{(y'\, m')|y}[1 - d(y', m', b')]/[1 + r_f]$, which is increasing by Proposition 2. But when $\lambda < 1$ (bonds have maturity longer than 1 period), the payoff under repayment depends on the value of the outstanding bonds next period, which, in turn, depends on the sovereign's output next period and the sovereign's borrowing decision next period. The following Proposition establishes that if the decision rule $a(y, m, b)$ is increasing in b, then $q(y, b')$ is increasing in b'.

Proposition 3: Suppose $\lambda \in [0, 1)$. If $a(y, m, b)$ is increasing in b, then the equilibrium pricing equation (4) implies that $q(y, b')$ is increasing in b'.

Proof: Let $d^*(y, m, b)$ and $a^*(y, m, b)$ be the equilibrium decision rules corresponding to the equilibrium pricing function $q^*(y, b')$. For any function $q : Y \times B \rightarrow \mathbb{R}_+$, define the operator $T(q)(y, b')$ as

$$E_{(y'\, m')|y} \left[[1 - d^*(y', m', b')] \frac{\lambda + [1 - \lambda][z + q(y', a^*(y', m', b'))]}{1 + r_f} \right].$$

Then the equilibrium pricing function $q^*(y, b')$ solves the equation $q^*(y, b') = T(q^*)(y, b')$.

Next, we will show that the operator T is a contraction mapping. Observe that (i) $q(y, b') \geq q(y, b')$ implies $T(q^1) \geq T(q^0)$ and (ii) for any positive constant θ and any $q(y, b')$, $T(q + \theta) \leq T(q) + \theta[1 - \lambda]/[1 + r_f]$ (this follows because $1 - d^*(y', m', b') \leq 1$). Since $[1 - \lambda]/[1 + r_f] < 1$, the operator T satisfies Blackwell's sufficiency conditions for a contraction mapping.

Next, we will show if $q(y, b')$ is an increasing function of b', then $T(q)(y, b')$ is also an increasing function of b'. Fix y, y' and m'. Let $b'^0 < b'^1$. Proposition 2 implies that $(1 - d^*(y', m', b'^0) \leq (1 - d^*(y', m', b'^1)$. Since $a^*(y, m, b')$ is increasing in b' by hypothesis and $q(y, b')$ is increasing in b' by assumption, it follows that $q(y', a^*(y', m', b'^0)) \leq q(y', a^*(y', m', b'^1))$. Since y' and m' were arbitrary, it follows that $T(q)(y, b'^0) \leq T(q)(y, b'^1)$.

Finally, to establish the result, let $\bar{q}$ be any number such that $q^*(y, b') \leq \bar{q}$ and let Q be the set of all nonnegative functions $q(y, b')$ that are increasing in b' and bounded above by $\bar{q}$. Define the norm of any function $q \in Q$ as $||q|| = \sup q(y, b')$. Then $(Q, || \cdot ||)$ is a complete metric space. By the previous step, $T(q) \in Q$ for any $q \in Q$. Since T is a contraction mapping, it follows from the Banach contraction mapping principle that there exits a unique $\hat{q}(y, b') \in Q$ such that $T(\hat{q}) = \hat{q}$. But then $q^*(y, b')$ must coincide with $\hat{q}(y, b')$. Hence, $q^*(y, b')$ must be increasing in b'.

$\square$

Proposition 3 assumed that the bond decision rule conditional on repayment was increasing in b and showed that the pricing function is increasing in b'. The next proposition shows that if the pricing function is increasing in b', then the bond decision rule is increasing in b. Thus Proposition 3 and 4 are "dual" of each other.

Proposition 4: If $q(y, b')$ is increasing in b', then in the region where repayment is feasible $a(y, m, b)$ is increasing in b.

Proof: Fix m and y and suppose that $b^1 < b^0$.

Denote $a(y, m, b^0)$ by b'^0 and the associated consumption level by c^0. Let $\hat{b}'$ be some other feasible choice greater than b'^0 and let $\hat{c}$ be the associated consumption level. Then, by optimality, we have

$$u(c^0 - m) + \beta E_{(y' \, m')|y} \max \left\{ V(y', m', b'^0), W(y', \bar{m}) \right\}$$

$$\geq$$

$$u(\hat{c} - m) + \beta E_{(y' \, m')|y} \max \left\{ V(y', m', \hat{b}'), W(y', \bar{m}) \right\} \tag{5}$$

Since $\hat{b}' > b'^0$, the fact that $V(y, m, b)$ is strictly increasing in b (Proposition 1) implies

$$E_{(y' \, m')|y} \max \left\{ V(y', m', \hat{b}'), W(y', \bar{m}) \right\} > E_{(y' \, m')|y} \max \left\{ V(y', m', b'^0), W(y', \bar{m}) \right\}.$$

Hence (5) implies $c^0 > \hat{c}$. Let $\Delta = c^0 - \hat{c} > 0$ denote the loss in current consumption from choosing

$\hat{b}'$ over b'^0 when the beginning-of-period borrowing is b^0. Then from the budget constraint we have

$$-q(y, b'^0)[b'^0 - [1-\lambda]b^0] - \Delta = -q(y, \hat{b}')[\hat{b}' - [1-\lambda]b^0],$$

or

$$-q(y, b'^0)b'^0 - \Delta = -q(y, \hat{b}')\hat{b}' + [q(y, \hat{b}') - q(y, b'^0)][1-\lambda]b^0.$$

Observe that since $\hat{b}' > b'^0$ and, by hypothesis, $q(y, b')$ is increasing in b', the term $[q(y, \hat{b}') - q(y, b'^0)] \geq 0$.

Holding fixed $\hat{b}'$ and b'^0, let $\Delta(b^1)$ be the value of Δ that solves:

$$-q(y, b'^0)b'^0 - \Delta(b^1) = -q(y, \hat{b}')\hat{b}' + [q(y, \hat{b}') - q(y, b'^0)][1-\lambda]b^1.$$

Then $\Delta(b^1)$ is the change in current consumption from choosing $\hat{b}'$ over b'^0 when the beginning-of-period asset level is b^1. Since $[q(y, \hat{b}') - q(y, b'^0)] \geq 0$, $b^1 < b^0$ implies $\Delta(b^1) \geq \Delta$. Thus the loss in current consumption from choosing $\hat{b}'$ over b'^0 is at least as large when the beginning-of-period asset holding is b^1 as compared to b^0. Next, note that since $y - m + [\lambda + [1-\lambda]z]b^1 < y - m + [\lambda + [1-\lambda]z]b^0$, consumption under the choice of b'^0 when the beginning-of-period of bond holdings is b^1, denoted c^1, is strictly less than c^0. It follows from strict concavity of $u(\cdot)$ that

$$u(c^1 - m) - u(c^1 - \Delta(b^1) - m) > u(c^0 - m) - u(c^0 - \Delta - m).$$

Therefore,

$$u(c^1 - m) - u(c^1 - \Delta(b^1) - m)$$

$$>$$

$$\beta E_{(y'\, m')|y} \max \left\{ V(y', m', \hat{b}'), W(y', m') \right\} - \beta E_{(y'\, m')|y} \max \left\{ V(y', m', b'^0), W(y', m') \right\}.$$

Since $\hat{b}'$ is any asset choice greater than b'^0, the optimal choice of b' (under repayment) when beginning-of-period bond holding is b^1 cannot be greater than b'^0. Therefore, $a(y, m, b^1) \leq a(y, m, b^0)$.

□

Proposition 5: In the region where repayment is feasible, $a(y, m, b)$ is decreasing in m.

Proof: Fix y and b and let $m^0 < m^1$. Denote $a(y, m^0, b)$ by b'^0 and the associated consumption by c^0. Let $\hat{b}' > b'^0$ be some other feasible choice of b' greater than $\hat{b}'^0$ and denote the associated consumption by $\hat{c}$. Then, by optimality,

$$u\left(c^0 - m^0\right)) + \beta E_{(y'\,m')|y} \max\left\{V(y', m', b'^0), W(y', \bar{m})\right\}$$

$$\geq$$

$$u\left(\hat{c} - m^0\right)) + \beta E_{(y'\,m')|y} \max\left\{V(y', m', \hat{b}'), W(y', \bar{m})\right\}$$

Since $V(y, m, b)$ is strictly increasing in b (Proposition 1), the above inequality implies $c^0 > \hat{c}$ (the implied inequality is strict as long as probability of repayment of b'^0 is strictly positive). Since $u(c - m^1) = u(c - m^0) - \int_{m^0}^{m^1} u'(c - x)dx$, we have that $u(c^0 - m^1) - u(\hat{c} - m^1) = u(c^0 - m^0) - u(\hat{c} - m^0) - \int_{m^0}^{m^1} \left[u'^0 - x) - u'(\hat{c} - x)dx\right]$. Since $c^0 > \hat{c}$, it follows from the strict monotonicity and concavity of $u(\cdot)$ (diminishing marginal utility) that $u(c^0 - m^1) - u(\hat{c} - m^1) > u(c^0 - m^0) - u(\hat{c} - m^0)$. Therefore, b'^0 strictly dominates $\hat{b}'$ when the minimum consumption requirement is m^1. Since $\hat{b}'$ was any asset choice greater than b'^0, it follows that $a(y, m^1, b)$ cannot exceed b'^0. Hence $a(y, m^1, b) \leq a(y, m^0, b)$.

□

Proposition 5 suggests that there is a *unique* threshold value of m at which the sovereign will be indifferent between its current choice of borrowing and its next best choice, and, as m crosses this unique threshold, it will switch from one borrowing level to another. The following lemma makes this suggestion explicit. It establishes that the sovereign can be indifferent between any two borrowing levels at exactly one particular value of m. The Lemma is useful in developing an algorithm for the computation of the equilibrium as well as in proving the existence of an equilibrium.

Lemma 1 : Let b'^0 and b'^1 be two different borrowing levels. There can be at most one value of m for which the two borrowing levels give the same lifetime utility.

Proof :   Denote $E_{(y'\,m')|y}\max\{V(y',m',b'),W(y',\bar{m})\}$ by $X_y(b')$. Since $b'^0 \neq b'^1$, it follows from the strict monotonicity of $V(y,m,b)$ that $X_y(b'^0) \neq X_y(b'^1)$. Now suppose that there is an m for which $u(c^0-m)+\beta X_y(b'^0) = u(c^1-m)+\beta X_y(b'^1)$, where c^0 and c^1 are the level of consumption when b'^0 and b'^1 are chosen, respectively. Clearly, $c^0 \neq c^1$. Suppose, to get a contradiction, that there is another $\tilde{m}$ such that $u(c^0-\tilde{m})+\beta X_y(b'^0) = u(c^1-\tilde{m})+\beta X_y(b'^1)$. Let $\tilde{m}-m = \Delta$. Then, we must have

$$u(c^0-m) - u(c^0-\Delta-m) = u(c^1-m) - u(c^1-\Delta-m).$$

But since $c^0 \neq c^1$ the above equality violates strict concavity of $u(\cdot)$. Hence there can only be at most one m for which $u(c^0-m)+\beta X_y(b'^0) = u(c^1-m)+\beta X_y(b'^1)$.

$\square$

The next proposition relates the set of m values for which there is default. Let $D(y,b) = \{m \in M : d(y,m,b) = 1\}$. Then we have:

Proposition 6: $D(y,b)$ is either the empty set, the whole interval M or a semi-open interval $(m^*,\bar{m}]$ where $m^* \in [0,\bar{m})$.

Proof: Three cases are possible. (i) $V(y,0,b) < W(y,\bar{m})$. Since V is strictly decreasing in m (Proposition 1), it follows that $V(y,m,b) < W(y,\bar{m})$ for all $m \in M$. Therefore, there will be default for every realization of m. In this case the default interval is the whole interval $[0,\bar{m}] = M$ (ii) $V(y,m,b) < W(y,\bar{m}) \leq V(y,0,b)$. Then, by the continuity and strict monotonicity of V with respect to m, there exists a unique $m^* \in [0,\bar{m})$ such that $V(y,m^*,b) = W(y,\bar{m})$. Then the default interval is $(m^*,\bar{m}]$ (iii) $W(y,\bar{m}) \leq V(y,\bar{m},b)$. In this case, $V(y,m,b)$ is at least as large as $W(y,\bar{m})$ for every realization of m and there will never be any default. Then, the default set is the empty set.

$\square$

The final proposition concerns the existence of an equilibrium with the property that the equilibrium price function $q(y,b')$ is increasing in b'. It relies on the following Lemma whose proof is given in the

appendix (for notational ease, in the proofs of Lemma 2 and Proposition 7 we will denote $q(y, b')$ as simply $q(y, b')$).

Lemma 2 : Let $q^n(y, b')$ be a sequence of pricing functions converging to $\hat{q}(y, b')$. Let $d(y, m, b; q^n)$, $a(y, m, b; q^n)$ and $d(y, m, b; \hat{q})$, $a(y, m, b; \hat{q})$ be the corresponding optimal decision rules. If $u(0)$ is sufficiently small, $d(y, m, b; q^n)$ converges pointwise to $d(y, m, b; \hat{q})$ and $a(y, m, b; q)$ converges pointwise to $a(y, m, b; \hat{q})$ except, possibly, at a finite number of points.

Proposition 7: There exists an equilibrium price function $q^*(y, b')$ that is increasing in b'.

Proof : Let $\bar{q} = [\lambda + [1 - \lambda]z]/[\lambda + r_f]$. Then $\bar{q}$ is the present discounted value of a bond with coupon payment z and probability of maturity λ on which there is no risk of default. Let S be the set of all non-negative functions $q(y, b')$ defined on $Y \times B$ and let $Q \subset S$ be the subset of functions that are increasing in b' and bounded above by $\bar{q}$.

Define the $H(q)(y, b') : Q \to S$ as

$$E_{(y' \, m')|y} \left[[1 - d(y', m', b'; q)] \frac{\lambda + [1 - \lambda][z + q(y', a(y', m', b'; q))]}{1 + r_f} \right].$$

Then H has the following properties:

(i) $H(q)(y, b') \in Q$. Non-negativity is obvious. We will show that $H(q)(y, b') \leq \bar{q}$. Observe that $\bar{q}$ satisfies the equation $\bar{q} = [\lambda + (1 - \lambda)[z + \bar{q}]]/(1 + r_f)$. Then, since $1 - d(y', m', b') \leq 1$ and $q(y', a(y', m', b'; q)) \leq \bar{q}$ for every (y', m', b'), it follows that

$$\left[[1 - d(y', m', b'; q)] \frac{\lambda + [1 - \lambda][z + q(y', a(y', m', b'; q))]}{1 + r_f} \right] \leq \bar{q} \text{ for every } y', m', b'.$$

Hence $H(q)(y, b') \leq \bar{q}$. Next, we will show that $H(q)(y, b')$ is increasing in b'. Fix y' and m'. Since $q(y, b') \in Q$, $q(y, b')$ is increasing in b'. Then, by Proposition 4, $a(y', m', b'; q)$ is increasing in b'. Thus, $q(y', a(y', m', b'; q))$ is increasing in b'. And, by Proposition 2, $[1 - d(y', m', b'; q)]$ is also increasing in b'. Hence $H(q)(y, b')$ is increasing in b'. Thus $H(q)(y, b') \in Q$.

(iii) $H(q)(y, b')$ is continuous. Let $\{q^n\}$ be a sequence in Q converging to $\hat{q} \in Q$ and let $\{d(y, m, b; q^n),$

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$a(y, m, b; q^n)\}$ and $\{d(y, m, b; \hat{q}), a(y, m, b; \hat{q})\}$ be the corresponding optimal decision rules. Then

$$H(q^n)(y, b') = E_{(y'\,m')|y}\left[[1 - d(y', m', b'; q^n)]\frac{\lambda + [1 - \lambda][z + q^n(y', a(y', m', b'; q^n))]}{1 + r_f}\right].$$

Or,

$$H(q^n)(y, b') =$$
$$\sum_{y'}\left[\frac{\int_M [1 - d(y', m', b'; q^n)]\,[\lambda + [1 - \lambda][z + q^n(y', a(y', m', b'; q^n))]]\,dG(m')}{1 + r_f}\right]F(y', y).$$

Fix y' and b'. By Lemma 2, $\lim_{n\to\infty}[1 - d(y', m', b'; q^n)] = [1 - d(y', m', b'; \hat{q})]$ for all but a finite number of points (possibly) of m'. Since individual points of m have probability zero, $[1 - d(y', m', b'; q^n)]$ converge almost surely to $[1 - d(y', m', b'; \hat{q})]$ with respect to the measure induced by $G(m)$.

Also, by Lemma 2, $\lim_{n\to\infty} a(y', m', b'; q^n)) = a(y', m', b'; \hat{q}))$ for all but a finite number of points (possibly) of m'. If convergence holds then, since $a(\cdot; q^n)$ takes values in a finite set B, there must exist N such that for all $n > N$ $a(y', m', b'; q^n)) = a(y', m', b'; \hat{q}))$. Therefore, for $n > N$, $q^n(y', a(y', m', b'; q^n)) = q^n(y', a(y', m', b'; \hat{q}))$. Since $q^n \to \hat{q}$, it follows that $\lim_{n\to\infty} q^n(y', a(y', m', b'; \hat{q})) = \hat{q}(y', a(y', m', b'; \hat{q}))$. Thus, viewed as a function of m', $q^n(y', a(y', m', b'; q^n))$ converges almost surely to $\hat{q}(y', a(y', m', b'; \hat{q}))$. Therefore, we have that

$$\lim_{n\to\infty}[1 - d(y', m', b'; q^n)]\left[\lambda + [1 - \lambda][z + q^n(y', a(y', m', b'; q^n))]\right]$$
$$=$$
$$[1 - d(y', m', b'; \hat{q})]\left[\lambda + [1 - \lambda][z + \hat{q}(y', a(y', m', b'; \hat{q}))]\right]$$

except, possibly, at a finite number of points.

Now observe that each function in the sequence is non-negative and bounded above by $\lambda + (1 -$

$\lambda)[z + \bar{q}]$. Thus, by the Lebesgue Dominated Convergence Theorem, we have that

$$\lim_{n\to\infty} \int_M [1 - d(y', m', b'; q^n)] \left[\lambda + [1 - \lambda][z + q^n(y', a(y', m', b'; q^n))]\right] dG(m')$$
$$=$$
$$\int_M [1 - d(y', m', b'; \hat{q})] \left[\lambda + [1 - \lambda][z + \hat{q}(y', a(y', m', b'; \hat{q}))]\right] dG(m').$$

Putting these result together, we get

$$\lim_{n\to\infty} H(q^n)(y, b') =$$
$$\sum_{y'} \left[\frac{\int_M [1 - d(y', m', b'; q^n)] \left[\lambda + [1 - \lambda][z + q^n(y', a(y', m', b'; q^n))]\right] dG(m')}{1 + r_f} \right] F(y', y).$$
$$= H(\hat{q})(y, b').$$

Thus $H(q)$ is continuous.

To complete the proof note that Q is a compact and convex set and, since $H(q)$ is continuous, by Brouwer's Fixed Point Theorem there exists $q^* \in Q$ such that $q^*(y, b') = H(q^*)(y, b')$. That is,

$$q^*(y, b') = E_{(y' \, m')|y} \left[[1 - d(y', m', b'; q^*)] \frac{\lambda + [1 - \lambda][z + q^*(y', a(y', m', b'; q^*))]}{1 + r_f} \right]$$

This establishes the existence of an equilibrium price function that is increasing in b'.

$\square$

5   Computation

Computing the equilibrium price function for bonds with maturity greater than one period is challenging. In this section we discuss the nature of the challenge and how this challenge is met in our paper.

To understand the new computational issues introduced by long-maturity bonds, it is useful to begin with the case of one-period bonds ($z = 0$ and $\lambda = 1$) and no m shocks. In this case, the

equilibrium pricing function is computed via the following iteration

$$q^{k+1}(y, b') = E_{y'|y} \left[[1 - d(y', b'; q^k)] \frac{1}{1 + r_f} \right]. \tag{6}$$

Here q^k denotes the k-th iterate of the price function and $d(\cdot, \cdot, q^k)$ is the optimal default function given the price function q^k. Since there are no preference shocks, the state variables in this decision rule are simply endowments and beginning-of-period bond-holdings. For the iteration to converge, it is important that small changes between the k-th and the $k + 1$-st iterate of the price function not imply a large change between the $k + 1$-st and the $k + 2$-nd iterate. However, because default is a discrete choice, a small change in the price function can lead to a switch in behavior from default to repayment (or vice versa). This will happen if for some q^k the sovereign is very close to indifference between the choice of default and repayment for some (y, b) pair. If a switch happens for a small change in the pricing function, the expectation in (6) may change discretely. The reason for this is that the number of grid points on y is typically not large in applications and so each point on the y grid has significant probability mass. Thus, a discrete change in behavior for some y can change the expectation discretely. However, note that for any given y the indifference between default and repayment will happen for a very specific value of b. If this b value is not part of the grid, this problem may not arise. Indeed, in the simulations done with the parameters used in this paper there is never a convergence problem with one-period bonds.[4]

The difficulty is compounded when bonds can last more than one period. In this case, ignoring preference shocks, the pricing equation is solved by iterating on

$$q^{k+1}(y, b') = E_{y'|y} \left[[1 - d(y', b'; q^k)] \frac{\lambda + [1 - \lambda][z + q^k(y', a(y', b'; q^k))]}{1 + r_f} \right] \tag{7}$$

Now, the calculation of the $k + 1$-st iterate depends on the $a(\cdot, \cdot; q^k)$ decision rule as well. This creates two problems. First, it is no longer possible to have a coarse grid on bond holdings. A coarse grid would imply that whenever a small change in the pricing function induces the sovereign to switch its desired level of bond-holdings, the switch would affect the expectation in (7) discretely.

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[4]Nevertheless, it remains true that the problem associated with indifference and switching is more likely to arise for a fine set of grid points on b because a fine set is more likely to contain a grid point at which there is near-indifference. In this sense, there is a trade-off between an accurate solution to the sovereign's decision problem and the ease with which the equilibrium pricing function can be computed.

But making the grid finer does not overcome this problem because of the possible nonconvex nature of the optimization problem. This is so because the budget set under repayment may not be convex (because $q(y, b')$ is a nonlinear function of b') or because future expected utility (as a function of b') has nonconcave segments (see Figures 1 and 2). These nonconvexities imply that, given (y, b), the sovereign may be indifferent between two widely separated values of b'. Thus there may be jumps in the decision rule $a(y, b; q^k)$ viewed as a function of q^k. These jumps in decisions then cause large changes in the future value of the outstanding bonds and therefore in the current price.

To summarize, the jumps (or discontinuities) in the decision rules stem from the possibility of default and the resulting non-convexity of the budget set under repayment and non-concavity of the value function. Thus the jumps are an intrinsic part of the decision problem being studied here and cannot be avoided. Given this, the only approach to solving the problem is to arrange matters so that the jumps do not affect the expected value in (7) too much. For that purpose we introduce the continuous i.i.d variable m, and the solution of the model is found by calculating the thresholds over m where a switch occurs between different choices of non-dominated asset holdings (the algorithm will be described in more detail below). This implies that when there is a small change in price, the change in the threshold will also be small, even though at the threshold there might be a big jump in asset holdings. Since the probability mass over which there is a change in behavior approximates to zero with a continuous distribution (when the change in threshold is small), the change in the expected value in (7) is negligible.

Note that it is not possible to apply the above method over y because the asset choice is not monotonic in y, and we cannot calculate thresholds for y in order to solve the model. In contrast, as shown in Propositions 5 the choice of assets is monotonic in m and it is possible to calculate thresholds over which a switch might occur from a choice of lower asset level to a higher asset level. This difference is due to the fact that the m shock is i.i.d while the y shock is not. Of course, it is possible to simply increase the number of grid points over y in order to reduce the impact of jumps. Increasing the grid points on y does help with convergence. Nevertheless, convergence to an error of 0.01 or less in the pricing of q was not achieved even with the finest grids used that were computationally possible. And increasing the fineness of the grid on y slows down the program disproportionately. For these reasons we added the variable m and we chose its volatility to be as

low as possible so that there is convergence of the pricing function and at the same time the impact of adding m on model properties is negligible. We verified that the impact of m on quantitative results is negligible for the one-period bond case, since in that case the model without the m shock can be solved (these results are discussed later in the paper).

We now describe our solution algorithm. To solve the model, we use the characterization of behavior with respect to m, namely, Propositions 5 and 6. Proposition 6 indicates that we need to locate one value of m at which the sovereign is indifferent between repayment and default, if such a point exists. Proposition 5 indicates that the choice of b' is decreasing in m; as m increases, the optimal decision switches to a lower value of b'.

However, Proposition 5 does *not* imply that as m increases, the next optimal choice of b' will be *adjacent* to the current optimal choice. As noted earlier, the nonlinearity of the pricing function or nonconcavity of the value function may imply that the sovereign finds it optimal to switch between widely separated values of b'. This fact raises a challenge in the computation. Evidently, Proposition 5 implies that there exists $\{m^1 < m^2 < \ldots < m^{K-1} < \bar{m}\}$ and $\{b'^1 > b'^2 > \ldots > b'^K\}$ such that b'^1 is chosen for all $m \in [0, m^1)$, b'^2 is chosen for all $m \in [m^1, m^2)$, $\ldots$, b'^K is chosen for all $m \in (m^{K-1}, \bar{m}]$. But since b'^{k-1} need not be adjacent to b'^k, how does one find the values of b'^k and the associated values of m^k?

To understand the algorithm we use to determine the $\{(m^k, b'^k)\}$ pairs, imagine that we have located the pairs $\{(m^1, b'^1), (m^2, b'^2), \ldots (m^g, b'^g)\}$. To proceed, compare the utility from choosing b'^g with the utility from choosing the next lower asset level b'^- (higher debt level; $b'^- < b'^g$) for different values of m. Two cases are possible.

1. b'^g is better than b'^- for all $m \in M$. That might happen if the asset choice b'^- does not increase consumption today. Then, we drop b'^- from further consideration and move to comparing b'^g to the next lower asset level.

2. There is a value of m denoted $\tilde{m}$, for which the two choices give the same utility. By Lemma 1 there can be exactly one such m. Here two cases are possible:

   (a) If $\tilde{m} \geq m^g$, then we add $(\tilde{m}, b'^-)$ to the list of pairs and proceed to compare the utility

between b'^- with the next lower asset level.

(b) If $\tilde{m} < m^g$, we drop b'^g from further consideration and proceed backwards to compare b'^- with b'^{g-1}. The reason is that $\tilde{m} < m^g$ implies that b'^- is preferred to b'^g for any $m > \tilde{m}$ and at the same time b'^{g-1} is preferred to b'^g for any $m < m^g$. This implies that b'^g is dominated by the choices of b'^{g-1} and b'^- and will never be chosen. Thus it can be dropped from further consideration. When this is the case, b'^- needs to be compared to b'^{g-1}. The process is continued by finding a new $\widetilde{m}_2$ between the choices of b'^- and b'^{g-1}. If $\tilde{m}_2 \geq m^{g-1}$, then we add $(\tilde{m}_2, b'^-)$ to the list of pairs and proceed to compare the utility between b'^- with the next lower level of assets. If $\tilde{m}_2 < m^{g-1}$, we drop b'^{g-1} from further consideration and continue to go backwards through the list. This process will either end in finding m^{g-j} that is less than or equal to $\tilde{m}_{j+1}$ or in the exhaustion of all pairs in the list $\{m^k, b'^k\}$. If the latter, we conclude that b'^- dominates any $b' > b'^-$ for all m and proceed to compare b'^- with the next lower level of debt.

To implement this algorithm we start off with the list $\{(\bar{m}, 0)\}$ (meaning that no borrowing is optimal for all m) and then proceed to compare 0 with the next level of debt. The algorithm is applied until every element of B has been picked up and compared to the existing list.

6 Maturity, Indebtedness, and Spreads: The Argentine Case

We apply the framework developed in the previous sections to the Argentine case. Our objective is to simultaneously account for the average default spreads on Argentine bonds as well as the average indebtedness of Argentina over the 10-year period between 1991:Q1 and 2001:Q4. This is also the time period analyzed in Arellano (2008).[5] Arellano focused on understanding the average default spreads on Argentine bonds but did not attempt to match Argentina's average debt level. The

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[5]Arellano (2008) restricted her sample period to 1993-2001 on grounds of data availability. We use the data presented in Neumeyer and Perri (2005), which are easily available and widely used and cover all quarters between 1980 and 2002. We chose to examine the period between 1991-2001 because prior to 1991 Argentina was experiencing hyperinflation. Inflation fell when Argentina adopted a currency board in March 1991 that fixed the value of the peso in terms of the dollar at parity (1 to 1). Argentina remained on parity all the way through to its default in December 2001. Parity was abandoned in January 2002. Thus, the period 1991-2001 had a stable fixed foreign exchange regime in place. Since neither we nor Arellano model the determination of the exchange rate, the period 1991-2001 is a natural period for us to focus on.

main contribution of our quantitative work is to establish that allowing for long-duration bonds, besides being a closer fit with reality, helps to match both the level of spreads and the level of average indebtedness.

For the quantitative work we make the following specific functional form or distributional assumptions.

- Endowment process: The stochastic evolution of y_t is governed by an AR-1 process in logs:

$$\ln y_t = \rho \ln y_{t-1} + \epsilon_t, \text{ where } 0 < \rho < 1 \text{ and } \epsilon_t \text{ distributed } N\left(0, \sigma_\epsilon^2\right).$$

- Preference shock process: The preference shock m is drawn from a truncated normal distribution with support $[0, \bar{m}]$ centered at $\bar{m}/2$ and with variance σ_m^2.

- Utility function: The utility function is assumed to be the CRRA form $(c - m)^{1-\gamma}/(1-\gamma)$.

- Following Arellano (2008), the output loss in the event of default or exclusion is assumed to be of the form:

$$\phi(y) = \begin{cases} 0 & \text{if } y \leq \overline{y} \\ y - \overline{y} & \text{if } y > \overline{y} \end{cases}.$$

With these assumptions, the numerical specification of the model requires giving values to 11 parameters. These are (i) two endowment process parameters ρ and σ_ϵ^2, (ii) four preference parameters β, γ, $\bar{m}$ and σ_m^2, (iii) two parameters describing the bond, the maturity parameter λ and the coupon payment z, (iv) the default output loss parameter $\bar{y}$, (v) the probability of re-entry following default, ξ, and (vi) the risk-free rate r_f.

The parameter selection proceeds as follows. The endowment process was determined by fitting the AR-1 process to the quarterly real GDP data over the period 1980:1-2001:4.[6] The estimation yields a value of $\rho = 0.93$ and $\sigma_\epsilon = 0.0272$. Of the preference parameters, the value of the γ is set equal to 2, which is the standard value used in this literature. The precise value of $\bar{m}$ is not

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[6]The quarterly data series on real GDP, real aggregate consumer expenditure, real exports, real imports and the (nominal) interest rate on Argentine sovereign debt is taken from Neumeyer and Perri (2005). All the quantity variables were deseasonalized using the multiplicative X-12 routine in Eviews.

important to the results and so it was set at a relatively small number, 0.054, which implies that the mean m is 0.027. The value of σ_m^2 was set at 0.009, which is one-third of the variance of the innovation to income. For this value the search routine employed to match moments converged easily.

The parameters describing the bond were determined to match the maturity and coupon information for Argentina reported in Broner, Lorenzoni and Schumkler (2007). The average coupon rate is about 12 percent per annum, or 0.03 per quarter, and the median maturity of Argentine bonds is 5 years or 20 quarters. Thus, $z = 0.03$ and $\lambda = 1/20 = 0.05$. Re-entry into the financial market following default usually occurs between 2 to 3 years and so $\xi = 0.10$, which gives an average period of exclusion of 10 quarters or 2.5 years. The risk-free rate, r_f, was set at 0.01, which is roughly the real rate of return on a 3-month (one quarter) U.S. Treasury bill.

The two remaining parameters β and $\bar{y}$ are picked to match as closely as possible 70 percent of the average debt-to-output ratio and the average default spreads over the period 1990-2001.[7] We seek to match only a portion of the Argentine debt because we do not model repayment. In reality, most sovereign debt that goes into default pays off something. In Argentina's case, the repayment on debt defaulted on in 2001 has been around 30 cents to the dollar. Thus, we treat only 70 cents out of each dollar of debt as the truly unsecured portion of the debt.

We need to determine what in the model corresponds to the observed debt-to-output ratio and the observed default spreads. In the data, the value of a bond is a weighted average of its market value at time of issue and the face value of the bond, where the weight on the latter is higher the closer the bond is to maturity. The value of total debt is just the sum of the value of individual bonds calculated in this way. Since the bonds in our model do not have a maturity date, we cannot replicate this procedure exactly. What we do instead is value the promised stream of payments on a bond at the risk-free rate so that we can capture the averaging between the market value at the date of issue and the face value.

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[7]Debt is total public and publicly guaranteed long-term debt outstanding and disbursed and owed to both private and official creditors at the end of each year, as reported in the World Bank's Global Development Finance Database. The average debt-to-output ratio is the average ratio of debt to GNP measured at a quarterly rate. The spread was calculated as the difference between the interest rate data reported in Neumeyer and Perri (which is the same as the EMBI data) and the 3-month T-bill rate. The T-bill rate series used is the TB3MS series available at `http://research.stlouisfed.org/fred2/categories/116`. Both the interest rate data and the T-bill rate are reported in annualized terms.

The default spread in the model is calculated as follows. Given the unit price $q(y, b)$ of the outstanding bonds, we calculate an internal rate of return that makes the present discounted value of the promised sequence of future payments on a unit bond equal to the unit price. If the interest rate is r, the present discounted value of the promised sequence of payments on the bond is $[\lambda + (1 - \lambda)z]/[\lambda + r]$. The required internal rate of return is given by $r(y, b')$ such that $q(y, b') = [\lambda + (1 - \lambda)z]/[\lambda + r(y, b')]$. The difference between $(1 + r(y, b'))^4 - 1$ and $(1 + r_f)^4 - 1$ is the annualized default spread in the model.

The parameter selections are summarized in the following two tables. Table 1 lists the values of the parameters that are selected directly without solving for the equilibrium of the model. Table 2 lists the parameter values that are selected by solving the equilibrium of the model and choosing the parameters so as to make the model moments come as close as possible to the data moments.

Table 1: Parameters Selected Directly

Parameter	Description	Value
γ	risk aversion	2
$\bar{m}$	upper bound on m	0.054
σ_m	standard deviation of m	0.009
σ_ϵ	standard deviation of ϵ	0.0272
ρ	autocorrelation	0.93
ξ	probability of reentry	0.10
r_f	risk free return	0.01
λ	reciprocal of ave. maturity	0.05
z	coupon payments	0.03

Table 2: Parameters Selected by Matching Moments

Parameter	Description	Value
β	discount factor	0.968
$\bar{y}$	default punishment	0.879

The results are reported in Table 3. The top row reports the data for Argentina. The average risk spread over the period 1991-2001 is 8.8 percent. The average debt-to-output ratio is 1.00. The debt

service ratio, which is the ratio of the average annual payment on debt (interest and principal) to output, is 5.5 percent.

Table 3: Results and Comparison

	Def. Freq.	Spread	Debt-to-Y	Debt Service
Data	0.0568	0.0876	1.00	0.055
Baseline $\lambda = 0.05, z = 0.03$	0.0594	0.0877	0.70	0.041
Arellano ($\lambda = 1$)	0.0300	0.0358	0.06	0.053
Baseline with $\lambda = 1$	0.0033	0.0036	0.48	0.480

The second row reports the same moments in the model. Recall that we search over the values of β and $\bar{y}$ to match the average spread and 70 percent of the average debt-to-output ratio. It is evident that the matching exercise is fully successful. We do not target the debt service ratio but the agreement between model and data is close for this statistic, given that we are matching only a 70 percent of the total debt outstanding.

The third row of the table reports, for comparison, the statistics as given in Arellano (2008, Table 4, p. 706) for the variable in question. Notice that both the average spreads as well as the average debt-to-output ratio depart significantly from the data, the latter greatly so. Arellano did not attempt to match these statistics, but it nevertheless remains true that there is a major discrepancy between the quantitative predictions of her model and the data.

Table 3 also reports the frequency of default in the model. This is a quantity that Arellano targets but we do not. The evidence on the frequency of default comes from Reinhart, Rogoff and Savastano (2003, Table 1). On average, emerging markets defaulted 5.2 times in 126 years, which implies an annual default frequency of about 4 percent. Including the most recent default, the comparable statistic for Argentina is little under 4 percent. Thus this evidence seems closer to Arellano's 3 percent than our 6 percent. However, our statistic is calculated conditional on the country being in good standing (i.e., not in autarky due to default) and in debt. Reinhart et al. note that, on average, emerging markets spent 27.3 percent of 126 years in default or in restructuring. Presumably, these

countries did not have access to new credit during these times. Excluding these years from the calculation of default frequency, the average annual default frequency is 0.0568. The corresponding frequency for Argentina, including the most recent default, is 0.0525. These statistics are very close to the default frequency generated by our model. Thus, the correspondence between model predictions and data is excellent along this dimension as well.

Table 3 also indicates that in our model the default frequency is less than the default spread. At first sight this seems surprising since we expect the default frequency to be close to the default spread. To understand why the two deviate, recall that

$$[1 + r(y, b')] \cdot q(y, b') = \lambda + (1 - \lambda)[z + q(y, b')]$$

and that

$$[1 + r_f] \cdot q(y, b') = E_{(y' m')|y}[1 - d(y', m', b'; q)][\lambda + (1 - \lambda)[z + q(y', a(y', m', b'))]]$$
$$= \sum_{y'} G(m(y', b'))[\lambda + (1 - \lambda)[z + \int_0^{m(y',b')} q(y', a(y', m', b')) \frac{dG(m')}{G(m(y', b'))}]]F(y', y),$$

where $m(y', b')$ is the threshold value of m below which there is repayment conditional on y' and b' and $G(m(y', b'))$ is the probability of repayment conditional on y' and b'. Thus,

$$\frac{1 + r_f}{1 + r(y, b')} =$$
$$\frac{\sum_{y'} G(m(y', b'))[\lambda + (1 - \lambda)[z + \int_0^{m(y',b')} q(y', a(y', m', b')) \frac{dG(m')}{G(m(y',b'))}]]F(y', y)}{\lambda + (1 - \lambda)[z + q(y, b')]}.$$

Now observe that in the case of one-period bonds (i.e., $\lambda = 1$), the expression simplifies to

$$\frac{1 + r_f}{1 + r(y, b')} = \sum_{y'} G(m(y', b'))F(y'), y$$
$$= 1 - \sum_{y'}[1 - G(m(y', b'))]F(y', y)$$
$$= 1 - \pi(y, b')$$

where we have denoted the probability of default next period on a loan of size b' conditional on the

current period's output being y by $\pi(y, b')$. Taking logs on both sides and treating all interest rates and probabilities as small numbers we get $r(y, b') - r_f \approx \pi(y, b')$. Thus, default spread is closely tied to default frequency.

When bonds mature in more than one period (i.e., $\lambda < 1$), there is no simple connection between default spreads and default frequency because the sovereign's behavior under repayment matters also. To see how default spreads can exceed the default probability, suppose that the sovereign increases borrowing in the current period, i.e., $b' < b$. Given that y is persistent, i.e., $F(y' = y, y) \approx 1$ and m is iid with most of the mass in and around the mean m (which is equal to $\bar{m}/2$), we have that

$$\frac{1 + r_f}{1 + r(y, b')} \approx G(m(y, b')) \frac{[\lambda + (1 - \lambda)[z + q(y, a(y, \bar{m}/2, b'))]]}{\lambda + (1 - \lambda)[z + q(y, b')]}.$$

If we further assume that current m is also $\bar{m}/2$, then we know from Proposition 4 that $a(y, \bar{m}/2, b') \leq a(y, \bar{m}/2, b) = b'$. And, so, by Proposition 3, $q(y, a(y, \bar{m}/2, b')) \leq q(y, b')$. Thus, the ratio multiplying $G(m(y, b')$ will tend to be less than 1. If we denote this ratio by $(1 - \theta)$, where θ is some positive number between 0 and 1, then $(1 + r_f)/(1 + r) \approx [1 - \pi(y, b')](1 - \theta)$. Or taking logs and treating all numbers as small, $r - r_f \approx \pi(y, b') + \theta$. Basically, if the debt level is expected to increase tomorrow, there will be a capital loss on the value of bonds, which creates a wedge between default probability and spreads, shown by $\theta > 0$ above. If debt is expected to decrease in the future, the same logic applies in the other direction, with $\theta < 0$. Because the sovereign starts with zero debt and increases its debt gradually over time (because it is impatient relative to the rest of the world), on average $\theta > 0$ and we see a positive wedge between spreads and default probability.

The role of maturity in creating a wedge between default spreads and default frequency is a useful insight. In the past, researchers working with one-period debt models have sought to explain the existence of this gap in the data (a gap that does not arise in one-period debt models) in terms of risk aversion of the lenders. For instance, noting the large gap between the default frequency predicted by her model and average spreads, Arellano suggested that the difference comes about because the aggregate states in which a sovereign defaults are also the states in which the lender's marginal utility of wealth is high. This may happen because lenders have finite wealth or because there is correlated defaults or both. While there is probably truth to these assertions, our model

shows that it is not logically necessary to invoke risk aversion to account for a gap between default spreads and default frequency. A gap can arise as a consequence of the long-term nature of sovereign debt and the dynamics of debt accumulation and de-accumulation.

Although we made no attempt to target cyclical properties of the data, we report these properties in Table 4. Following Arellano, we report correlations and standard deviations for deviations from a linear trend for the data.[8]

Table 4: Cyclical Properties, Data and Models

Variable	Data (1980:Q1-2001:Q4)	Model	Arellano**
$\sigma(c)/\sigma(y)$	1.09	1.10	1.10
$\sigma(NX)/\sigma(y)$	0.31	0.22	0.26
$\sigma(r-r_f)^*/\sigma(y)$	0.58	1.43	1.09
$\sigma(c,y)$	0.98	0.97	0.97
$\sigma(NX,y)$	−0.86	−0.33	−0.25
$\sigma(r-r_f,y)^*$	−0.77	−0.68	−0.29
*Data for 1991:Q1-2001:Q4 only is used.			
*Data adapted from Arellano (2008), Table 4.			

The first column reports the data. Three features of the data stand out. First, the relative volatility of consumption is about the same as output – in stark contrast to developed countries. Second, the trade balance is countercyclical – net exports decline during periods of above-trend output and rise during periods of below-trend output. This is also in contrast to developed countries. And, third, spreads on sovereign debt are countercyclical.

The second column reports the same statistics for the model. The model gets most of the qualitative patterns of the data right: model consumption and trade balance have about the right level of volatility relative to output and the trade balance and spreads are countercyclical while consumption

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[8]In calculating these statistics for our model, we exclude the first 20 periods following re-entry into the international capital markets. In the period following re-entry, the debt and spread statistics are atypical in the model because the country has no debt and begins to accumulate debt pretty much regardless of the level of output. This debt accumulation phase attenuates the negative correlation between net exports and output. In reality, Argentina emerged from re-structuring episodes in the 1980s and 1990s with debt so it did not have the same incentive for debt accumulation as exists in our model.

is highly procyclical. The volatility of spreads is much greater in the model than in the data but this discrepancy may be more apparent than real and is discussed more below. The forces in the model that lead to these patterns are the ones emphasized in Aguiar and Gopinath (2006) and Arellano (2008). When output is below trend, the probability of default on new loans rises. If this rise is sharp enough, it is optimal for the sovereign to *reduce* debt rather than to increase it (so as to smooth consumption). Thus, there is a tendency for consumption to decline more than the decline in output and for the trade balance to improve with a fall in output.

The third column reports the results for the benchmark model in Arellano (2008). Evidently our model has about the same level of success as Arellano's in accounting for the cyclical facts. The main differences appear to be with regard to the behavior of spreads. Compared to that in Arellano's model, the volatility of spreads in our model is further away from the data, while the correlation of spreads with output is closer. The reason model volatility is high is due to very sharp upward spikes in spreads just prior to default. Aside from these spikes, the behavior of spreads is reasonably close to the data. This can be seen in Figure 3 which plots of the actual and simulated spreads for the period 1991:Q1-2001:Q4. As is evident, except for the period right before default, the model spreads track the data reasonably well. Thus, the prediction of excess volatility of spreads is confined to only a single period – the period immediately prior to default. The rest of the time, the volatility of model spreads is subdued relative to data.[9]

The close concordance between the data and model moments shown in Table 3 and 4 raises the question: to what extent is this concordance the result of including long-term debt? There are two ways of answering this question. One way is to simply ask what happens to the equilibrium of the model if all debt is restricted to be one period. The results from this exercise are shown in the bottom row of Table 3. Observe that both the default premium (spreads) and the debt-to-income ratio are much lower. The average spread is now around 0.4 percent and the debt-to-output ratio drops to 48 percent. The default frequency is very close to the default spread (as one would expect in a one-period debt model) and implies that there will be 1 default episode, on average, every 333 years. The debt service is very close to the debt-to-output ratio because all debt is due in one-period. Finally, the volatility of consumption rises as the sovereign engages less in consumption

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[9]Excluding the spike in spreads in the period immediately before default also lowers the average spread in the model. This can be seen in the fact that the model spread is generally below the actual spread in the figure.

smoothing – the standard deviation of consumption is around 115 percent of the standard deviation of output. Evidently, both spreads and the debt-to-output ratio are very sensitive to the maturity of the bond. Lengthening the maturity (and keeping all else constant) raises the equilibrium value of these variables.

A second way to answer this question is to ask what is the best that a model with one-period debt can do in terms of matching the debt and spread statistics? We answer this question by fixing the values of all parameters mentioned in Table 1 at their Table 1 values, except for the values λ and z, which are now set to 1 (only short-term debt) and 0 (no coupon payments), respectively. As before, the value of β and $\bar{y}$ are chosen to match the average spreads and 70 percent of the average debt-to-output ratio. Tables 5-7 report the results.

Table 5

Parameter	Description	Value
β	discount factor	0.690
$\bar{y}$	default punishment	0.844

Table 6

	Def. Freq.	Spread	Debt-to-Output	Debt Service
Data		0.0876	1.00	0.055
$\lambda = 1, z = 0$	0.0776	0.0874	0.70	0.693
$\lambda = 0.05, z = 0.03$	0.0594	0.0877	0.70	0.041

It turns out that it is possible to match the debt-to-output ratio and the average spreads pretty closely even in a model with one-period debt. But the default frequency is close to the average spread and the debt service is close to the debt-to-output ratio. Both of these statistics are too high relative to the data, but they are entirely expected given the one-period nature of bonds. Observe that the discount factor needed to match the spread and debt-to-output ratio is very low relative to the long-term debt model. The value of $\bar{y}$ is comparable.

Table 7: Cyclical Properties, One-Period Debt Model

Variable	Data	One-Period Debt	Long-term Debt
$\sigma(c)/\sigma(y)$	1.09	1.83	1.10
$\sigma(NX)/\sigma(y)$	0.31	1.29	0.22
$\sigma(r-r_f)^*/\sigma(y)$	0.58	0.81	1.43
$\sigma(c,y)$	0.98	0.69	0.97
$\sigma(NX,y)$	-0.86	-0.21	-0.33
$\sigma(r-r_f,y)^*$	-0.77	-0.58	-0.68
*Data for 1991:Q1-2001:Q4 only is used.			

The way in which this matching exercise fails is with regard to the implied cyclical properties. The relative volatility of consumption and trade balance is huge compared to the data. Essentially, the model matches the debt and spread statistics by making the sovereign very impatient. Greater impatience generates more borrowing and higher spreads. But the volatility in spreads, because all debt must be "rolled over" each period, implies a much higher level of volatility in net exports and, therefore, consumption.[10]

We wrap up the discussion of the one-period debt model by discussing whether the results of the matching exercise with one-period debt is sensitive to the way the default punishment was modeled with respect to m and so the existence of m shocks themselves. We considered the case where the decision to default is determined by comparing the value of repayment $V(y,m,b)$ to $W(y,m)$ as opposed to $W(y,\bar{m})$. That is, we assumed that the act of default leaves the value of m unchanged.[11] Once again, it was possible find values of β and $\bar{y}$ that matched the average spreads and 70 percent of the average debt-to-output ratio (all other parameters were chosen as in Table 1 with the exception of λ and z, which were set at 1 and 0, respectively). These values turned out to be 0.71 and 0.841, respectively. Since these are very close to the values found earlier, the cyclical properties of the model turned out to be very similar as well. In particular, the relative volatility of

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[10]If we expanded the set of targets to include the relative volatility of consumption and insisted that the model output match the relative volatility very closely, the results would begin to resemble those in Arellano (2008). The model will match the relative volatility closely but the average spread and the average debt-to-output ratio will drop significantly from their target values.

[11]For this specification the set of m values for which the sovereign would choose to default is an interval of the form (m^L, m^U). A proof of this statement can be constructed along the lines of the proof given for Theorem 3 in Chatterjee et al.(2007). Thus, the computation must now locate two thresholds rather than one.

consumption remained high at 1.86. Since it is not necessary to assume the existence of m shocks to compute the one-period debt model, we also checked to see how the results change if we shut off the m shocks completely. The effects of doing this were minor. The average spread and the average debt-output ratio rose slightly to 8.81 and 0.71, respectively, and the relative volatility of consumption also rose slightly to 1.88.

To understand why long-term debt improves model performance we can look at Figures 4 and 5. These figures show how spreads behave with respect to borrowing in the models with long-term and short-term debt. Figure 4 shows these relationships when output is above trend. Observe that spreads are higher and rise more gradually for the long-term bond case than the short-term bond case. The spreads are higher because lenders anticipate that the country will borrow more in the future and thereby inflict capital losses on them. The spreads rise gradually because the incentive to default does not rise rapidly with debt – the fact that the country needs to "roll over" only a portion of its debt makes repayment an attractive option. The same gradual increase in spreads is evident in Figure 5 as well which shows the situation for below-trend output (observe that the scale for spreads is very different).

We can obtain some further intuition on how the model works by considering the sovereign's decision to issue additional debt. Although this choice is discrete in the model, we may think of it as being continuous for the moment. Also, for the moment, ignore the preference shock. Then, the marginal gain from borrowing is given by:

$$\left(-q(y,b') - \frac{\partial q(y,b')}{\partial b'}\left[b' - (1-\lambda)b\right]\right)u'\left(y + \lambda b - q(y,b')\left[b' - (1-\lambda)b\right]\right)$$

To understand the expression, it is easier to think of the sovereign as a "monopolistic" supplier of bonds. When the sovereign issues an extra unit of bond, it gets revenue from that extra unit sold, but at the same time the decrease in the price of the bond (because of increased default probability) decreases the revenue on all bonds that are currently issued. The sovereign borrows more with long-term bond because both the price schedule is flatter with long-term bonds (as seen in the figures), and also because the decrease in the price of the bond due to extra borrowing affects only the currently issued bonds, and with long-term bonds, the currently issued bonds are just a small portion of the outstanding debt.

The average spreads are much higher with long-term bonds. There are two reasons for this. One is, as we see in the above equation, the sovereign more readily enters these high probability of default regions, since it does not internalize the effect that increasing the level of debt has on decreasing the value of bonds issued in previous periods (which explains why the lenders take into account that the country will increase its debt in future periods and charge high spreads from the start). The second reason is that, with long-term bonds, just before defaulting, the sovereign issues high levels of debt at very high spreads. Typically, with long-term bonds, it is not very costly to delay default by one more period (by just paying the portion λ of the debt). But by borrowing at high spreads the sovereign can disproportionately tax the high-income outcomes next period (where it will not default), and consume more today. And it does borrow at very high spreads just before defaulting with long-term bonds. In contrast, with short-term debt this opportunity never arises, since raising enough money at very high spreads to refinance all existing debt becomes impossible.

The final issue we address is the welfare effects of lengthening maturity. In particular, we compare the welfare effects of moving from an environment with one-period bonds to one with long-term bonds. This comparison is done in the following way. We imagine that at some randomly chosen period, the sovereign is offered the option to issue long-term debt of the type ($\lambda = 0.05, z = 0.03$). This option is offered after the sovereign has chosen to repay whatever one-period debt is due that period but before it gets to choose its new level of short-term debt. We calculate the percentage of consumption the sovereign would be willing to pay in perpetuity to have this option. Formally, this is the value $\phi(y, m, b)$ that solves the equation

$$\phi(y, m, b)^{1-\sigma} V^{0,1}(y, m, b) = \max_{b'} u(y - b + q_{0.03, 0.05}(y, b')b' - m) + E_{(y', m')|y} V^{0.03, 0.05}(y', m', b')$$

for those states in which $V^{0,1}(y, m, b) \geq W^{0,1}(y, \bar{m})$, where $V^{z, \lambda}(y, m, b)$ is the value function when debt has coupon z and maturity λ. We simulate our model economy with one-period debt over many periods and calculate the mean ϕ over all such no-default states. It turns out that the mean ϕ is 0.995, meaning that the option to issue long-term debt is *not* valuable. The sovereign would be willing to pay, on average, 0.5 percent of consumption in perpetuity to not be forced to issue long-term debt of type $(0.03, 0.05)$.

Why is this? The welfare-decreasing effect of long-term bonds comes from the fact that the sovereign

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cannot commit to limit its future borrowing. The sovereign imposes a negative externality on international lenders who hold the sovereign's bonds when it increases its debt level (by inflicting a capital loss on the lenders bond holdings). With long-term bonds, the sovereign does not internalize this externality. Nevertheless, it pays the cost in previous periods because lenders take into account the sovereign's future actions when they purchase the sovereign's bonds. In contrast, when the sovereign issues all its bonds simultaneously each period, it internalizes the effect of capital loss incurred by additional borrowing. Although long-maturity debt provides more insurance with respect to adverse output shocks (only a small portion of the outstanding debt needs to be refinanced at a higher interest rate), this insurance does not turn out to be very valuable in the simulations. We have already seen that with one-period debt, the average level of debt and the average default spread is low (recall the last row of Table 3). In other words, with one-period debt the sovereign avoids getting into regions to default. Since the sovereign is not saddled with a lot of debt to begin with, the flexibility of repayment does not have much value, and thus there is no welfare gain from the option to issue long-term debt. However, this logic leaves open the possibility that for some other model parameters and some other choice of coupon payment and maturity, the option to issue long-term bonds may be valuable. Also, note that our model does not include features that might make long-term bonds more attractive relative to short-term bonds. In particular, there are no transactions costs of participating in the international credit market and there are no coordination issues between lenders that might lead to a "run" on short-term debt; i.e., we do not model the possibility that lenders become unwilling to roll over short-term debt because other lenders are reluctant to do the same.

7   Conclusion

In this paper, we developed a tractable model of long-term sovereign debt and applied it to understanding the behavior of debt and default spreads in an emerging market, namely, Argentina. We showed that introducing long-term debt improves upon existing quantitative models of sovereign debt in matching the average default spreads and the average level of the debt-to-output ratio, without sacrificing the model's ability to account for the key cyclical patterns in emerging market economies, namely, the relatively high volatility of consumption and the countercyclicality of

spreads (on sovereign debt) and the trade balance. We also provided a fairly complete set of characterization results regarding the default and borrowing behavior of the sovereign as well as a proof of the existence of an equilibrium pricing function with the property that the price of debt is decreasing in the amount of debt issued. We also addressed issues pertaining to the computation of long-term debt models, providing a novel and useful algorithm for computing the decision rule for debt in the presence of non-convexities.

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9   Appendix A: Proof of Lemma 2

The proof proceeds in 4 steps. The first step is to establish that if $u(0)$ is low enough, optimal consumption is always bounded below by a strictly positive number. Given this result, the second step shows that for two pricing functions sufficiently close to each other, any action feasible under one pricing function is also feasible under the other pricing function. Given this result, the third step shows that value functions are continuous in prices. The lemma follows easily from this step.

Step 1 (Optimal consumption is bounded below by a positive number). Since $u(\cdot)$ is bounded above, let M be such that $\sup_{x \geq 0} u(x) = M$. Let $\underline{y} = \min Y$. Assume that $u(0)$ is low enough so that $u(0) + \beta M/(1 - \beta) < u(\underline{y} - \phi(\underline{y}) - \bar{m})/(1 - \beta)$. Then, by continuity of u there exists $\underline{c} > 0$ such that $u(\underline{c}) + \beta M/(1 - \beta) < u(\underline{y} - \phi(\underline{y}) - \bar{m})/(1 - \beta)$. Then optimal consumption can never fall below $\underline{c}$. Since the sovereign can always consume at least $\underline{y} - \phi(\underline{y})$ each period it can guarantee itself a lifetime utility of at least $u(\underline{y} - \phi(\underline{y}) - \bar{m})/(1 - \beta)$. The highest utility possible from selecting an action that leads to consumption $\underline{c}$ or less is $u(\underline{c}) + \beta M/(1 - \beta)$. By assumption the former dominates the latter. Hence it is never optimal to choose to consume $\underline{c}$ or less.

Step 2 (Optimal decision rules under $\hat{q}$ are feasible under q^n and optimal decision rules under q^n are feasible under $\hat{q}$). Define $c(y, m, b; q)$ as the optimal consumption under prices q.

That is, $c(y, m, b; q) = y - \phi(y)$ if $d(y, m, b; q) = 1$ and $c(y, m, b; q) = y + [\lambda + (1 - \lambda)z]b + q(y, a(y, m, b; q))[a(y, m, b; q) - (1 - \lambda)b]$ if $d(y, m, b; q) = 0$. Define $c^n(y, m, b; \hat{q})$ as $c^n(y, m, b; \hat{q}) = y - \phi(y)$ if $d(y, m, b; \hat{q}) = 1$ and $c^n(y, m, b; \hat{q}) = y + [\lambda + (1 - \lambda)z]b + \hat{q}(y, a(y, m, b; q^n))[a(y, m, b; q^n) - (1 - \lambda)b]$ if $d(y, m, b; \hat{q}) = 0$. And define $\hat{c}(y, m, b; q^n)$ as $\hat{c}(y, m, b; q^n) = y - \phi(y)$ if $d(y, m, b; q^n) = 1$ and $\hat{c}(y, m, b; q^n) = y + [\lambda + (1 - \lambda)z]b + q^n(y, a(y, m, b; \hat{q}))[a(y, m, b; \hat{q}) - (1 - \lambda)b]$ if $d(y, m, b; q^n) = 0$.

Consider the difference $c^n(y, m, b; \hat{q}) - c(y, m, b; q^n)$. The difference is 0 for y, m, b such that $d(y, m, b; q^n) = 1$ and it is $\hat{q}(y, a(y, m, b; q^n))a(y, m, b; q^n) - q^n(y, a(y, m, b; q^n))a(y, m, b; q^n)$ otherwise. Therefore

$$|c^n(y, m, b; \hat{q}) - c(y, m, b; q^n)| \leq |\hat{q}(y, a(y, m, b; q^n)) - q^n(y, a(y, m, b; q^n))||a(y, m, b; q^n)|$$
$$\leq \max_{b'} |\hat{q}(y, b') - q^n(y, b')||b_I|$$

Since $q^n \to \hat{q}$, $\max_{b'} |\hat{q}(y, b') - q^n(y, b')| \to 0$. Hence $c^n(y, m, b; \hat{q})$ converges to $c(y, m, b; q^n)$. This implies that there exists N such that for all $n \geq N$, $|c^n(y, m, b; \hat{q}) - c(y, m, b; q^n)| < \underline{c}$. Hence $c^n(y, m, b; \hat{q}) > c(y, m, b; q^n) - \underline{c}$. By Step 1 $c(y, m, b; q^n) - \underline{c} > 0$ and therefore $c^n(y, m, b; \hat{q})$ is feasible for all $n \geq N$.

Next, consider the difference $\hat{c}(y, m, b; q^n) - c(y, m, b; \hat{q})$. We have

$$|\hat{c}(y, m, b; q^n) - c(y, m, b; \hat{q})| \leq |q^n(y, a(y, m, b; \hat{q})) - \hat{q}(y, a(y, m, b; \hat{q}))||a(y, m, b; \hat{q})|.$$

Since $q^n \to \hat{q}$, $c(y, m, b; q^n)$ converges to $c(y, m, b; \hat{q})$. Therefore, there exists N such that for all $n \geq N$ $|\hat{c}(y, m, b; q^n) - c(y, m, b; \hat{q})| < \underline{c}$. Therefore, $\hat{c}(y, m, b; q^n) > c(y, m, b; \hat{q}) - \underline{c} > 0$. Thus $\hat{c}(y, m, b; q^n)$ is feasible for all $n \geq N$.

Step 3 (Continuity of $V(y, m, b; q)$, $W(y, \bar{m}; q)$ and $X_y(b'; q)$ with respect to q). Let $q^n \to \hat{q}$. Define $\hat{V}(y, m, b; q^n)$ as the utility under repayment from following the decision rules $d(y, m, b; \hat{q})$ and $a(y, m, b; \hat{q})$ when the price function is q^n. And define $V^n(y, m, b; \hat{q})$ as the utility under repayment from following the decision rules $d(y, m, b; q^n)$ and $a(y, m, b; q^n)$ when the price function is $\hat{q}$. By Step 2 these constructs are well-defined. Then $V(y, m, b; q^n) \geq \hat{V}(y, m, b; q^n)$ and $V(y, m, b; \hat{q}) \geq V^n(y, m, b; \hat{q})$. Observe that the first inequality implies $V(y, m, b; q^n) \geq \hat{V}(y, m, b; q^n) - V(y, m, b; \hat{q})] +$

$V(y, m, b; \hat{q})$. By Step 2 $\lim_{n\to\infty} \hat{c}(y, m, b; q^n) = c(y, m, b; \hat{q})$. Hence $\lim_{n\to\infty}[\hat{V}(y, m, b; q^n) - V(y, m, b; \hat{q})] = 0$. Therefore, $\liminf_{n\to\infty} V(y, m, b; q^n) \geq V(y, m, b; \hat{q})$. Next, observe that the second inequality implies $V(y, m, b; \hat{q}) \geq [V^n(y, m, b; \hat{q}) - V(y, m, b; q^n)] + V(y, m, b; q^n)$. By Step 2, $\lim_{n\to\infty} c^n(y, m, b; \hat{q}) = c(y, m, b; q^n)$. Therefore, $\lim_{n\to\infty}[V^n(y, m, b; \hat{q}) - V(y, m, b; q^n)] = 0$. Therefore $V(y, m, b; \hat{q}) \geq \limsup_{n\to\infty} V(y, m, b; q^n)$. Since the limit superior of any sequence is at least as large as the limit inferior, $\limsup_{n\to\infty} V(y, m, b; q^n) = \liminf_{n\to\infty} V(y, m, b; q^n) = V(y, m, b; \hat{q})$. Therefore $\lim_{n\to\infty} V(y, m, b; q^n)$ exists and is equal to $V(y, m, b; \hat{q})$. Thus, $V(y, m, b; q)$ is continuous in q.

Define $\hat{W}(y, m; q^n)$ as the utility under default from following the decision rules $d(y, m, b; \hat{q})$ and $a(y, m, b; \hat{q})$ when the price function is q^n. And define $W^n(y, m, b; \hat{q})$ as the utility under repayment from following the decision rules $d(y, m, b; q^n)$ and $a(y, m, b; q^n)$ when the price function is $\hat{q}$. By Step 2, these constructs are well-defined. The steps in the previous paragraph can then be repeated to conclude that $\lim_{n\to\infty} W(y, m; q^n) = W(y, m; \hat{q})$.

Finally, $X_y(b'; q^n) = E_{(y', m'|y)} \max\{W(y', \bar{m}; q^n), V(y', m', b'; q^n)\}$. Continuity of $W(y, m; q)$ and $V(y, m, b; q)$ implies $\lim_{n\to\infty} \max\{W(y', \bar{m}; q^n), V(y', m', b'; q^n)\} = \max\{W(y', \bar{m}; \hat{q}), V(y', m', b'; \hat{q})\}$. Since the functions $\max\{W(y', \bar{m}; q^n), V(y', m', b'; q^n)\}$ are non-negative and bounded above by $M/(1 - \beta)$, by the Lebesgue Dominated Convergence Theorem

$$\lim_{n\to\infty} \int \max\{W(y', \bar{m}; q^n), V(y', m', b'; q^n)\} dG(m') = \int \max\{W(y', \bar{m}; \hat{q}), V(y', m', b'; \hat{q})\} dG(m').$$

Therefore,

$$\lim_{n\to\infty} E_{(y', m'|y)} \max\{W(y', \bar{m}; q^n), V(y', m', b'; q^n)\} = E_{(y', m'|y)} \max\{W(y', \bar{m}; \hat{q}), V(y', m', b'; \hat{q})\}.$$

Step 4A (Convergence of $a(y, m, b; q^n)$). Let $q^n \to \hat{q}$. Fix y and b. For a given m, let $b'^0 = a(y, m, b; \hat{q})$. Let $V_{b'}(y, m, b; \hat{q})$ denote the lifetime utility if the sovereign chooses to borrow b' in the current period but follows the optimal plan in all future periods. Two cases are possible: (i) $V(y, m, b; \hat{q}) > V_{b'}(y, m, b; \hat{q})$ for all $b' \neq b'^0$ and (ii) $V(y, m, b; \hat{q}) = V_{b'}(y, m, b; \hat{q})$ for some $b' \neq b'^0$. Consider case (i). Let $V(y, m, b; \hat{q}) - V_{b'}(y, m, b; \hat{q}) = \Delta$. Since $V(y, m, b; q)$ is continuous in q there exists N_1 such that for all $n \geq N_1$ $V(y, m, b; q^n) > V(y, m, b; \hat{q}) - \Delta/2$. Next, note

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that $V_{b'}(y, m, b; q^n) = u(y - m + [\lambda + (1 - \lambda)z]b + q^n(y, b')[b' - (1 - \lambda)b]) + \beta X_y(b'; q^n)$. Since $X_y(b'; q)$ is continuous in q it follows that there exists N_2 such that for all $n \geq N_2$ $V_{b'}(y, m, b; q^n) < V_{b'}(y, m, b; \hat{q}) + \Delta/2$. Therefore $V(y, m, b; q^n) - V_{b'}(y, m, b; q^n) > V(y, m, b; \hat{q}) - \Delta/2 - V_{b'}(y, m, b; \hat{q}) - \Delta/2 = 0$ for all $n \geq \max\{N_1, N_2\}$. Hence $a(y, m, b; q^n) = b'^0$ for all $n > \max\{N_1, N_2\}$. Now consider case (ii). In this case, convergence may fail because $a(y, m, b; q^n)$ may converge to b' rather than b'^0. However, by Lemma 1 there can be only a finite number of m values for which case (ii) can hold. Therefore, given y and b, we may conclude that the functions $a(y, m, b; q^n)$ converge pointwise to $a(y, m, b; \hat{q})$ except, possibly, for a finite number of m values.

Step 4B (Convergence of $d(y, m, b; q^n)$). Let $q^n \to \hat{q}$. Fix y and b. Again, two cases are possible. (i) $W(y) \neq V(y, m, b; \hat{q})$ and (ii)$W(y) = V(y, m, b; \hat{q})$. Consider case (i). For concreteness, suppose that $W(y) - V(y, m, b; \hat{q}) = \Delta > 0$. Then, by continuity of $V(y, m, b; q)$ there exists N such that for all $n \geq N$, $V(y, m, b; q^n) < V(y, m, b; \hat{q}) + \Delta$. For all such n, $W(y) - V(y, m, b; q^n) > W(y) - V(y, m, b; \hat{q}) - \Delta = 0$. Hence $d(y, m, b; q^n) = d(y, m, b; \hat{q}) = 1$ for all $n \geq N$. If $\Delta < 0$ then there exists N such that for all $n \geq N$, $V(y, m, b; q^n) > V(y, m, b; \hat{q}) + \Delta$. For all such n, $W(y) - V(y, m, b; q^n) < W(y) - V(y, m, b; \hat{q}) - \Delta = 0$. Hence $d(y, m, b; q^n) = d(y, m, b; \hat{q}) = 0$ for all $n \geq N$. Now consider case (ii). Again, convergence may fail in this case because $d(y, m, b; q^n)$ may converge to 1 or 0 while $d(y, m, b; \hat{q})$ is 0 or 1. However, by Proposition 2, there can only be one value of m for which (ii) can hold. Therefore, we may conclude that the functions $d(y, m, b; q^n)$ converge pointwise to $d(y, m, b; \hat{q})$ except, possibly, at one value of m.

$\square$

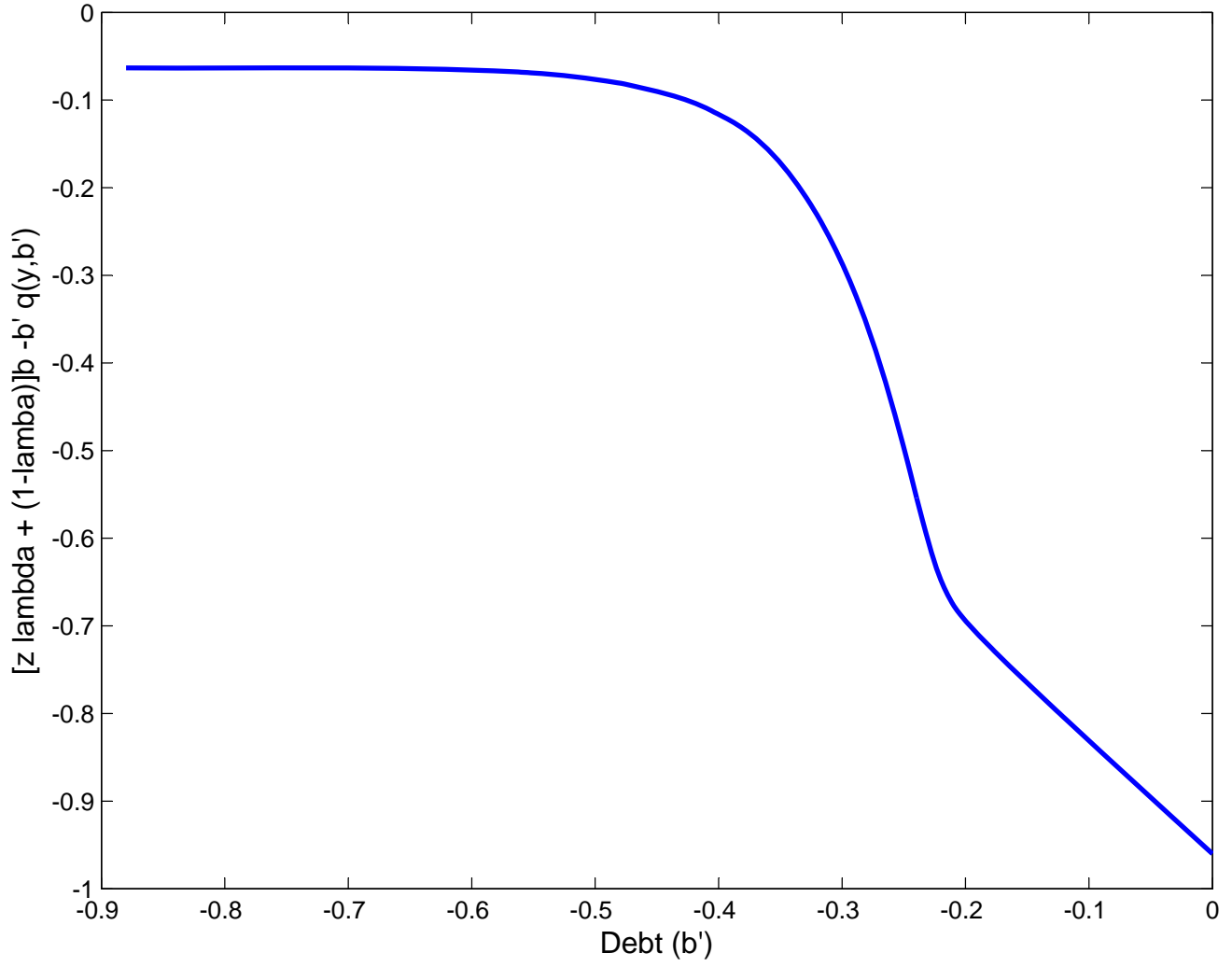Figure1 : Nonconvexity of the Budget Set

Figure 2: Nonconcavity of Next Period's Expected Utility
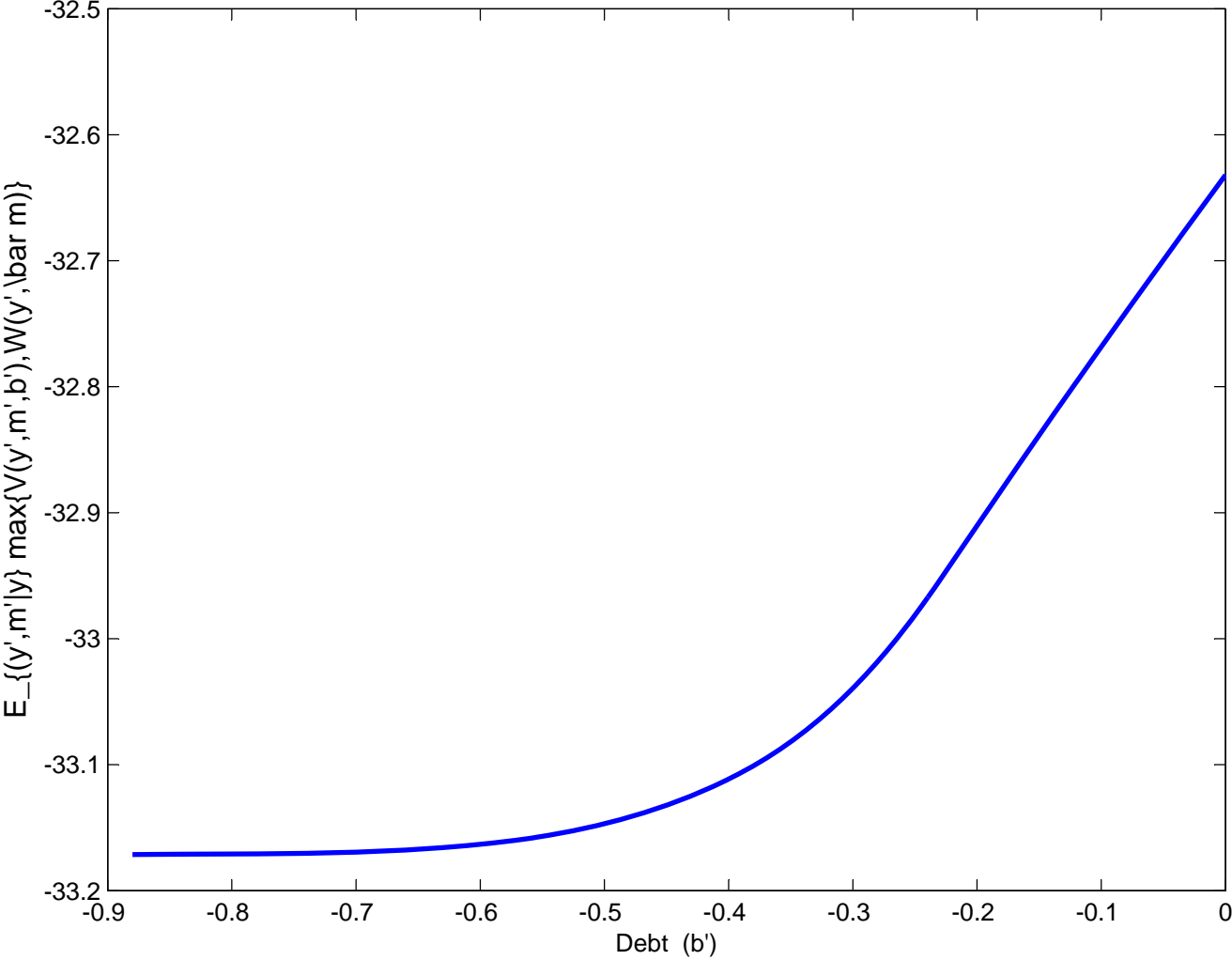
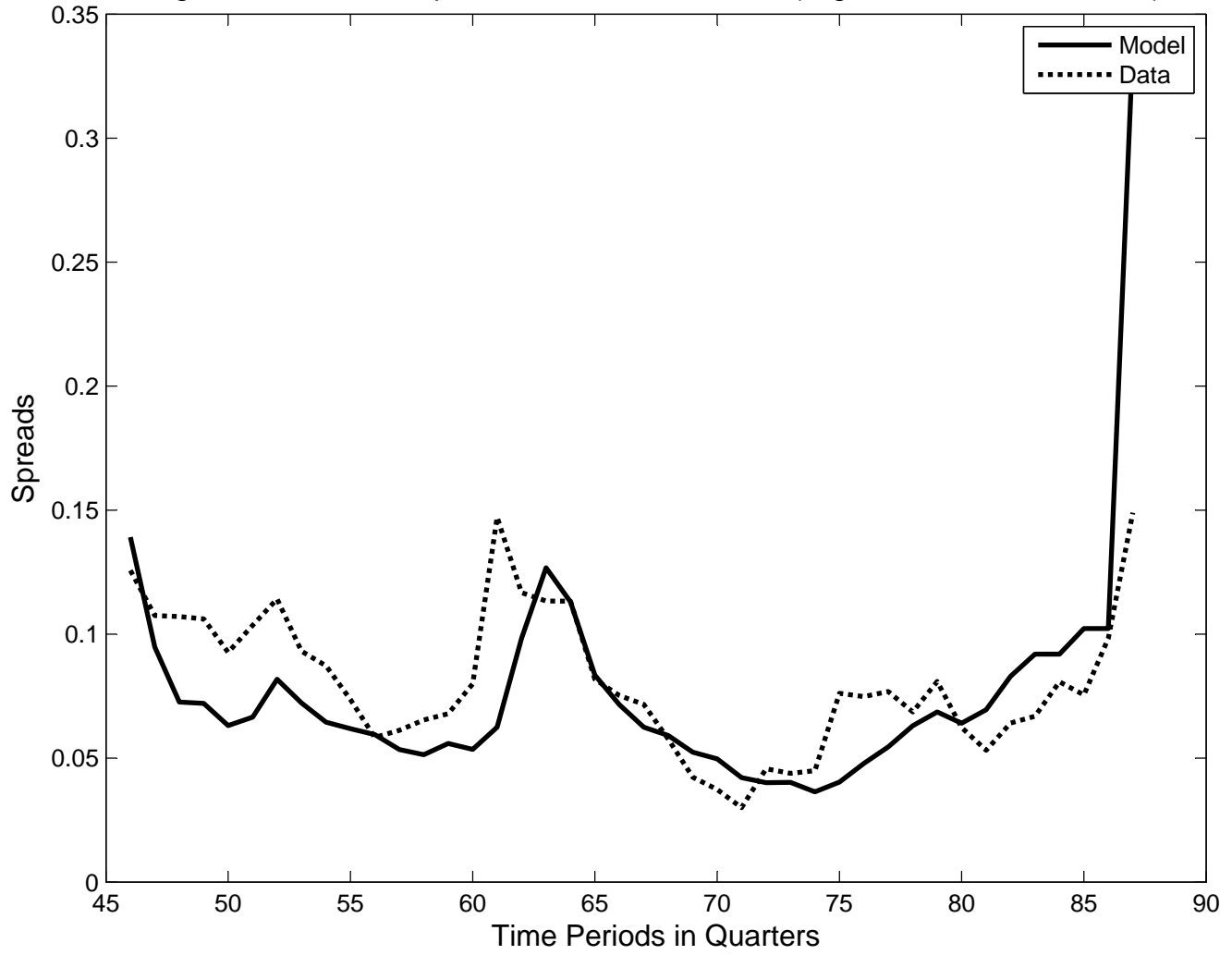Figure 3: Simulated Spreads for Model and Data (Argentina, 1991:1 - 2001:4)

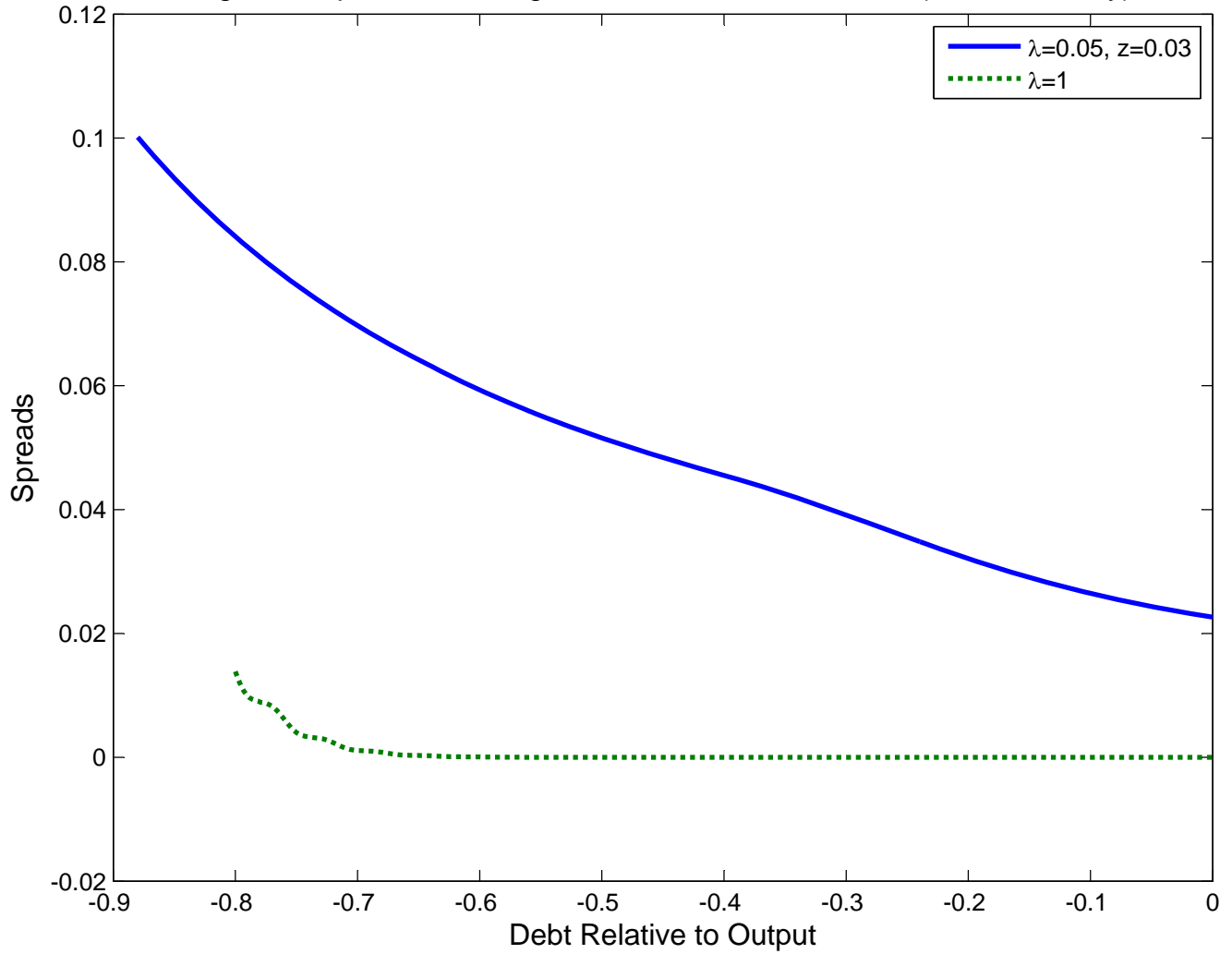Figure 4: Spreads for Long-term and Short-term Bonds (Above-Trend y)

Figure 5: Spreads for Long-term and Short-term Bonds (Below-Trend y)