Online Appendix to "Optimal Monetary Policy during a Cost-of-Living Crisis"

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A Model derivations

A.1 Households

In this section, we present the household problem. The economy is populated by a mass 1 of households, uniformly distributed and indexed by i. Household preferences are additively separable across time, have a common discount factor β and a common death rate δ, while intratemporal preferences – over consumption and labor – can depend on their type i. In addition to intratemporal preferences, the type of the household determines its initial wealth and permanent labor productivity. When a household dies, it is replaced by a household of the same type. We will assume throughout the paper that the dividend shares are equal to the steady-state labor income shares ($\varsigma_k(i) = Wn(i)/WN$). For each type i, a fraction $\varphi(i)$ is Hand-to-Mouth (HtM) while the remaining $1-\varphi(i)$ is unconstrained. The birth date of a household is denoted by a superscript t_0.

A.1.1 Unconstrained households

Unconstrained households can save and borrow in a risk-free bond b_t and are subject to the natural borrowing constraint (or equivalently a no Ponzi condition). We decompose the household problem into an intertemporal saving problem and an intratemporal consumption/labor supply problem. For now, we omit the superscript t_0 to simplify the notation.

1. Given time separability, the **intertemporal problem** of unconstrained households born at t is to maximize the flow of intratemporal indirect utility $\mathcal{V}_i(\mathcal{I}, W, \mathbf{P})$, defined below. $\mathcal{V}_i(\mathcal{I}, W, \mathbf{P})$ depends on intratemporal income $\mathcal{I}$, the wage W and sectoral prices $\mathbf{P} = \{P_1, ..., P_K\}$. The intertemporal problem is given by:

$$V_t(i) = \sup_{\{\mathcal{I}_{t+s}, b_{t+s+1}\}_{s=0}^{\infty}} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta(1-\delta))^s \mathcal{V}_i(\mathcal{I}_{t+s}, W_{t+s}, \mathbf{P}_{t+s}) \tag{1}$$

$$s.t. \quad \mathcal{I}_{t+s} = b_{t+s} + \sum_{k=1}^{K} \varsigma_k(i)\, Div_{k,t+s} - \frac{b_{t+s+1}}{R_{t+s}}, \tag{2}$$

$$b_t = b(i)\left(1 + \sum_{k=1}^{K} \bar{s}_k \frac{P_{k,t} - P_k^*}{P_k^*}\right), \quad \lim_{s\to\infty} b_{t+s} / \prod_{u=0}^{s-1} R_{t+u} = 0 \ a.s.$$

Household i earns a constant share, $\varsigma_k(i)$, of dividends in sector k ($Div_{k,t}$). The initial real wealth (in terms of the CPI) of household i, born at time t is $b_t(i) = b(i)\left(1 + \sum_k \bar{s}_k \frac{P_{k,t}-P_k^*}{P_k^*}\right)$– where P_k^* denotes (initial) steady state prices and $\bar{s}_k$ is the steady-state expenditure share of k. This assumption implies that the initial real wealth of a household i born at t is set to its steady-state value. By assumption $\int_0^1 b(i)\, di = 0$.

2. The **intratemporal problem** determines the optimal allocation of labor n and consumption expenditure e and defines the function $\mathcal{V}_i(\mathcal{I}, W, \mathbf{P})$:

$$\mathcal{V}_i(\mathcal{I}, W, \mathbf{P}) = \sup_{e,n} v_i(e, \mathbf{P}) - \chi_i\left(\frac{n}{\vartheta(i)}\right) \quad s.t. \quad e - Wn = \mathcal{I}. \tag{3}$$

Here, $\vartheta(i)$ denotes (constant) labor productivity, the function χ_i is an increasing, C^2 and strictly convex function, and $v_i(e, \mathbf{P})$ is the indirect utility of consumption, defined below.

3. Finally, the indirect utility of consumption $v_i(e, \mathbf{P})$ is defined by the **consumption problem** which determines the optimal allocation of expenditure e across sectors. Household preferences for varieties within sector k are defined by a CES aggregator (common across households): $c_k = \left(\int c_k(j)^{\frac{\epsilon_k - 1}{\epsilon_k}} dj\right)^{\frac{\epsilon_k}{\epsilon_k - 1}}$. Utility maximization implies that the demand for variety j is given by $c_k(j) = (p_k(j)/P_k)^{-\epsilon_k} c_k$ where the sectoral price index is $P_k = \left(\int p_k(j)^{1-\epsilon_k} dj\right)^{\frac{1}{1-\epsilon_k}}$. The consumption problem can therefore be written as:

$$v_i(e, \mathbf{P}) = \sup_{\{c_k\}_{k=1}^{K}} u_i(c_1, c_2, \dots, c_K), \quad s.t. \quad \sum_{k=1}^{K} P_k c_k = e,\ P_k = \left(\int p_k(j)^{1-\epsilon_k} dj\right)^{\frac{1}{1-\epsilon_k}}, \tag{4}$$

where u_i is increasing, C^2 and strictly concave.

First-order conditions and log-linearization We now derive the optimal responses of consumption and labor supply to changes in prices (variety prices, wage, and interest rate) near a steady state where variety prices are constant and equal within sectors, and the real interest rate satisfies $R_t^{-1} = (1-\delta)\beta$. For the sake of readability, for the consumption and intratemporal problems, we omit time subscripts in this section of the appendix, unless stated otherwise. We denote log deviations of variables – from the steady state – with a hat.

Consumption. The consumption problem stated in Equation (4) defines a Marshallian demand function $c_{k,i}(e, \boldsymbol{P})$ and an expenditure function $e_{k,i}(e, \boldsymbol{P}) = P_k c_{k,i}(e, \boldsymbol{P})$ which are C^2. The demand functions are defined implicitly from the first-order conditions of the consumption problem ($\partial_{c_k} u_i / P_k = \partial_e v_i$ and $\sum P_k c_k = e$) and could be expressed in terms of the utility function. However, it is equivalent and more tractable to take them as primitives of the household problem. The response of sectoral consumption to a price change is then given by:

$$\hat{c}_{k,i}(e,\boldsymbol{P}) = \partial_e e_k(i)\left(\hat{e} - \sum_{l=1}^{K} s_l(i)\hat{P}_l\right)\frac{1}{s_k(i)} + \sum_{l=1}^{K}\rho_{k,l}(i)\hat{P}_l,$$

where $\rho_{k,l}(i) = \left(P_k \partial_{\ln P_l} c_{k,i}(e,\boldsymbol{P}) + e_{l,i}(e,\boldsymbol{P})\partial_e e_{k,i}(e,\boldsymbol{P})\right)/e_{k,i}(e,\boldsymbol{P})$ is the compensated price elasticity of c_k, and $s_k(i) = e_{k,i}(e,\boldsymbol{P})/e$, and $\partial_e e_k(i) = \partial_e e_{k,i}(e,\boldsymbol{P})$ are the expenditure share and marginal expenditure of sector k, respectively.
In addition, we record in the following lemma the derivatives of the marginal value of income $\partial_e v_i(e,\boldsymbol{P})$, which are crucial to characterize the optimal choice of labor supply and consumption expenditure in the intratemporal problem.

Lemma 1. *The derivatives of the marginal value of income $\partial_e v_i(e,\boldsymbol{P})$ with respect to prices are given by:*

$$P_k \partial_{P_k}\{\partial_e v_i(e,\boldsymbol{P})\} = -\partial_e v_i(e,\boldsymbol{P})\left(\partial_e e_{k,i}(e,\boldsymbol{P}) - \frac{1}{\sigma(i)}s_{k,i}(e,\boldsymbol{P})\right),$$

with $\sigma(i) \equiv -\partial_e v_i(e,\boldsymbol{P})/(e\partial_{ee}v_i(e,\boldsymbol{P}))$. In addition, we can always renormalize the utility function u_i so that, at initial prices, $\sigma(i)$ is common across households and equal to σ without changing the demand functions $c_{k,i}(e,\boldsymbol{P})$.

Proof. We have:

$$\begin{aligned} P_k \partial_{P_k}\{\partial_e v_i(e,\boldsymbol{P})\} &= P_k \partial_e\left\{\partial_{P_k} v_i(e,\boldsymbol{P})\right\}, \\ &= -P_k \partial_e\left\{\partial_e v_i(e,\boldsymbol{P})\, c_{k,i}(e,\boldsymbol{P})\right\}, \\ &= -\partial_e v_i(e,\boldsymbol{P})\left(\partial_e e_{k,i}(e,\boldsymbol{P}) - \frac{1}{\sigma_i}s_{k,i}(e,\boldsymbol{P})\right), \end{aligned}$$

where the first equality is Schwarz's identity and the second is Roy's identity. In addition, redefining $\tilde{u}_i = \Upsilon_i(u_i)$ with $\Upsilon_i(\cdot) = \left(v_i^{-1}(\cdot,\boldsymbol{P})\right)^{1-\frac{1}{\sigma}}/\left(1-\frac{1}{\sigma}\right)$, it is straightforward to verify that Υ_i is increasing so that the demand functions are left unchanged and that $\tilde{v}_i(e,\boldsymbol{P}) = \Upsilon_i(v_i(e,\boldsymbol{P}))$ satisfies $-\partial_e \tilde{v}_i(e,\boldsymbol{P})/(e\partial_{ee}\tilde{v}_i(e,\boldsymbol{P})) = \sigma$. □

Let us briefly comment on the result of Lemma 1. The second term in the expression, $\partial_e v_i(e,\boldsymbol{P})\frac{1}{\sigma_i}s_{k,i}(e,\boldsymbol{P})$, is standard: as the price of sector k increases, the consumption of agent i decreases, increasing the marginal value of income according to the curvature of v_i summarized by $1/\sigma_i$. The first term is more novel. Recall that $\partial_e v_i(e,\boldsymbol{P})$ is the utility value of an additional dollar of income. With this additional dollar, agent i spends on good k according to the marginal budget share $\partial_e e_{k,i}(e,\boldsymbol{P})$. An increase in the price of good k therefore lowers the amount the household can consume with the additional dollar by $-\partial_e e_{k,i}(e,\boldsymbol{P})$. To take an extreme example, for a necessity such that $\partial_e e_{k,i}(e,\boldsymbol{P}) = 0$, a compensated increase in P_k does not change the value of an additional dollar of income as the agent does not spend on k at the margin.

Intratemporal Problem. We now derive the optimal choice of labor supply. The first-order condition for the intratemporal problem (3) determining optimal labor supply and expenditure is:

$$W\partial_e v_i(e,\boldsymbol{P}) = \frac{1}{\vartheta(i)}\chi_i'\left(\frac{n}{\vartheta(i)}\right).$$

Together with the intratemporal budget constraint, this determines labor supply $n(i) = n_i(\mathcal{I},W,\boldsymbol{P})$ and expenditure $e(i) = n_i(\mathcal{I},W,\boldsymbol{P})$. Log-linearizing the above equation near the optimal allocation we obtain the following lemma:

Lemma 2. *The change in labor supply in response to a change in wages, prices and expenditures is given by:*

$$\hat{n}(i) = \psi(i)\left\{\hat{W} - \sum_{k=1}^{K}\partial_e e_k(i)\hat{P}_k\right\} - \frac{\psi(i)}{\sigma(i)}\left(\hat{e}(i) - \sum_{k=1}^{K}s_k(i)\hat{P}_k\right).$$

where $\psi(i) \equiv \chi_i'\left(\frac{n(i)}{\vartheta(i)}\right)/\left(\frac{n(i)}{\vartheta(i)}\chi_i''\left(\frac{n(i)}{\vartheta(i)}\right)\right)$ is the Frisch elasticity of labor supply. In addition, using $\chi_i\left(\frac{n}{\vartheta(i)}\right) = \kappa_i\left(\frac{n}{\vartheta(i)}\right)^{1+\frac{1}{\psi}}$ with $\kappa_i > 0$, we can impose that ψ_i is common across households and equal to ψ (while the scaling constant $\kappa_i/\vartheta(i)^{1+\frac{1}{\psi}}$ can be used to impose the steady-state level of labor supply). Finally, the envelope condition gives $\partial_{\mathcal{I}}\mathcal{V}_i(\mathcal{I},W,\mathbf{P}) = \partial_e v_i(e(i),\boldsymbol{P})$.

Proof. The proof is direct using Lemma 1, we have:

$$\begin{aligned} \frac{\frac{n(i)}{\vartheta(i)}\chi''\left(\frac{n(i)}{\vartheta(i)}\right)}{\chi'\left(\frac{n(i)}{\vartheta(i)}\right)}\hat{n}(i) &= \frac{e(i)\partial_{ee}v_i}{\partial_e v_i}\hat{e}(i) + \frac{1}{\partial_e v_i}\sum_k \frac{\partial}{\partial e}\left\{P_k \partial_{P_k} v_i\right\}\hat{P}_k + \hat{W}, \\ &= \frac{e(i)\partial_{ee}v_i}{\partial_e v_i}\hat{e}(i) - \frac{1}{\partial_e v_i}\sum_k\left\{\partial_e v_i \partial_e e_{k,i} + e_{k,i}\partial_{ee}v_i\right\}\hat{P}_k + \hat{W}, \\ \Rightarrow \hat{n}(i) &= \psi(i)\left\{\hat{W} - \sum_{k=1}^{K}\partial_e e_k(i)\hat{P}_k\right\} - \frac{\psi(i)}{\sigma(i)}\left(\hat{e}(i) - \sum_{k=1}^{K}s_k(i)\hat{P}_k\right). \end{aligned}$$

The other items of the lemma are trivial.□

Intertemporal Problem. We now reintroduce time subscripts and derive the optimal choice of expenditure across time by log-linearizing the first-order conditions of the intertemporal problem (1) given by:

$$\frac{\partial \mathcal{V}_i\left(\mathcal{I}_{t+s}, W_{t+s}, \mathbf{P}_{t+s}\right)}{\partial \mathcal{I}_{t+s}} = \beta(1-\delta) R_{t+s} \mathbb{E}_{t+s} \frac{\partial \mathcal{V}_i\left(\mathcal{I}_{t+s+1}, W_{t+s+1}, \mathbf{P}_{t+s+1}\right)}{\partial \mathcal{I}_{t+s+1}},$$
$$\Rightarrow \partial_e v_{i,t+s} = \beta(1-\delta) R_{t+s} \mathbb{E}_{t+s} \partial_e v_{i,t+s+1},$$

where the second line uses the envelope result of Lemma 2 and for simplicity we denote $\partial_e v_{i,t} = \partial_e v_i\left(e\left(\mathcal{I}_t, W_t, \mathbf{P}_t\right), \boldsymbol{P}_t\right)$. We record the result in the following lemma:

Lemma 3. *Expenditure near a steady state where $R_t^{-1} = (1-\delta)\beta$ satisfies the following Euler equation:*

$$\left(\hat{e}_t(i) - \sum_k s_k(i) \hat{P}_{k,t}\right) = \mathbb{E}_t\left(\hat{e}_{t+1}(i) - \sum_k s_k(i) \hat{P}_{k,t+1}\right) - \sigma \mathbb{E}_t\left(\hat{R}_t - \sum_k \partial_e e_k(i) \pi_{k,t+1}\right),$$

where we used the fact (from Lemma 1) that the EIS can be common across households.

Proof. The proof is direct. Using Lemma 1, we have:

$$\left(\partial_{ee} v_i\right) de_t(i) + \sum_k \left(\partial_{eP_k} v_i\right) dP_{k,t} = \beta(1-\delta) dR_t \mathbb{E}_t \partial_e v_i + \beta(1-\delta) R \mathbb{E}_t\left(\left(\partial_{ee} v_i\right) de_{t+1}(i) + \sum_k \left(\partial_{eP_k} v_i\right) dP_{k,t+1}\right),$$
$$\Rightarrow \left(\hat{e}_t(i) - \sum_k s_k(i) \hat{P}_{k,t}\right) = \mathbb{E}_t\left(\hat{e}_{t+1}(i) - \sum_k s_k(i) \hat{P}_{k,t+1}\right) - \sigma \mathbb{E}_t\left(\hat{R}_t - \sum_k \partial_e e_k(i) \pi_{k,t+1}\right). \square$$

Taking stock, the log-linearized version of the unconstrained household problem are summarized by the following three equations (in addition to the transversality condition). The first equation summarizes optimal consumption choices in terms of income and substitution effects. The second is the marginal rate of substitution between leisure and consumption expenditures. The third is the individual Euler equation.

$$\hat{c}_{k,t+s}(i) = \partial_e e_k(i)\left(\hat{e}_{t+s}(i) - \sum_{l=1}^K s_l(i) \hat{P}_{l,t+s}\right) \frac{1}{s_k(i)} + \sum_{l=1}^K \rho_{k,l}(i) \hat{P}_{l,t+s},$$
$$\hat{W}_{t+s} - \sum_{k=1}^K \partial_e e_k(i) \hat{P}_{k,t+s} = \frac{1}{\psi} \hat{n}_{t+s} - \frac{1}{\sigma}\left(\hat{e}_{t+s}(i) - \sum_{k=1}^K s_k(i) \hat{P}_{k,t+s}\right),$$
$$\left(\hat{e}_{t+s}(i) - \sum_k s_k(i) \hat{P}_{k,t+s}\right) = \mathbb{E}_t\left(\hat{e}_{t+s+1}(i) - \sum_k s_k(i) \hat{P}_{k,t+s+1}\right) - \sigma \mathbb{E}_t\left(\hat{R}_{t+s} - \sum_k \partial_e e_k(i) \pi_{k,t+s+1}\right).$$

A.1.2 Hand-to-Mouth households

HtM households consume all their current income, i.e. they never adjust their nominal bond holdings. Recall that a household of type i born in t has initial bond holdings $b_t = b(i)\left(1 + \sum_{k=1}^K \bar{s}_k \frac{P_{k,t} - P_k^*}{P_k^*}\right)$. The problem of HtM households can be rewritten as:

$$\mathcal{V}_i\left(\mathcal{I}_{t+s}, W_{t+s}, \mathbf{P_{t+s}}\right) = \sup_{e,n} v_i\left(e, \mathbf{P}_{t+s}\right) - \chi_i\left(\frac{n}{\vartheta(i)}\right) \quad s.t. \quad e - W_{t+s} n = \mathcal{I}_{t+s},\ \mathcal{I}_{t+s} = \left(1 - \frac{1}{R_{t+s}}\right) b_t + \sum_{k=1}^K \varsigma_k(i) Div_{k,t+s}.$$

Using the results of Lemma 1 and 2, we obtain:

$$\hat{c}_{k,t+s}(i) = \partial_e e_k(i)\left(\hat{e}_{t+s}(i) - \sum_{l=1}^K s_l(i) \hat{P}_{l,t+s}\right) \frac{1}{s_k(i)} + \sum_{l=1}^K \rho_{k,l}(i) \hat{P}_{l,t+s},$$
$$\hat{W}_{t+s} - \sum_{k=1}^K \partial_e e_k(i) \hat{P}_{k,t+s} = \frac{1}{\psi} \hat{n}_{t+s} - \frac{1}{\sigma}\left(\hat{e}_{t+s}(i) - \sum_{k=1}^K s_k(i) \hat{P}_{k,t+s}\right),$$
$$\hat{e}_{t+s}(i) - \sum_{k=1}^K s_k(i) \hat{P}_{k,t+s} = \frac{1}{e(i) + \frac{\psi}{\sigma} Wn(i)}\left(b(i)\left(\frac{\hat{R}_{t+s}}{R} + \left(1 - \frac{1}{R}\right) \sum_{k=1}^K \bar{s}_k\left(\hat{P}_{k,t} - \hat{P}_{k,t+s}\right)\right) - \sum_{k=1}^K \left(e(i)\left(s_k(i) - \bar{s}_k\right)\right) \hat{P}_{k,t+s}\right)$$
$$+ \frac{1}{e(i) + \frac{\psi}{\sigma} Wn(i)}\left(Wn(i)\left(\hat{W}_{t+s} - \sum_{k=1}^K \bar{s}_k \hat{P}_{k,t+s} + \psi\left(\hat{W}_{t+s} - \sum_{k=1}^K \partial_e e_k(i) \hat{P}_{k,t+s}\right)\right) + \sum_k \varsigma_k(i) dDiv_{k,t+s}\right).$$

A.2 Firms

In this section, we derive the sectoral New Keynesian Phillips Curves. In each sector, ex-ante identical firms with constant-return-to-scale (CRS) technology produce varieties of good k using labor and a bundle of sector l goods, aggregated by a representative intermediary as inputs. We first derive the firm's pricing equation away from steady-state as a function of the change in unit marginal cost. We then study the firm's intratemporal problem to derive changes in demand for intermediate inputs and labor.

Finally, using market clearing conditions for goods and labor, we derive the sectoral NKPCs in terms of sectoral prices and the output gap.

A.2.1 Intermediate inputs producers

We start with competitive intermediaries producing intermediate inputs. They aggregate differentiated varieties into $\tilde{Y}_k$, using the same CES aggregator as the household aggregator, and sell them to firms at a price P_k^I:

$$\sup_{\{\tilde{y}_k(j)\}_{0\le j\le 1}} P_k^I \tilde{Y}_k - \int_0^1 p_k(j)\tilde{y}_k(j)dj \quad s.t. \quad \tilde{Y}_k = \left(\tilde{y}_k(j)^{\frac{\epsilon_k-1}{\epsilon_k}} dj\right)^{\frac{\epsilon_k}{\epsilon_k-1}}.$$

We therefore have $P_k^I = \left(\int_0^1 p_k^{1-\epsilon_k}(j)dj\right)^{\frac{1}{1-\epsilon_k}} = P_k$, $\tilde{y}_k(j) = (p_k(j)/P_k)^{-\epsilon_k}\tilde{Y}_k$.

A.2.2 Goods varieties firms: price setting

We now turn to the firms producing individual goods varieties. Denoting $Y_k = \tilde{Y}_k + C_k$ total demand for sector k, demand for variety j is simply $y_k(j) = (p_k(j)/P_k)^{-\epsilon_k} Y_k$. We can write the present value of firm profits in terms of the reset price and using the fact that production of firms in k has CRS:*

$$\max_{p_{k,t}(j^*)} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left\{ (p_{k,t}(j^*) - (1-\tau_k)MC_k(W_{t+s}, \boldsymbol{P}_{t+s})) \left(\frac{p_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k} Y_{k,t+s} - T_{k,t+s} \right\},$$

where MC_k is the marginal cost, to be specified below. The first-order optimality condition gives the standard pricing condition with CES demand:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left(p_{k,t}(j^*) - \frac{\epsilon_k}{\epsilon_k-1}(1-\tau_k)MC_k(W_{t+s}, \boldsymbol{P}_{t+s})\right)\left(\frac{p_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k} Y_{k,t+s} = 0.$$

In steady state, we have $p_{k,ss}(j^*) = P_{k,ss} = \frac{\epsilon_k}{\epsilon_k-1}(1-\tau_k)MC_k(W_{ss}, \boldsymbol{P}_{ss})$. Taking a first-order approximation of the first-order optimality condition and using the expressions above around the steady state we obtain:

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left(dp_{k,t}(j^*) - \frac{\epsilon_k}{\epsilon_k-1}(1-\tau_k)dMC_{k,t+s}\right)$$

$$\Rightarrow \hat{p}_{k,t}(j^*) = \left(1-\tilde{\beta}\theta_k\right)\mathbb{E}_t \sum_{s=0}^{\infty}\left(\tilde{\beta}\theta_k\right)^s \widehat{MC}_{k,t+s},$$

where we used that, in the steady state, $\Lambda_{t,t+s} = \tilde{\beta}^s$, and $\tilde{\beta} = (1-\delta)\beta = 1/R$ and the subsidy is chosen such that $\frac{\epsilon_k}{\epsilon_k-1}(1-\tau_k) = 1$. Finally, noting that all reset prices at t are identical so that $\hat{P}_{k,t} = (1-\theta_k)\hat{P}_{k,t}^* + \theta_k \hat{P}_{k,t-1}$, and denoting $\hat{mc}_{k,t+s} \equiv \hat{MC}_{k,t+s} - \hat{P}_{k,t+s}$, we obtain the NKPC in terms of marginal costs:

$$\pi_{k,t} = \lambda_k \hat{mc}_{k,t+s} + \tilde{\beta}\mathbb{E}_t \pi_{k,t+1},$$

with $\lambda_k = \left(1-\tilde{\beta}\theta_k\right)(1-\theta_k)/\theta_k$.

A.2.3 Goods varieties firms: intermediate input choice

The cost-minimization problem of the firm is given by: $\min W n_k(j) + \sum_l P_l \tilde{Y}_{l,k}(j) \quad s.t. \quad A_k F_k\left(n_k(j), \tilde{Y}_{1,k}(j), \tilde{Y}_{2,k}(j), \ldots, \tilde{Y}_{K,k}(j)\right) \geq y_k(j)$. The production function F_k has constant return-to-scale, so demand for input can be written $\tilde{Y}_{l,k}(j) = \mathcal{Y}_{l,k}(\boldsymbol{P}, W)\frac{y_k(j)}{A_k}$, $n_k(j) = \mathcal{N}_k(\boldsymbol{P}, W)\frac{y_k(j)}{A_k}$. $\mathcal{Y}_{l,k}(\boldsymbol{P}, W)$ is the unit demand for input l by firms in k, which is common to all firms in k. $\mathcal{N}_k(\boldsymbol{P}, W)$ is the unit demand for labor by firms in k, common to all firms in k. With these definitions, we can express the change in the marginal cost as:

$$dMC_k = \frac{W\mathcal{N}_k(\boldsymbol{P}, W)}{A_k}\hat{W} + \sum_{l=1}^{K}\frac{P_l \mathcal{Y}_{l,k}(\boldsymbol{P}, W)}{A_k}\hat{P}_l - MC_k \hat{A}_k,$$

$$\Rightarrow \hat{MC}_k = \left(\Omega_{N,k}\hat{W} + \sum_l \Omega_{k,l}\hat{P}_l\right) - \hat{A}_k.$$

Ω is the matrix of intermediate input shares ($\Omega_{k,l} = \frac{P_l \mathcal{Y}_{l,k}(\boldsymbol{P},W)}{W\mathcal{N}_k(\boldsymbol{P},W) + \sum_{l=1}^{K} P_l \mathcal{Y}_{l,k}(\boldsymbol{P},W)} = \frac{P_l \tilde{Y}_{l,k}}{P_k Y_k}$ in a steady state with equal variety prices), Ω_N a column vector of length K of labor shares ($\Omega_{N,k} = 1 - \sum_{l=1}^{K}\Omega_{k,l}$). Using again that F_k has CRS, we can derive the change in aggregate demand for input bundle l, around the steady state with equal variety prices, as:

$$\frac{d\tilde{Y}_l}{\tilde{Y}_l} = \sum_k \mathcal{Q}_{l,k}\left(\hat{Y}_k - \hat{A}_k\right) + \tilde{\mathcal{T}}_{l,W}\hat{W} + \sum_k \tilde{\mathcal{T}}_{l,k}\hat{P}_k.$$

$\tilde{\mathcal{T}}$ is the matrix of aggregate input price elasticities such that $\tilde{\mathcal{T}}_{l,k} = \sum_m \frac{\tilde{Y}_{l,m}}{\tilde{Y}_l}\frac{\partial \mathcal{Y}_{l,m}}{\partial P_k}\frac{P_k}{\mathcal{Y}_{l,m}}$, $\tilde{\mathcal{T}}_{l,W} = \sum_m \frac{\tilde{Y}_{l,m}}{\tilde{Y}_l}\frac{\partial \mathcal{Y}_{l,m}}{\partial W}\frac{W}{\mathcal{Y}_{l,m}}$ is the column vector of wage elasticities and $\mathcal{Q}_{l,k} = \frac{Y_k}{\tilde{Y}_l}\mathcal{Y}_{l,k}$. Since intermediary input producers have a CRS technology we can write the (aggregated) market clearing equation for variety k as $\hat{Y}_k = s_k^C \hat{C}_k + (1 - s_k^C)\tilde{Y}_k$. We have, denoting $\mathcal{D}[s^c]$ and $\mathcal{D}[PY]$ as the diagonal matrices with share of consumption demand and sectoral revenue on the diagonal ($s_k^C = \frac{E_k}{P_k Y_k}$and $P_k Y_k$), $\hat{\boldsymbol{Y}}$,$\hat{\boldsymbol{C}}$, $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{P}}$ the column vectors of sectoral output, consumption, TFP shocks and prices:

$$\hat{\boldsymbol{Y}} = \mathcal{D}[s^c]\,\hat{\boldsymbol{C}} + (Id - \mathcal{D}[s^c])\left(\mathcal{Q}\left(\hat{\boldsymbol{Y}} - \hat{\boldsymbol{A}}\right) + \tilde{\mathcal{T}}_W \hat{W} + \tilde{\mathcal{T}}\hat{\boldsymbol{P}}\right),$$
$$\Leftrightarrow \left(Id - \mathcal{D}[PY]^{-1}\Omega^T \mathcal{D}[PY]\right)\hat{\boldsymbol{Y}} = \mathcal{D}[s^c]\,\hat{\boldsymbol{C}} - \mathcal{D}[PY]^{-1}\Omega^T\mathcal{D}[PY]\,\hat{\boldsymbol{A}} + \mathcal{T}_W \hat{W} + \mathcal{T}\hat{\boldsymbol{P}}.$$

where we use the fact that $[(Id - \mathcal{D}[s^c])\,\mathcal{Q}]_{k,l} = \frac{P_k\tilde{Y}_k}{P_k Y_k}\frac{\tilde{Y}_{k,l}}{\tilde{Y}_k} = \frac{P_k Y_k}{P_l Y_l}\Omega_{l,k}$. Note that $\mathcal{T}_W = (Id - \mathcal{D}[s^c])\,\tilde{\mathcal{T}}_W$ and similarly $\mathcal{T} = (Id - \mathcal{D}[s^c])\,\tilde{\mathcal{T}}$.

A.2.4 Labour demand response

We can similarly express demand for labor by a firm j in sector k as $n_k(j) = \mathcal{N}_k(\boldsymbol{P}, W)\frac{y_k(j)}{A_k}$. Differentiating and aggregating this function, we can express the percentage change in aggregate labor demand as:

$$\hat{N} = \boldsymbol{s^N}\left(\left(\hat{Y} - \hat{A}\right) + \tilde{\mathcal{T}}_W^N \hat{W} + \tilde{\mathcal{T}}^N \hat{\mathbf{P}}\right),$$

where $\boldsymbol{s^N} = \left[\frac{W\int n_1(j)dj}{WN}, ..., \frac{W\int n_K(j)dj}{WN}\right]$, $\tilde{\mathcal{T}}_W^N = \left[\partial_{\ln(W)}\ln(\mathcal{N}_1), ..., \partial_{\ln(W)}\ln(\mathcal{N}_K)\right]$, $\tilde{\mathcal{T}}_{k,l}^N = \partial_{\ln(P_l)}\ln(\mathcal{N}_k)$. One can show that the change in labor demand will only depend on the change in consumption and productivities as follows:

$$\hat{N} = \boldsymbol{s^N}\left(\left(Id - \mathcal{D}[PY]^{-1}\Omega^T\mathcal{D}[PY]\right)^{-1}\left(\mathcal{D}[s^c]\,\hat{\mathbf{C}} - \mathcal{D}[PY]^{-1}\Omega^T\mathcal{D}[PY]\,\hat{\mathbf{A}} + \mathcal{T}_W\hat{W} + \mathcal{T}\hat{\mathbf{P}}\right) - \hat{A} + \tilde{\mathcal{T}}_W^N\hat{W} + \tilde{\mathcal{T}}^N\hat{\mathbf{P}}\right),$$
$$= \boldsymbol{s^N}\left(\left(Id - \mathcal{D}[PY]^{-1}\Omega^T\mathcal{D}[PY]\right)^{-1}\left(\mathcal{D}[s^c] - \hat{\mathbf{A}}\right)\right) + \boldsymbol{s^N}\left(\mathcal{T}_W^N\hat{W} + \mathcal{T}^N\hat{\mathbf{P}} + \left(Id - \mathcal{D}[PY]^{-1}\Omega^T\mathcal{D}[PY]\right)^{-1}\left(\mathcal{T}_W\hat{W} + \mathcal{T}\hat{\mathbf{P}}\right)\right).$$

Note that, as $(1 + \mu_k)(1 - \tau_k) = 1$, we have

$$[WN_1, ..., WN_K]\left(Id - \mathcal{D}[PY]^{-1}\Omega^T\mathcal{D}[PY]\right)^{-1} = [PY_1, ..., PY_K],$$
$$\partial_{ln(W)}\mathcal{N}_k\hat{W} + \sum_l \partial_{ln(P_l)}\mathcal{Y}_{l,k}\hat{W} = 0,$$
$$\partial_{ln(P_l)}\mathcal{N}_k\hat{P}_l + \sum_m \partial_{ln(P_l)}\mathcal{Y}_{m,k}\hat{P}_l = 0,$$

where Id denotes the diagonal matrix. We thus obtain:

$$\hat{N} = \boldsymbol{s^N}\left(Id - \mathcal{D}[PY]^{-1}\Omega^T\mathcal{D}[PY]\right)^{-1}\left(\mathcal{D}[s^c]\,\hat{\mathbf{C}} - \hat{\mathbf{A}}\right),$$
$$= \sum_{k=1}^K \bar{s}_k\hat{C}_k - \sum_{k=1}^K \frac{P_k Y_k}{E}\hat{A}_k.$$

A.2.5 Aggregate consumption response

Let us re-introduce time subscripts. In addition, we denote by a superscript t_0 the birth date of the household. We can derive aggregate spending in sector k by simply aggregating individual decisions from the consumption problem (in equation (4)):

$$\hat{C}_{k,t} = \frac{1}{C_k}\mathbb{E}_{\delta,t}\int dc_{k,t}^{t_0}(i)\,di = \mathbb{E}_{\delta,t}\int \frac{e(i)}{E_k}\partial_e e_k(i)\left(\hat{e}_t^{t_0}(i) - \sum_l s_l(i)\,\hat{P}_l\right)di + \sum_{l=1}^K \bar{\rho}_{k,l}\hat{P}_{l,t},$$

where $\bar{\rho}_{k,l} = \int \rho_{k,l}(i)\,\frac{e_k(i)}{E_k}di$ is the aggregate compensated price elasticity of sector k with respect to P_l. Finally, the operator $\mathbb{E}_{\delta,t}$ sums over cohorts at time t. For a variable X^{t_0} which depends on t_0, we have:

$$\mathbb{E}_{\delta,t}(X^{t_0}) = \delta\sum_{t_0=-\infty}^{t}(1-\delta)^{t-t_0}X^{t_0}.$$

A.2.6 Labor market clearing

Recall from Lemma 2 that:

$$\hat{n}_t^{t_0}(i) = \psi\left\{\hat{W}_t - \sum_{k=1}^K \partial_e e_k(i)\,\hat{P}_{k,t}\right\} - \frac{\psi}{\sigma}\left(\hat{e}_t^{t_0}(i) - \sum_{k=1}^K s_k(i)\,\hat{P}_{k,t}\right).$$

Aggregating over all households labor supply is:

$$\hat{N}_t^s = \psi\left(\hat{W}_t - \sum_{k=1}^{K}\int \frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t}\right) - \frac{\psi}{\sigma}\mathbb{E}_{\delta,t}\int \frac{Wn(i)}{WN}\left(\hat{e}_t^{t_0}(i) - \sum_{k=1}^{K}s_k(i)\hat{P}_{k,t}\right)di.$$

Recall that labor demand is:

$$\hat{N}_t = \sum_{k=1}^{K}\bar{s}_k\hat{C}_{k,t} - \sum_{k=1}^{K}\frac{P_kY_k}{E}\hat{A}_{k,t},$$

so labor market clearing becomes:

$$\begin{aligned}\psi\left(\hat{W}_t - \sum_{k=1}^{K}\int \frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t}\right) - \frac{\psi}{\sigma}\mathbb{E}_{\delta,t}\int \frac{Wn(i)}{WN}\left(\hat{e}_t^{t_0}(i) - \sum_{k=1}^{K}s_k(i)\hat{P}_{l,t}\right)di &= \sum_{k=1}^{K}\left(\bar{s}_k\left(\hat{E}_{k,t}-\hat{P}_{k,t}\right) - \frac{P_kY_k}{E}\hat{A}_{k,t}\right),\\ &= \sum_{k=1}^{K}\bar{s}_k\left(\mathbb{E}_{\delta,t}\int\frac{e(i)}{E_k}\partial_e e_k(i)\left(\hat{e}_t^{t_0}(i) - \sum_{l=1}^{K}s_l(i)\hat{P}_{l,t}\right)di + \sum_{l=1}^{K}\bar{\rho}_{k,l}\hat{P}_{l,t}\right) - \sum_{k=1}^{K}\frac{P_kY_k}{E}\hat{A}_{k,t},\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}\psi\hat{W}_t - \psi\sum_{k=1}^{K}\int\frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t} + \sum_{k=1}^{K}\frac{P_kY_k}{E}\hat{A}_{k,t} &= \mathbb{E}_{\delta,t}\int\frac{e(i)}{E}\left(\sum_k\partial_e e_k(i) + \frac{\psi Wn(i)}{\sigma e(i)}\right)\left(\hat{e}_t^{t_0}(i) - \sum_{k=1}^{K}s_k(i)\hat{P}_{k,t}\right)di\\ &\quad + \sum_{l=1}^{K}\sum_{k=1}^{K}\bar{s}_k\bar{\rho}_{k,l}\hat{P}_{l,t},\\ &= \mathbb{E}_{\delta,t}\int\frac{e(i)}{E}\left(1+\frac{\psi Wn(i)}{e(i)\sigma}\right)\left(\hat{e}_t(i)^{t_0} - \sum_{k=1}^{K}s_k(i)\hat{P}_{k,t}\right)di,\end{aligned}$$

where we have used the fact that $\sum_{k=1}^{K}c_k(i)\rho_{k,l}(i) = 0$ for all l,i so $\sum_{k=1}^{K}\bar{s}_k\bar{\rho}_{k,l} = 0$. Finally, we rescale the technology shocks, $\hat{A}_{k,t}$, according to:

$$\tilde{\mathbf{A}}_t \equiv (Id-\Omega)^{-1}\hat{\mathbf{A}}_t.$$

With this rescaling, the aggregate demand index $\hat{\mathcal{Y}}_t$ can be written:

$$\begin{aligned}\hat{\mathcal{Y}}_t &\equiv \frac{\sigma}{\sigma+\psi}\left(\psi\hat{W}_t - \psi\sum_{k=1}^{K}\overline{\partial_e e}_k\hat{P}_{k,t} + \sum_{k=1}^{K}\frac{P_kY_k}{E}\hat{A}_{k,t}\right),\\ &= \frac{\sigma}{\sigma+\psi}\left(\psi\hat{W}_t - \psi\sum_{k=1}^{K}\overline{\partial_e e}_k\hat{P}_{k,t} + \sum_{k=1}^{K}\bar{s}_k\tilde{A}_{k,t}\right),\end{aligned}$$

where the last line uses the fact that $[E_1,...,E_k](Id-\Omega)^{-1} = [P_1Y_1,...,P_kY_k]$. Rewriting the labor market condition in terms of $\hat{\mathcal{Y}}_t$, we have:

$$\hat{\mathcal{Y}}_t = \frac{\sigma}{\sigma+\psi}\mathbb{E}_{\delta,t}\int\frac{e(i)}{E}\left(1+\frac{\psi Wn(i)}{e(i)\sigma}\right)\left(\hat{e}_t^{t_0}(i) - \sum_{k=1}^{K}s_k(i)\hat{P}_{k,t}\right)di + \frac{\sigma\psi}{\sigma+\psi}\sum_{k=1}^{K}\int\frac{Wn(i)-e(i)}{E}\partial_e e_k(i)di\hat{P}_{k,t}.$$

Defining the natural level of aggregate demand as the level that prevails in the absence of markups distortions (i.e. with flexible prices) we obtain our formula for the output gap:

$$\begin{aligned}\hat{\mathcal{Y}}_t^* &\equiv \frac{\sigma}{\sigma+\psi}\sum_{k=1}^{K}\left(\psi\overline{\partial_e e}_k + \bar{s}_k\right)\tilde{A}_{k,t},\\ \tilde{\mathcal{Y}}_t &= \frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t - \sum_{k=1}^{K}\overline{\partial_e e}_k\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right).\end{aligned}$$

See the optimal policy section for a justification of the efficiency of $\hat{\mathcal{Y}}_t^*$. Note that in the absence of markup distortions, it holds that $\hat{P}_{k,t} = \hat{W}_t - \tilde{A}_{k,t}$. We will show later that the output gap shows up in the social welfare function.

A.3 Sectoral NKPC

Recall that:

$$\begin{aligned}\pi_{k,t} &= \lambda_k\hat{mc}_{k,t} + \tilde{\beta}\mathbb{E}_t\pi_{k,t+1},\\ \hat{mc}_{k,t} &= \Omega_{N,k}\hat{W}_t + \sum_{l=1}^{K}\Omega_{k,l}\hat{P}_{l,t} - \hat{A}_{k,t} - \hat{P}_{k,t},\\ \tilde{\mathcal{Y}}_t &= \frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t - \sum_{k=1}^{K}\overline{\partial_e e}_k\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right).\end{aligned}$$

Combining these equations, we obtain:

$$\pi_{k,t} = \lambda_k \left\{ \Omega_{N,k} \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t + \Omega_{N,k} \sum_{l=1}^{K} \overline{\partial_e e_l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} \right) + \sum_{l=1}^{K} \Omega_{k,l} \hat{P}_{l,t} - \hat{A}_{k,t} - \hat{P}_{k,t} \right\} + \tilde{\beta} \mathbb{E}_t \pi_{k,t+1},$$

$$= \lambda_k \left\{ \Omega_{N,k} \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t + \Omega_{N,k} \sum_{l=1}^{K} \overline{\partial_e e_l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} - \left(\hat{P}_{k,t} + \tilde{A}_{k,t} \right) \right) + \sum_{l=1}^{K} \Omega_{k,l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} - \left(\hat{P}_{k,t} + \tilde{A}_{k,t} \right) \right) \right\} + \tilde{\beta} \mathbb{E}_t \pi_{k,t+1},$$

Finally, defining

$$\mathcal{NH}_t \equiv \sum_{l=1}^{K} (\overline{\partial_e e_l} - \bar{s}_l)(\hat{P}_{l,t} + \tilde{A}_{l,t}),$$

$$\bar{\mathcal{P}}_{k,t} \equiv (\hat{P}_{k,t} - \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t}) + (\tilde{A}_{k,t} - \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t}),$$

$$\mathcal{P}_{k,t} \equiv \Omega_{N,k} \bar{\mathcal{P}}_{k,t} - \sum_{l=1}^{K} \Omega_{k,l} \left(\bar{\mathcal{P}}_{l,t} - \bar{\mathcal{P}}_{k,t} \right),$$

and noting that $\Omega_{N,k} \sum_{l=1}^{K} \overline{\partial_e e_l} + \sum_{l=1}^{K} \Omega_{k,l} = 1$, gives the formula in the model equation appendix B. To obtain the equations of the main text without the Input-Output structure, we simply set $\Omega_{N,k} = 1$ and $\mathcal{P}_{k,t} = \bar{\mathcal{P}}_{k,t}$, and obtain:

$$\pi_{k,t} = \lambda_k \left(\left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t + \mathcal{NH}_t - \mathcal{P}_{k,t} \right) + \tilde{\beta} \mathbb{E}_t \pi_{k,t+1}.$$

A.4 Distributional Responses

In this section, we derive the dynamic equations characterizing the evolution of averages of individual households expenditures for arbitrary weights, taking into account the death/birth process. We use these equations to compute the distribution of consumption expenditures in section 4.2. In the next subsection, we use these equations to derive the dynamic equation for the output gap.

We define the response of real consumption expenditures of household i , born at t_0 as $\hat{c}_t^{t_0}(i) = \hat{e}_t^{t_0}(i) - \sum_{k=1}^{K} s_k(i) \widehat{P}_{k,t}$. Moreover, let $\boldsymbol{\omega}$ be a vector defining a weight $\omega(i)$ on each household i, with $\int \omega(i) di = 1$. We can thus use $\boldsymbol{\omega}$ to select and weight any arbitrary subset of households.

Now consider some moment of the consumption distribution, $\hat{C}_t(\boldsymbol{\omega}) = \delta \sum_{t_0=-\infty}^{t} (1-\delta)^{t-t_0} \int \omega(i) \hat{c}_t^{t_0}(i) di$. For instance, if we set $\omega(i) = e(i)/E$, then this moment corresponds to the response of aggregate real expenditures. We could also set $\omega(i) = 1$ for only one specific household i, and zero for all others. In that case, $\hat{C}_t(\boldsymbol{\omega})$ corresponds to the individual consumption response of a particular household. Alternatively, one can choose $\boldsymbol{\omega}$ to compute the average response among households with certain characteristics. In this section, we show that $\hat{C}_t(\boldsymbol{\omega})$ is characterized with the following equations:

$$\mathbb{E}_t \hat{C}_{t+1}(\boldsymbol{\omega}) - \hat{C}_t(\boldsymbol{\omega}) = \sigma \int_0^1 \omega(i) di \hat{R}_t - \sigma \sum_{k=1}^{K} \int_0^1 \omega(i) \partial_e e_k(i) di \, \mathbb{E}_t \pi_{k,t+1} + \frac{\delta}{1-\delta} \hat{C}_t^0(\boldsymbol{\omega}),$$

where

$$\hat{C}_t^0(\boldsymbol{\omega}) - \frac{\mathbb{E}_t \hat{C}_{t+1}^0(\boldsymbol{\omega})}{(1-\delta) R} = \int_0^1 \omega^0(i) \frac{b(i)}{RE} di (\hat{R}_t - \mathbb{E}_t \sum_{k=1}^{K} \bar{s}_k \pi_{k,t+1}) + (1 + \frac{\psi}{\sigma}) \int_0^1 \omega^0(i) \frac{Wn(i)}{WN} di \hat{\mathcal{Y}}_t$$
$$- \sum_{k=1}^{K} \int_0^1 \omega^0(i) \left(\frac{e(i)}{E} (s_k(i) - \bar{s}_k) + \frac{Wn(i)}{WN} \psi \left(\partial_e e_k(i) - \overline{\partial_e e_k} \right) \right) di \hat{P}_{k,t} - \frac{R-1}{R} \hat{C}_t(\boldsymbol{\omega}),$$

and where we defined $\omega^0(i) = \frac{R-1}{R} \frac{\omega(i)}{e(i)/E + Wn(i)/WN \frac{\psi}{\sigma}}$.

To derive these equations , let us denote by $C_{t+1}^u(\boldsymbol{\omega}) = \delta \sum_{t_0=-\infty}^{t} (1-\delta)^{t-t_0} \int (1-\varphi(i)) \omega(i) \left(\hat{e}_t^{t_0}(i) - \sum_{k=1}^{K} s_k(i) \hat{P}_{k,t} \right) di$ the contribution of unconstrained (non-HtM) households to the demand index. We have:

$$\mathbb{E}_t C_{t+1}^u(\boldsymbol{\omega}) = (1-\delta) C_t^u(\boldsymbol{\omega}) + (1-\delta) \sigma \mathbb{E}_t \int (1-\varphi(i)) \omega(i) \left(\hat{R}_t - \sum_{k=1}^{K} \partial_e e_k(i) \pi_{k,t+1} \right) di + \delta \mathbb{E}_t \tilde{C}_{t+1}^{u,0}(\boldsymbol{\omega}).$$

Here, we use the individual Euler equation, as derived in Lemma 3:

$$\left(\hat{e}_t^{t_0}(i) - \sum_{k=1}^{K} s_k(i) \hat{P}_{k,t} \right) = \mathbb{E}_t \left(\hat{e}_{t+1}^{t_0}(i) - \sum_{k=1}^{K} s_k(i) \hat{P}_{k,t+1} \right) - \sigma \mathbb{E}_t \left(\hat{R}_t - \sum_{k=1}^{K} \partial_e e_k(i) \pi_{k,t+1} \right),$$

for households "born" before $t+1$. $\tilde{C}_{t+1}^{u,0}(\boldsymbol{\omega})$ is the consumption of the households born at $t+1$. Note that the lifetime budget constraint of the households born at t with wealth $b(i)\left(1+\sum_{k=1}^{K}\bar{s}_k\hat{P}_{k,t}\right)$ is

$$-b(i)\sum_{k=1}^{K}\bar{s}_k\hat{P}_{k,t}=\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{b(i)}{R}\hat{R}_{t+s}+Wn(i)\left(\hat{W}_{t+s}-\hat{n}^t_{t+s}(i)\right)+dDiv_{t+s}(i)-e(i)\sum_{k=1}^{K}s_k(i)\hat{P}_{k,t+s}\right)$$
$$-e(i)\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{e}^t_{t+s}(i)-\sum_{k=1}^{K}s_k(i)\hat{P}_{k,t+s}\right).$$

Using labor supply decisions, derived in lemma 2, and $dDiv_t(i)=\frac{Wn(i)}{WN}\sum_k E_k\left(\hat{P}_{k,t}+\tilde{A}_{k,t}-\hat{W}_t\right)$ we obtain:

$$-b(i)\sum_{k=1}^{K}\bar{s}_k\hat{P}_{k,t}+\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{e}^t_{t+s}(i)-\sum_{k=1}^{K}s_k(i)\hat{P}_{k,t+s}\right)=$$
$$\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{b(i)}{R}\hat{R}_{t+s}+\left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_{t+s}-\sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t+s}-\left(1-\frac{1}{R}\right)b(i)\sum_{k=1}^{K}\bar{s}_k\hat{P}_{k,t+s}\right)$$

$\Leftrightarrow$

$$\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{e}^t_{t+s}-\sum_{k=1}^{K}s_k(i)\hat{P}_{k,t+s}\right)=$$
$$\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{b(i)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+s+1}\right)+\left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_{t+s}-\sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t+s}\right).$$

Using the Euler equation, $\hat{e}_{t+u,t}(i)-\sum_{k=1}^{K}s_k(i)\hat{P}_{k,t+u}=\hat{e}_{t,t}(i)-\sum_{k=1}^{K}s_k(i)\hat{P}_{k,t}+\sigma\mathbb{E}_t\sum_{s=0}^{u-1}\left(\hat{R}_{t+s}-\sum_{k=1}^{K}\partial_e e_k(i)\pi_{k,t+s+1}\right)$ so we obtain:

$$\hat{e}^t_t(i)-\sum_k s_k(i)\hat{P}_{k,t}=\frac{1-\frac{1}{R}}{e(i)+\frac{\psi}{\sigma}Wn(i)}\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{b(i)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+s+1}\right)+\left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_{t+s}\right)$$
$$-\frac{1-\frac{1}{R}}{e(i)+\frac{\psi}{\sigma}Wn(i)}\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^s}\sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t+s}$$
$$-\sigma\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^{s+1}}\left(\hat{R}_{t+s}-\sum_k\partial_e e_k(i)\pi_{k,t+s+1}\right).$$

Averaging across households with the arbitrary weights $\boldsymbol{\omega}$, we have:

$$\tilde{C}^{u,0}_t(\boldsymbol{\omega})-\frac{1}{R}\mathbb{E}_t\tilde{C}^{u,0}_{t+1}(\boldsymbol{\omega})=-\sigma\frac{1}{R}\mathbb{E}_t\int(1-\varphi(i))\,\omega(i)\left(\hat{R}_t-\sum_{k=1}^{K}\partial_e e_k(i)\pi_{k,t+1}+\frac{Wn(i)\left(1-\frac{1}{R}\right)\left(1+\frac{\psi}{\sigma}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\hat{\mathcal{Y}}_{t+s}\right)di$$
$$\mathbb{E}_t\int\frac{(1-\varphi(i))\,\omega(i)\left(1-\frac{1}{R}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left(\frac{b(i)}{R}\left(\hat{R}_t-\pi_{cpi,t+1}\right)-\sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right)di.$$

Defining $C^{u,0}_t(\boldsymbol{\omega})\equiv\tilde{C}^{u,0}_t(\boldsymbol{\omega})-C^u_t(\boldsymbol{\omega})$, we have:

$$\mathbb{E}_tC^u_{t+1}(\boldsymbol{\omega})=C^u_t(\boldsymbol{\omega})+\sigma\mathbb{E}_t\int(1-\varphi(i))\,\omega(i)\left(\hat{R}_t-\sum_{k=1}^{K}\partial_e e_k(i)\pi_{k,t+1}\right)di+\frac{\delta}{1-\delta}\mathbb{E}_tC^{u,0}_{t+1}(\boldsymbol{\omega}),$$

$$C^{u,0}_t(\boldsymbol{\omega})-\frac{1}{R}\mathbb{E}_tC^{u,0}_{t+1}(\boldsymbol{\omega})+C_t(\boldsymbol{\omega})-\frac{1}{R}\mathbb{E}_tC^u_{t+1}(\boldsymbol{\omega})=$$
$$\mathbb{E}_t\int\frac{(1-\varphi(i))\,\omega(i)\left(1-\frac{1}{R}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left(\frac{b(i)}{R}\left(\hat{R}_t-\pi_{cpi,t+1}\right)+\left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_t-\sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right)di$$
$$-\sigma\frac{1}{R}\mathbb{E}_t\int(1-\varphi(i))\,\omega(i)\left(\hat{R}_t-\sum_k\partial_e e_k(i)\pi_{k,t+1}\right)di,$$

$$C^{u,0}_t(\boldsymbol{\omega})-\frac{1}{R(1-\delta)}\mathbb{E}_tC^{u,0}_{t+1}(\boldsymbol{\omega})+\left(1-\frac{1}{R}\right)C^u_t(\boldsymbol{\omega})=$$
$$\mathbb{E}_t\int\frac{(1-\varphi(i))\,\omega(i)\left(1-\frac{1}{R}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left(\frac{b(i)}{R}\left(\hat{R}_t-\pi_{cpi,t+1}\right)+\left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_t-\sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right)di.$$

Now we consider the contribution of the HtMs. Using our derivations of section A.1.2, we have:

$$\mathbb{E}_t C^{HtM}_{t+1}(\boldsymbol{\omega}) = (1-\delta) C^{HtM}_t(\boldsymbol{\omega}) + \delta \mathbb{E}_t \tilde{C}^{HtM,0}_{t+1}(\boldsymbol{\omega})$$

$$+ (1-\delta)\mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{ \Delta\hat{R}_{t+1}\frac{b(i)}{R} + \left(1+\frac{\psi}{\sigma}\right) Wn(i)\Delta\hat{\mathcal{Y}}_{t+1} - \sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\pi_{k,t+1}\right\} di$$

$$- (1-\delta)\mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{ \left(1-\frac{1}{R}\right) b(i)\pi_{cpi,t+1}\right\} di,$$

$$\tilde{C}^{HtM,0}_t(\boldsymbol{\omega}) - \frac{1}{R}\mathbb{E}_t \tilde{C}^{HtM,0}_{t+1}(\boldsymbol{\omega}) =$$

$$-\frac{1}{R}\mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{ \Delta\hat{R}_{t+1}\frac{b(i)}{R} + \left(1+\frac{\psi}{\sigma}\right) Wn(i)\Delta\hat{\mathcal{Y}}_{t+1} - \sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\pi_{k,t+1}\right\} di$$

$$-\frac{1}{R}\mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{ -\left(1-\frac{1}{R}\right) b(i)\pi_{cpi,t+1}\right\} di$$

$$+\left(1-\frac{1}{R}\right)\mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{ \hat{R}_t\frac{b(i)}{R} + \left(1+\frac{\psi}{\sigma}\right) Wn(i)\hat{\mathcal{Y}}_t - \sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right\} di$$

$$+\left(1-\frac{1}{R}\right)\mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{ -\left(1-\frac{1}{R}\right) b(i)\hat{P}_{cpi,t}\right\} d$$

$$+\mathbb{E}_t \int \varphi(i)\omega(i)\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(1-\frac{1}{R}\right) b(i) di \left(\hat{P}_{cpi,t} - \frac{1}{R}\hat{P}_{cpi,t+1}\right).$$

Defining $C^{HtM,0}_t(\boldsymbol{\omega}) \equiv \tilde{C}^{HtM,0}_t(\boldsymbol{\omega}) - C^{HtM}_t(\boldsymbol{\omega})$, we have:

$$\mathbb{E}_t C^{HtM}_{t+1}(\boldsymbol{\omega}) = C^{HtM}_t(\boldsymbol{\omega}) + \frac{\delta}{1-\delta}\mathbb{E}_t \tilde{C}^{HtM,0}_{t+1}(\boldsymbol{\omega}) + \mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{ -(1-\frac{1}{R}) b(i)\pi_{cpi,t+1}\right\} di$$

$$\mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{ \Delta\hat{R}_{t+1}\frac{b(i)}{R} + \left(1+\frac{\psi}{\sigma}\right) Wn(i)\Delta\hat{\mathcal{Y}}_{t+1} - \sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\pi_{k,t+1}\right\} di,$$

$$\tilde{C}^{HtM,0}_t(\boldsymbol{\omega}) - \frac{1}{R(1-\delta)}\mathbb{E}_t \tilde{C}^{HtM,0}_{t+1}(\boldsymbol{\omega}) + \left(1-\frac{1}{R}\right) C^{HtM}_t(\boldsymbol{\omega}) = \left(1-\frac{1}{R}\right)\mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left(1+\frac{\psi}{\sigma}\right) Wn(i)\hat{\mathcal{Y}}_t di$$

$$\left(1-\frac{1}{R}\right)\mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{ \frac{b(i)}{R}\left(\hat{R}_t - \pi_{cpi,t+1}\right) - \sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right\} di.$$

Putting everything together, we obtain:

$$\mathbb{E}_t C_{t+1}(\boldsymbol{\omega}) = C_t(\boldsymbol{\omega}) + \sigma\mathbb{E}_t \int (1-\varphi(i))\,\omega(i)\left(\hat{R}_t - \sum_k \partial_e e_k(i)\pi_{k,t+1}\right) di + \frac{\delta}{1-\delta}\mathbb{E}_t C^0_{t+1}(\boldsymbol{\omega}) +$$

$$\mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left(\Delta\hat{R}_{t+1}\frac{b(i)}{R} + \left(1+\frac{\sigma}{\psi}\right) Wn(i)\Delta\hat{\mathcal{Y}}_{t+1} - \sum_{k=1}^{K}\left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\pi_{k,t+1}\right) di$$

$$- \mathbb{E}_t \int \frac{\varphi(i)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left(\left(1-\frac{1}{R}\right) b(i)\pi_{cpi,t+1}\right) di,$$

$$C^0_t(\boldsymbol{\omega}) - \frac{1}{R(1-\delta)}\mathbb{E}_t C^0_{t+1}(\boldsymbol{\omega}) + \left(1-\frac{1}{R}\right) C_t(\boldsymbol{\omega})$$

$$= \mathbb{E}_t \int \frac{\omega(i)\left(1-\frac{1}{R}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left(\frac{b(i)}{R}\left(\hat{R}_t - \pi_{cpi,t+1}\right) + \left(1+\frac{\psi}{\sigma}\right) Wn(i)\hat{\mathcal{Y}}_t - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right) di,$$

with

$$C^0_0(\boldsymbol{\omega}) = (1-\delta)\left(\int \frac{\omega(i)\left(1-\frac{1}{R}\right)}{\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)} b(i) di\right) P_{cpi,0}.$$

A.4.1 Euler Equation for the output gap.

We now derive the evolution of the output gap. Recall the definition $\tilde{\mathcal{Y}}_t = \hat{\mathcal{Y}}_t - \hat{\mathcal{Y}}_t^*$, with $\hat{\mathcal{Y}}_t = \frac{\sigma}{\sigma+\psi}\left(\psi \hat{W}_t - \psi \sum_k \overline{\partial_e e_k} \hat{P}_{k,t} + \sum_k \bar{s}_k \tilde{A}_{k,t}\right)$. Using the labor market condition, $\hat{\mathcal{Y}}_t$ can be expressed as a special case of the demand index $C_t(\omega)$, with $\omega(i) = \frac{e(i)}{E} + \frac{\psi}{\sigma}\frac{Wn(i)}{WN}$: $\hat{\mathcal{Y}}_t = \frac{\sigma}{\sigma+\psi} C_t - \frac{\sigma\psi}{\sigma+\psi}\sum_k \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN}\right) \partial_e e_k(i) di \hat{P}_{k,t}$. Therefore, applying the formulas derived above, we have:

$$
\begin{aligned}
\mathbb{E}_t \hat{\mathcal{Y}}_{t+1} - \hat{\mathcal{Y}}_t = \sigma \mathbb{E}_t \int (1-\varphi(i)) \frac{\sigma \frac{e(i)}{E} + \psi \frac{Wn(i)}{WN}}{\sigma+\psi} \left(\hat{R}_t - \sum_k \partial_e e_k(i) \pi_{k,t+1}\right) di = \\
- \frac{\sigma\psi}{\sigma+\psi} \mathbb{E}_t \sum_k \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN}\right) \partial_e e_k(i) di \pi_{k,t+1} + \frac{\delta}{1-\delta} \mathbb{E}_t \hat{\mathcal{Y}}^0_{t+1} \\
+ \frac{\sigma}{\sigma+\psi} \mathbb{E}_t \int \varphi(i) \left\{ \Delta \hat{R}_{t+1} \frac{b(i)}{RE} + \left(1 + \frac{\psi}{\sigma}\right) \frac{Wn(i)}{WN} \Delta \hat{\mathcal{Y}}_{t+1} \right\} di \\
- \frac{\sigma}{\sigma+\psi} \mathbb{E}_t \int \varphi(i) \left\{ \sum_k \left(\frac{e(i)}{E} (s_k(i) - \bar{s}_k) + \psi \frac{Wn(i)}{WN} \left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right) \pi_{k,t+1} + \left(1 - \frac{1}{R}\right) \frac{b(i)}{E} \pi_{cpi,t+1} \right\} di,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathcal{Y}}^0_t - \frac{1}{R(1-\delta)} \mathbb{E}_t \hat{\mathcal{Y}}^0_{t+1} + \left(1 - \frac{1}{R}\right) \hat{\mathcal{Y}}_t + \frac{\sigma\psi}{\sigma+\psi} \sum_{k=1}^K \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN}\right) \partial_e e_k(i) di \hat{P}_{k,t} = \\
\frac{\sigma}{\sigma+\psi}\left(1 - \frac{1}{R}\right) \mathbb{E}_t \int \left(\frac{b(i)}{RE} \left(\hat{R}_t - \pi_{cpi,t+1}\right) + \left(1 + \frac{\psi}{\sigma}\right) \frac{Wn(i)}{WN} \hat{\mathcal{Y}}_t - \sum_{k=1}^K \left(\frac{e(i)}{E} (s_k(i) - \bar{s}_k) + \psi \frac{Wn(i)}{WN} \left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right) \hat{P}_{k,t} \right) di.
\end{aligned}
$$

Using

$$
\int \frac{b(i)}{RE} di = \int \sum_{k=1}^K \frac{e(i)}{E} (s_k(i) - \bar{s}_k) di = 0,
$$

we obtain:

$$
\begin{aligned}
\hat{\mathcal{Y}}^0_t - \frac{1}{R(1-\delta)} \mathbb{E}_t \hat{\mathcal{Y}}^0_{t+1} + \left(1 - \frac{1}{R}\right) \hat{\mathcal{Y}}_t + \frac{\sigma\psi}{\sigma+\psi} \sum_{k=1}^K \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN}\right) \partial_e e_k(i) di \hat{P}_{k,t} = \\
\frac{\sigma}{\sigma+\psi}\left(1 - \frac{1}{R}\right) \int \left(\left(1 + \frac{\psi}{\sigma}\right) \frac{Wn(i)}{WN} \hat{\mathcal{Y}}_t - \sum_{k=1}^K \psi \frac{Wn(i)}{WN} \left(\partial_e e_k(i) - \overline{\partial_e e_k}\right) \hat{P}_{k,t} \right) di, \\
\hat{\mathcal{Y}}^0_t - \frac{1}{R(1-\delta)} \mathbb{E}_t \hat{\mathcal{Y}}^0_{t+1} = 0.
\end{aligned}
$$

Using $\hat{\mathcal{Y}}^0_0 = 0$ and $\frac{1}{R(1-\delta)} > 1$, we have $\hat{\mathcal{Y}}^0_t = 0$ for all t. Defining $\varphi^E \equiv \int \varphi(i) \frac{e(i)}{E} di$, $\varphi^N \equiv \int \varphi(i) \frac{Wn(i)}{WN} di$, we obtain:

$$
\begin{aligned}
\left(1 - \varphi^N\right) \left(\mathbb{E}_t \hat{\mathcal{Y}}_{t+1} - \hat{\mathcal{Y}}_t\right) = \left(1 - \varphi^N\right) \sigma \mathbb{E}_t \left(\hat{R}_t - \sum_{k=1}^K \overline{\partial_e e_k} \pi_{k,t+1}\right) \\
+ \mathbb{E}_t \int \frac{\sigma \varphi(i)}{\sigma+\psi} \left\{ \frac{b(i)}{RE} \left(\Delta \hat{R}_{t+1} - \sigma(R-1) \hat{R}_t\right) - \frac{e(i)}{E} \sum_{k=1}^K \left((s_k(i) - \bar{s}_k) - \sigma \left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right) \pi_{k,t+1} \right\} di \\
- \mathbb{E}_t \int \frac{\sigma \varphi(i)}{\sigma+\psi} \left\{ - \left(1 - \frac{1}{R}\right) \frac{b(i)}{E} \left(\pi_{cpi,t+1} - \sigma \pi_{mcpi,t+1}\right) \right\} di.
\end{aligned}
$$

By definition, we have $r^*_t = \frac{1}{\sigma}\left(\mathbb{E}_t \hat{\mathcal{Y}}^*_{t+1} - \hat{\mathcal{Y}}^*_t\right)$, so $r^*_t \equiv \mathbb{E}_t \frac{1}{\sigma+\psi}\left(\left(\sum_{k=1}^K \psi \overline{\partial_e e_k} + \bar{s}_k\right)\left(\tilde{A}_{k,t+1} - \tilde{A}_{k,t}\right)\right)$.
The evolution of the output gap $\tilde{\mathcal{Y}}_t = \hat{\mathcal{Y}}_t - \hat{\mathcal{Y}}^*_t$ is given by:

$$
\begin{aligned}
\left(1 - \varphi^N\right) \left(\mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t\right) = \left(1 - \varphi^N\right) \sigma \mathbb{E}_t \left(\hat{R}_t - \sum_{k=1}^K \overline{\partial_e e_k} \pi_{k,t+1} - r^*_t\right) + \\
\mathbb{E}_t \int \frac{\sigma \varphi(i)}{\sigma+\psi} \left\{ \frac{b(i)}{RE} \left(\Delta \hat{R}_{t+1} - \sigma(R-1) \hat{R}_t\right) - \frac{e(i)}{E} \sum_{k=1}^K \left((s_k(i) - \bar{s}_k) - \sigma(\partial_e e_k(i) - \overline{\partial_e e_k})\right) \pi_{k,t+1} \right\} di \\
- \mathbb{E}_t \int \frac{\sigma \varphi(i)}{\sigma+\psi} \left\{ \left(1 - \frac{1}{R}\right) \frac{b(i)}{E} \left(\pi_{cpi,t+1} - \sigma \pi_{mcpi,t+1}\right) \right\} di,
\end{aligned}
$$

which gives the equation of the main text.

A.5 Alternative formulations of the NKPC

In this section, we discuss alternative representation of the NKPC. Instead of expressing the NKPC in terms of the output gap, we express it in turn in terms of the average marginal cost or in terms of the unemployment gap.

A.5.1 NKPC in terms of marginal cost

As shown in section A.3, the real marginal cost in sector k is given by:

$$\hat{mc}_{k,t} = \left(\Omega_{N,k}\hat{W}_t + \sum_{l=1}^{K}\Omega_{k,l}\hat{P}_{l,t}\right) - \hat{A}_{k,t} - \hat{P}_{k,t}.$$

We define the average marginal cost as the average of sectoral marginal costs weighted by sectoral sales:

$$\begin{aligned}\hat{mc}_t \equiv \sum_{k=1}^{K}\frac{P_kY_k}{E}\hat{mc}_{k,t} &= \sum_{k,l=1}^{K}\frac{P_kY_k}{E}\left(\Omega_{N,k}\hat{W}_t + \sum_{l}\Omega_{k,l}\hat{P}_{l,t}\right) - \sum_{k=1}^{K}\frac{P_kY_k}{E}\left(\hat{A}_{k,t}+\hat{P}_{k,t}\right),\\ &= \hat{W}_t + \sum_{k=1}^{K}\left(\frac{P_lY_l}{E} - \bar{s}_l\right)\hat{P}_{l,t} - \sum_{k=1}^{K}\frac{P_kY_k}{E}\left(\hat{A}_{k,t}+\hat{P}_{k,t}\right),\\ &= \hat{W}_t - \sum_{k=1}^{K}\bar{s}_k\left(\tilde{A}_{k,t}+\hat{P}_{k,t}\right).\end{aligned}$$

Where the last line uses $\sum_{k=1}^{K}\frac{P_kY_k}{E}\hat{A}_{k,t} = \sum_{k=1}^{K}\bar{s}_k\tilde{A}_{k,t}$. Recall that the output gap is defined as:

$$\tilde{\mathcal{Y}}_t = \frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t - \sum_{k=1}^{K}\overline{\partial_e e}_k\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right).$$

We therefore have that

$$\hat{mc}_t = \left(\frac{1}{\sigma}+\frac{1}{\psi}\right)\tilde{\mathcal{Y}}_t + \mathcal{NH}_t,$$

where $\mathcal{NH}_t \equiv \sum_{l=1}^{K}(\overline{\partial_e e_l} - \bar{s}_l)(\hat{P}_{l,t}+\tilde{A}_{l,t})$ is our non-homothetic wedge. Note that when preferences are homothetic, we have $\overline{\partial_e e_l} = \bar{s}_l$ for all l so $\hat{mc}_t = \left(\frac{1}{\sigma}+\frac{1}{\psi}\right)\tilde{\mathcal{Y}}_t$: marginal cost is simply a scaled version of the output gap.
With our definition of the marginal cost, the sectoral NKPC can be rewritten as:

$$\pi_{k,t} = \lambda_k\left(\hat{mc}_t - \mathcal{P}_{k,t}\right) + \tilde{\beta}\mathbb{E}_t\pi_{k,t+1},$$

with the definition of $\mathcal{P}_{k,t}$ unchanged from before. The non-homothetic wedge simply drops from the Phillips Curve. The Euler equation for the average marginal cost becomes:

$$\begin{aligned}\left(1-\varphi^N\right)\left(\mathbb{E}_t\hat{mc}_{t+1} - \hat{mc}_t\right) &= \left(1-\varphi^N\right)\left(1+\frac{\sigma}{\psi}\right)\mathbb{E}_t\left(\hat{R}_t - \sum_{k=1}^{K}\overline{\partial_e e}_k\pi_{k,t+1} - r_t^*\right) + \left(1-\varphi^N\right)\left(\mathbb{E}_t\mathcal{NH}_{t+1} - \mathcal{NH}_t\right)\\ &+\mathbb{E}_t\int\frac{\varphi(i)}{\psi}\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1} - \sigma(R-1)\hat{R}_t\right) - \frac{e(i)}{E}\sum_{k=1}^{K}\left((s_k(i)-\bar{s}_k) - \sigma(\partial_e e_k(i) - \overline{\partial_e e}_k)\right)\pi_{k,t+1}\right\}di\\ &-\mathbb{E}_t\int\frac{\varphi(i)}{\psi}\left\{\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\left(\pi_{cpi,t+1} - \sigma\pi_{mcpi,t+1}\right)\right\}di,\end{aligned}$$

so the non-homothetic wedge now appears in the Euler equation.

A.5.2 NKPC in terms of employment

We first show that without wealth heterogeneity, the employment gap and the output gap coincide. For completeness we then provide formulas for the employment gap with wealth heterogeneity and express the NKPCs in terms of the employment gap. Recall from section A.1 and A.2 that individual and aggregate hours worked satisfy:

$$\begin{aligned}\hat{n}_t(i) &= \psi\left\{\hat{W}_t - \sum_{k=1}^{K}\partial_e e_k(i)\hat{P}_{k,t}\right\} - \frac{\psi}{\sigma}\left(\hat{e}_t(i) - \sum_{k=1}^{K}s_k(i)\hat{P}_{k,t}\right),\\ \hat{N}_t &= \psi\left\{\hat{W}_t - \sum_{k=1}^{K}\int\frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t}\right\} - \mathbb{E}_t\int\frac{e(i)}{E}\frac{\psi Wn(i)}{\sigma e(i)}\left(\hat{e}_t(i) - \sum_{k=1}^{K}s_k(i)\hat{P}_{k,t}\right)di.\end{aligned}$$

From the labor market clearing condition, we have

$$\psi\left\{\hat{W}_t - \sum_{k=1}^{K}\int\frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t}\right\} + \sum_{k=1}^{K}\bar{s}_k\tilde{A}_{k,t} = \int\frac{e(i)}{E}\left(1+\frac{\psi Wn(i)}{\sigma e(i)}\right)\left(\hat{e}_t(i) - \sum_{k=1}^{K}s_k(i)\hat{P}_{k,t}\right)di.$$

Without wealth heterogeneity ($b(i) = 0$), we have $Wn(i) = e(i)$ and we obtain:

$$\int \frac{e(i)}{E}\frac{\psi Wn(i)}{\sigma e(i)}\left(\hat{e}_t(i) - \sum_{k=1}^{K} s_k(i)\hat{P}_{k,t}\right)di = \frac{\frac{\psi}{\sigma}}{1+\frac{\psi}{\sigma}}\left(\psi\left\{\hat{W}_t - \sum_{k=1}^{K}\overline{\partial_e e_k}\hat{P}_{k,t}\right\} + \sum_{k=1}^{K}\bar{s}_k\tilde{A}_{k,t}\right),$$

so total employment is given by

$$\hat{N}_t = \frac{\psi}{1+\frac{\psi}{\sigma}}\left\{\hat{W}_t - \sum_{k=1}^{K}\overline{\partial_e e_k}\hat{P}_{k,t}\right\} - \frac{\frac{\psi}{\sigma}}{1+\frac{\psi}{\sigma}}\sum_{k=1}^{K}\bar{s}_k\tilde{A}_{k,t}.$$

In a flexible price economy, we have $\hat{N}_t^* = \frac{\psi}{1+\frac{\psi}{\sigma}}\sum_{k=1}^{K}\overline{\partial_e e_k}\tilde{A}_{k,t} - \frac{\frac{\psi}{\sigma}}{1+\frac{\psi}{\sigma}}\sum_{k=1}^{K}\bar{s}_k\tilde{A}_{k,t}$, so the employment gap is given by:

$$\tilde{N}_t \equiv \hat{N}_t - \hat{N}_t^* = \frac{\psi}{1+\frac{\psi}{\sigma}}\left\{\hat{W}_t - \sum_{k=1}^{K}\overline{\partial_e e_k}\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\} = \tilde{\mathcal{Y}}_t.$$

Without wealth heterogeneity, the employment gap and the output gap coincide.

With wealth heterogeneity, the employment gap depends on the lump sum transfers which are implemented in the flexible price economy. We assume that the transfers in the flexible price economy are efficient (they minimize the loss function derived in Appendix E). In that case, $\hat{N}_t^*$ is the efficient level of employment. The employment gap is then given by:[1]

$$\tilde{N}_t = \left(1 - \sum_k\int\frac{\left(1-\frac{1}{R}\right)b(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\frac{e(i)}{E}di\right)\tilde{\mathcal{Y}}_t + \frac{\psi}{1+\frac{\psi}{\sigma}}\sum_k\int\frac{\left(1-\frac{1}{R}\right)b(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\frac{e(i)}{E}\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)di\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right) + \tilde{\mathcal{W}\mathcal{E}}_t,$$

To express the evolution of the heterogeneous wealth effects $\tilde{\mathcal{W}\mathcal{E}}_t$, we define

$$\tilde{b}(i) \equiv \frac{\left(1-\frac{1}{R}\right)b(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} - \int\frac{\left(1-\frac{1}{R}\right)b(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\frac{(1-\varphi(i))\frac{Wn(i)}{WN}}{(1-\varphi^N)}.$$

With this notation, the evolution of $\tilde{\mathcal{W}\mathcal{E}}_t$ is given by:

$$\begin{aligned}
&\mathbb{E}_t\tilde{\mathcal{W}\mathcal{E}}_{t+1} - \tilde{\mathcal{W}\mathcal{E}}_t - \frac{\delta}{1-\delta}\mathbb{E}_t\tilde{\mathcal{W}\mathcal{E}}^0_{t+1} = \sum_k\psi_k^{\mathcal{W}\mathcal{E}}\left(\hat{R}_t - \hat{R}_t^* - \mathbb{E}_t\left(\pi_{k,t+1}+\Delta\tilde{A}_{k,t+1}\right)\right)\\
&+\frac{\frac{\psi}{\sigma}}{1+\frac{\psi}{\sigma}}\mathbb{E}_t\int\varphi(i)\tilde{b}(i)\left(\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1}-(R-1)\pi_{cpi,t+1}\right) - \sum_k\left(\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)\right)\pi_{k,t+1}\right)di\\
&-\frac{\frac{\psi}{\sigma}}{1+\frac{\psi}{\sigma}}\mathbb{E}_t\int\varphi(i)\tilde{b}(i)\left(\frac{\sigma e(i)}{E}+\frac{\psi Wn(i)}{WN}\right)\sum_l\left(\frac{1}{\int g^{-1}(i)}\bar{s}_l - \int\frac{g^{-1}(i)}{\int g^{-1}(i)}\frac{\sigma e(i)}{\sigma e(i)+Wn(i)\psi}\partial_e e_l(i)\,di\right)\Delta\tilde{A}_{l,t+1}di\\
&+\frac{\frac{\psi}{\sigma}}{1+\frac{\psi}{\sigma}}\mathbb{E}_t\int\varphi(i)\tilde{b}(i)\left\{\sum_l\frac{Wn(i)}{WN}\bar{s}_l\Delta\tilde{A}_{l,t+1} - \sum_l\sigma\frac{e(i)}{E}\partial_e e_l(i)\Delta\tilde{A}_{l,t+1}\right\}di
\end{aligned}$$

$$\begin{aligned}
&\mathcal{W}\mathcal{E}^0_t - \frac{1}{R(1-\delta)}\mathbb{E}_t\mathcal{W}\mathcal{E}^0_{t+1} + \left(1-\frac{1}{R}\right)\mathcal{W}\mathcal{E}_t =\\
&\frac{\frac{\psi}{\sigma}}{1+\frac{\psi}{\sigma}}\left(1-\frac{1}{R}\right)\mathbb{E}_t\int\frac{\left(1-\frac{1}{R}\right)b(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left\{\frac{b(i)}{RE}\left(\hat{R}_t-\pi_{cpi,t+1}\right) - \sum_k\left(\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)\right)\hat{P}_{k,t}\right\}di\\
&-\frac{\frac{\psi}{\sigma}}{1+\frac{\psi}{\sigma}}\left(1-\frac{1}{R}\right)\mathbb{E}_t\int\frac{\left(1-\frac{1}{R}\right)b(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left(\frac{g^{-1}(i)}{\int g^{-1}(i)} - \frac{Wn(i)}{WN}\right)\sum_k\bar{s}_k\tilde{A}_{k,t}di\\
&-\frac{\frac{\psi}{\sigma}}{1+\frac{\psi}{\sigma}}\left(1-\frac{1}{R}\right)\mathbb{E}_t\int\frac{\left(1-\frac{1}{R}\right)b(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}g(i)^{-1}\left(\frac{\sigma e(i)}{\sigma e(i)+Wn(i)\psi}\partial_e e_l(i) - \int\frac{g^{-1}(i)}{\int g^{-1}(i)}\frac{\sigma e(i)}{\sigma e(i)+Wn(i)\psi}\partial_e e_l(i)\,di\right)\tilde{A}_{l,t}di
\end{aligned}$$

[1]Derivations can be provided upon request.

With:

$$\psi_k^{\mathcal{WE}} = \int \frac{b(i)}{E}\left((1-\varphi(i))\,\partial_e e_k(i) - \frac{\int (1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,\partial_e e_k(i)\,di}{\sigma e(i)+\psi Wn(i)}\frac{(1-\varphi(i))\frac{Wn(i)}{WN}}{(1-\varphi^N)}\right)di$$
$$-\left(1-\frac{1}{R}\right)\frac{\psi\sigma}{\sigma+\psi}\int \frac{b(i)}{E}\frac{\psi Wn(i)}{\sigma e(i)+\psi Wn(i)}(1-\varphi(i))\left\{\partial_e e_k(i) - \frac{\int (1-\varphi(i))\,Wn(i)\partial_e e_k(i)\,di}{1-\varphi^N}\right\}di$$
$$\hat{R}_t^* = \frac{1}{\int (1-\varphi(i))\left(\sigma\frac{e(i)}{E}+\psi\frac{Wn(i)}{WN}\right)di}\mathbb{E}_t\left(\psi\sum_k \overline{\partial_e e_k}\Delta\tilde{A}_{k,t+1} + \sum_k \bar{s}_k \Delta\tilde{A}_{k,t+1}\right)$$
$$-\frac{\int \varphi(i)\left(\sigma\frac{e(i)}{E}+\psi\frac{Wn(i)}{WN}\right)di}{\int (1-\varphi(i))\left(\sigma\frac{e(i)}{E}+\psi\frac{Wn(i)}{WN}\right)di}\sum_l\left(\frac{1}{\int g^{-1}(i)}\bar{s}_l - \int\frac{g^{-1}(i)}{\int g^{-1}(i)}\frac{\sigma e(i)}{\sigma e(i)+Wn(i)\,\psi}\partial_e e_l(i)\,di\right)\mathbb{E}_t\Delta\tilde{A}_{l,t+1}$$

With our definition of the employment gap, we can rewrite the sectoral NKPC as:

$$\pi_{k,t} = \lambda_k\left\{\Omega_{N,k}\frac{\frac{1}{\sigma}+\frac{1}{\psi}}{1-\sum_k\int\frac{\left(1-\frac{1}{R}\right)b(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\frac{e(i)}{E}di}\left(\tilde{N}_t - \tilde{\mathcal{WE}}_t - \frac{\psi}{1+\frac{\psi}{\sigma}}\sum_k\int\frac{\left(1-\frac{1}{R}\right)b(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\frac{e(i)}{E}\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)di\,(\hat{P}_{k,t}+\tilde{A}_{k,t})\right)\right\}$$
$$+\lambda_k\left\{\Omega_{N,k}\mathcal{NH}_t - \mathcal{P}_{k,t}\right\} + \tilde{\beta}\mathbb{E}_t\pi_{k,t+1}$$

When the NKPC is expressed in terms of the employment gap (which does not appear directly in the loss function), an additional wedge $\tilde{\mathcal{WE}}_t$ is introduced. The evolution of $\tilde{\mathcal{WE}}_t$ is more complex and depends on the heterogeneous impacts of wealth effects on labor supply.

A.6 Model with a Representative Agent

We derive here the representative agent version of the model. For simplicity, we assume that labor is the only input in production (no input–output linkages) and that pricing frictions are homogeneous across sectors. Let E_t denote total household expenditure, $C_{k,t}$ consumption of good k, and N_t total labor supply. As in the main text, we consider a steady state with efficient subsidies, no inflation and a constant interest rate $R = (\beta(1-\delta))^{-1}$, steady-state variables do not have a time subscript. We denote, as in the main text, $\bar{s}_k$ and $\overline{\partial_e e_k}$ to be the steady state average and marginal budget share of good k respectively.

Household Behavior. Optimal labor supply is determined by the marginal rate of substitution between labor and expenditure:

$$W_t v'(E_t, \mathbf{P_t}) = \chi'(N_t).$$

Log-linearizing around the steady state gives:

$$N\chi''\hat{N}_t = Wv'\hat{W}_t + W\left(Ev''\hat{E}_t - \sum_{k=1}^K P_k\partial_{P_k}v'\hat{P}_{k,t}\right).$$

Next we have:

$$P_k\partial_{P_k}v' = P_k\partial_e\left(\partial_{P_k}v\right) = -Ev''\bar{s}_k - v'\overline{\partial_e e_k}.$$

So labor supply satisfies:

$$\hat{N}_t = \psi\left(\hat{W}_t - \sum_{k=1}^K\overline{\partial_e e_k}\,\hat{P}_{k,t}\right) - \frac{\psi}{\sigma}\left(\hat{E}_t - \sum_{k=1}^K\bar{s}_k\,\hat{P}_{k,t}\right),$$

where $\sigma = -v'(E,\mathbf{P})\,/\,(Ev''(E,\mathbf{P}))$, $\psi = \chi'(N)\,/\,(N\chi''(N))$ are the EIS and Frisch elasticity.

Consumption growth is governed by the Euler equation:

$$v'(E_t,\mathbf{P_t}) = \beta(1-\delta)R_t\mathbb{E}_t v'(E_{t+1},\mathbf{P_{t+1}}).$$

Log-linearizing gives:

$$\mathbb{E}_t\left(\hat{E}_{t+1} - \sum_{k=1}^K\bar{s}_k\,\hat{P}_{k,t+1}\right) - \left(\hat{E}_t - \sum_{k=1}^K\bar{s}_k\,\hat{P}_{k,t}\right) = \sigma\mathbb{E}_t\left(\hat{R}_t - \sum_{k=1}^K\overline{\partial_e e_k}\,\pi_{k,t+1}\right).$$

Labor Market Clearing. Labor demand from firms in sector k is $N_{k,t}^d = C_{k,t}/A_{k,t}$, with $A_{k,t}$ denoting sector k productivity. So total labor demand, $N_t^d = \sum_{k=1}^K N_{k,t}^d$ satisfies:

$$\hat{N}_t^d = \sum_{k=1}^K\bar{s}_k\left(\hat{C}_{k,t} - \hat{A}_{k,t}\right) = \hat{E}_t - \sum_{k=1}^K\bar{s}_k\,\hat{P}_{k,t} - \sum_{k=1}^K\bar{s}_k\,\hat{A}_{k,t}.$$

From labor market clearing, we obtain:

$$\frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t - \sum_{k=1}^K\overline{\partial_e e_k}\hat{P}_{k,t}\right) = \hat{E}_t - \sum_{k=1}^K s_k\hat{P}_{k,t} - \frac{\sigma}{\sigma+\psi}\sum_{k=1}^K\bar{s}_k\,\hat{A}_{k,t}.$$

So labor supply satisfies:

$$\hat{N}_t = \frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t - \sum_{k=1}^{K}\overline{\partial_e e_k}\,\hat{P}_{k,t}\right) - \frac{\psi}{\sigma+\psi}\sum_{k=1}^{K}\bar{s}_k\,\hat{A}_{k,t}.$$

With flexible prices $\hat{W}_t - \hat{P}_{k,t} = -\hat{A}_{k,t}$, so efficient labor supply, $\hat{N}_t^*$, satisfies:

$$\hat{N}_t^* = \frac{\sigma\psi}{\sigma+\psi}\sum_{k=1}^{K}\overline{\partial_e e_k}\hat{A}_{k,t} - \frac{\psi}{\sigma+\psi}\sum_{k=1}^{K}\bar{s}_k\hat{A}_{k,t}.$$

Output Gap and Phillips Curve. Defining the output gap as $\tilde{\mathcal{Y}}_t = \hat{N}_t - \hat{N}_t^* = \frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t - \sum_{k=1}^{K}\overline{\partial_e e_k}\left(\hat{P}_{k,t}+\hat{A}_{k,t}\right)\right)$ the Euler equation becomes:

$$\mathbb{E}_t\left(\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t\right) = \sigma\mathbb{E}_t\left(\hat{R}_t - \sum_{k=1}^{K}\overline{\partial_e e_k}\,\pi_{k,t+1} - r_t^*\right),$$

with $r_t^* = \frac{1}{\sigma+\psi}\mathbb{E}_t\left(\sum_{k=1}^{K}\psi\overline{\partial_e e_k} + \bar{s}_k\right)\left(\hat{A}_{k,t+1} - \hat{A}_{k,t}\right)$.
The New Keynesian Phillips Curve (NKPC) for sector k is:

$$\pi_{k,t} = \lambda\left(\hat{W}_t - \hat{P}_{k,t} - \hat{A}_{k,t}\right) + \frac{1}{R}\mathbb{E}_t\pi_{k,t+1}.$$

Averaging across sectors yields the CPI inflation equation:

$$\begin{aligned}\pi_{CPI,t} &= \lambda\left(\hat{W}_t - \sum_{k=1}^{K}\bar{s}_k\left(\hat{P}_{k,t}+\hat{A}_{k,t}\right)\right) + \frac{1}{R}\mathbb{E}_t\pi_{CPI,t+1}\\ &= \lambda\left(\frac{\sigma+\psi}{\sigma\psi}\tilde{\mathcal{Y}}_t - \sum_{k=1}^{K}\left(\bar{s}_k - \overline{\partial_e e_k}\right)\left(\hat{P}_{k,t}+\hat{A}_{k,t}\right)\right) + \frac{1}{R}\mathbb{E}_t\pi_{CPI,t+1}.\end{aligned}$$

B Model summary

This section collects the coefficients and equations governing the dynamic evolution of the economy, as derived in Appendix A. It is intended to serve as a reference for the accompanying MATLAB code.

B.1 Model equations

Coefficients - households

Individual coefficients:

$$\begin{aligned} s_k(i) &= \frac{e_k(i)}{e(i)}, \\ \partial_e e_k(i) &= \frac{\partial e_k(i)}{\partial e(i)} \overset{(NHCES)}{=} s_k(i)\left(\eta + (1-\eta)\frac{\zeta_k}{\bar{\zeta}(i)}\right) \quad \text{where} \quad \bar{\zeta}(i) = \sum_l s_l(i)\zeta_l, \\ \rho_{k,l}(i) &= \partial_{P_l} e_k(i)/P_k + e_l(i)/P_l \partial_e e_k(i)/P_k \overset{(NHCES)}{=} (s_l(i) - 1 \cdot \mathbb{I}[k=l])\,\eta, \end{aligned}$$

where the second equality sign imposes the assumed preferences in the calibration.

Aggregate coefficients:

$$\begin{aligned} \bar{s}_k &= \frac{E_k}{E} = \frac{\int e_k(i)di}{\int e(i)di}, \\ \bar{s}_k^u &= \int \frac{(1-\varphi(i))\,e(i)}{(1-\varphi^E)\,E}\frac{e_k(i)}{e(i)}di, \\ \overline{\partial_e e_k} &= \int \frac{e(i)}{E}\partial_e e_k(i)di, \\ \overline{\partial_e e_k}^u &= \int \frac{(1-\varphi(i))\,e(i)}{(1-\varphi^E)\,E}\partial_e e_k(i)di, \\ s_k^C &= \frac{E_k}{P_k Y_k}, \\ \bar{\rho}_{k,l} &= \int \frac{e_k(i)}{E_k}\rho_{k,l}(i)di, \\ \varphi^E &= \frac{\int \varphi(i)e(i)di}{E}, \\ \varphi^N &= \frac{\int \varphi(i)Wn(i)di}{WN}, \\ R &= \frac{1}{\beta(1-\delta)}. \end{aligned}$$

Coefficients - firms

$$\begin{aligned} \Omega_{N,k} &= \frac{WN_k}{P_k Y_k}, \\ \Omega_{k,l} &= \frac{P_l \mathcal{Y}_{l,k}}{P_k Y_k}, \\ \tilde{\Omega} &= (Id - \Omega)^{-1}. \end{aligned}$$

Coefficients - equations

NKPC:

$$\lambda_k = \frac{(1-\theta_k)(1-\beta\theta_k)}{\theta_k}.$$

Output Gap:

$$ygap\_htm\_\pi_l = -\left(\sigma\left(\overline{\partial_e e_l}^u - \overline{\partial_e e_l}\right)\left(1-\varphi^E\right) - (s_l^u - \bar{s}_l)\left(1-\varphi^E\right) - \left(\varphi^E - \varphi^N\right)\left(\sigma\overline{\partial_e e_l} - \bar{s}_l\right)\right).$$

Sectoral equations.

For every sector $k = 1, ..., K$ we have:

$$
\begin{aligned}
\pi_{k,t} &= \hat{P}_{k,t} - \hat{P}_{k,t-1}, \\
\pi_{k,t} &= \lambda_k \left(\left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \Omega_{N,k} \tilde{\mathcal{Y}}_t + \Omega_{N,k} \mathcal{N}\mathcal{H}_t - \mathcal{P}_{k,t} \right) + \beta (1-\delta) \mathbb{E}_t \pi_{k,t+1}, \\
\bar{\mathcal{P}}_{k,t} &= \left(\hat{P}_{k,t} - \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t} \right) + \left(\tilde{A}_{k,t} - \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} \right), \\
\mathcal{P}_{k,t} &= \Omega_{N,k} \bar{\mathcal{P}}_{k,t} - \sum_{l=1}^{K} \Omega_{k,l} \left(\bar{\mathcal{P}}_{l,t} - \bar{\mathcal{P}}_{k,t} \right), \\
\hat{P}^*_{k,t} &= -\tilde{A}_{k,t}, \\
\hat{A}_{k,t} &= \rho_A \hat{A}_{k,t-1} + \varepsilon^A_{k,t}, \\
\tilde{A}_{k,t} &= \sum_{l=1}^{K} \tilde{\Omega}_{k,l} \hat{A}_{l,t}.
\end{aligned}
$$

Aggregate equations

$$
\begin{aligned}
\tilde{\mathcal{Y}}_t &= \mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \sigma \mathbb{E}_t \left(\hat{R}_t - \pi_{mcpi,t+1} - \hat{r}^*_t \right) \\
&\quad - \frac{1}{1-\varphi^N} \frac{\sigma}{\sigma+\psi} \left(\frac{\varphi^E - \varphi^N}{R-1} \left(\mathbb{E}_t \hat{R}_{t+1} - \left(1 + \sigma (R-1)\right) \hat{R}_t \right) + \sum_l ygap\_htm\_\pi_l \cdot \mathbb{E}_t \pi_{l,t+1} \right), \\
\hat{r}^*_t &= \frac{1}{\sigma+\psi} \sum_{l=1}^{K} \left(\psi \overline{\partial_e e_l} + \bar{s}_l \right) \left(\mathbb{E}_t \tilde{A}_{l,t+1} - \tilde{A}_{l,t} \right), \\
\hat{\mathcal{Y}}^*_t &= \frac{1}{1+\frac{\psi}{\sigma}} \sum_{l=1}^{K} \left(\psi \overline{\partial_e e_l} + \bar{s}_l \right) \tilde{A}_{l,t}, \\
\mathcal{N}\mathcal{H}_t &= \sum_{l=1}^{K} \left(\overline{\partial_e e_l} - \bar{s}_l \right) \left(\hat{P}_{l,t} - \hat{P}^*_{l,t} \right), \\
P_{cpi,t} &= \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t}, \\
P^*_{cpi,t} &= \sum_{l=1}^{K} \bar{s}_l \hat{P}^*_{l,t}, \\
\pi_{cpi,t} &= \sum_{l=1}^{K} \bar{s}_l \pi_{l,t}, \\
\pi_{mcpi,t} &= \sum_{l=1}^{K} \overline{\partial_e e_l} \pi_{l,t}, \\
\hat{R}_t &= \phi \pi_{cpi,t} + u^R_t, \\
u^R_t &= \rho_R u^R_{t-1} + \varepsilon^R_t.
\end{aligned}
$$

Equations for demand indices.

Coefficients:

$$
\begin{aligned}
fracu_{\omega} &= \int (1-\varphi(i))\,\omega(i)di, \\
msu_{\omega,l} &= \int (1-\varphi(i))\,\omega(i)\partial_e e_l(i)di, \\
chtm\_R_{\omega} &= \int \left(\varphi(i)\frac{\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\frac{b(i)}{R}\right)di, \\
chtm\_Y_{\omega} &= \left(1+\frac{\psi}{\sigma}\right)\int \left(\varphi(i)\frac{\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}Wn(i)\right)di, \\
chtm\_\pi_{\omega,l} &= \int \left(\varphi(i)\frac{\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left(\frac{(R-1)\,b(i)}{R}\bar{s}_l + e(i)\left(s_l(i)-\bar{s}_l\right)+\psi Wn(i)\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right)\right)\right)di, \\
c0\_r_{\omega} &= \frac{R-1}{R}\int \frac{\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\frac{b(i)}{R}di, \\
c0\_P_{\omega,l} &= -\frac{R-1}{R}\int \frac{\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left(e(i)\left(s_l(i)-\bar{s}_l\right)+\psi Wn(i)\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right)\right)di, \\
c0\_Y_{\omega} &= \frac{R-1}{R}\left(1+\frac{\psi}{\sigma}\right)\int \frac{\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}Wn(i)di.
\end{aligned}
$$

Equations:

$$
\begin{aligned}
\mathbb{E}_t\hat{C}_{t+1}(\boldsymbol{\omega})-\hat{C}_t(\boldsymbol{\omega}) = \sigma\left(fracu_{\omega}\hat{R}_t-\sum_l msu_{\omega,l}\pi_{l,t+1}\right)+\frac{\delta}{1-\delta}\mathbb{E}_t\hat{C}^0_{t+1}(\boldsymbol{\omega}) \\
+chtm\_R_{\omega}\left(\mathbb{E}_t\hat{R}_{t+1}-\hat{R}_t\right)+chtm\_Y_{\omega}\left(\mathbb{E}_t\hat{\mathcal{Y}}_{t+1}-\hat{\mathcal{Y}}_t\right)-\sum_{l=1}^{K} chtm\_\pi_{\omega,l}\mathbb{E}_t\pi_{l,t+1},
\end{aligned}
$$

$$
\hat{C}^0_{t-1}(\boldsymbol{\omega})-\frac{1}{(1-\delta)\,R}\mathbb{E}_t\hat{C}^0_t(\boldsymbol{\omega}) = c0\_r_{\omega}\left(\hat{R}_{t-1}-\mathbb{E}_t\pi_{cpi,t}\right)+\sum_{l=1}^{K} c0\_P_{\omega,l}\hat{P}_{l,t-1}+c0\_Y_{\omega}\hat{\mathcal{Y}}_{t-1}-\frac{R-1}{R}\hat{C}_{t-1}(\boldsymbol{\omega}).
$$

B.2 Non-homothetic utility: examples

This section briefly discusses three widely-used parametric specifications of non-homothetic preferences, including the non-homothetic CES case that is used in the quantitative application. The other two cases that we discuss here are the log-linear, and Stone-Geary preferences. The table below collects the functional forms and the main sufficient statistics that arise in the model: the budget share and the marginal budget share.

Preferences	Functional Form	Average Shares	Marginal Budget Shares
Log-linear	$\frac{1}{\tilde{\sigma}}\left(c_1 + \ln(c_2)\right)^{\tilde{\sigma}}$	$s_1 = 1 - \frac{P_1}{e}, \quad s_2 = \frac{P_1}{e}$	$\partial_e e_1 = 1, \quad \partial_e e_2 = 0$
Stone-Geary	$\frac{1}{\tilde{\sigma}}\left(\prod_{k=1}^{K}(c_k - \underline{c}_k)^{\alpha_k}\right)^{\tilde{\sigma}}$	$\frac{P_k \underline{c}_k + \alpha_k\left(e - \sum_{l=1}^{K} P_l \underline{c}_l\right)}{e}$	α_k
NHCES	$\sum_{k=1}^{K} \mathcal{V}_k \left(\frac{c_k}{g(U)^{\zeta_k}}\right)^{\frac{\eta-1}{\eta}} = 1$	s_k	$s_k\left(\eta + (1-\eta)\frac{\zeta_k}{\sum_{l=1}^{K} s_l \zeta_l}\right)$

Log-linear preferences have the feature that the marginal utility of consumption is constant for one good, while decreasing for the others. In the table, we illustrate the case with only two goods, though the logic easily extends to any arbitrary number of goods. These preferences imply that for very low levels of expenditure ($e < P_1$), everything is spent on the good with initially high marginal utility (good 2 in our example). Once a certain level of consumption of the necessity good is reached, the household starts consuming good 1. From that point onward, any additional expenditure is devoted entirely to good 1, since its marginal utility remains constant. As a result, for $e \geq P_1$ the budget share is increasing for good 1 and falling for good 2. These preferences generate the 'extreme' case that we use in the main text to illustrate the role of non-homotheticities.
Finally, note that the functional form that we have specified also includes a curvature parameter $\tilde{\sigma}$. This term is added explicitly here to show that by choosing $\tilde{\sigma}$ one can get any value for the elasticity of intertemporal substitution, while leaving the average and marginal budget shares unchanged.

Stone-Geary preferences feature minimum consumption levels for each good ($\underline{c}_k \geq 0$) that represent subsistence requirements. Otherwise, the preferences take a 'standard' Cobb-Douglas form, with the power terms α_k normalized to sum up to one. Note that like the log-linear case, we have added a curvature parameter $\tilde{\sigma}$ that can be used to control the EIS without affecting the goods demand functions for a given level of expenditure and prices. The solution is well-defined only for high-enough expenditure levels $e \geq \sum_{k=1}^{K} P_k \underline{c}_k$. In words, the household must afford at least the subsistence level of consumption at the given prices. Beyond this subsistence bundle, the household will allocate their additional expenditure according to the Cobb-Douglas parameters: goods with higher α_k will have a higher marginal budget share. In the limit, as expenditure increases towards infinity, average budget shares converge to α_k, so that marginal and average shares coincide. This implies that non-homotheticities are strongest for poor households and diminish as households become wealthier.

Non-homothetic CES (NHCES) preferences feature a constant (compensated) elasticity of substitution given by η,[2] and non-homotheticities in demand that are generated by the good-specific parameters ζ_k. The terms $\mathcal{V}_k$ are demand-shifters: they affect the overall level of demand for each good but do not appear directly in the expressions for marginal budget shares or the elasticity of substitution. These preferences do not allow for an explicit solution to the budget shares, so instead we denote these objects by s_k. The marginal budget shares can, however, be expressed as functions of the average budget shares and the parameters of the utility function. If the elasticity of substitution is below one, goods with a higher ζ_k are classified as luxuries, since their marginal share is above their average. Conversely, for $\eta > 1$, luxury goods are those with lower values of the ζ_k parameter. It is also straightforward to see from the formula that when the non-homotheticity parameters are identical for all goods, marginal and average budget shares coincide, reducing the utility function to the standard CES case.
Finally, we show how one can parametrize the function $g(\cdot)$ in order to set an arbitrary value of the elasticity of intertemporal substitution. Under the non-homothetic CES assumption, the indirect utility function is the solution to the following problem:

$$v(\boldsymbol{P}, e) \equiv \max_{\{c_1, c_2, \ldots c_K, U\}} U \quad \text{st} \sum_{k=1}^{K} \mathcal{V}_k \left(\frac{c_k}{g(U)^{\zeta_k}}\right)^{\frac{\eta-1}{\eta}} = 1, \text{ and } \sum_k P_k c_k \leq e,$$

where $g(\cdot)$ is a monotonically increasing and continuously differentiable function.

[2]Using the first-order conditions of the cost-minimization problem, it follows that for any two goods k and m $\frac{\partial \ln\left(c_k^H / c_m^H\right)}{\partial \ln(P_k/P_m)} = -\eta$, where the superscript H denotes the Hicksian demand.

The optimal consumption bundle $\{c_k\}_{k=1}^K$, together with $\{U\}$ is the solution to the following $K+1$ system of equations:

$$\mathcal{V}_k \left(\frac{c_k}{g(U)^{\zeta_k}}\right)^{\frac{\eta-1}{\eta}} \frac{e}{P_k c_k} = 1, \text{ for all } k,$$
$$\sum_{k=1}^K \mathcal{V}_k \left(\frac{c_k}{g(U)^{\zeta_k}}\right)^{\frac{\eta-1}{\eta}} = 1.$$

Let λ be the Lagrange multiplier of the budget constraint, which must satisfy the condition $\lambda e \frac{g'(U)}{g(U)} \sum_{k=1}^K \zeta_k s_k = 1$. Differentiating and re-arranging terms, we get that

$$1 + e\partial_e v \left(\frac{g''(U)}{g'(U)} - \frac{g'(U)}{g(U)}\right) + e\frac{\sum_{k=1}^K \zeta_k \partial_e s_k}{\sum_{k=1}^K \zeta_k s_k} = \frac{1}{\sigma},$$

where we have used the definition of the EIS as $\sigma \equiv -\frac{\partial_e v}{e\partial_{ee} v}$ and the fact that the Lagrange multiplier λ is equal to the marginal utility of expenditure, $\lambda = \partial_e v$.

Assume that $g(U) = e^{U^{1+\theta}}$, and using the fact that $\partial_e s_k = \frac{1}{e}(1-\eta) s_k \left(\frac{\zeta_k}{\sum_{k=1}^K \zeta_k s_k} - 1\right)$, we have that for any marginal utility of income, we can get the desired value of the EIS by setting the parameter θ such that

$$\theta = \frac{1}{\sigma} - 1 + \frac{U}{e\partial_e v}(1-\eta) \frac{\sum_{k=1}^K \zeta_k^2 s_k - \left(\sum_{k=1}^K \zeta_k s_k\right)^2}{\left(\sum_{k=1}^K \zeta_k s_k\right)^2}.$$

Log-Linearizing with Stone-Geary

Below we show how to derive the Euler equation when preferences are Stone-Geary. This example is for illustration purposes, as Lemma 1-3 apply to all well-behaved utility functions, irrespective of whether the indirect utility function has a closed-form solution or not. Given that this particular non-homothetic utility entails a closed-form expression, a more direct approach can be taken. We will show how the resulting expression aligns with the more general result.

Using the the solution to the demand functions $c_k = \underline{c}_k + \frac{\alpha_k\left(e - \sum_{l=1}^K P_l \underline{c}_l\right)}{P_k}$, one can explicitly write the indirect utility function as

$$v(e, \boldsymbol{P}) = \frac{1}{\tilde{\sigma}} \left(\prod_{k=1}^K \left(\frac{\alpha_k\left(e - \sum_{l=1}^K P_l \underline{c}_l\right)}{P_k}\right)^{\alpha_k}\right)^{\tilde{\sigma}}.$$

Taking the derivative with respect to expenditure and simplifying, we have that

$$\partial_e v = \frac{\tilde{\sigma} v}{e - e^*},$$

where $e^* = \sum_{l=1}^K P_l \underline{c}_l$ denotes the subsistence level of expenditure. Taking another derivative to evaluate $\partial_{ee} v$, and using the definition of the elasticity of intertemporal substitution, we get that

$$EIS = \frac{1}{1-\tilde{\sigma}} \left(\frac{e - e^*}{e}\right).$$

To obtain a first-order approximation of the Euler Equation we first need to evaluate the change in the marginal utility of expenditure. For simplicity, we omit time subscripts at this stage. Using the previous result for $\partial_e v$, and totally differentiating with respect to prices and expenditure we have

$$\begin{aligned}
d\ln\partial_e v &= d\ln v - \frac{de - \sum_{l=1}^K dP_l \underline{c}_l}{e - e^*}, \\
&= \tilde{\sigma}\left(\frac{de - \sum_{l=1}^K dP_l c_l}{e - e^*}\right) - \frac{de - \sum_{l=1}^K dP_l\left(c_l - \alpha_l\left(e - e^*\right)\right)}{e - e^*}, \\
&= -\left(1-\tilde{\sigma}\right)\frac{e}{e - e^*}\left(\hat{e} - \sum_{l=1}^K s_l \hat{P}_l\right) - \sum_{l=1}^K \hat{P}_l \alpha_l.
\end{aligned}$$

The term that multiplies the real expenditure change is the inverse of the EIS, while the second term weighs price changes according to the α parameters, which are equal to the marginal budget shares. Finally, we apply this result to the Euler equation which in log-deviation terms around a stationary steady state is given by $d\ln\partial_e v_t = \hat{R}_t + \mathbb{E}_t\left[d\ln\partial_e v_{t+1}\right]$. Denoting the elasticity of intertemporal substitution by σ, we have that

$$\left(\hat{e}_t - \sum_{k=1}^K s_k \hat{P}_{k,t}\right) = \mathbb{E}_t\left(\hat{e}_{t+1} - \sum_{k=1}^K s_k \hat{P}_{k,t+1}\right) - \sigma\mathbb{E}_t\left(\hat{R}_t - \sum_{k=1}^K \alpha_k \pi_{k,t+1}\right).$$

This is exactly the Euler equation that we derived under general preferences in Lemma 3.

C Proofs of Propositions in Section 3

In this appendix we provide a general statement of Proposition 1 and 2 in economies with Input-Output linkages as well as proofs of the results and a proof of Proposition 3. In addition, in Propositions C1 we provide closed form solutions for inflation, the output gap and the non-homothetic wedge under assumption A1 and without Input-Output linkages.

C.1 Proposition 1

Proposition 1 (policy invariance of the wedges). *Assume that $\kappa_k = \lambda_k\left(\frac{1}{\sigma}+\frac{1}{\psi}\right)\Omega_{N,k}$ is constant across sectors, equal to κ. Then $\mathcal{NH}_t$ and $\mathcal{P}_{k,t}$ evolve independently of monetary policy. Without Input-Output, assuming that κ_k is constant across sectors is equivalent to **A.1**.*

Proof. Assume without loss of generality that $\bar{s}_K \neq 0$ and denote for $1 \leq k \leq K-1$, $\tilde{P}_{k,t} = \hat{P}_{k,t} - \hat{P}_{cpi,t}$ and $\tilde{\pi}_{k,t} = \pi_{k,t} - \pi_{cpi,t}$, the sectoral price and inflation relative to the CPI. We have:

$$\pi_{k,t} = \kappa\tilde{\mathcal{Y}}_t + \frac{\sigma\psi}{\sigma+\psi}\kappa\sum_{l=1}^{K}\overline{\partial_e e_l}\left((P_{l,t}+\tilde{A}_{l,t}) - (P_{k,t}+\tilde{A}_{k,t})\right) + \lambda_k\sum_{l=1}^{K}\Omega_{k,l}\left((P_{l,t}+\tilde{A}_{l,t}) - (P_{k,t}+\tilde{A}_{k,t})\right) + \tilde{\beta}\mathbb{E}_t\pi_{k,t+1},$$

$$\pi_{cpi,t} = \kappa\tilde{\mathcal{Y}}_t + \frac{\sigma\psi}{\sigma+\psi}\kappa\sum_{l=1}^{K}\overline{\partial_e e_l}\left(P_{l,t}+\tilde{A}_{l,t} - \left(P_{cpi,t} + \sum_{k=1}^{K}\bar{s}_k\tilde{A}_{k,t}\right)\right) + \sum_{1\leq k,l\leq K}\lambda_k\bar{s}_k\Omega_{k,l}\left(P_{l,t}+\tilde{A}_{l,t} - (P_{k,t}+\tilde{A}_{k,t})\right) + \tilde{\beta}\mathbb{E}_t\pi_{cpi,t+1}.$$

This implies, since $\hat{P}_{K,t} - \hat{P}_{cpi,t} = -\sum_{k=1}^{K-1}\frac{\bar{s}_k}{\bar{s}_K}\tilde{P}_{k,t}$, that we can write the NKPC for the relative inflation and relative price of k as:

$$\begin{aligned}
\tilde{\pi}_{k,t} = &-\left(\lambda_k\tilde{P}_{k,t} - \sum_{l=1}^{K-1}\bar{s}_l\left(\lambda_l-\lambda_K\right)\tilde{P}_{l,t}\right) + \sum_{l=1}^{K-1}\left(\lambda_k\left(\Omega_{k,l} - \frac{\bar{s}_l}{\bar{s}_K}\Omega_{k,K}\right) - \sum_{m=1}^{K}\bar{s}_m\lambda_m\left(\Omega_{m,l} - \frac{\bar{s}_l}{\bar{s}_K}\Omega_{m,K}\right)\right)\tilde{P}_{l,t} + \tilde{\beta}\mathbb{E}_t\tilde{\pi}_{k,t+1}\\
&-\left(\lambda_k\tilde{A}_{k,t} - \sum_{k=1}^{K}\bar{s}_l\lambda_l\tilde{A}_{l,t}\right) + \sum_{l=1}^{K}\left(\lambda_k\Omega_{k,l} - \sum_{m=1}^{K}\bar{s}_m\lambda_m\Omega_{m,l}\right)\tilde{A}_{l,t},\\
\tilde{P}_{k,t} = &\ \tilde{\pi}_{k,t} + \tilde{P}_{k,t-1}.
\end{aligned}$$

Since $\tilde{A}_{k,t}$ are exogenous, the $K-1$ relative prices (and inflation) $\tilde{P}_{k,t}, \tilde{\pi}_{k,t}$ are pinned down by a system of $2(K-1)$ equations which does not involve $\hat{R}_t$ (and initial and final conditions $\tilde{P}_{k,0} = \tilde{\pi}_{k,0}$, $\lim_{t\to\infty}\tilde{\pi}_{k,t} = 0$). These variables are therefore independent of monetary policy. As the non-homotheticity and relative price wedge can be written in terms of our relative prices:

$$\mathcal{P}_{k,t} = \tilde{P}_{k,t} + \tilde{A}_{k,t} - \sum_{k=1}^{K}\bar{s}_l\tilde{A}_{l,t},$$

$$\mathcal{NH}_t = \sum_{l=1}^{K-1}\left(\overline{\partial_e e_l} - \frac{\bar{s}_l}{\bar{s}_K}\overline{\partial_e e_K}\right)\tilde{P}_{l,t} + \sum_{l=1}^{K}\left(\overline{\partial_e e_l} - \bar{s}_l\right)\tilde{A}_{l,t},$$

they are independent from monetary policy. □

C.2 Proposition 2

Proposition 2 (divine coincidence index). *Assume that for all $1 \leq k_0 \leq K$, either $\Omega_{N,k_0} > 0$ or there exists a sequence $k_1, ..., k_n$ such that $\Omega_{k_i,k_{i+1}} > 0$ and $\Omega_{N,k_n} > 0$ (labor is an essential input). Then, the matrix $Id - \Omega$ is invertible. Fluctuations in the output gap can be eliminated by stabilizing the divine coincidence index $\pi_{DC,t} \equiv \sum_{k=1}^{K}\frac{\omega_k^*}{\sum_l \omega_l^*}\pi_{k,t}$ where the weights ω_k^* are non-negative and are defined as $[\omega_1^* \quad ... \quad \omega_K^*] \equiv [\overline{\partial e_1} \quad ... \quad \overline{\partial e_K}]\left(Id-\Omega\right)^{-1}\Delta_\lambda^{-1}$ (with Δ_λ the diagonal matrix of λ_k). In particular, without Input-Output linkages and under assumption **A.1**, the divine coincidence index coincides with the Marginal CPI index $\pi_{DC,t} = \sum_{k=1}^{K}\overline{\partial_e e_k}\pi_{k,t}$.*

Proof. Suppose that we have $(Id-\Omega)v = 0$ for some vector $v \neq 0$ and denote $k_0 = \text{argmax}_k|v_k|$. If $\Omega_{N,k_0} > 0$, we have $\left(1-\Omega_{k_0,k_0}\right)|v_{k_0}| = \left|\sum_{l\neq k_0}\Omega_{k_0,l}v_l\right| \leq \sum_{l\neq k_0}\Omega_{k_0,l}|v_l| \leq \sum_{l\neq k_0}\Omega_{k_0,l}|v_{k_0}|$ which implies $\Omega_{N,k_0} = 1 - \sum_{l=1}^{K}\Omega_{k_0,l} \leq 0$, a contradiction. If $\Omega_{N,k_0} = 0$ then $|v_{k_0}| = |v_{k_1}|$, if not, we have $\left(1-\Omega_{k_0,k_0}\right)|v_{k_0}| = \left|\sum_{l\neq k_0}\Omega_{k_0,l}v_l\right| < \sum_{l\neq k_0}\Omega_{k_0,l}|v_{k_0}|$ which implies $|v_{k_0}| < \sum_{l=1}^{K}\Omega_{k_0,l}|v_{k_0}| \leq |v_{k_0}|$ a contradiction. By direct induction, we then have $|v_{k_0}| = |v_{k_n}|$ which implies $\Omega_{N,k_n} \leq 0$ a contradiction. Therefore, $Id - \Omega$ is invertible.
Note that by definition of ω_k^*, we have $\lambda_k\omega_k^* - \sum_{l=1}^{K}\Omega_{l,k}\lambda_l\omega_l^* = \overline{\partial e_k}$. Summing over k, we obtain $\sum_{k=1}^{K}\Omega_{N,k}\lambda_k\omega_k^* = 1$ so we have $\lambda_k\omega_k^* = \sum_{l=1}^{K}\left(\Omega_{l,k} + \Omega_{N,l}\overline{\partial e_k}\right)\lambda_l\omega_l^*$. Therefore, $[\lambda_1\omega_1^* \quad ... \quad \lambda_K\omega_K^*]$ is the Perron-Frobenius (left) eigenvector of the probability matrix $\tilde{\Omega}_{k,l} = \Omega_{k,l} + \Omega_{N,k}\overline{\partial e_l}$ and therefore has non negative components so $\omega_k^* \geq 0$ and $\sum_{l=1}^{K}\omega_l^* > 0$.
Now recall that the sector k NKPC is given by:

$$\pi_{k,t} = \lambda_k\left\{\Omega_{N,k}\left(\hat{W}_t - (\hat{P}_{k,t}+\tilde{A}_{k,t})\right) + \sum_{l=1}^{K}\Omega_{k,l}\left((\hat{P}_{l,t}+\tilde{A}_{l,t}) - (\hat{P}_{k,t}+\tilde{A}_{k,t})\right)\right\} + \tilde{\beta}\mathbb{E}_t\pi_{k,t+1}.$$

Given that $\Omega_{N,k} = 1 - \sum_{l=1}^{K} \Omega_{k,l}$, we can rewrite it as:

$$\pi_{k,t} = \lambda_k \left\{ \left(\hat{W}_t - \left(\hat{P}_{k,t} + \tilde{A}_{k,t}\right)\right) - \sum_{l=1}^{K} \Omega_{k,l} \left(\hat{W}_t - \left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right)\right) \right\} + \tilde{\beta}\mathbb{E}_t \pi_{k,t+1}.$$

Or in matrix form:

$$\left[\pi_{1,t}, ..., \pi_{K,t}\right]' = \Delta_\lambda \left(Id - \Omega\right) \left[\hat{W}_t - \left(\hat{P}_{1,t} + \tilde{A}_{1,t}\right), ..., \hat{W}_t - \left(\hat{P}_{K,t} + \tilde{A}_{K,t}\right)\right]' + \tilde{\beta}\mathbb{E}_t \left[\pi_{1,t+1}, ..., \pi_{K,t+1}\right]'.$$

Therefore, we have:

$$\begin{aligned}
\pi_{DC,t} &= \frac{1}{\sum_l \omega_l^*} \left[\overline{\partial e}_1 \quad ... \quad \overline{\partial e}_K\right] \left(Id - \Omega\right)^{-1} \Delta_\lambda^{-1} \Delta_\lambda \left(Id - \Omega\right) \left[\hat{W}_t - \left(\hat{P}_{1,t} + \tilde{A}_{1,t}\right), ..., \hat{W}_t - \left(\hat{P}_{K,t} + \tilde{A}_{K,t}\right)\right]' + \tilde{\beta}\mathbb{E}_t \pi_{DC,t+1}, \\
&= \frac{1}{\sum_l \omega_l^*} \left(\hat{W}_t - \sum_{k=1}^{K} \overline{\partial e}_k \left(\hat{P}_{k,t} + \tilde{A}_{k,t}\right) \right) + \tilde{\beta}\mathbb{E}_t \pi_{DC,t+1}, \\
&= \frac{1}{\sum_l \omega_l^*} \left(\frac{1}{\sigma} + \frac{1}{\psi}\right) \tilde{\mathcal{Y}}_t + \tilde{\beta}\mathbb{E}_t \pi_{DC,t+1}.
\end{aligned}$$

Therefore, stabilizing $\pi_{DC,t}$ eliminates fluctuations in the output gap. Finally, without IO, we have $\Omega = 0$ so $\pi_{DC,t} = \sum_{k=1}^{K} \frac{\lambda_k^{-1} \overline{\partial e}_k}{\sum_l \lambda_l^{-1} \overline{\partial e}_l} \pi_{k,t}$. If in addition $\lambda_k = \lambda$ for all k, $\pi_{DC,t} = \sum_{k=1}^{K} \overline{\partial_e e}_k \pi_{k,t}$. $\square$

C.3 Proposition 3

Proposition 3 (Non-Homotheticity Wedge). *Assume that (A.1.) holds and that there are no Input-Output linkages. Following a negative productivity shock in a necessity (luxury) sector $\mathcal{NH}_t$ rises (falls) on impact.*

Proof. Denote the gap between MCPI and CPI inflation by $\pi_{\Delta,t} = \sum_l \left(\overline{\partial_e e_l} - \bar{s}_l\right) \pi_{l,t}$, and analogously define $\hat{P}_{\Delta,t}$ and $\hat{A}_{\Delta,t}$. Note that $\mathcal{NH}_t = \hat{P}_{\Delta,t} + \hat{A}_{\Delta,t}$. Under (**A.1**) $\lambda_k = \lambda$ for all k. We can write the NKPC for $\pi_{\Delta,t}$ as:

$$R\pi_{\Delta,t} = -\lambda R \left(\hat{P}_{\Delta,t} + \hat{A}_{\Delta,t}\right) + \pi_{\Delta,t+1}$$

$\Leftrightarrow$

$$\hat{P}_{\Delta,t+1} - \left(1 + R + R\lambda\right) \hat{P}_{\Delta,t} + R\hat{P}_{\Delta,t-1} = \lambda R \hat{A}_{\Delta,t}$$

The eigenvalues of the system are:

$$\mu_{\pm} = \frac{R + R\lambda + 1 \pm \sqrt{\left(R + R\lambda - 1\right)^2 + 4R\lambda}}{2},$$

with $\mu_+ > R + R\lambda$, $\mu_- < 1$. We obtain:

$$\hat{P}_{\Delta,t} = -\lambda \sum_{s=0}^{t} \mu_-^{t-s+1} \sum_{u=0}^{\infty} \frac{1}{\mu_+^u} \hat{A}_{\Delta,u+s}.$$

Therefore, we have:

$$\mathcal{NH}_t = -\lambda \sum_{s=0}^{t} \mu_-^{t-s+1} \sum_{u=0}^{\infty} \frac{1}{\mu_+^u} \hat{A}_{\Delta,u+s} + \hat{A}_{\Delta,t}.$$

Now suppose that we have a negative shock in a necessity (luxury) sector, in that case $\hat{A}_{\Delta,t} \geq 0$ ($\hat{A}_{\Delta,t} \leq 0$). Assume in addition that $\left|\hat{A}_{\Delta,t}\right| \leq \left|\hat{A}_{\Delta,0}\right|$ (the shock is larger on impact), then we have for a shock in a necessity sector

$$\begin{aligned}
\mathcal{NH}_0 &\geq \left(1 - \lambda\mu_- \sum_{u\geq 0} \frac{1}{\mu_+^u}\right) \hat{A}_{\Delta,0}, \\
&\geq \left(1 - \frac{\lambda\mu_-\mu_+}{\mu_+ - 1}\right) \hat{A}_{\Delta,0}, \\
&\geq \left(1 - \frac{\lambda R}{R + R\lambda - 1}\right) \hat{A}_{\Delta,0} \geq 0.
\end{aligned}$$

Similarly for a shock in a luxury sector, we have:

$$\mathcal{NH}_0 \leq \left(1 - \frac{\lambda R}{R + R\lambda - 1}\right) \hat{A}_{\Delta,0} \leq 0. \square$$

In the following Proposition, we provide closed form solutions for the non-homotheticity wedge, CPI and MCPI inflation and the output gap when shocks are AR(1). We show in particular that under CPI targeting, the output gap falls and CPI inflation rises on impact following a negative necessity shock. In addition, the derivations are used in the proof of Proposition 6 in Appendix E.

Proposition C1 (AR(1) Shocks). *Assume no Input-Output linkages and that A.1. holds. If shocks vanish at a constant rate ρ_a then following a negative shock in a necessity sector:*

i. *There exists a time $t_{\mathcal{NH}}$ ($t_{\mathcal{NH}}=0$ if $\rho_a=0$, $t_{\mathcal{NH}}=\infty$ if $\rho_a=1$) such that for $\mathcal{NH}_t \geq 0$ for $t \leq t_{\mathcal{NH}}$, and $\mathcal{NH}_t \leq 0$ for $t > t_{\mathcal{NH}}$.*

ii. *The gap $\pi_{cpi,t}-\pi_{mcpi,t}$ evolves independently of the policy rule. There exists t^* ($t^*=0$ if $\rho_a=0$, $t^*=\infty$ if $\rho_a=1$) such that $\pi_{cpi,t} \geq \pi_{mcpi,t}$ for $t \leq t^*$, and $\pi_{cpi,t} \leq \pi_{mcpi,t}$ for $t > t^*$.*

iii. *Under the MCPI rule $\hat{R}_t=\phi\pi_{mcpi,t}+\hat{r}^*_t$ (with $\phi>1$), we have that $\pi_{mcpi,t}=\tilde{\mathcal{Y}}_t=0$, and so $\pi_{cpi,t}\geq 0$ for $t\leq t^*$, and $\pi_{cpi,t}\leq 0$ for $t>t^*$.*

iv. *Under the CPI rule $\hat{R}_t=\phi\pi_{cpi,t}+\hat{r}^*_t$, there exists a time $t_{\mathcal{Y}}$($t_{\mathcal{Y}}=0$ if $\rho_a=0$, $t_{\mathcal{Y}}=\infty$ if $\rho_a=1$) such that $\tilde{\mathcal{Y}}_t\leq 0$ for $t\leq t_{\mathcal{Y}}$, and $\tilde{\mathcal{Y}}_t\geq 0$ for $t>t_{\mathcal{Y}}$.*

v. *Under the CPI rule $\hat{R}_t=\phi\pi_{cpi,t}+\hat{r}^*_t$, there exists a level of persistence ρ^*such that for $\rho_a\leq\rho^*$, $\pi_{cpi,t}\geq 0$ for all t. For $\rho_a>\rho^*$, there exists t_{CPI} ($t_{CPI}=\infty$ if $\rho_a=1$) such that $\pi_{cpi,t}\leq 0$ for $t\leq t_{CPI}$, and $\pi_{cpi,t}\geq 0$ for $t>t_{CPI}$.*

vi. *Under the alternative rule $\hat{R}_t=\phi\pi_{mcpi,t}$ or $\hat{R}_t=\phi\pi_{cpi,t}$, the response of the output gap and both inflation indices at t are simply shifted up proportionally to $\rho_a^t\hat{r}^*_0$. Normalizing shocks such that $\hat{r}^*_0=-1$ (equal impact of sectoral shocks on efficient output), we have that for $t\leq t_{\mathcal{Y}}$ ($t>t_{\mathcal{Y}}$) and CPI targeting the output gap will be higher (lower) following a shock in a luxury sector rather than in a necessity sector. In addition, for high enough persistence the output gap will be negative under CPI targeting following a shock in a necessity sector.*

Proof. We now prove each of the items in Proposition C1:

Dynamics of the $\mathcal{NH}$ wedge.

Rewriting $\mathcal{NH}_t=-\lambda\sum_0^t\mu_-^{t-s+1}\sum\frac{1}{\mu_+^u}\hat{A}_{\Delta,u+s}+\hat{A}_{\Delta,t}$, with $\hat{A}_{\Delta,t}=\rho_a^t\hat{A}_{\Delta,0}$ we have:

$$\mathcal{NH}_t=\frac{1}{(\mu_+-\rho_a)(\mu_--\rho_a)}\left((R-\rho_a)(1-\rho_a)\rho_a^t-(R-\mu_-)(1-\mu_-)\mu_-^t\right)\hat{A}_{\Delta,0}.$$

Define $t^*=ln\left(\frac{(R-\mu_-)(1-\mu_-)}{(R-\rho_a)(1-\rho_a)}\right)/ln\left(\frac{\rho_a}{\mu_-}\right)$. For $t\leq t^*$, $\mathcal{NH}_t$ has same sign as $A_{\Delta,0}$, and for $t>t^*$, $\mathcal{NH}_t$ same sign as $-A_{\Delta,0}$. For a transitory shock $t^*=0$, while for a permanent shock $t^*=\infty$.

We now derive the evolution of inflation (CPI and MCPI) and the output gap under some particular interest rules.

Case $\hat{R}_t=\phi\pi_{mcpi,t}+\hat{r}^*_t$.

The system of equations becomes:

$$R\pi_{mcpi,t}=R\kappa\tilde{\mathcal{Y}}_t+\mathbb{E}_t\pi_{mcpi,t+1},$$
$$\mathbb{E}_t\tilde{\mathcal{Y}}_{t+1}-\tilde{\mathcal{Y}}_t=\sigma\left(\phi\pi_{mcpi,t}-\mathbb{E}_t\pi_{mcpi,t+1}\right).$$

The eigenvalues of the system are:

$$\lambda_\pm=\frac{R+R\kappa\sigma+1\pm\sqrt{(R+R\kappa\sigma-1)^2-4R\kappa\sigma(\phi-1)}}{2}.$$

For $\phi>1$, the eigenvalues are larger than 1 in modulus, we therefore have $\pi_{mcpi,t}=\tilde{\mathcal{Y}}_t=0$ for all t. The evolution of CPI is then

$$R\pi_{cpi,t}=R\lambda\mathcal{NH}_t+\mathbb{E}_t\pi_{cpi,t+1},$$

$$\pi_{cpi,t}=\frac{R\lambda}{(\mu_+-\rho_a)(\mu_--\rho_a)}\left((1-\rho_a)\rho_a^t-(1-\mu_-)\mu_-^t\right)\hat{A}_{\Delta,0}.$$

We have that $\pi_{cpi,0}$ has the same sign as $A_{\Delta,0}$ (positive for a shock in a necessity sector, negative for a shock in a luxury sector). In addition, define $t^*=ln\left(\frac{(1-\mu_-)}{(1-\rho_a)}\right)/ln\left(\frac{\rho_a}{\mu_-}\right)$, for $t\leq t^*$, $\pi_{cpi,t}$ has same sign as $\hat{A}_{\Delta,0}$ and for $t>t^*$, $\pi_{cpi,t}$ has the same sign as $-\hat{A}_{\Delta,0}$. For transitory shock $t^*=0$, for a permanent shock $t^*=\infty$.

Case $\hat{R}_t=\phi\pi_{cpi,t}+\hat{r}^*_t$.

The system of equations becomes:

$$R\pi_{cpi,t}=R\kappa\tilde{\mathcal{Y}}_t+R\lambda\mathcal{NH}_t+\mathbb{E}_t\pi_{cpi,t+1},$$
$$\mathbb{E}_t\tilde{\mathcal{Y}}_{t+1}-\tilde{\mathcal{Y}}_t=\sigma\left(\phi\pi_{cpi,t}-\mathbb{E}_t\pi_{mcpi,t+1}\right).$$

In that case, we have:

$$\pi_{cpi,t}=\frac{R\lambda}{(\mu_+-\rho_a)(\mu_--\rho_a)}\left\{\left(1-\frac{R\kappa\sigma\phi}{(\lambda_+-\rho_a)(\lambda_--\rho_a)}\right)(1-\rho_a)\rho_a^t-\left(1-\frac{R\kappa\sigma\phi}{(\lambda_+-\mu_-)(\lambda_--\mu_-)}\right)(1-\mu_-)\mu_-^t\right\}\hat{A}_{\Delta,0},$$
$$\tilde{\mathcal{Y}}_t=-\frac{R\lambda\sigma\phi}{(\mu_+-\rho_a)(\mu_--\rho_a)}\left\{\frac{(1-\rho_a)(R-\rho_a)}{(\lambda_+-\rho_a)(\lambda_--\rho_a)}\rho_a^t-\frac{(1-\mu_-)(R-\mu_-)}{(\lambda_+-\mu_-)(\lambda_--\mu_-)}\mu_-^t\right\}\hat{A}_{\Delta,0},$$

$$\pi_{cpi,t}-\pi_{mcpi,t}=\frac{R\lambda}{(\mu_{+}-\rho_{a})(\mu_{-}-\rho_{a})}\left((1-\rho_{a})\rho_{a}^{t}-(1-\mu_{-})\mu_{-}^{t}\right)\hat{A}_{\Delta,0}.$$

Note that the fraction $\frac{(1-x)(R-x)}{(\mu_{+}-x)(\mu_{-}-x)}$ is decreasing in x. From this we deduce that $\tilde{\mathcal{Y}}_t$ initially has the same sign as $-\hat{A}_{\Delta,0}$ (for $t\leq t^{*}=t^{*}=ln\left(\frac{(R-\mu_{-})(1-\mu_{-})}{(R-\rho_{a})(1-\rho_{a})}\frac{(\lambda_{+}-\rho_{a})(\lambda_{-}-\rho_{a})}{(\lambda_{+}-\mu_{-})(\lambda_{-}-\mu_{-})}\right)/ln\left(\frac{\rho_{a}}{\mu_{-}}\right)$) then for $t>t^{*}$ has the same sign as $A_{\Delta,0}$ (for a transitory shock$\tilde{\mathcal{Y}}_t$ has the same sign as $\hat{A}_{\Delta,0}$ for $t>0$, for a permanent shock, $\tilde{\mathcal{Y}}_t$ has the same sign of $-A_{\Delta,0}$ for all t). This implies that the output gap is always negative on impact in response to a negative shock in the necessity sector, positive for a shock in a luxury sector. The response of CPI is more ambiguous and depends on the persistence of the shock. There exist a persistence $0<\nu=\frac{R+R\kappa\sigma+1-\sqrt{(R+R\kappa\sigma-1)^{2}+4R\kappa\sigma}}{2}<\rho^{*}<\frac{R+R\lambda+1-\sqrt{(R+R\lambda-1)^{2}+4R\lambda}}{2}=\mu_{-}$ such that for $\rho_a\leq\rho^{*}$, $\pi_{cpi,t}$ always has the same sign as $\hat{A}_{\Delta,0}$. In that case, $\pi_{cpi,t}$ and the output gap initially move in opposite direction. If $\rho_a>\rho^{*}$, initially cpi inflation has the same sign as $-A_{\Delta,0}$ and then switches sign (keeping the sign of $-\hat{A}_{\Delta,0}$ if $\rho_a=1$). In that case, $\pi_{cpi,t}$ and the output gap initially co-move. To see this consider the polynomial $P(x)=((1-x)(R-x)-R\kappa\sigma x)(1-x)-(\lambda_{+}-x)(\lambda_{-}-x)\left(1-\frac{R\kappa\sigma\phi}{(\lambda_{+}-\mu_{-})(\lambda_{-}-\mu_{-})}\right)(1-\mu_{-})$. It is a third order polynomial with a negative dominant term. It is direct to check that $P(x)\geq0$ for $x\leq\nu$, $P(\mu_{-})=0$, $P(1)=0$ and $P'(\mu_{-})=0$. This implies $P(x)\geq0$ for $x\in[0,\rho^{*}]\cup[\mu_{-},1]$, $P(x)\leq0$ for $x\in[\rho^{*},\mu_{-}]$ with $\nu<\rho^{*}<\mu_{-}$. Inspecting the formula for $\pi_{cpi,t}$ then gives the result. In the extreme case where $\phi\to\infty$, we have $\pi_{cpi,t}=0$, $\tilde{\mathcal{Y}}_t=-\frac{\sigma\psi}{\sigma+\psi}\mathcal{NH}_t$: stabilizing CPI inflation comes at the cost of distorting the output gap. Finally, since by Proposition 1 $\pi_{cpi,t}-\pi_{mcpi,t}$ is independent of monetary policy, we have as in the previous case that for a negative shock in a necessity sector, $\pi_{cpi,t}$ is initially higher than $\pi_{mcpi,t}$ and then lower and the opposite is true for a negative shock in a luxury sector.

Case $\hat{R}_t=\phi\pi_{mcpi,t}$**.**

The system of equations becomes:

$$R\pi_{mcpi,t}=R\kappa\tilde{\mathcal{Y}}_t+\mathbb{E}_t\pi_{mcpi,t+1},$$
$$\mathbb{E}_t\tilde{\mathcal{Y}}_{t+1}-\tilde{\mathcal{Y}}_t=\sigma\mathbb{E}_t\left(\phi\pi_{mcpi,t}-\pi_{mcpi,t+1}-\hat{r}_t^{*}\right).$$

In that case, we have:

$$\pi_{mcpi,t}=\frac{R\kappa\sigma}{(\mu_{+}-\rho_{a})(\mu_{-}-\rho_{a})}\rho_a^t\hat{r}_0^{*},$$
$$\tilde{\mathcal{Y}}_t=\frac{\sigma(R-\rho_a)}{(\mu_{+}-\rho_{a})(\mu_{-}-\rho_{a})}\rho_a^t\hat{r}_0^{*}.$$

The response is as in the standard model with $\pi_{mcpi,t}$ and $\tilde{\mathcal{Y}}_t$ both increasing in response to a negative shock (sectoral or aggregate) and increase is smaller the stronger the Taylor rule. In addition

$$\pi_{cpi,t}=\frac{R\lambda}{(\mu_{+}-\rho_{a})(\mu_{-}-\rho_{a})}\left((1-\rho_{a})\rho_{a}^{t}-(1-\mu_{-})\mu_{-}^{t}\right)\hat{A}_{\Delta,0}+\frac{R\kappa\sigma}{(\mu_{+}-\rho_{a})(\mu_{-}-\rho_{a})}\rho_a^t\hat{r}_0^{*}.$$

$\pi_{cpi,t}$ increases relatively more than $\pi_{mcpi,t}$ for $t\leq t^{*}=ln\left(\frac{(1-\mu_{-})}{(1-\rho_{a})}\right)/ln\left(\frac{\rho_{a}}{\mu_{-}}\right)$, (less for $t>t^{*}$) for a negative shock in a necessity sector, relatively less for a negative shock in a luxury sector.

Case $\hat{R}_t=\phi\pi_{cpi,t}$**.**

The system of equations becomes:

$$R\pi_{cpi,t}=R\kappa\tilde{\mathcal{Y}}_t+R\lambda\mathcal{NH}_t+\mathbb{E}_t\pi_{cpi,t+1},$$
$$\mathbb{E}_t\tilde{\mathcal{Y}}_{t+1}-\tilde{\mathcal{Y}}_t=\sigma\mathbb{E}_t\left(\phi\pi_{cpi,t}-\pi_{mcpi,t+1}-\hat{r}_t^{*}\right).$$

We have:

$$\pi_{cpi,t}=\frac{R\lambda}{(\mu_{+}-\rho_{a})(\mu_{-}-\rho_{a})}\left\{\left(1-\frac{R\kappa\sigma\phi}{(\lambda_{+}-\rho_{a})(\lambda_{-}-\rho_{a})}\right)(1-\rho_a)\rho_a^t-\left(1-\frac{R\kappa\sigma\phi}{(\lambda_{+}-\mu_{-})(\lambda_{-}-\mu_{-})}\right)(1-\mu_{-})\mu_{-}^{t}\right\}\hat{A}_{\Delta,0}+\frac{R\kappa\sigma}{(\lambda_{+}-\rho_{a})(\lambda_{-}-\rho_{a})}\rho_a^t\hat{r}_0^{*},$$

$$\tilde{\mathcal{Y}}_t=-\frac{R\lambda\bar{\sigma}\phi}{(\mu_{+}-\rho_{a})(\mu_{-}-\rho_{a})}\left\{\frac{(1-\rho_a)(R-\rho_a)}{(\lambda_{+}-\rho_{a})(\lambda_{-}-\rho_{a})}\rho_a^t-\frac{(1-\mu_{-})(R-\mu_{-})}{(\lambda_{+}-\mu_{-})(\lambda_{-}-\mu_{-})}\mu_{-}^{t}\right\}\hat{A}_{\Delta,0}+\frac{\sigma(R-\rho_a)}{(\lambda_{+}-\rho_{a})(\lambda_{-}-\rho_{a})}\rho_a^t\hat{r}_0^{*}.$$

Using the results of the previous cases, we can directly see that following a negative shock in a necessity sector, the output gap is lower under targeting than under MCPI targeting. In addition, if we compare the response of a negative shock in a luxury sector and a necessity sector which have the same impact on efficient output ($\mathcal{Y}_t^{*}=\frac{1}{1+\frac{\psi}{\sigma}}\sum_l\left(\psi\overline{\partial_e e_l}+\bar{s}_l\right)\hat{A}_{l,t}$), the output gap is relatively lower in response to the shock in the necessity sector. If the shock is sufficiently persistent the output gap is negative in response to a shock in a necessity sector (as $\hat{r}_0^{*}\to0$ when $\rho_a\to1$). $\square$

D Calibration procedure and numerical details

D.1 Calibration procedure

Non-homothetic preferences

To calibrate the non-homothetic CES preferences we use the LCF survey, which is the most comprehensive survey on household spending in the UK. Each member of the household keeps a detailed spending diary for a period of two weeks, while expenditure information on bigger items (like cars, vacations, housing etc.) are collected during interviews with the household head. We map these highly disaggregated consumption data into the standard 3-digit COICOP categories using a mapping table provided by the ONS. Aggregating these to the COICOP division level forms the basis of our definition of sectors for the UK economy.

We exclude housing costs from household expenditures by redefining the relevant consumption category (COICOP4) to only include expenditure on Electricity, Gas and Other Fuels.[3] Furthermore, we exclude the following four sectors from our model: Alcohol & Tobacco, Health, Communication and Education. Health and Education are largely publicly provided in the UK, and only a very small fraction of households report any private spending in these sectors. The other two sectors account for a small budget share so overall we still capture the vast majority of private expenditure, with the notable exception of housing.[4]

We construct household-specific price indices using the observed consumption shares in the 3-digit subcategories of each COICOP group so that $\ln P_{k,t}(i) = \sum_{m \in M_k} s_{m,k,t}(i) \ln P_{m,k,t}$. Whenever indices of 3-digit COICOP categories are not available (only occurring before 2015 and for a small subset of categories), we use the 2-digit price index of the corresponding group. To guard against any potential endogeneity of prices (similarly to what is done in **?**) we construct Hausman-type price instruments by using the shares of all other households in the same region and for any given sector. To instrument for total expenditure we use log disposable income as well as the expenditure quintile of the household.

We impose that the individual parameter shifters take the following form:

$$\ln \mathcal{V}_{i,k} = x_i \beta_k + v_i^k,$$

where x_i are household demographic characteristics, and v_i^k is an idiosyncratic and time invariant preference shifter that satisfies $\mathbb{E}\left[v_i^k | x_i\right] = 0$. The specific demographic controls include the size of the household $(1, 2+\text{adults})$, number of children $(0, 1+)$ and the age of the household head $(18-37, 38-50, 51-64, 65+)$. Note that since households are surveyed at different points during the year, we also include quarter dummies to allow for potential seasonal effects in the consumption of different goods. The table below shows the results across a different set of specifications, with the first column showing our baseline version. For all estimates in this table, we use Clothing as the base sector and assume an elasticity of substitution equal to 0.1. We conduct different robustness checks to show that our results do not qualitatively change with the specific assumptions made in the baseline. The other columns show in turn the GMM results from winsorizing the data, adding regional controls (there are 12 regions in the UK) and expanding the sample to include all years available. For the winsorized sample we mark the households that are in the bottom or top 2% of expenditure shares in each of the eight COICOP categories and then drop them from the estimation. The GMM results are quite robust to outliers so the exact cut-off does not matter much. Note also that in specification 4 we add year dummies on top of the quarter dummies that are present in all specifications. We have also run other robustness checks where we use different instruments or weight the observations by household expenditure and qualitatively the results are unchanged.

[3]Note that these are not the only direct expenditure on energy as households who own vehicles will also spend on diesel and petrol, included in the Transport category.

[4]The correlation between the three different measures of total expenditure (i.e. the original variable, excluding housing and excluding housing plus the four sectors) is always greater than 0.966.

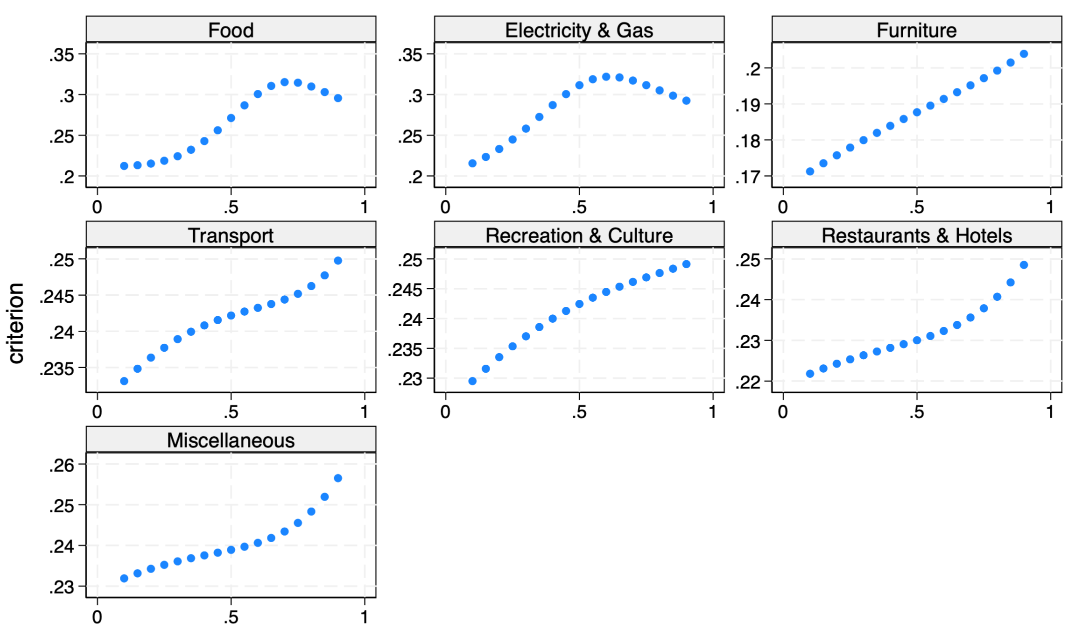
	(1)	(2)	(3)	(4)
Food	0.50	0.46	0.49	0.37
	(0.02)	(0.02)	(0.02)	(0.00)
Electricity & Gas	0.52	0.47	0.51	0.30
	(0.02)	(0.02)	(0.02)	(0.00)
Furniture	1.21	1.12	1.19	1.20
	(0.05)	(0.05)	(0.05)	(0.01)
Transport	0.90	0.89	0.88	1.10
	(0.04)	(0.04)	(0.04)	(0.01)
Recreation	1.23	1.15	1.20	1.11
	(0.05)	(0.04)	(0.04)	(0.01)
Restaurants & Hotels	0.98	0.97	0.96	0.99
	(0.04)	(0.04)	(0.03)	(0.01)
Miscellaneous	0.86	0.82	0.84	0.90
	(0.03)	(0.03)	(0.03)	(0.01)
N	3,164	2,815	3,164	56,538

These estimates allows us in turn to construct the marginal budget share $\partial_e e_k(i) = \eta + (1-\eta)\frac{\zeta_k}{\bar{\zeta}(i)}$, where $\bar{\zeta}(i)$ is the household specific 'average' non-homotheticity measure given by $\bar{\zeta}(i) = \sum_k s_k(i)\zeta_k$. This implies that richer households that spend more on luxury goods will have a higher $\bar{\zeta}(i)$. These preferences also imply that the compensated price elasticities take the following form:

$$\rho_{k,l}(i) = \begin{cases} \eta s_l(i) & \text{if } k \neq l, \\ -\eta\,(1 - s_l(i)) & \text{if } k = l. \end{cases}$$

Elasticity of Substitution Parameter. We set the elasticity of substitution parameter equal to 0.1, following the **?** estimation for their 10-sector model. Here we show that increasing the value of η worsens the fit of the model, as measured by the criterion function of the GMM procedure. Figure 8 plots the criterion value as we vary the value of the elasticity parameter between 0.05 and 0.9. Regardless which of the sectors we choose as the base, the fit of the model worsens with higher values of η. [5]

Figure A1. Criterion Value for different values of the elasticity parameter.



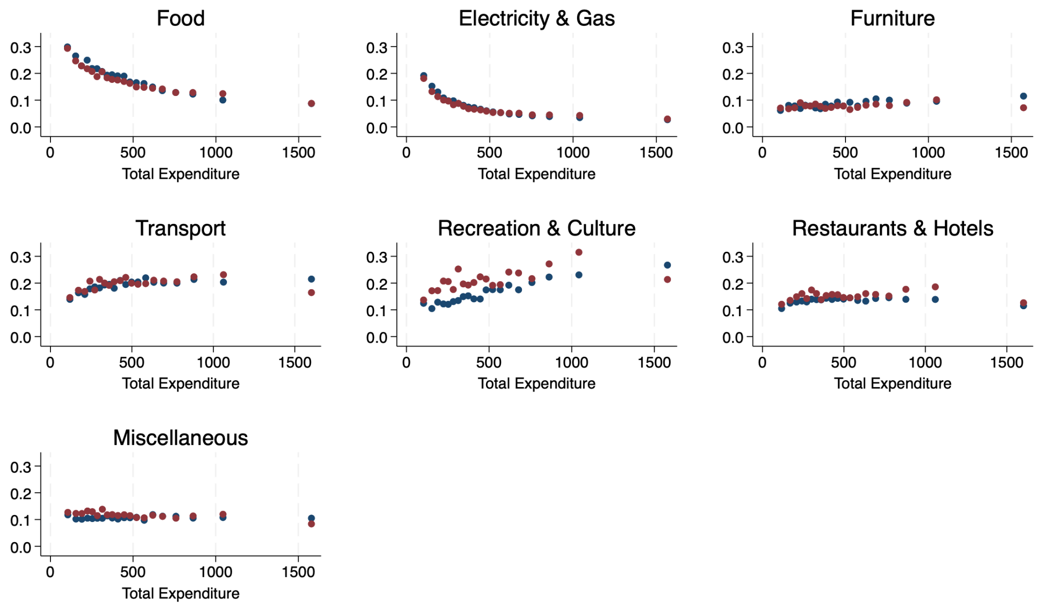
Notes: Each panel plots the minimized criterion function for the same GMM procedure for a given base sector.

Finally, figure A2 shows that the estimated preferences reproduce reasonably well the non-homotheticities observed in the data. In particular, we plot each category's budget share against total household expenditure. The data are binned in 20 equally sized groups by total expenditure, and each dot represents the

[5] Note that we do not estimate the parameter η jointly with the ζ's because as the figure shows the estimation would demand an η that goes to zero and so the procedure is not well behaved.

average budget share in that bin. In blue we have the observed budget shares while the red dots represent the fitted budget shares, and they align quite well for almost all sectors.

Figure A2. Actual vs. Predicted budget shares by household total expenditure.



Notes: Each point represents the average expenditure share on a given sector by total expenditure bin. The data has been binned into 20 equally sized groups.

Input-Output. To calibrate the parameters relating to the IO part of the model, we use the tables of intermediate input consumption provided by the ONS. These tables of input flows are constructed based on the CPA classification that defines 105 industries/products and which are different from the COICOP classification that we use in our model. To bridge this gap, we construct a mapping between the CPA classification and the COICOP one starting from the most disaggregated list of product classification (CPC10) of which there are more than 2000 products, although only 832 are for final consumption. The mapping consists in two steps. The first is to use the CPC10 to COICOP tables and assign weights to each product using the CPI weights available from ONS data. For example, if there are four CPC10 goods for a given COICOP category (we use the most disaggregated one for which we observe consumption weights) that has a weight of 1, each good will receive a weight of 0.25. Also note that the vast majority of CPC10 goods (more than 80%) map to a single COICOP category. Another 12% maps to two categories and only less than 5% maps to 3-5 COICOP categories.

Similarly in the other direction, we map the COICOP10 consumption goods to the CPA industry definitions using the concordance tables available from the UN's Statistics Division. [6] Unsurprisingly, the mapping of consumption goods to industries contains fewer one-to-one cases than with COICOP. Nonetheless, about 60% of goods only map to one or two CPA industries and another 30% map to 3 or 4.

Closed economy adjustment. The intermediate consumption tables provided by the ONS do not specify the share of inputs produced domestically vs what is imported. In our closed-economy world it must be the case that final demand (private consumption) plus intermediate consumption equals to total domestic output $[PY]$. To make this identity hold when we calibrate the model to the real-world data we adjust the vector of domestic total outputs with weights $\{\alpha_1, \alpha_2, \ldots, \alpha_K\}$ such that the following holds

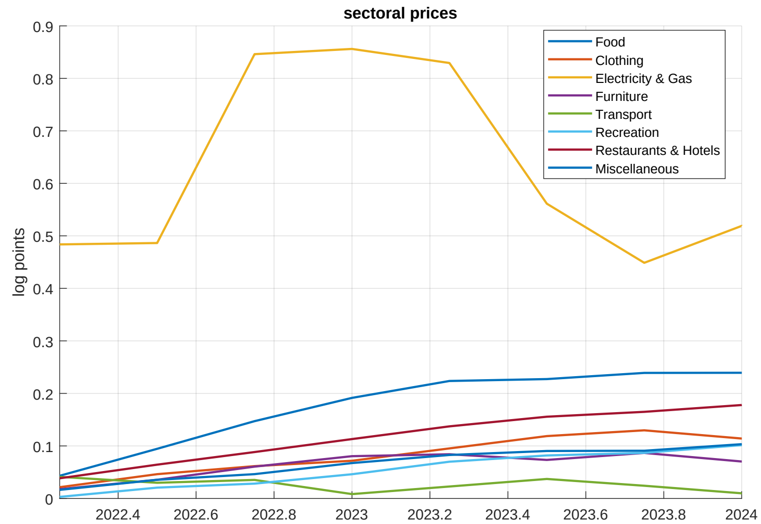
$$[PC]_k + T\,[\alpha]_k = \mathcal{D}\,[\alpha]\,[PY]_k\,,$$

where the matrix T gives the flow of intermediate inputs and specifically $T_{i,j}$ is the amount of product i used in industry j. [7] This correction imposes that all production is done domestically (while not distort-

[6] Note that this has to be done in a few steps that consists of the following chain of mapping CPC10 $\rightarrow$ ISIC3 $\rightarrow$ ISIC3.1 $\rightarrow$ ISIC4 $\rightarrow$ NACE2. That final classification contains 626 categories that can be aggregated to the 105 sectors used in the UK's IO tables.

[7] Note that in terms of the Ω matrix one can write the flow matrix as $T = (\mathcal{D}\,[PY]\,\Omega)^T$.

Figure A3. Sectoral prices.



Notes: Source: Office for National Statistics. Data are relative to 2022q1.

ing the input mix used by different industries as given by T) and hence sectors in which the UK imports (exports) a lot will have a higher (lower) adjustment factor α.

The table below shows the IO matrix Ω for the eights sectors in our model. As is standard, we observe that sectors mostly tend to use goods produced by their own sector and so the diagonal entries dominate.

0.200	0.009	0.023	0.019	0.031	0.049	0.006	0.043
0.003	0.024	0.016	0.024	0.023	0.028	0.001	0.040
0.006	0.011	0.322	0.055	0.036	0.036	0.001	0.094
0.005	0.019	0.060	0.108	0.047	0.064	0.001	0.086
0.008	0.011	0.057	0.039	0.239	0.066	0.003	0.089
0.019	0.011	0.051	0.042	0.068	0.180	0.008	0.109
0.090	0.002	0.043	0.007	0.008	0.014	0.014	0.029
0.005	0.010	0.055	0.029	0.042	0.073	0.007	0.239

D.2 2022 experiment: details

In this section, we describe the details of the 2022 experiment presented in Sections 4 and 5. Our goal is to feed into the model shocks to the *Food* and *Electricity & Gas* sectors resulting from the invasion of Ukraine in the first quarter of 2022. We model these as one-time, unexpected shocks, with the full time paths of productivity in the two sectors revealed in the initial quarter.

We recover the shocks using the sectoral New Keynesian Phillips Curves (NKPCs) in terms of marginal cost, as presented in Appendix A.5.1. Specifically, we input time series for sectoral prices ($\hat{P}_{k,t}$ for $k = 1, \ldots, K$) and nominal wages ($\hat{W}_t$) into the sectoral NKPCs for the two sectors –accounting for input-output linkages– and then recover the corresponding time paths for productivity ($\hat{A}_{k,t}$). Sectoral price inflation data are provided by the Consumer Price Index series of the ONS, and are shown in Figure A3. Nominal wages are measured as the average labor compensation per hour worked. All data are measured over the period 2022q2:2024q3 and normalized relative to 2021q1.

Having backed out the productivity shocks, we smooth using a one-year moving average filter and extrapolate them five years beyond 2024q3, assuming a decay rate of 30 percent per quarter, i.e. assuming $\hat{A}_{k,t+1} = 0.7\hat{A}_{k,t}$ for $t > 2024q3$. The productivity paths are then fed into the model under the assumption that they are fully revealed in 2022Q2.

E Additional numerical results

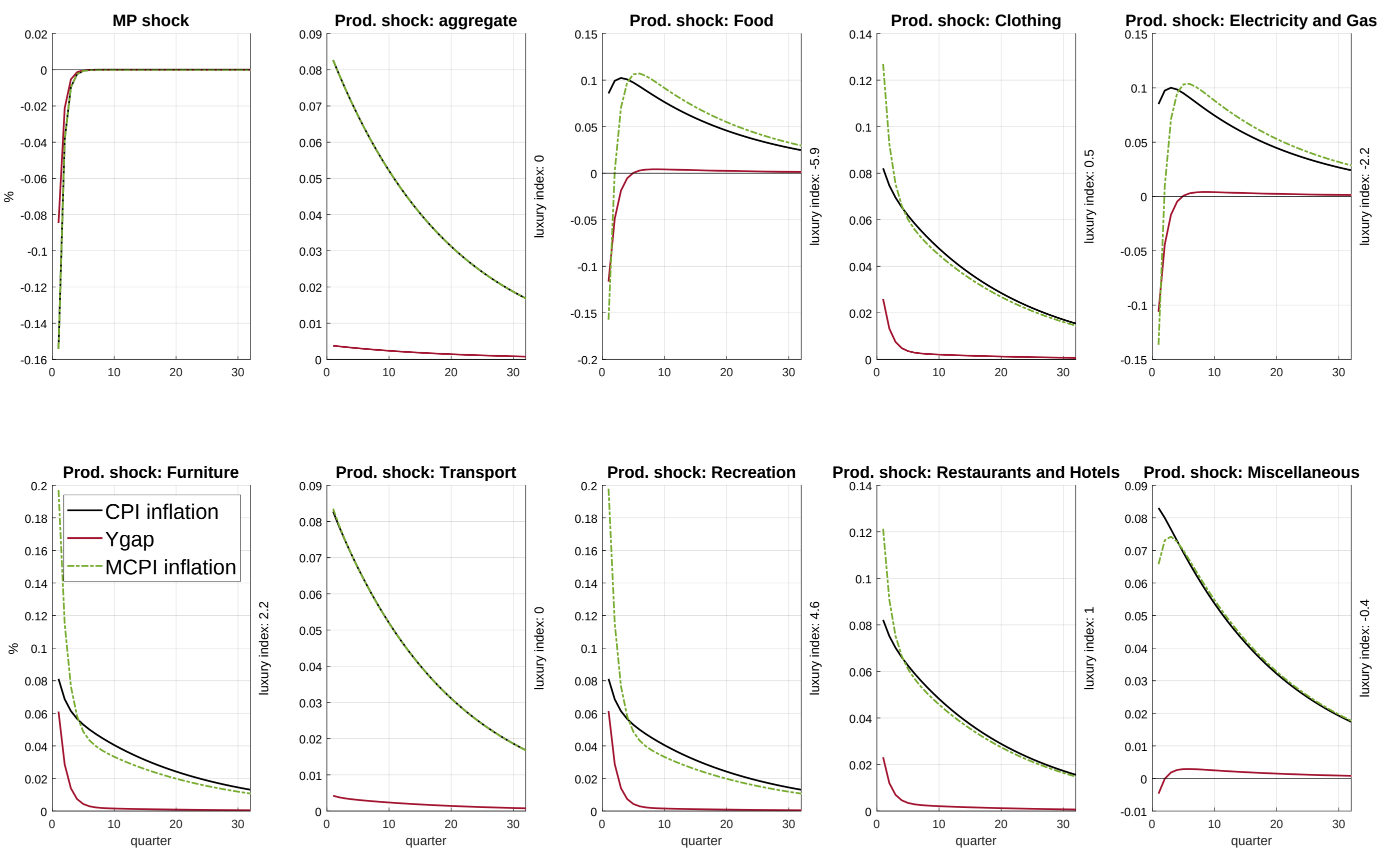
E.1 Model without heterogeneity in price stickiness and markups, and without I-O linkages

The baseline model includes various features other than non-homotheticities. In this appendix we study their quantitative importance. First, we shut down sectoral heterogeneity in prices stickiness and steady-state markups, as well as Input-Output linkages. Concretely, we achieve this by targeting in the calibration the (unweighted) average markup across sectors, setting all Calvo parameters equal to the average across sectors, and by setting intermediate input shares to zero.

Figure A4 shows impulse responses under a Taylor rule, with the aforementioned features shut down. As shown by the figure, we preserve the key result that the output gap declines in the two necessity sectors: *Food* and *Electricity & Gas*. In sector *Transport*, the output gap now increases. The increase observed in the baseline model is thus driven by the features that we shut down in this appendix. This is consistent with the fact that Transport is neither a luxury nor a necessity sector (the luxury index equals zero for this sector).

Overall these results underscore the importance of non-homotheticities and show that our main results in the baseline model are not driven by sectoral heterogeneity in price setting, markups and I-O linkages.

Figure A4. Responses in the baseline model, but without heterogeneity in prices stickiness and steady-state markups across sectors, and without Input-Output linkages.



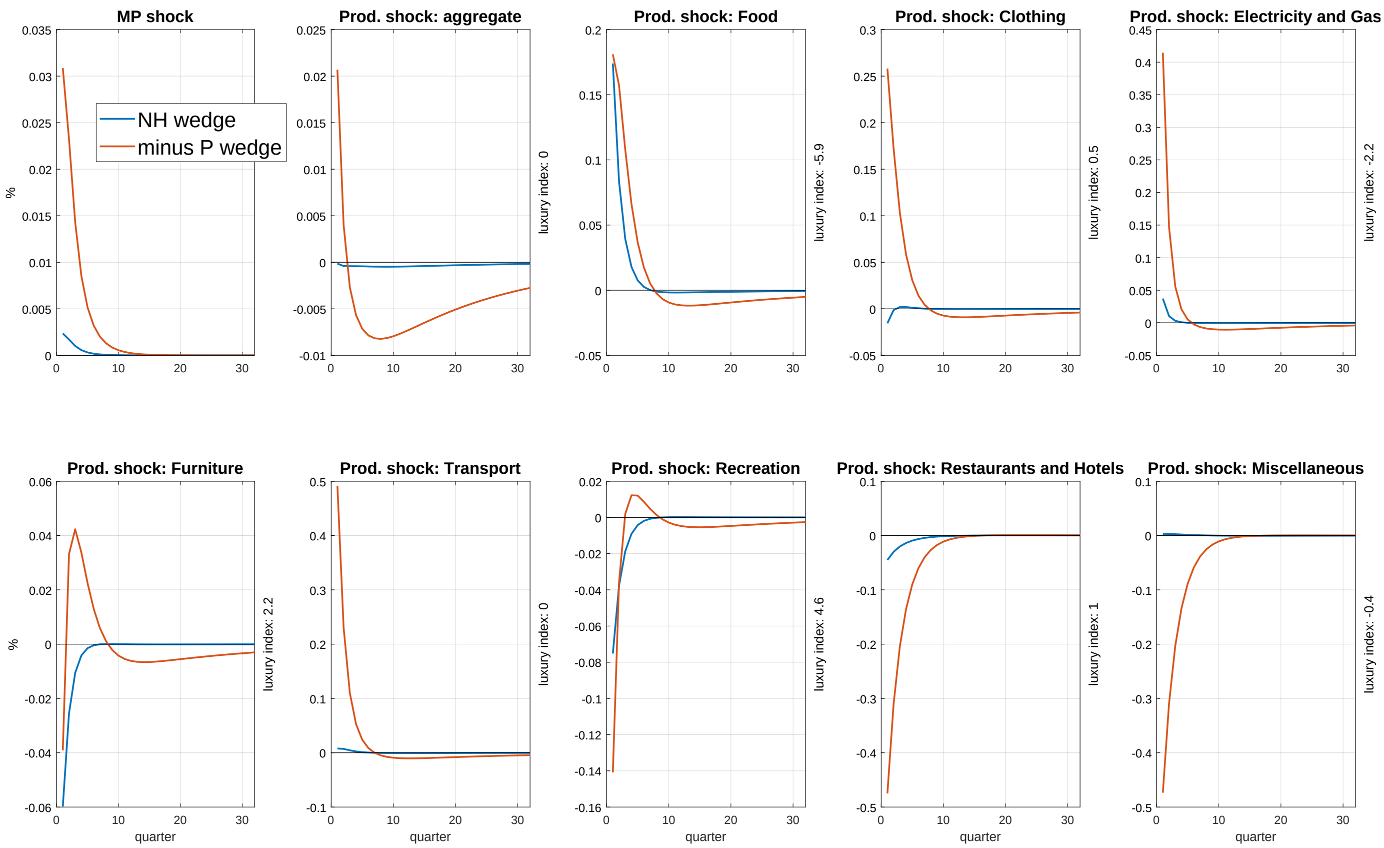
Notes: Responses for productivity shocks are for a 1 percent decline in productivity where scaled for comparability (see main text). On the right axis, the luxury index is defined as $100(\overline{\partial_e e_l} - \bar{s}_k)$.

Next, we study the wedges quantitatively. Figure A5 plots the baseline model. We observe that the $\mathcal{NH}$ wedge increases following negative shocks productivity shocks in the sectors *Food* and *Electricity & Gas*, and decreases following shocks to *Clothing*, *Furniture*, *Recreation*, and *Restaurants & Hotels*. We also observe that the movements tend to be relatively transitory. Finally, we observe that the relative price wedge $\mathcal{P}$ plays a quantitatively important role as well. Recall that this wedge enters the NKPC with a negative sign. The negative of the wedge increases for following negative shocks to *Food*, *Clothing*, *Electricity & Gas* and

Transport as these are sectors with relatively flexible prices. For the remaining sectoral shocks, the negative of the wedge falls (on impact). From Table 4 in the main text it can be seen luxury sectors tend to have stickier prices. Thus, the $\mathcal{NH}$ and the $\mathcal{P}$ wedge tend to reinforce each other, although there are exceptions (e.g. *Clothing* shocks).

Figure A6 shows the wedges for the baseline model, but without heterogeneity in Calvo probabilities and without I-O linkages. In this model version, the weighted relative price wedge is zero. The movements in the $\mathcal{NH}$ wedge is qualitatively similar to the baseline model.

Figure A5. Responses of the wedges in the baseline model.



Notes: wedges are aggregated using NKPC weights given by $\frac{\bar{s}_k \lambda_k}{\sum_l \bar{s}_l \lambda_l}$.

E.2 Implementing optimal policy with a Taylor rule plus guidance

We now explore quantitatively the optimal policy responses to various sectoral shocks, and study to what extent the optimal policy response to shocks in sectors like *Food* or *Electricity & Gas* is indeed relatively loose, compared to a typical policy rule $\hat{R}_t = \phi \pi_{cpi,t}$ (with $\phi = 1.5$) and compared to the optimal response to other shocks. In order to make this comparison quantitatively, we exploit the fact that one can always implement the optimal interest rate path $\{\hat{R}_t\}_{t=0}^{\infty}$ as a rule $\hat{R}_t = \phi \pi_{cpi,t} + u_t^R$ where $\{u_t^R\}_{t=0}^{\infty}$ is a specific time path for the deviation from the rule ("optimal guidance"), announced when the productivity shock initially hits. We simulate the model both under such an interest rate rule and under optimal policy, and then numerically solve for the guidance path that implements the optimal policy. This path then quantifies how tight or loose the optimal policy is relative to the simple CPI-based rule. In Appendix E.2 we provide the details of this procedure.

Guidance is defined as a series of interest rate rule residuals, $\{u_{t+s}^R\}_{s=0}^{\infty}$, where $u_{t+s}^R = \hat{R}_{t+s} - \phi \pi_{t+s}$. These residuals are announced at the moment a certain shock hits (this could be e.g. a sectoral or aggregate productivity shock). The guidance may vary across shocks.

Our goal is to solve for the guidance which, for a certain shock, implements the optimal policy. Let IRF_{OP} be a column vector containing the Impulse Response Function (IRF) of some variable under optimal monetary policy, IRF_{TR} the IRF under a Taylor rule, and $IRF_{MP(s)}$ be the IRF to a purely transitory, unit news shock to the Taylor rule, hitting at date s and announced at date 0. We want to solve for $\{u^R_{t+s}\}_{s=0}^{S-1}$ such that

$$IRF_{OP} = IRF_{TR} + u^R_{t+s} \sum_{s=0}^{S} IRF_{MP(s)} = IRF_{TR} + \boldsymbol{IRF}_{MP}\boldsymbol{u}$$

where S is a truncation date, $\boldsymbol{IRF}_{MP}$ is an $S \times S$ matrix containing the IRFs to the monetary policy shocks on its columns, and $\boldsymbol{u^R}$ is a column vector containing the guidance. We solve for the guidance vector as:

$$\boldsymbol{u^R} = \boldsymbol{IRF}_{MP}^{-1} \left(IRF_{OP} - IRF_{TR} \right).$$

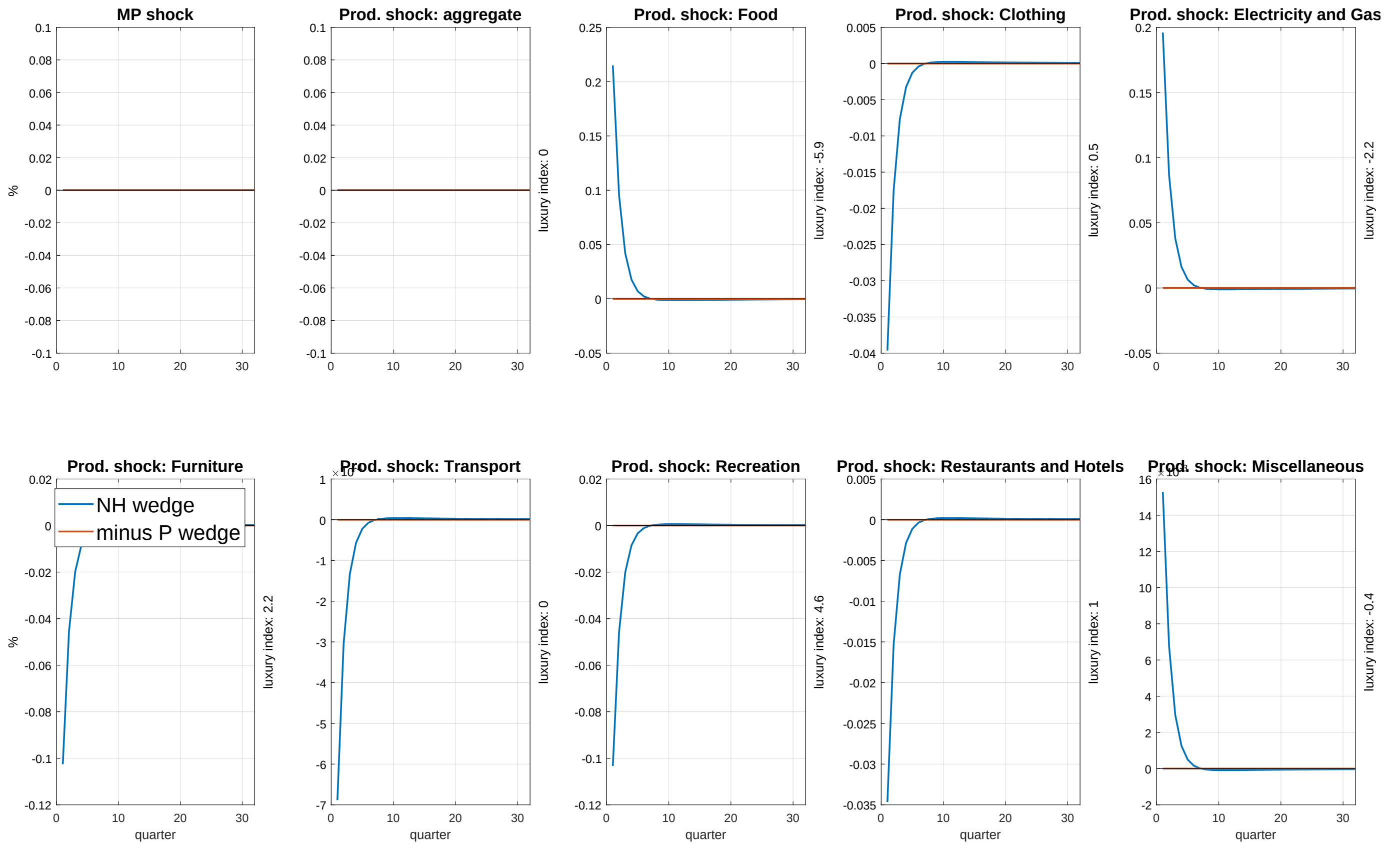
In our implementation, we use the IRF of CPI inflation to aggregate and sector-level shocks. We set the truncation horizon to 75 quarters. We verify ex post that the IRFs of variables are close to identical under optimal policy and the interest rate rule plus guidance.

The left panel in Figure A7 plots the optimal guidance for the aggregate and sectoral productivity shocks in the full model. In line with the analytical results, optimal policy is initially significantly looser than the rule following a negative necessity shock. Following negative shocks to luxuries, the optimal policy is also initially looser than the rule, but less so than following necessity shocks. We thus find that the sectoral source of the shock indeed has significant quantitative consequences for optimal policy, in line with the analytical results.[8]

How important are redistributive motives in driving the optimal policy? In the right panel of Figure A7 we shut down the redistributive motives of monetary policy. Qualitatively, the results are unchanged, in the sense that the optimal policy response to negative necessity shocks is significantly looser than the response to luxury shocks. Quantitatively however, the monetary policy response is tighter without the redistribution motive. Intuitively this motive pushes towards a front-loaded accommodative (i.e. looser) policy for all shocks, as this helps to redistribute towards poorer households, who tend to be more heavily affected in utility terms.

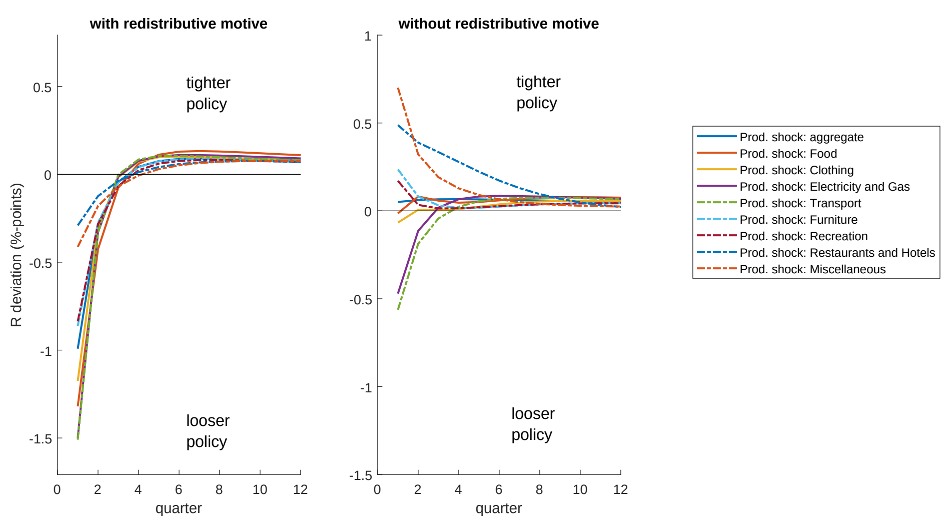
[8] Consistent with the quantitative results in the previous section, we find that negative shocks to *Transport* (neither a necessity nor a luxury) call for a relatively loose optimal policy, which is again due to the low degree of price rigidity in this sector and the position of this sector in the I-O matrix. Appendix E.1 repeats the exercise shutting down I-O linkages and heterogeneity in price rigidity. In this case, the optimal policy response to a transport shock is similar to the response to an aggregate productivity shock, i.e. much less loose.

Figure A6. Responses of the wedges in the baseline model, but without heterogeneity in prices stickiness and steady-state markups across sectors, and without Input-Output linkages.



Notes: wedges are aggregated using NKPC weights given by $\frac{\bar{s}_k \lambda_k}{\sum_l \bar{s}_l \lambda_l}$.

Figure A7. Optimal policy relative to Taylor rule.



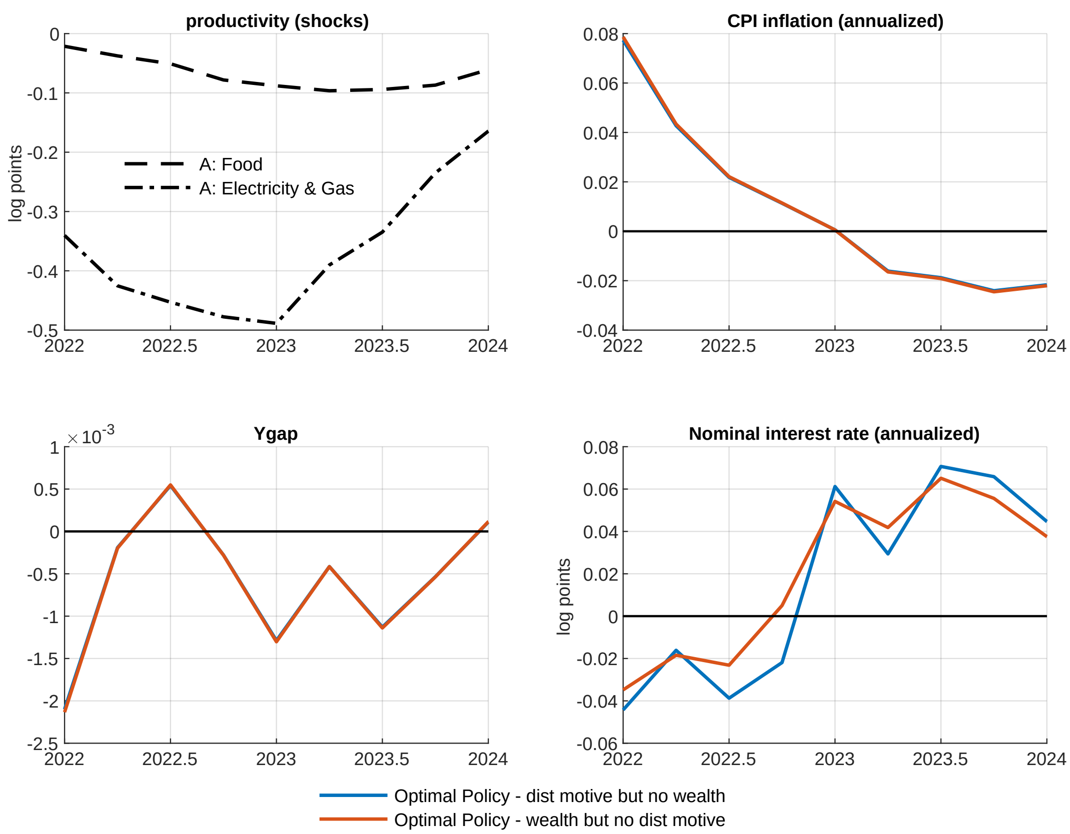
Notes: Deviations from the Taylor rule $\hat{R}_t = 1.5\pi_{cpi,t}$ which implement Optimal Policy ("optimal guidance"). Higher values mean that optimal monetary policy is tight relative to this rule. See the main text for details. All productivity shocks are negative.

E.3 2022 experiment: additional results

In this section, we present additional numerical results to explore the distributional motive of monetary policy. To this end, we consider several variations on the 2022 experiment presented in Section 5.

In the first variation, we compare outcomes under optimal policy without a distributional motive to outcomes under optimal monetary policy that includes a distributional motive, but without any steady-state wealth. The latter is achieved by simply setting $b(i) = 0$ for any household i in the calibration. Figure A8 shows that outcomes are quite similar. This implies that the distributional motive of monetary policy is implemented mostly via wealth distributions (as opposed to, e.g., changes in relative prices).

Figure A8. 2022 experiment: variation 1



In the second variation, shown in Figure A9, we consider outcomes under fully optimal policy (including the distributional motive) in the baseline versus a version in which we shut down any heterogeneity in consumption baskets, but preserve non-homotheticities. This is achieved by setting the regular and marginal budget shares equal to their aggregate counterparts (formally: $s_k(i) = \bar{s}_k$ and $\partial_e e_k(i) = \overline{\partial_e e_k}$). The simulated paths turn out to be very similar, which implies that while non-homotheticities are important, heterogeneity in consumption baskets itself has limited implications for optimal policy. Intuitively, the central bank has only one instrument, which makes it difficult to manipulate relative sectoral prices.
In the third variation, we compare optimal policy (with and without a distributional motive) in the baseline model to optimal policy (with a distributional motive) in a version of the model without Hand-to-Mouth (HtM) households. Figure A10 shows thatâexcept for the initial period of the shockâthe latter version is quite similar to the baseline without a distributional motive. This implies that HtM households are important in shaping the distributional motive of monetary policy. Intuitively, large fluctuations in interest rates create large fluctuations in interest expenses (or income) among HtM households, which is especially costly for them. Optimal policy therefore has a motive to avoid interest rate volatility.

Figure A9. 2022 experiment: variation 2

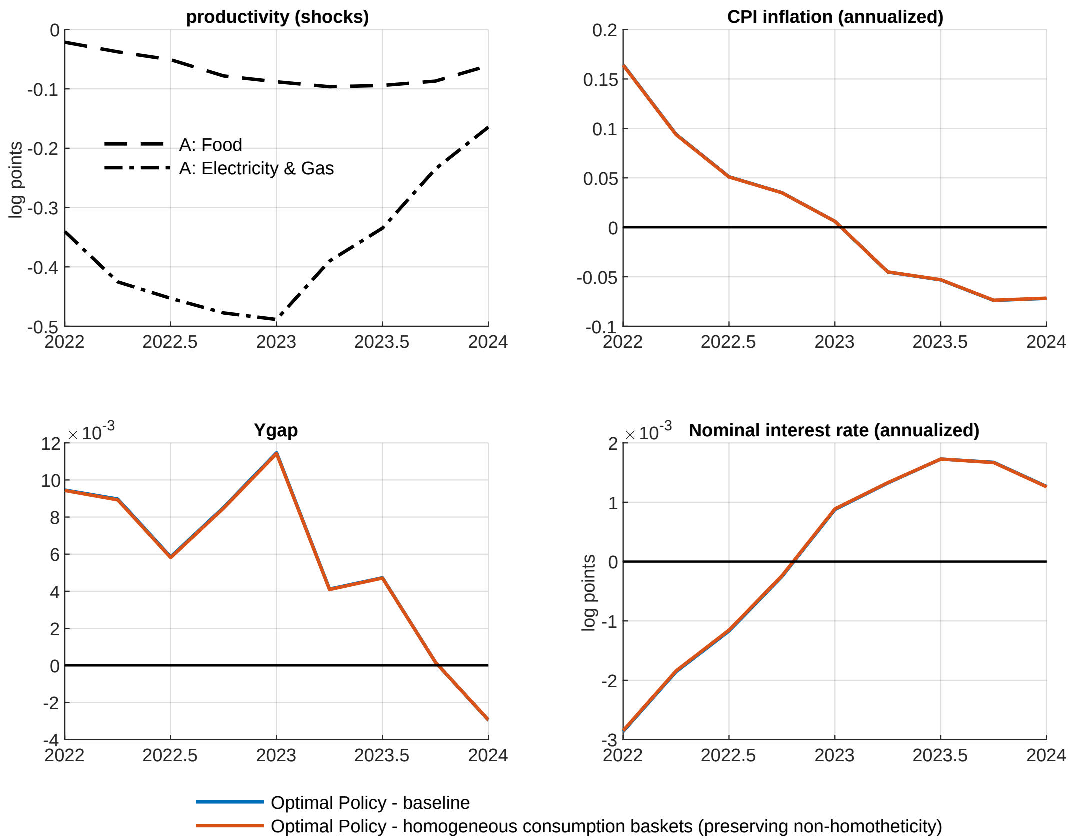
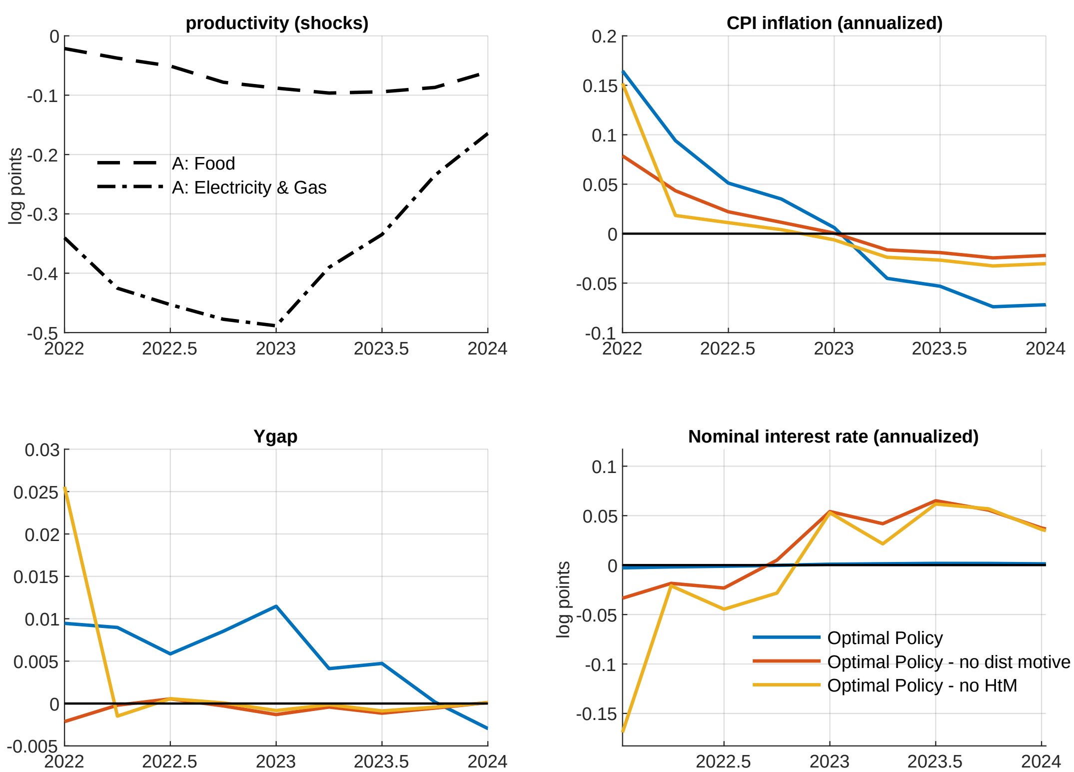


Figure A10. 2022 experiment: variation 3



F Optimal Policy

This section of the Appendix presents the derivations for the optimal policy results. Subsection F.1 provides the solution to the Ramsey problem of optimally setting the interest rates under commitment. With these results at hand, we then present in subsection F.2 the proofs to the analytical results for optimal policy which are given by Proposition 4, 5 and 6 in the main text. Finally, in subsection F.3 we derive the second-order approximation of the loss function, and conclude by showing that the initial steady state is efficient.

F.1 Optimal policy: derivations

As noted in the main text, the Central Bank (CB) values the utility of households according to the social welfare function $\mathcal{W}$ defined as:

$$\mathcal{W} = (1-\delta)\int G\left(V^{-}(i), i\right) di + \delta \mathbb{E}_0 \sum_{t_0=0}^{\infty} \beta^{t_0} \int G\left(V^{t_0}(i), i\right) di.$$

Here, a superscript t_0 denotes the birth date of a cohort (within a household type i) and a superscript $_-$ denotes cohorts born before $t = 0$.[9] The value of a cohort t_0 in type i is given by:

$$V^{t_0}(i) = \mathbb{E}_{t_0} \sum_{s=0}^{\infty} \left((1-\delta)\beta\right)^s \left\{ (1-\varphi(i)) \left[u_i\left(c^{t_0,u}_{1,t_0+s}(i), \ldots, c^{t_0,u}_{K,t_0+s}(i)\right) - \chi\left(\frac{n^{t_0,u}_{t_0+s}(i)}{\vartheta(i)}\right)\right] + \varphi(i)\left[u_i\left(c^{t_0,HtM}_{1,t_0+s}(i), \ldots, c^{t_0,HtM}_{K,t_0+s}(i)\right) - \chi\left(\frac{n^{t_0,HtM}_{t_0+s}(i)}{\vartheta(i)}\right)\right]\right\},$$

and note that within each cohort/type a fraction $\varphi(i)$ is HtM, and recall that non-HtM households are denoted by a superscript u. The value of pre-existing cohorts, $V_{-}(i)$, is defined analogously. The CB maximizes $\mathcal{W}$ under the following set of constraints (for any i, j, k, t, t_0):

- Optimality of intratemporal consumption decisions:

$$c^{t_0,h}_{k,t}(i,j) = \left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k} c^*_k\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right),$$

$$P_{k,t} = \left(\int p_{k,t}(j)^{1-\epsilon_k} dj\right)^{\frac{1}{1-\epsilon_k}},$$

$$v_i\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right) = u_i\left(c^*_1\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right), \ldots, c^*_K\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right)\right),$$

  for $h \in \{u, HtM\}$. Here, c^*_k is the solution of the consumption problem defined in the previous sections.

- Optimality of labor supply decisions, for $h \in \{u, HtM\}$:

$$\chi'\left(\frac{n^{t_0,h}_t(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)} = W_t \partial_e v_{i,t}\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right).$$

- Optimality of intertemporal expenditure decisions for non-HtM households (Euler equation and budget constraint):

$$\partial_e v_{i,t}\left(e^{t_0,u}_t(i), \boldsymbol{P}_t\right) = \beta(1-\delta)R_t \mathbb{E}_t\left[\partial_e v_{i,t+1}\left(e^{t_0,u}_{t+1}(i), \boldsymbol{P}_t\right)\right],$$

$$\frac{b^{t_0,u}_{t+1}(i)}{R_t} = b^{t_0,u}_t(i) + n^{t_0,u}_t(i) W_t + \sum_k \varsigma_k(i) Div_{k,t} - e^{t_0,u}_t(i),$$

  with $b^{t_0,u}_{t_0}(i) = b^{t_0,HtM}_{t_0}(i) = \left(1 + \sum_l \bar{s}_l\left(\frac{P_{l,t_0}-P^*_l}{P^*_l}\right)\right) b(i).$

[9] Note that it would be equivalent – to a first order approximation – to differentiate households born before t_0 according to their date of birth, that is consider the social welfare function $\mathcal{W} = \delta \mathbb{E}_0 \sum_{t_0=-\infty}^{\infty} \beta^{t_0} \int G\left(V^{t_0}(i), i\right) di$.

- HtM consumption:
$$\left(\frac{1}{R_t}-1\right)b_{t_0}^{t_0,HtM}(i)=n_t^{t_0,HtM}(i)W_t+\sum_k \varsigma_k(i)Div_{k,t}-e_t^{t_0,HtM}(i).$$
- Optimal Price resetting:
$$\mathbb{E}_t\sum_{s=0}^{\infty}\tilde{\beta}^s\theta_k^s\left(\frac{p_{k,t}^*(j)}{P_{k,t+s}}\right)^{-\epsilon_k}Y_{k,t+s}\left(p_{k,t}^*(j)-\frac{\epsilon_k}{\epsilon_k-1}(1-\tau_k)MC_{k,t+s}(j^*)\right)=0,$$
where the aggregate demand for varieties is defined in the previous section.
- Labor market clearing:
$$(1-\delta)^{t+1}\left(\int(1-\varphi(i))\,n_t^{-,u}(i)+\varphi(i)\,n_t^{-,HtM}(i)di\right)+\delta\sum_{t_0=0}^{t}(1-\delta)^{t-t_0}\left(\int(1-\varphi(i))\,n_t^{t_0,u}(i)+\varphi(i)\,n_t^{t_0,HtM}(i)di\right)=\sum_{k=1}^{K}\mathcal{N}_k(\boldsymbol{P}_t,W_t)\int\frac{\left(\frac{p_{k,t}(j)}{P_{k,t+s}}\right)^{-\epsilon_k}Y_{k,t}}{A_{k,t}}dj.$$

The firm's optimal choice of input, the market clearing conditions for intermediate goods and consumption goods, and the government budget constraint will be used implicitly.
We denote by $\mathbb{E}_{\delta,t}\left(X_t^{t_0}\right)\equiv(1-\delta)^{t+1}X_t^-+\delta\sum_{t_0=0}^{t}(1-\delta)^{t-t_0}X_t^{t_0}$ the inter-generational average of variable $X_t^{t_0}$ at t. We further denote by $\breve{\Xi}_t$ and $\tilde{\mu}_{k,t}$ the Lagrange multipliers on the labor market clearing constraint and optimal price setting constraints, and by $\check{\lambda}_t^{t_0}(i)$, $\check{\zeta}_t^{t_0,u}(i)$,$\check{\zeta}_t^{t_0,HtM}(i)$, $\check{\alpha}_t^{t_0}(i)$ and $\check{\aleph}_t^{t_0}(i)$, the Lagrange multipliers on the Euler equation of unconstrained households ($\check{\lambda}_t^{t_0}(i)$), on the optimality of labor supply decisions ($\check{\zeta}_t^{t_0,u}(i)$, $\check{\zeta}_t^{t_0,HtM}(i)$) and on the budget constraints of households ($\check{\alpha}_t^{t_0}(i)$ for unconstrained households and $\check{\aleph}_t^{t_0}(i)$ for HtM households). The Lagrangian of the optimal policy problem is:

$$\begin{aligned}
&(1-\delta)\int\frac{1}{E}G\left(V^-(i)(i),i\right)di+\delta\mathbb{E}_0\sum\beta^{t_0}\int G\left(V^{t_0}(i),i\right)di\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\partial_e v_{t,i}\left(e_t^{t_0,u}(i),\boldsymbol{P}\right)\left(\check{\lambda}_t^{t_0}(i)-R_{t-1}\check{\lambda}_{t-1}^{t_0}(i)\right)di\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\check{\zeta}_t^{t_0,u}(i)\left(W_t\partial_e v_{t,i}\left(e_t^{t_0,u}(i),\boldsymbol{P}_t\right)-\chi'\left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)}\right)di+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\left(W_t\partial_e v_{t,i}\left(e_t^{t_0,HtM}(i),\boldsymbol{P}_t\right)-\chi'\left(\frac{n_t^{t_0,HtM}(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)}\right)di\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\check{\alpha}_t^{t_0}(i)\left(\frac{b_{t+1}^{t_0,u}(i)}{R_t}-\left(b_t^{t_0,u}(i)+n_t^{t_0,u}(i)W_t+\sum_k\varsigma_k(i)Div_{k,t}-e_t^{t_0,u}(i)\right)\right)di\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\check{\aleph}_t^{t_0}(i)\,\varphi(i)\left(\left(\frac{1}{R_t}-1\right)b_{t_0}^{t_0,HtM}(i)-\left(n_t^{t_0,HtM}(i)W_t+\sum_k\varsigma_k(i)Div_{k,t}-e_t^{t_0,HtM}(i)\right)\right)\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\breve{\Xi}_tW_t\left((1-\delta)^{t+1}\left(\int(1-\varphi(i))\,n_t^{-,u}(i)+\varphi(i)\,n_t^{-,HtM}(i)di\right)+\delta\sum_{t_0=0}^{t}(1-\delta)^{t-t_0}\left(\int(1-\varphi(i))\,n_t^{t_0,u}(i)+\varphi(i)\,n_t^{t_0,HtM}(i)di\right)-\sum_{k=1}^{K}\mathcal{N}_k(\boldsymbol{P}_t,W_t)\int\frac{\left(\frac{p_{k,t}(j)}{P_{k,t+s}}\right)^{-\epsilon_k}Y_{k,t}}{A_{k,t}}dj\right)\\
&-\sum_{k=1}^{K}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\tilde{\mu}_{k,t}\left(\sum_{s=0}^{\infty}\tilde{\beta}^s\theta_k^s\left(\frac{p_{k,t}^*(j)}{P_{k,t+s}}\right)^{-\epsilon_k}Y_{k,t+s}\left(p_{k,t}^*(j)-\frac{\epsilon_k}{\epsilon_k-1}(1-\tau_k)MC_{k,t+s}(j^*)\right)\right).
\end{aligned}$$

First-order conditions

Let us consider a steady state in which the CB targets zero inflation (and all goods prices and wages are constant), setting $R_t=1/\tilde{\beta}$. Recall also that we normalized $A_{k,t}=1$ and that we assumed that elasticities of substitution across varieties are equal for households and intermediate input producers (equal to ϵ_k). In such a steady state, wealth, expenditure and labor supply of households is constant across time and identical for unconstrained and HtM households of the same type i. We first show that, given the presence of a subsidy undoing markups, $(1-\tau_k)\frac{\epsilon_k}{\epsilon_k-1}=1$, and the first assumption on the social welfare function, $G'(V_{t_0}(i),i)\partial_e v(i)=1$, this steady state is efficient. We do so by first showing that the first-order conditions to the optimal policy problem hold at the steady state.[10] To streamline the derivation, we use the fact (shown at the end of the subsection) that $\tilde{\mu}_{k,t}=\check{\Xi}_t=\check{\zeta}_t^{t_0,u}=\check{\lambda}_t^{t_0}$ in steady state. After doing so, we perturb the first-order conditions around the steady state, in order to solve for the optimal dynamics.

- First-order conditions for $b_t^{t_0,u}(i)$:

$$\tilde{\beta}\check{\alpha}_t^{t_0}(i)=\frac{1}{R_{t-1}}\check{\alpha}_{t-1}^{t_0}(i).$$
$$\Rightarrow \check{\alpha}_t^{t_0}(i)=\check{\alpha}^{t_0}(i)$$

  where the second line gives the necessary optimality condition in a steady state with constant prices and $R_t=1/\tilde{\beta}$.

- First-order conditions for the interest rate, R_t :

$$-\beta\mathbb{E}_{\delta,t+1}\left(\int(1-\varphi(i))\check{\lambda}_t^{t_0}(i)\,\partial_e v_{t+1,i}\left(e_{t+1}^{t_0,u}(i),\boldsymbol{P}\right)di\right)-\mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\check{\alpha}_t^{t_0}(i)\frac{1}{R_t^2}b_{t+1}^{t_0,u}(i)\,di\right)-\mathbb{E}_{\delta,t}\left(\int\varphi(i)\check{\aleph}_t^{t_0}(i)\frac{1}{R_t^2}b_{t_0}^{t_0,HtM}(i)\,di\right)=0$$
$$\Rightarrow -\mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\check{\lambda}_t^{t_0}(i)\,\partial_e v_i(e(i),\boldsymbol{P})\,di\right)-\mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\check{\alpha}^{t_0}(i)\frac{b(i)}{R}di\right)-\mathbb{E}_{\delta,t}\left(\int\varphi(i)\check{\aleph}_t^{t_0}(i)\frac{1}{R}b(i)\,di\right)=0,$$

  where the second line gives the necessary optimality condition in a steady state with constant prices and $R_t=1/\tilde{\beta}$ (so wealth is constant across time and generations).

- First-order conditions for W_t. Denoting $\mathcal{Q}_{l,k}=\mathcal{Y}_{l,k}\frac{Y_k}{\tilde{Y}_l A_k}$ the matrix of intermediate shares, we have:

$$\begin{aligned}
0=\;&\mathbb{E}_{\delta,t}\int(1-\varphi(i))\check{\zeta}_t^{t_0,u}(i)\,\partial_e v_{t,i}\left(e_t^{t_0,u}(i),\boldsymbol{P}_t\right)di+\mathbb{E}_{\delta,t}\int\varphi(i)\check{\zeta}_t^{t_0,HtM}(i)\,\partial_e v_{t,i}\left(e_t^{t_0,HtM}(i),\boldsymbol{P}_t\right)di\\
&+\sum_{k=1}^{K}(1-\tau_k)\sum_{s=0}^{t}\left((1-\delta)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\frac{\epsilon_k}{\epsilon_k-1}(1-\tau_k)\mathcal{N}_k(\boldsymbol{P}_t,W_t)\left(\frac{p_{k,s}^*(j)}{P_{k,t}}\right)^{-\epsilon_k}Y_{k,t}\\
&-\mathbb{E}_{\delta,t}\int(1-\varphi(i))\check{\alpha}_t^{t_0}(i)\left(n_t^{t_0,u}(i)-\varsigma(i)\sum_k\frac{\mathcal{N}_k(\boldsymbol{P}_t,W_t)Y_{k,t}}{A_{k,t}}\right)di-\mathbb{E}_{\delta,t}\int\check{\aleph}_t^{t_0}(i)\varphi(i)\left(n_t^{t_0,HtM}(i)-\varsigma(i)\sum_k\frac{\mathcal{N}_k(\boldsymbol{P}_t,W_t)Y_{k,t}}{A_{k,t}}\right)\\
&-\mathbb{E}_{\delta,t}\int\left((1-\varphi(i'))\check{\alpha}_t^{t_0}(i')+\varphi(i')\check{\aleph}_t^{t_0}(i')\right)\varsigma(i')\sum_k\int\left(\frac{p_{k,s}(j)}{P_{k,t}}\right)^{-\epsilon_k}\partial_W\tilde{Y}_{k,t}\left(p_{k,t}(j)-MC_{k,t}\right)dj\,di'\\
&-\check{\Xi}_t\sum_{k=1}^{K}\partial_W\mathcal{N}_k(\boldsymbol{P}_t,W_t)\frac{Y_k}{A_{e_k,t}}-\check{\Xi}_t\left[\frac{\mathcal{N}_1(\boldsymbol{P}_t,W_t)Y_1}{A_{1,t}},...,\frac{\mathcal{N}_K(\boldsymbol{P}_t,W_t)Y_K}{A_{K,t}}\right](Id-Q)^{-1}\left[\sum_k\frac{\partial\mathcal{Y}_{1,k}}{\partial W}\frac{Y_k}{A_{k,t}Y_1},...,\sum_k\frac{\partial\mathcal{Y}_{K,k}}{\partial W}\frac{Y_k}{A_{k,t}Y_K}\right]^T,
\end{aligned}$$

  where the change in demand for intermediary in response to a change in wage solves:

$$\partial_W\tilde{Y}_{l,t}=\sum_k\partial_W\mathcal{Y}_{l,k}(\boldsymbol{P_t},W_t)\int\left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k}dj\,Y_{k,t}+\sum_k\mathcal{Y}_{l,k}(\boldsymbol{P_t},W_t)\int\left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k}dj\,\partial_W\tilde{Y}_{k,t}.$$

  We use the market clearing condition for intermediary and the optimal input demand from firms to obtain the expression on the last line. Using the fact that

[10] When we derive the loss function, we also show that the second-order conditions are satisfied.

$(1-\tau_k)\frac{\epsilon_k}{\epsilon_k-1}=1$, we can use:

$$\left(\frac{p^*_{k,s}(j)}{P_{k,t}}\right)^{-\epsilon_k} Y_{k,t}\left(p^*_{k,t}(j)-\frac{\epsilon_k}{\epsilon_k-1}(1-\tau_k)MC_{k,t+s}(j^*)\right)=0 \quad \forall k,$$
$$\left[W\mathcal{N}_1(\boldsymbol{P},W)Y_1,...,W\mathcal{N}_K(\boldsymbol{P},W)Y_K\right](Id-Q)^{-1}=\left[P_1Y_1,...,P_KY_K\right],$$
$$W_t\partial_W\mathcal{N}_k(\boldsymbol{P},W)+\sum_l P_l\frac{\partial\mathcal{Y}_{l,k}}{\partial W}=0 \quad \forall k.$$

In addition, we define $\check{\mu}_{k,T}$ which constrains the growth rate of sectoral inflation:

$$\check{\mu}_{k,T}\equiv-\frac{\theta_k}{1-\theta_k}\sum_{t=0}^{T}\left((1-\delta)\theta_k\right)^{T-t}\tilde{\mu}_{k,t}\sum_{s=0}^{\infty}\tilde{\beta}^s\theta_k^s\frac{\epsilon_k}{\epsilon_k-1}(1-\tau_k)\frac{MC_{k,t+s}(j^*)}{p_{k,t}(j^*)}\left(\frac{p_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k},$$

and note that around our steady steady state:

$$\begin{aligned}\check{\mu}_{k,T}&=-\frac{\theta_k}{(1-\theta_k)\left(1-\tilde{\beta}\theta_k\right)}\frac{\epsilon_k}{\epsilon_k-1}(1-\tau_k)\sum_{t=0}^{T}\left((1-\delta)\theta_k\right)^{T-t}\tilde{\mu}_{k,t}\\&=-\frac{1}{\lambda_k}\sum_{t=0}^{T}\left((1-\delta)\theta_k\right)^{T-t}\tilde{\mu}_{k,t}.\end{aligned}$$

Using this, we rewrite the first order condition as:

$$0=\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\check{\zeta}_t^{t_0,u}(i)\,\partial_e v_{t,i}^{u,t_0}di+\mathbb{E}_{\delta,t}\int\varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\,\partial_e v_{t,i}^{HtM,t_0}di-\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,s}\mathcal{N}_k(\boldsymbol{P}_t,W_t)$$
$$-\mathbb{E}_{\delta,t}\beta^t\int(1-\varphi(i))\,\check{\alpha}_t^{t_0}(i)\left(n(i)-\varsigma(i)\sum_k\mathcal{N}_k(\boldsymbol{P}_t,W_t)Y_{k,t}\right)di-\mathbb{E}_{\delta,t}\int\check{\aleph}_t^{t_0}(i)\,\varphi(i)\left(n(i)-\varsigma(i)\sum_k\mathcal{N}_k(\boldsymbol{P}_t,W_t)Y_{k,t}\right)di.$$

Using the fact that $\varsigma(i)=n(i)/N$, the steady state equation is:

$$0=\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\check{\zeta}_t^{t_0,u}(i)\,\partial_e v_i di+\mathbb{E}_{\delta,t}\int\varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\,\partial_e v_i di-\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,s}\mathcal{N}_k(\boldsymbol{P}_t,W_t).$$

- First order conditions with respect to labor supply, $n_t^{t_0,u}(i)$ and , $n_t^{t_0,HtM}(i)$:

$$\begin{aligned}&\check{\zeta}_t^{t_0,u}(i)\,\partial_e v_{t,i}^{u,t_0}=\psi n_t^{t_0,u}(i)\left\{\check{\Xi}_t-\check{\alpha}_t^{t_0}(i)-G'\left(V_{t_0}(i)\right)\partial_e v_{t,i}^{u,t_0}\right\},\\&\Rightarrow\check{\zeta}_t^{t_0,u}(i)\,\partial_e v_i=\psi n(i)\left\{\check{\Xi}_t-\check{\alpha}^{t_0}(i)-1\right\}.\\&\check{\zeta}_t^{t_0,HtM}(i)\,\partial_e v_{t,i}^{HtM,t_0}=\psi n_t^{t_0,HtM}(i)\left\{\check{\Xi}_t-\check{\aleph}_t^{t_0}(i)-G'\left(V_{t_0}(i)\right)\partial_e v_{t,i}^{HtM,t_0}\right\},\\&\Rightarrow\check{\zeta}_t^{t_0,HtM}(i)\,\partial_e v_i=\psi n(i)\left\{\check{\Xi}_t-\check{\aleph}_t^{t_0}(i)-1\right\},\end{aligned}$$

where we used the definition $\psi=\chi'\left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)}\right)/\left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)}\chi''\left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)}\right)\right)$ and the optimality of labor supply decisions. The second and fourth line are the steady state equations.

- First order condition with respect to expenditure of the non-HtM, $e_t^{u,t_0}(i)$

$$
\begin{aligned}
G'\left(V_{t_0}(i), i\right)\partial_e v_{t,i}^{u,t_0} + \left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_{ee} v_{t,i}^{u,t_0} + \check{\zeta}_t^{t_0}(i)\, W_t \partial_{ee} v_{t,i}^{u,t_0} \\
+\check{\alpha}_t^{t_0}(i) - \mathbb{E}_{\delta,t}\int\left(\left(1-\varphi(i')\right)\check{\alpha}_t^{t_0}(i') + \varphi(i')\check{\aleph}_t^{t_0}(i')\right)\varsigma(i')\sum_k\int\left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k}\left(p_{k,t}(j) - MC_{k,t}\right) dj di'\left(\partial_e c_{k,t} + \partial_e \tilde{Y}_{k,t}\right) \\
-\check{\Xi}_t\sum_{k=1}^K \frac{W_t \mathcal{N}_k(\boldsymbol{P}_t, W_t)}{A_k}\int\left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k} dj\left(\partial_e c_{k,t} + \partial_e\tilde{Y}_{k,t}\right) = 0
\end{aligned}
$$

With some abuse of notation, $\partial_e\tilde{Y}_k$ is the Gateaux derivative (keeping prices fixed) of demand for intermediary output with respect to a change in $e_t^{t_0,u}(i)$. Note that we use $(1-\tau_k)\frac{\epsilon_k}{\epsilon_k-1} = 1$ and the adjustment of the lump sum tax to express the total change in dividends. We have, denoting $\tilde{Q}_{l,k} = \mathcal{Y}_{l,k}/A_k\left(\frac{p_k(j)}{P_k}\right)^{-\epsilon_k}$:

$$
\begin{aligned}
\left[\partial_e\tilde{Y}_1, ..., \partial_e\tilde{Y}_K\right]^T &= (Id-\tilde{Q})^{-1}\left[\sum_k \frac{\mathcal{Y}_{1,k}(\boldsymbol{P}, W)}{A_k}\int\left(\frac{p_k(j)}{P_k}\right)^{-\epsilon_k} dj\partial_e c_k, ..., \sum_k \frac{\mathcal{Y}_{K,k}(\boldsymbol{P}, W)}{A_k}\int\left(\frac{p_k(j)}{P_k}\right)^{-\epsilon_k} dj\partial_e c_k\right]^T \\
&= (Id-\tilde{Q})^{-1}\tilde{Q}\left[\partial_e c_1, ..., \partial_e c_K\right]^T.
\end{aligned}
$$

Using the fact that variety prices are equal in steady state, we have:

$$
\sum_{k=1}^K \frac{W\mathcal{N}_k(\boldsymbol{P}, W)}{A_k}\int\left(\frac{p_k(j)}{P_k}\right)^{-\epsilon_k} dj\left(\partial_e c_k + \partial_e\tilde{Y}_k\right) = \sum_{k=1}^K P_k\partial_e c_k = 1.
$$

Simplifying we have, in steady state:

$$
1 + \left(\check{\lambda}_t^{t_0}(i) - R\lambda_{t-1}^{t_0}(i)\right)\partial_{ee} v + \check{\zeta}_t^{t_0,u}(i)\, W\partial_{ee} v + \check{\alpha}^{t_0}(i) - \check{\Xi}_t = 0.
$$

Following the same steps, the first order conditions for the expenditure of HtM households, $e_t^{HtM}(i)$, is

$$
\begin{aligned}
G'\left(V_{t_0}(i), i\right)\partial_e v_{t,i}^{HtM,t_0} + \check{\zeta}_t^{t_0,u}(i)\, W_t\partial_{ee} v_{t,i}^{HtM,t_0} + \check{\aleph}_t^{t_0}(i) - \mathbb{E}_{\delta,t}\int\left(\left(1-\varphi(i')\right)\check{\alpha}_t^{t_0}(i') + \varphi(i')\check{\aleph}_t^{t_0}(i')\right)\varsigma(i')\sum_k\int\left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k}\left(p_{k,t}(j) - MC_{k,t}\right) dj\left(\partial_e c_{k,t} + \partial_e\tilde{Y}_{k,t}\right) di' \\
-\check{\Xi}_t\sum_{k=1}^K \frac{W_t\mathcal{N}_k(\boldsymbol{P}_t, W_t)}{A_{e_k,t}}\int\left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k} dj\left(\partial_e c_{k,t} + \partial_e\tilde{Y}_{k,t}\right) = 0,
\end{aligned}
$$

and in steady state simplifies to:

$$
1 + \check{\zeta}_t^{t_0}(i)\, W\partial_{ee} v + \check{\aleph}_t^{t_0}(i) - \check{\Xi}_t = 0.
$$

- Finally, consider the first order conditions for a compensated change in resetted prices $p_{k,t}^*(j^*)$ (That is, each household receives a transfer in period $t+s$, $s \geq 0$ which cancels the income effect of the price change. Note that we can alternatively consider an uncompensated change in prices, but the terms corresponding to the income effects can then be simplified using the first-order condition corresponding to the optimality of expenditure of unconstrained and HtM households.):

$$
\begin{aligned}
0 = & -\tilde{\mu}_{k,t}\sum_{s=0}^{\infty}\tilde{\beta}^s\theta_k^s\left(\frac{p^*_{k,t}(j)}{P_{k,t+s}}\right)^{-\epsilon_k}Y_{k,t+s}+(1-\theta_k)\sum_{T=t}^{\infty}(\beta\theta_k)^{T-t}\sum_{l=1}^{K}\sum_{s=0}^{T}((1-\delta)\theta_l)^{T-s}\tilde{\mu}_{l,s}\frac{\mathcal{Y}_{k,l}}{A_{l,T}}\left(\frac{p^*_{k,t}(j^*)}{P_{k,T}}\right)^{-\epsilon_k}\left(\frac{p_{l,s}(j^*)}{P_{l,T}}\right)^{-\epsilon_l}Y_{l,T} \\
& -(1-\theta_k)\sum_s(\beta\theta_k)^s\check{\Xi}_{t+s}\left(\sum_{l=1}^{K}\partial_{P_k}\mathcal{N}_l\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}\frac{Y_l}{A_{l,t}}\right) \\
& -(1-\theta_k)\sum_s(\beta\theta_k)^s\check{\Xi}_{t+s}\left(\left[\frac{\mathcal{N}_1Y_1}{A_{1,t+s}},...,\frac{\mathcal{N}_KY_K}{A_{K,t+s}}\right](Id-Q)^{-1}\left[\sum_l\frac{\partial\mathcal{Y}_{1,l}}{\partial P_k}\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}\frac{Y_l}{A_{l,t}Y_1},...,\sum_k\frac{\partial\mathcal{Y}_{K,l}}{\partial P_k}\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}\frac{Y_l}{A_{l,t}Y_K}\right]^T\right) \\
& -(1-\theta_k)\sum_s(\beta\theta_k)^s\check{\Xi}_{t+s}\sum_{l=1}^{K}\frac{\mathcal{N}_l(\boldsymbol{P}_{t+s},W_{t+s})}{A_{l,t}}\mathbb{E}_{\delta,t+s}\int\int\left(\frac{p_{l,t+s}(j)}{P_{l,t+s}}\right)^{-\epsilon_l}\left(\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}\partial_{P_k}c^h_{l,t+s}(i)+\partial_{p_k(j^*)}\tilde{Y}_{l,t+s}\right)djdi \\
& +(1-\theta_k)\sum_s(\beta\theta_k)^s\check{\Xi}_{t+s}\frac{\mathcal{N}_k(\boldsymbol{P}_{t+s},W_{t+s})}{A_{e_k,t}}\mathbb{E}_{\delta,t+s}\epsilon_k\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}\left(\frac{1}{p_{k,t+s}(j^*)}-\int\left(\frac{p_{k,t+s}(j)}{P_{k,t+s}}\right)^{-\epsilon_k}\frac{1}{P_{k,t+s}}dj\right)Y_{k,t+s} \\
& -(1-\theta_k)\sum_s(\beta\theta_k)^s\mathbb{E}_{\delta,t+s}\left(\int\left[(1-\varphi(i))\left(\check{\lambda}^{t_0}_{t+s}(i)-R\check{\lambda}^{t_0}_{t+s-1}(i)\right)+\check{\zeta}^{t_0}_{t+s}(i)W\right]\partial_e v^{t_0,u}_{t+s,i}\partial_e e_{k_{t+s,i}}^{t_0,u}\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}di\right) \\
& -(1-\theta_k)\sum_s(\beta\theta_k)^s\mathbb{E}_{\delta,t+s}\left(\int\varphi(i)\check{\zeta}^{t_0,HtM}_{t+s}(i)W\partial_e v^{t_0,HtM}_{t+s,i}\partial_e e_{k_{t+s,i}}^{t_0,HtM}\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}di\right) \\
& +(1-\theta_k)\sum_s(\beta\theta_k)^s\mathbb{E}_{\delta,t+s}\left(\int(1-\varphi(i))\check{\alpha}^{t_0}_t(i)\left[\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}c_{k,t+s}(i)-\varsigma(i)\partial_{p(j^*)}Div_t\right]di\right)-\delta\int(1-\varphi(i))\check{\alpha}^{t+s}_{t+s}(i)b^{t+s,u}_{t+s}(i)di\bar{s}_k\frac{1}{P_k}\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k} \\
& +(1-\theta_k)\sum_{s=0}(\beta\theta_k)^s\left(-\mathbb{E}_{\delta t+s}\left(\int\varphi(i)\check{\aleph}^{t_0}_{t+s}\left[\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}c_{k,t+s}(i)-\varsigma(i)\partial_{p(j^*)}Div_t\right]\right)di\right) \\
& +(1-\theta_k)\sum_{s=0}(\beta\theta_k)^s\left(-\delta\sum_{u=0}((1-\delta)\beta)^u\int\varphi(i)\check{\aleph}^{t+s}_{t+u+s}(i)b^{t+s,HtM}_{t+s}(i)di\bar{s}_k\frac{R_{t+u}-1}{R_{t+u}}\frac{1}{P_k}\left(\frac{p^*_{k,t}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}\right)
\end{aligned}
$$

with

$$
\begin{aligned}
\left[\partial_{p_k(j^*)}\tilde{Y}_1,...,\partial_{p_k(j^*)}\tilde{Y}_K\right]^T = & \left(\frac{p_{k,t+s}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}(Id-\tilde{Q})^{-1}\tilde{Q}\left[\int\int\partial_{P_k}c^h_1djdi,...,\int\int\partial_{P_k}c^h_Kdjdi\right]^T \\
& +\left(\frac{p_{k,t+s}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}(Id-\tilde{Q})^{-1}\tilde{Q}\left[0,...,-\epsilon_k\mathbb{E}_{\delta,t+s}\left(\left(\frac{1}{p_k(j^*)}-\int\left(\frac{p_k(j)}{P_k}\right)^{-\epsilon_k}\frac{1}{P_k}dj\right)Y_k\right),...,0\right]
\end{aligned}
$$

Using the fact that variety prices are equal in steady state, we have:

$$
\sum_{l=1}^{K}\frac{W\mathcal{N}_l}{A_{l,t}}\int\left(\partial_{P_k}c^h_l(i)+\partial_{p_k(j^*)}\tilde{Y}_l\right)di=\sum_{l=1}^{K}P_l\int\partial_{P_k}c^h_l(i)di=0,
$$

$$\partial_{p(j^*)}Div_{t+s} = \sum_l \int\int \left(\partial_{p_k(j^*)}c^h_{l,t+s} + \partial_{p_k(j^*)}\tilde{Y}_{l,t+s}\right)\left(\frac{p_{l,t+s}(j)}{P_{l,t+s}}\right)^{-\epsilon_k}(p_{l,t+s}(j) - MC_{l,t+s})\,didj - \epsilon_k \int \left(\frac{p_{k,t+s}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}\left(1 - \frac{MC_{k,t+s}}{p_{k,t+s}(j^*)}\right)Y_{l,t+s}di$$
$$+\epsilon_k \int\int \left(\frac{p_{k,t+s}(j)}{P_{k,t+s}}\right)^{-\epsilon_k}\left(\frac{p_{k,t+s}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}\left(\frac{p_{k,t+s}(j)}{P_{k,t+s}} - \frac{MC_{k,t+s}}{P_{k,t+s}}\right)Y_{l,t+s}didj + y_{k,t+s}(j) - \left(\frac{p_{k,t+s}(j^*)}{P_{k,t+s}}\right)^{-\epsilon_k}\sum_l \mathcal{Y}_{k,l,t+s}Y_{l,t+s}$$

Recall that we defined $\check{\mu}_{k,T}$, which constrains the growth rate of sectoral inflation:

$$\check{\mu}_{k,T} \equiv -\frac{1}{\lambda_k}\sum_{t=0}^{T}\left((1-\delta)\,\theta_k\right)^{T-t}\tilde{\mu}_{k,t}.$$

Using the properties of the steady state, we get:

$$0 = \frac{(1-\theta_k)}{\theta_k}\frac{1}{P_k}\left(\check{\mu}_{k,t} - ((1-\delta)\,\theta_l)\,\check{\mu}_{k,t-1}\right) - (1-\theta_k)\sum_{T=t}(\beta\theta_k)^{T-t}\left(\frac{(1-\theta_k)\left(1-\tilde{\beta}\theta_k\right)}{\theta_k}\check{\mu}_{k,T}\frac{1}{P_k} + \frac{1}{P_k}\lambda_k\check{\mu}_{k,T}\right) - (1-\theta_k)\sum_{T=t}^{\infty}(\beta\theta_k)^{T-t}\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,T}}{P_lY_l}\frac{\mathcal{Y}_{k,l}Y_l}{A_{l,T}}$$
$$-(1-\theta_k)\sum_s(\beta\theta_k)^s\,\mathbb{E}_{\delta,t+s}\left(\int\left[(1-\varphi(i))\left(\check{\lambda}^{t_0}_{t+s}(i) - R\check{\lambda}^{t_0}_{t+s-1}(i)\right) + \check{\zeta}^{t_0,u}(i)\,W\right]\partial_e v(i)\,\frac{1}{P_k}\partial_e e_k(i) + \varphi(i)\,\check{\zeta}^{t_0,HtM}(i)\,W\partial_e v(i)\,\frac{1}{P_k}\partial_e e_k(i)\,di\right)$$
$$+(1-\theta_k)\sum_s(\beta\theta_k)^s\,\mathbb{E}_{\delta,t+s}\left(\int(1-\varphi(i))\,\check{\alpha}^{t_0}(i)\,\frac{1}{P_k}\left[e_k(i) - \varsigma(i)\,E_k\right]di\right) - \delta\int(1-\varphi(i))\,\check{\alpha}^{t+s}(i)\,b^{t+s}(i)\,di\bar{s}_k\frac{1}{P_k}$$
$$+(1-\theta_k)\sum_s(\beta\theta_k)^s\left(-\mathbb{E}_{\delta,t+s}\left(\int\varphi(i)\,\check{\aleph}^{t_0}_{t+s}(i)\,\frac{1}{P_k}\left[e_k(i) - \varsigma(i)\,E_k\right]di\right) - \delta\frac{R-1}{R}\sum_{u=0}\frac{1}{R^u}\int\varphi(i)\,\check{\aleph}^{t+s}_{t+u+s}(i)\,b(i)\,di\bar{s}_k\frac{1}{P_k}\right).$$

Taking the difference between the equation at $t+1$ times $\beta\theta_k$ and the equation at t we obtain

$$\left(\beta\check{\mu}_{k,t+1} - (1 + ((1-\delta)\,\beta))\,\check{\mu}_{k,t} + (1-\delta)\,\check{\mu}_{k,t-1}\right) = \lambda_k\check{\mu}_{k,t} - \sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{P_lY_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l}$$
$$-\mathbb{E}_{\delta,t}\left(\int\left[(1-\varphi(i))\left(\check{\lambda}^{t_0}_t(i) - R\lambda^{t_0}_{t-1}(i)\right) + \check{\zeta}^{t_0,u}_t(i)\right]\partial_e v(i)\,\partial_e e_k(i) + \varphi(i)\,\check{\zeta}^{t_0,HtM}_t(i)\,\partial_e v(i)\,\partial_e e_k(i)\,di\right)$$
$$+\mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))\,\check{\alpha}^{t_0}(i) + \varphi(i)\,\check{\aleph}^{t_0}_t\right)\left[e_k(i) - \varsigma(i)\,E_k\right]di\right) - \delta\int\left((1-\varphi(i))\,\check{\alpha}^{t}(i) + \varphi(i)\,\frac{R-1}{R}\sum_{u=0}\frac{1}{R^u}\check{\aleph}^t_{t+u}(i)\right)b(i)\,di\bar{s}_k.$$

We can now verify that a steady state with $R = 1/\tilde{\beta}$, constant wages and prices (chosen such that the good markets and labor market clear, recall that this implies that wealth, expenditure and labor supply of households is constant across time and identical for unconstrained and HtM households) and $\check{\zeta}^{t_0,u}_t = \check{\zeta}^{t_0,HtM}_t = \check{\Xi}_t = \check{\mu}_{k,t} = \check{\lambda}^{t_0}_t = 0$,

$\check{\alpha}^{t}(i) = \check{\aleph}_{t}^{t}(i) = \check{\aleph}^{t}(i) = -1$ solves the set of first-order conditions[11]

$$-\mathbb{E}_{\delta,t}\left(\left(1-\varphi(i)\right)\check{\lambda}_{t}^{t_0}(i)\,\partial_e v\left(e(i),\boldsymbol{P}\right)\right) - \mathbb{E}_{\delta,t}\left(\left(1-\varphi(i)\right)\check{\alpha}^{t_0}(i)\frac{b(i)}{R}\right) - \mathbb{E}_{\delta,t}\left(\varphi(i)\,\check{\aleph}^{t_0}(i)\frac{1}{R}b(i)\right) = 0 \Rightarrow \mathbb{E}_{\delta,t}\left(\frac{b(i)}{R}\right) = 0,$$

$$\check{\zeta}_{t}^{t_0,u}(i)\,\partial_e v(i) = \psi n(i)\left\{\check{\Xi}_t - \check{\alpha}^{t_0}(i) - 1\right\} \quad \check{\zeta}_{t}^{t_0,HtM}(i)\,\partial_e v(i) = \psi n(i)\left\{\check{\Xi}_t - \check{\alpha}^{t_0}(i) - 1\right\} \Rightarrow 0 = \psi n(i)\left\{1-1\right\} \quad 0 = \psi n(i)\left\{1-1\right\},$$

$$\mathbb{E}_{\delta,t}\int\left(1-\varphi(i)\right)\check{\zeta}_{t}^{t_0,u}(i)\,\partial_e v(i)\,di + \mathbb{E}_{\delta,t}\int\varphi(i)\,\check{\zeta}_{t}^{t_0,HtM}(i)\,\partial_e v(i)\,di - \sum_{k=1}^{K}\lambda_k\check{\mu}_{k,s}\mathcal{N}_k(\boldsymbol{P}_t,W_t) = 0 \Rightarrow 0 = 0,$$

$$1 + \left(\check{\lambda}_{t}^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_{ee}v(i) + \check{\zeta}_{t}^{t_0,u}(i)\,W\partial_{ee}v(i) + \check{\alpha}^{t_0}(i) - \check{\Xi}_t = 0 \Rightarrow 1-1 = 0,$$

$$1 + \check{\zeta}_{t}^{t_0}(i)\,W\partial_{ee}v(i) + \check{\aleph}^{t_0}(i) - \check{\Xi}_t = 0 \Rightarrow 1-1 = 0,$$

$$\left(\beta\check{\mu}_{k,t+1} - \left(1+(1-\delta)\beta\right)\check{\mu}_{k,t} + (1-\delta)\check{\mu}_{k,t-1}\right) = \lambda_k\check{\mu}_{k,t} \Rightarrow 0 = 0,$$

$$-\mathbb{E}_{\delta,t}\left(\int\left[\left(1-\varphi(i)\right)\left(\check{\lambda}_{t}^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right) + \check{\zeta}_{t}^{t_0,u}(i)\,W\right]\partial_e v(i)\,\partial_e e_k(i) + \varphi(i)\,\check{\zeta}_{t}^{t_0,HtM}(i)\,W\partial_e v(i)\,\partial_e e_k(i)\,di\right) + \mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi(i)\right)\check{\alpha}^{t_0}(i) + \varphi(i)\,\check{\aleph}^{t_0}\right)\left[e_k(i) - \varsigma(i)\,E_k\right]di\right)$$

$$-\delta\int\left(\left(1-\varphi(i)\right)\check{\alpha}^{t}(i) + \varphi(i)\,\check{\aleph}^{t}(i)\right)b(i)\,di\bar{s}_k P_k \Rightarrow 0 = -\mathbb{E}_{\delta,t}\left(\int\left[e_k(i) - E_k\right]di\right) + \delta\int b(i)\,di\bar{s}_k P_k.$$

Differentiating the first-order conditions

We now differentiate the first-order conditions around the steady state constructed in the previous section. Prices are in log-deviation while Lagrange multipliers are in absolute deviations.

- First order conditions with respect to $b_t^{t_0,u}(i)$:

$$\begin{aligned}\check{\alpha}_{t}^{t_0}(i) &= \check{\alpha}_{t-1}^{t_0}(i) + \hat{R}_{t-1}\\ &= \check{\alpha}^{t_0}(i) + \sum_{s=0}^{t-t_0-1}\hat{R}_{t_0+s}\end{aligned}$$

- First order conditions for the interest rate

$$\begin{aligned}-\mathbb{E}_{\delta,t}\left(\int\left(1-\varphi(i)\right)\check{\lambda}_{t}^{t_0}(i)\,\partial_e v(i)\,di\right) &= \mathbb{E}_{\delta,t}\left(\int\left(1-\varphi(i)\right)\check{\alpha}^{t_0}(i)\frac{b(i)}{R}di\right) + \mathbb{E}_{\delta,t}\left(\int\varphi(i)\,\check{\aleph}_{t}^{t_0}(i)\frac{1}{R}b(i)\,di\right)\\ &+ \mathbb{E}_{\delta,t}\left(\int\left(1-\varphi(i)\right)\sum_{s=0}^{t-t_0-1}\hat{R}_{t_0+s}\frac{b(i)}{R}di\right)\end{aligned}$$

- First order conditions for W_t:

$$0 = \mathbb{E}_{\delta,t}\int\left(\left(1-\varphi(i)\right)\check{\zeta}_{t}^{t_0,u}(i) + \varphi(i)\,\check{\zeta}_{t}^{t_0,HtM}(i)\right)\partial_e v(i)\,di - \sum_{k=1}^{K}\lambda_k\check{\mu}_{k,s}\mathcal{N}_k(\boldsymbol{P}_t,W_t) + \sum_{k,l}\frac{P_l\partial_W\mathcal{Y}_{l,k}}{A_k}Y_k\left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_{k}\frac{W\partial_W\mathcal{N}_k}{A_k}Y_k\hat{W}_t$$

[11] Note that to solve the steady state system we only need $\check{\Xi} - \check{\alpha}^{t_0} = 1$ and $\check{\Xi} - \check{\aleph}^{t_0} = 1$. It's direct to verify that choosing any values for $\check{\Xi}$, $\check{\alpha}^{t_0}$, and $\check{\aleph}^{t_0}$ that satisfy this would give the same system of differentiated first order conditions.

- First-order conditions with respect to labor supply:

$$\check{\zeta}_t^{t_0,u}(i)\,\partial_e v(i) = \psi n(i)\left\{\check{\Xi}_t - \sum_{s=0}^{t-t_0-1}\hat{R}_{t_0+s} - \check{\alpha}^{t_0}(i) - \frac{V(i)\,G''(V(i),i)}{G'(V(i),i)}\hat{V}_{t_0}(i) + \frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}(i) - \sum_l s_l(i)\,\hat{P}_{l,t}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t}\right\}$$

$$= \psi n(i)\left\{\check{\Xi}_t - \check{\alpha}^{t_0}(i) - \frac{V(i)\,G''(V(i),i)}{G'(V(i),i)}\hat{V}_{t_0}(i) + \frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\,\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\right\},$$

$$\check{\zeta}_t^{t_0,HtM}(i)\,\partial_e v(i) = \psi n(i)\left\{\check{\Xi}_t - \check{\aleph}_t^{t_0}(i) - \frac{V(i)\,G''(V(i),i)}{G'(V(i),i)}\hat{V}_{t_0}(i) + \frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}(i) - \sum_l s_l(i)\,\hat{P}_{l,t}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t}\right\}.$$

- First-order condition with respect to expenditure. We only need to re-express the impact of individual consumption on profits.

- Log linearizing $\sum_k \int \left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k}(p_{k,t}(j) - MC_{k,t})\,dj\,\left(\partial_e c_k + \partial_e \tilde{Y}_k\right)$ we have:

$$\frac{d\sum_k \int [p_{k,t}(j) - MC_{k,t}]\left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k} dj\,\left(\partial_e c_k + \partial_e \tilde{Y}_k\right)}{\sum_k \int [p_{k,t}(j) - MC_{k,t}]\left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k} dj\,\left(\partial_e c_k + \partial_e \tilde{Y}_k\right)} = \left(\hat{P}_{k,t} + \hat{A}_{k,t} - \Omega_{N,k}\hat{W}_t + \sum_l \Omega_{k,l}\hat{P}_{l,t}\right)\mathcal{D}(P)\,(Id - \tilde{Q})^{-1}\mathcal{D}^{-1}(P)\,[\partial_e e_1, ..., \partial_e e_K]^T$$

$$= [\partial_e e_1, ..., \partial_e e_K]^T\,(Id - \Omega)^{-1}\left[\left(\hat{P}_{k,t} + \hat{A}_{k,t} - \Omega_{N,k}\hat{W}_t - \sum_l \Omega_{k,l}\hat{P}_{l,t}\right)\right]$$

$$= -\hat{W}_t + \sum_l \partial_e e_l\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)$$

So we have:

$$\frac{V(i)\,G''(V(i),i)}{G'(V(i),i)}\hat{V}_{t_0}(i) - \frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}(i) - \sum_l s_l(i)\,\hat{P}_{l,t}\right) - \sum_l \partial_e e_l(i)\,\hat{P}_{l,t} + \left(\lambda_t^{t_0}(i) - R\lambda_{t-1}^{t_0}(i)\right)\partial_{ee}v(i) + \check{\zeta}_t^{t_0,u}(i)\,W\partial_{ee}v(i)$$
$$+\check{\alpha}^{t_0}(i) + \sum_{s=0}^{t-t_0-1}\hat{R}_{t_0+s} - \check{\Xi}_t - \left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right) = 0,$$

$$\frac{V(i)\,G''(V(i),i)}{G'(V(i),i)}\hat{V}_{t_0}(i) - \frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\,\hat{P}_{l,t_0}\right) - \sum_l \partial_e e_l(i)\,\hat{P}_{l,t} + \left(\lambda_t^{t_0}(i) - R\lambda_{t-1}^{t_0}(i)\right)\partial_{ee}v(i) + \check{\zeta}_t^{t_0,u}(i)\,W\partial_{ee}v(i)$$
$$+\check{\alpha}^{t_0}(i) - \check{\Xi}_t - \left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right) = 0.$$

For the expenditure of HtM households:

$$\frac{V(i)\,G''(V(i),i)}{G'(V(i),i)}\hat{V}_{t_0}(i) - \frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}(i) - \sum_l s_l(i)\,\hat{P}_{l,t}\right) - \sum_l \partial_e e_l(i)\,\hat{P}_{l,t} + \check{\zeta}_t^{t_0,HtM}(i)\,W\partial_{ee}v(i)$$
$$+\check{\aleph}_t^{t_0}(i) - \check{\Xi}_t - \left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right) = 0$$

- Finally for resetted prices, note that we have:

$$d\left(\sum_s (\beta\theta_k)^s \partial_{p(j^*)} Div_{t+s} - \sum_s (\beta\theta_k)^{s+1} \partial_{p(j^*)} Div_{t+1+s}\right) = \sum_l \left(\int e_l(i)\,\rho_{l,k}(i)\,\frac{1}{P_l P_k} di + \partial_{p_k(j^*)} \tilde{Y}_l\right) P_l\left(\hat{P}_{l,t} + \hat{A}_{l,t} - \hat{MC}_{l,t+s}\right)$$
$$+\frac{Y_k}{P_k}\epsilon_k \frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)}\left(\beta\pi_{k,t+1} - \pi_{k,t}\right) + dE_{k,t}$$

Defining $\vartheta_k = \epsilon_k \frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)}$, we obtain:

$$\left(\beta\check{\mu}_{k,t+1} - (1 + ((1-\delta)\beta))\,\check{\mu}_{k,t} + (1-\delta)\,\check{\mu}_{k,t-1}\right) = \lambda_k \check{\mu}_{k,t} - \sum_{l=1}^{K} \lambda_l \frac{\check{\mu}_{l,t}}{P_l Y_l} \frac{P_k \mathcal{Y}_{k,l} Y_l}{A_l}$$
$$-\mathbb{E}_{\delta,t}\left(\int \left[(1-\varphi(i))\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right) + \check{\zeta}_t^{t_0,u}(i)\,W\right] \partial_e v(i)\,\partial_e e_k(i) + \varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\,W\partial_e v(i)\,\partial_e e_k(i)\,di\right)$$
$$+\mathbb{E}_{\delta,t}\left(\int \left((1-\varphi(i))\,\check{\alpha}^{t_0}(i) + \varphi(i)\,\check{\aleph}_t^{t_0}(i)\right)\left[e_k(i) - \varsigma(i)\,E_k\right] di\right) + \mathbb{E}_{\delta,t}\left(\int (1-\varphi(i))\left[e_k(i) - \varsigma(i)\,E_k\right] di \sum_{s=0}^{t-t_0-1} \hat{R}_{t_0+s}\right) - \delta \int (1-\varphi(i))\,\check{\alpha}^t(i)\,b(i)\,di\bar{s}_k$$
$$-\delta \sum_{u=0} ((1-\delta)\beta)^u \int \varphi(i)\,\check{\aleph}_{t+u}^t(i)\,b(i)\,di\bar{s}_k \frac{R-1}{R} + \delta \sum_{u=0} \frac{1}{R^u} \int \varphi(i)\,b(i)\,di\bar{s}_k \frac{\hat{R}_{t+u}}{R}$$
$$+\sum_{l,m} \frac{P_l P_k \partial_{P_k} \mathcal{Y}_{l,m}}{A_m} Y_m\left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_l \frac{W P_k \partial_{P_k} \mathcal{N}_l}{A_l} Y_l \hat{W}_t + \sum_l E_l \bar{\rho}_{l,k}\left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right) - P_k Y_k \vartheta_k\left(\pi_{k,t} - \beta\pi_{k,t+1}\right)$$

Solving the Labor market equation

The next step is to re-write the (infinite number of) linearized conditions, into a system of a limited number of equations and variables. Our first main equation is the optimality of the wage

$$0 = \mathbb{E}_{\delta,t} \int \left((1-\varphi(i))\,\check{\zeta}_t^{t_0,u}(i) + \varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\right) W\partial_e v(i)\,di - \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,s} W \mathcal{N}_k + \sum_{k,l} \frac{P_l W \partial_W \mathcal{Y}_{l,k}}{A_k} Y_k\left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_k \frac{W^2 \partial_W \mathcal{N}_k}{A_k} Y_k \hat{W}_t.$$

Let us define the first component as $\tilde{Z}_t \equiv \mathbb{E}_{\delta,t} \int \left((1-\varphi(i))\,\check{\zeta}_t^{t_0,u}(i) + \varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\right) W\partial_e v(i)\,di$. Substituting out the Lagrange multipliers gives:

$$\tilde{Z}_t = \psi W N \check{\Xi}_t - \mathbb{E}_{\delta,t} \int \psi n(i) \frac{V(i)\,G''(V(i),i)}{G'(V(i),i)} \hat{V}_{t_0}(i)\,di + \mathbb{E}_{\delta,t} \int (1-\varphi(i))\,\psi W n(i) \left\{\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\,\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0} - \check{\alpha}^{t_0}(i)\right\}$$
$$+\varphi(i)\,\psi W n(i) \left\{\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}(i) - \sum_l s_l(i)\,\hat{P}_{l,t}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t} - \check{\aleph}_t^{t_0}(i)\right\} di.$$

Our first goal is to solve for $\check{\Xi}_t$, $\check{\alpha}^{t_0}(i)$ and $\check{\aleph}_t^{t_0}(i)$. Using the optimality of household's expenditure and substituting the $\check{\zeta}_t^{t_0,u}(i)$ term

$$\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_e v(i) = \sigma e(i)\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) - \sigma e(i)\left(\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\hat{P}_{l,t_0}\right) - \check{\zeta}_t^{t_0,u}(i)\,W\partial_e v(i) + \sigma e(i)\check{\alpha}^{t_0}(i)$$
$$- \sigma e(i)\check{\Xi}_t - \sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right),$$
$$\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_e v(i) = (\sigma e(i) + \psi n(i))\left(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) - \left(\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\hat{P}_{l,t_0}\right)\right)$$
$$+ (\sigma e(i) + \psi n(i))\left(\check{\alpha}^{t_0}(i) - \check{\Xi}_t\right) - \sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right).$$

So using, $\check{\lambda}_{t_0-1}^{t_0}(i) = 0\,, \frac{1}{R^t}\check{\lambda}_t^{t_0}(i) \to 0$, we have:

$$0 = (\sigma e(i) + \psi W n(i))\left(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) - e(i)\left(\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\hat{P}_{l,t_0}\right)\right) + (\sigma e(i) + \psi W n(i))\check{\alpha}^{t_0}(i)$$
$$- \left(1 - \frac{1}{R}\right)(\sigma e(i) + \psi W n(i))\sum_{s=0}^{\infty}\frac{1}{R^s}\check{\Xi}_{t_0+s} - \sigma e(i)\left(1 - \frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s} - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t_0+s} + \hat{P}_{l,t_0+s}\right)\right),$$
$$\check{\alpha}^{t_0}(i) = -\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) + \left(\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\hat{P}_{l,t_0}\right)$$
$$+ \sum_{s=0}^{\infty}\frac{1}{R^{s+1}}\Delta\Omega_{t_0+1+s} + \Omega_{t_0} + \frac{\sigma e(i)}{\sigma e(i) + \psi n(i)}\left(1 - \frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s} - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t_0+s} + \hat{P}_{l,t_0+s}\right)\right).$$

Similarly for the budget constraint multiplier of HtM agents,

$$0 = (\sigma e(i) + \psi W n(i))\left(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) - \left(\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,HtM}(i) - \sum_l s_l(i)\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\hat{P}_{l,t_0}\right)\right)$$
$$+ (\sigma e(i) + \psi W n(i))\left(\check{\aleph}_t^{t_0}(i) - \check{\Xi}_t\right) - \sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right),$$
$$\check{\aleph}_t^{t_0}(i) = -\left(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) - \left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}(i) - \sum_l s_l(i)\hat{P}_{l,t}\right) + \sum_l \partial_e e_l(i)\hat{P}_{l,t}\right)\right)$$
$$+ \check{\Xi}_t + \frac{\sigma e(i)}{\sigma e(i) + \psi W n(i)}\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right).$$

Next, define

$$\tilde{\Lambda}_t \equiv \mathbb{E}_{\delta,t}\left((1 - \varphi(i))\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_e v(i)\right).$$

Using the first-order condition for the optimality of expenditure of unconstrained households to substitute $\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_e v(i)$ and the overlapping generation structure, the evolution of $\tilde{\Lambda}_t$ is given by:

$$\tilde{\Lambda}_{t+1}-(1-\delta)\tilde{\Lambda}_t=-(1-\delta)\left\{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di\Delta\check{\Xi}_{t+1}+\int(1-\varphi(i))\sigma e(i)\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l(i)\Delta\left(\tilde{A}_{l,t+1}+\hat{P}_{l,t+1}\right)\right)\right\}di$$
$$+\delta\int(1-\varphi(i))\check{\lambda}^{t+1}_{t+1}(i)\partial_e v(i)\,di.$$

Next to derive the evolution of $\int(1-\varphi(i))\check{\lambda}^{t+1}_{t+1}(i)\partial_e v(i)\,di$ we define

$$\tilde{\Lambda}^0_t\equiv\int(1-\varphi(i))\lambda^t_t(i)\partial_e v(i)\,di-\tilde{\Lambda}_t.$$

The evolution of $\tilde{\Lambda}_t$ in terms of $\tilde{\Lambda}^0_t$ is simply:

$$\tilde{\Lambda}_{t+1}-\tilde{\Lambda}_t=-\left\{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di\Delta\check{\Xi}_{t+1}+\int(1-\varphi(i))\sigma e(i)\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l(i)\Delta\left(\tilde{A}_{l,t+1}+\hat{P}_{l,t+1}\right)\right)\right\}di+\frac{\delta}{1-\delta}\tilde{\Lambda}^0_{t+1}.$$

While using the definition of $\lambda^t_t(i)$ from the FOC for expenditure, the evolution of $\tilde{\Lambda}^0_t$ is given by:

$$\begin{aligned}\int(1-\varphi(i))\lambda^t_t(i)\partial_e v(i)\,di&=(\sigma e(i)+\psi Wn(i))\left(\frac{V(i)G''(V(i))}{G'(V(i))}\hat{V}_t(i)-\left(\frac{1}{\sigma}\left(\hat{e}^{t,u}_t(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)+\sum_l\partial_e e_l(i)\hat{P}_{l,t}\right)\right)\\&+(\sigma e(i)+\psi Wn(i))\left(\check{\alpha}^t(i)-\check{\Xi}_t\right)-\sigma e(i)\left(\hat{W}_t-\sum_l\partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)\\&=\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di\sum_{s=0}^{\infty}\frac{1}{R^{s+1}}\Delta\Omega_{t+1+s}\\&+\int(1-\varphi(i))\sigma e(i)\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\Delta\hat{W}_{t+1+s}-\sum_l\partial_e e_l(i)\Delta\left(\tilde{A}_{l,t+1+s}+\hat{P}_{l,t+1+s}\right)\right)di\\\Rightarrow\tilde{\Lambda}^0_t-\frac{1}{(1-\delta)R}\tilde{\Lambda}^0_{t+1}&=-\left(1-\frac{1}{R}\right)\tilde{\Lambda}_t.\end{aligned}$$

Coming back to $\tilde{Z}_t$, and defining a new variable $\tilde{\tilde{Z}}_t$ which captures the contribution of the unconstrained households to the variable $\tilde{Z}_t$, we can write

$$\begin{aligned}\tilde{Z}_t&=\psi WN\check{\Xi}_t-\mathbb{E}_{\delta,t}\int\psi Wn(i)\frac{V(i)G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)\,di\\&+\mathbb{E}_{\delta,t}\int(1-\varphi(i))\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0}(i)-\sum_l s_l(i)\hat{P}_{l,t_0}\right)+\sum_l\partial_e e_l(i)\hat{P}_{l,t_0}-\check{\alpha}^{t_0}(i)\right\}+\varphi(i)\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}^{t_0,HtM}_t(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)+\sum_l\partial_e e_l(i)\hat{P}_{l,t}-\check{\aleph}^{t_0}_t(i)\right\}di,\\\tilde{\tilde{Z}}_t&\equiv\tilde{Z}_t+\int\varphi(i)\psi Wn(i)\left(\frac{\sigma e(i)}{\sigma e(i)+\psi n(i)}\left(\hat{W}_t-\sum_l\partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)\right)di\\&=\int(1-\varphi(i))\psi Wn(i)di\check{\Xi}_t+\mathbb{E}_{\delta,t}\int(1-\varphi(i))\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0}(i)-\sum_l s_l(i)\hat{P}_{l,t_0}\right)+\sum_l\partial_e e_l(i)\hat{P}_{l,t_0}-\frac{V(i)G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)-\check{\alpha}^{t_0}(i)\right\}.\end{aligned}$$

The evolution of $\tilde{\tilde{Z}}_t$ is given by (using the fact that the second term in the definition of $\tilde{\tilde{Z}}_t$ is independent of t):

$$\tilde{\tilde{Z}}_{t+1} - (1-\delta)\tilde{\tilde{Z}}_t = (1-\delta)\int (1-\varphi(i))\, Wn(i)\, \psi di \Delta\check{\Xi}_{t+1}$$
$$+\delta\left(\int (1-\varphi(i))\, \psi Wn(i) di \check{\Xi}_{t+1} + \int (1-\varphi(i))\, \psi Wn(i) \left\{\frac{1}{\sigma}\left(\hat{e}^{t+1,u}_{t+1}(i) - \sum_l s_l(i)\, \hat{P}_{l,t+1}\right) + \sum_l \partial_e e_l(i)\, \hat{P}_{l,t+1} - \frac{V(i)\, G''(V(i))}{G'(V(i))}\hat{V}_{t+1}(i) - \check{\alpha}^{t+1}(i)\right\}\right).$$

The second line correspond to the contribution of unconstrained households born at $t+1$ to $\tilde{\tilde{Z}}_{t+1}$. To characterize the dynamics of this term, we define:

$$\tilde{\tilde{Z}}^0_{t+1} \equiv \int (1-\varphi(i))\, \psi Wn(i) di \check{\Xi}_{t+1} + \int (1-\varphi(i))\, \psi Wn(i) \left\{\frac{1}{\sigma}\left(\hat{e}^{t+1,u}_{t+1}(i) - \sum_l s_l(i)\, \hat{P}_{l,t+1}\right) + \sum_l \partial_e e_l(i)\, \hat{P}_{l,t+1} - \frac{V(i)\, G''(V(i))}{G'(V(i))}\hat{V}_{t+1}(i) - \check{\alpha}^{t+1}(i)\right\} - \tilde{\tilde{Z}}_{t+1}.$$

Using the definition of $\tilde{\tilde{Z}}^0_{t+1}$, the joint evolution of $\tilde{\tilde{Z}}_t$ and $\tilde{\tilde{Z}}^0_t$ is given by:

$$\tilde{\tilde{Z}}_{t+1} - \tilde{\tilde{Z}}_t = \int (1-\varphi(i))\, Wn(i)\, \psi di \Delta\check{\Xi}_{t+1} + \frac{\delta}{1-\delta}\tilde{\tilde{Z}}^0_{t+1},$$
$$\tilde{\tilde{Z}}^0_t - \frac{1}{(1-\delta)R}\tilde{\tilde{Z}}^0_{t+1} = -\left(1-\frac{1}{R}\right)\int (1-\varphi(i))\, \psi Wn(i) \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)}\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right) di - \left(1-\frac{1}{R}\right)\tilde{\tilde{Z}}_t.$$

Finally, define

$$Z_t \equiv \tilde{\tilde{Z}}_t + \frac{\int (1-\varphi(i))\, Wn(i)\, \psi di}{\int (1-\varphi(i))\, (\sigma e(i) + \psi Wn(i))\, di}\tilde{\Lambda}_t$$
$$+\frac{\int (1-\varphi(i))\, Wn(i)\, \psi di}{\int (1-\varphi(i))\, (\sigma e(i) + \psi Wn(i))\, di}\int (1-\varphi(i))\, \sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right) di,$$
$$Z^0_t \equiv \tilde{\tilde{Z}}^0_t + \tilde{\Lambda}^0_t.$$

The dynamics of Z_t are characterized by the following two equations:

$$Z_{t+1} - Z_t = \frac{\delta}{1-\delta}Z^0_{t+1},$$
$$Z^0_t - \frac{1}{(1-\delta)R}Z^0_{t+1} + \left(1-\frac{1}{R}\right)Z_t = \left(1-\frac{1}{R}\right)\int (1-\varphi(i))\, \sigma e(i)\left(\frac{\int (1-\varphi(i))\, Wn(i)\, \psi di}{\int (1-\varphi(i))\, (\sigma e(i) + \psi Wn(i))\, di} - \frac{\psi Wn(i)}{\sigma e(i) + \psi Wn(i)}\right)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right) di.$$

And we can rewrite our original equation characterizing the optimality of the nominal wage as:

$$\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,s} W\mathcal{N}_k(\boldsymbol{P}_t, W_t) = Z_t - \frac{\int (1-\varphi(i))\, Wn(i)\, \psi di}{\int (1-\varphi(i))\, (\sigma e(i) + \psi Wn(i))\, di}\tilde{\Lambda}_t$$
$$-\frac{\int (1-\varphi(i))\, Wn(i)\, \psi di}{\int (1-\varphi(i))\, (\sigma e(i) + \psi Wn(i))\, di}\int (1-\varphi(i))\, \sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right) di$$
$$-\int \varphi(i)\, \psi Wn(i)\left(\frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)}\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right)\right) di + \sum_{k,l}\frac{P_l W \partial_W \mathcal{Y}_{l,k}}{A_k}Y_k\left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_k \frac{W^2 \partial_W \mathcal{N}_k}{A_k}Y_k \hat{W}_t.$$

Next, we derive the evolution of $\tilde{\Lambda}_t$. The optimality of $\hat{R}_t$ allows us to express $\tilde{\Lambda}_t$ in terms of the Lagrange multipliers of the budget constraints of unconstrained and HtM households:

$$-\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\check{\lambda}_{t}^{t_{0}}\left(i\right)\left(i\right)\partial_{e}v\left(i\right)\right)=\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t}^{t_{0}}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\check{\aleph}_{t}^{t_{0}}\left(i\right)\frac{1}{R}b\left(i\right)\right),$$

$$\begin{aligned}-\tilde{\Lambda}_{t}=&\ \mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t}^{t_{0}}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\check{\aleph}_{t}^{t_{0}}\left(i\right)\frac{1}{R}b\left(i\right)\right)\\&-\left(1-\delta\right)R\mathbb{E}_{\delta,t-1}\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t-1}^{t_{0}}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\check{\aleph}_{t-1}^{t_{0}}\left(i\right)\frac{1}{R}b\left(i\right)\right).\end{aligned}$$

Define

$$\tilde{A}_{b,t}\equiv\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t}^{t_{0}}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\check{\aleph}_{t}^{t_{0}}\left(i\right)\frac{1}{R}b\left(i\right)\right).$$

Using our formulas for $\check{\alpha}_{t}^{t_{0}}$ and $\check{\aleph}_{t}^{t_{0}}$ derived above, the evolution of $\tilde{A}_{b,t}$ is given by:

$$\begin{aligned}\tilde{A}_{b,t+1}-\left(1-\delta\right)\tilde{A}_{b,t}=&\left(1-\delta\right)\hat{R}_{t}\int\left(1-\varphi\left(i\right)\right)\frac{b\left(i\right)}{R}di+\left(1-\delta\right)\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di\Delta\check{\Xi}_{t+1}\\&+\left(1-\delta\right)\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\Delta\hat{W}_{t+1}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\Delta\tilde{A}_{l,t+1}+\Delta\hat{P}_{l,t+1}\right)\right)di\\&+\left(1-\delta\right)\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\frac{1}{R}b\left(i\right)\Delta\frac{1}{\sigma}\left(\hat{e}_{t+1}^{t_{0},HtM}\left(i\right)-\sum_{l}s_{l}\left(i\right)\hat{P}_{l,t+1}\right)+\Delta\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t+1}\right)\\&+\delta\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t+1}^{t+1}\left(i\right)+\varphi\left(i\right)\check{\aleph}_{t+1}^{t+1}\left(i\right)\right)\frac{1}{R}b\left(i\right)di\end{aligned}$$

The last line gives the contribution of the newborn households, to characterize its evolution, we define:

$$\tilde{A}_{b,t+1}^{0}=\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t+1}^{t+1}\left(i\right)+\varphi\left(i\right)\check{\aleph}_{t+1}^{t+1}\left(i\right)\right)\frac{1}{R}b\left(i\right)di-\tilde{A}_{b,t+1}$$

The decisions of households born at t in terms of expenditure at t and their change in welfare at t (using Roy's identity) are given by:

$$\hat{e}_{t}^{t_{0},HtM}\left(i\right)-\sum_{k}s_{k}\left(i\right)\hat{P}_{k,t}=\frac{\sigma}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left\{\hat{R}_{t}\frac{b\left(i\right)}{R}+Wn(i)\left(\psi\hat{W}_{t}+\sum\bar{s}_{k}\tilde{A}_{k,t}\right)-\sum_{k}\left(e(i)\left(s_{k}(i)-\bar{s}_{k}\right)+\psi Wn(i)\partial_{e}e_{k}(i)\right)\hat{P}_{k,t}+\left(1-\frac{1}{R}\right)b(i)\sum_{k}\bar{s}_{k}\left(\hat{P}_{k,t_{0}}-\hat{P}_{k,t}\right)\right\}$$

$$\begin{aligned}dV_{t_{0}}\left(i\right)&=\partial_{e}v\left(i\right)\sum_{s=0}\frac{1}{R_{s}}\left\{\frac{b\left(i\right)}{R}\hat{R}_{t+s}+\hat{W}_{t+s}Wn\left(i\right)-\sum_{k}e_{k}\left(i\right)\hat{P}_{k,t}+\varsigma\left(i\right)\sum_{k}E_{k}\left(\hat{P}_{k,t+s}+\tilde{A}_{k,t+s}-\hat{W}_{t+s}\right)+\frac{R-1}{R}b\left(i\right)\sum_{k}\bar{s}_{k}\hat{P}_{k,t_{0}}\right\}\\&=\partial_{e}v\left(i\right)\sum_{s=0}\frac{1}{R_{s}}\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_{k}\left(s_{k}\left(i\right)-\bar{s}_{k}\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_{k}\bar{s}_{k}\tilde{A}_{k,t+s}\right\}\end{aligned}$$

$$dV_{-}\left(i\right)=\partial_{e}v\left(i\right)\left(\sum_{s=0}\frac{1}{R_{s}}\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{s}-\pi_{cpi,1+s}\right)-e\left(i\right)\sum_{k}\left(s_{k}\left(i\right)-\bar{s}_{k}\right)\hat{P}_{k,s}+Wn\left(i\right)\sum_{k}\bar{s}_{k}\tilde{A}_{k,s}\right\}-b\left(i\right)\sum_{k}\bar{s}_{k}\hat{P}_{k,0}\right)$$

$$\hat{e}_0^{-,u}(i) - \sum_k s_k(i)\hat{P}_{k,t} = -\sigma \sum \frac{1}{R^{s+1}} \left(\hat{R}_{t+s} - \sum_k \partial_e e_k(i) \pi_{k,t+s+1} \right)$$

$$+ \frac{\left(1-\frac{1}{R}\right)\sigma}{(\sigma e(i) + \psi Wn(i))} \left\{ \sum \frac{1}{R^s} \left(\frac{b(i)}{R} \left(\hat{R}_{t+s} - \pi_{cpi,t+s+1} \right) + \psi Wn(i) \hat{W}_{t+s} - \sum_k \left(e(i)\left(s_k(i) - \bar{s}_k\right) + \psi Wn(i) \partial_e e_k(i) \right) \hat{P}_{k,t+s} + Wn(i) \sum_k \bar{s}_k \tilde{A}_{k,t+s} \right) - b(i) \sum_k \bar{s}_k \hat{P}_{k,0} \right\}.$$

Using these expressions, $\check{\alpha}_t^t$ can be rewritten as:

$$\check{\alpha}_t^t(i) = \left(-\frac{\partial_e v(i) G''(V(i))}{G'(V(i))} + \frac{\left(1-\frac{1}{R}\right)}{(\sigma e(i) + \psi Wn(i))} \right) \sum_{s=0} \frac{1}{R_s} \left\{ \frac{b(i)}{R} \left(\hat{R}_{t+s} - \pi_{cpi,t+1+s} \right) - e(i) \sum_k \left(s_k(i) - \bar{s}_k\right) \hat{P}_{k,t+s} + Wn(i) \sum_k \bar{s}_k \tilde{A}_{k,t+s} \right\}$$

$$+ \left(1 - \frac{1}{R}\right) \sum \frac{1}{R^s} \hat{W}_{t+s} - \sum \frac{1}{R^{s+1}} \hat{R}_{t+s} + \sum_{s=0}^{\infty} \frac{1}{R^{s+1}} \Delta\Omega_{t+1+s} + \check{\Xi}_t - \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} \left(1 - \frac{1}{R}\right) \sum_{s=0}^{\infty} \frac{1}{R^s} \sum_k \partial_e e_k(i) \tilde{A}_{k,t+s}.$$

The evolution of $\tilde{A}_{b,t}$ and $\tilde{A}_{b,t}^0$ is therefore characterized by:

$$\tilde{A}_{b,t+1} - \tilde{A}_{b,t} = \hat{R}_t \int (1 - \varphi(i)) \frac{b(i)}{R} di + \int \varphi(i) \frac{1}{R} b(i)\, di \Delta\check{\Xi}_{t+1} + \int \varphi(i) \frac{b(i)}{R} \Delta\hat{W}_{t+1} di - \int \varphi(i) \frac{b(i)}{R} \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} \sum_k \partial_e e_k(i) \Delta\tilde{A}_{k,t+1} di$$

$$+ \mathbb{E}_{\delta,t} \left(\varphi(i) \frac{b(i)}{R} \frac{1}{\sigma e(i) + \psi Wn(i)} \left\{ \frac{b(i)}{R} \left(\Delta\hat{R}_{t+1} - (R-1)\pi_{cpi,t+1} \right) + Wn(i) \sum \bar{s}_k \Delta\tilde{A}_{k,t+1} - \sum_k e(i) \left(s_k(i) - \bar{s}_k\right) \pi_{k,t+1} \right\} \right) + \frac{\delta}{1-\delta} \tilde{A}_{b,t+1}^0.$$

$$\tilde{A}_{b,t}^0 - \frac{1}{(1-\delta)R} \tilde{A}_{b,t+1}^0 + \left(1 - \frac{1}{R}\right) \tilde{A}_{b,t} = \int \frac{b(i)}{R} \left(-\frac{\partial_e v(i) G''(V(i))}{G'(V(i))} + \frac{\left(1-\frac{1}{R}\right)}{\sigma e(i) + \psi Wn(i)} \right) \left\{ \frac{b(i)}{R} \left(\hat{R}_{t+s} - \pi_{cpi,t+1+s} \right) - e(i) \sum_k \left(s_k(i) - \bar{s}_k\right) \hat{P}_{k,t+s} + Wn(i) \sum_k \bar{s}_k \tilde{A}_{k,t+s} \right\} di$$

$$- \left(1 - \frac{1}{R}\right) \int \frac{b(i)}{R} \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} \sum_k \partial_e e_k(i) \tilde{A}_{k,t} di.$$

Using the relationship between $\tilde{\Lambda}_t$ and $\check{\Xi}_t$, we obtain:

$$\left(1 - \frac{\int \varphi(i) \frac{1}{R} b(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di}\right) A_{b,t+1} - \left(1 - \frac{(1-\delta+R) \int \varphi(i) \frac{1}{R} b(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di}\right) A_{b,t} - \frac{(1-\delta)R \int \varphi(i) \frac{1}{R} b(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} A_{b,t-1} - \frac{\delta}{1-\delta} A_{b,t+1}^0 =$$

$$- \sum \left(\int \varphi(i) \frac{b(i)}{R} \frac{\sigma e(i) \partial_e e_l(i)}{\sigma e(i) + \psi Wn(i)} di + \frac{\int (1-\varphi(i)) \frac{b(i)}{R} di \int (1-\varphi(i)) \sigma e(i) \partial_e e_l(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \Delta\tilde{A}_{l,t+1}$$

$$+ \left(\int \varphi(i) \frac{\left(\frac{b(i)}{R}\right)^2}{\sigma e(i) + \psi Wn(i)} di + \frac{\left(\int (1-\varphi(i)) \frac{b(i)}{R} di\right)^2}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \left(\Delta\hat{R}_{t+1} - (R-1)\pi_{cpi,t+1} \right)$$

$$+ \left(\int \varphi(i) \frac{\frac{b(i)}{R} Wn(i)}{\sigma e(i) + \psi Wn(i)} di + \frac{\int (1-\varphi(i)) \frac{b(i)}{R} di \int (1-\varphi(i)) Wn(i) di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \sum \bar{s}_k \Delta\tilde{A}_{k,t+1}$$

$$- \sum \left(\int \varphi(i) \frac{\frac{b(i)}{R} e(i) \left(s_k(i) - \bar{s}_k\right)}{\sigma e(i) + \psi Wn(i)} di + \frac{\int (1-\varphi(i)) \frac{b(i)}{R} di \int (1-\varphi(i)) \frac{e(i)}{E} \left(s_k(i) - \bar{s}_k\right)}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \pi_{k,t+1}$$

$$A_{b,t}^{0}-\frac{1}{\left(1-\delta\right)R}A_{b,t+1}^{0}+\left(1-\frac{1}{R}\right)A_{b,t}=\int\frac{b\left(i\right)}{R}\left(-\frac{\partial_{e}v\left(i\right)G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_{k}\left(s_{k}\left(i\right)-\bar{s}_{k}\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_{k}\bar{s}_{k}\tilde{A}_{k,t+s}\right\}di$$
$$-\left(1-\frac{1}{R}\right)\int\frac{b\left(i\right)}{R}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\sum_{k}\partial_{e}e_{k}\left(i\right)\tilde{A}_{k,t}di$$

And the original wage equation in terms of Z_t and $A_{b,t}$ is

$$\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,s}W\mathcal{N}_{k}(\boldsymbol{P}_{t},W_{t})=Z_{t}+\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi n\left(i\right)\right)di}\left(A_{b,t}-(1-\delta)RA_{b,t-1}\right)$$
$$-\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi n\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_{t}-\sum_{k}\partial_{e}e_{k}\left(i\right)\left(\tilde{A}_{k,t}+\hat{P}_{k,t}\right)\right)di$$
$$-\int\varphi\left(i\right)\psi Wn(i)\left(\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\hat{W}_{t}-\sum_{k}\partial_{e}e_{k}\left(i\right)\left(\tilde{A}_{k,t}+\hat{P}_{k,t}\right)\right)\right)di+\sum_{k,l}\frac{P_{l}W\partial_{W}\mathcal{Y}_{l,k}}{A_{k}}Y_{k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_{k}\frac{W^{2}\partial_{W}\mathcal{N}_{k}}{A_{k}}Y_{k}\hat{W}_{t}$$

Solving the Price Setting equation

The second set of main equations are given by the optimality of price setting:

$$\left(\beta\check{\mu}_{k,t+1}-\left(1+\left(\left(1-\delta\right)\beta\right)\right)\check{\mu}_{k,t}+\left(1-\delta\right)\check{\mu}_{k,t-1}\right)=\lambda_{k}\check{\mu}_{k,t}-\sum_{l=1}^{K}\lambda_{l}\frac{\check{\mu}_{l,t}}{P_{l}Y_{l}}\frac{P_{k}\mathcal{Y}_{k,l}Y_{l}}{A_{l}}$$
$$-\mathbb{E}_{\delta,t}\left(\int\left[\left(1-\varphi\left(i\right)\right)\left(\check{\lambda}_{t}^{t_{0}}\left(i\right)-R\check{\lambda}_{t-1}^{t_{0}}\left(i\right)\right)+\check{\zeta}_{t}^{t_{0},u}\left(i\right)W\right]\partial_{e}v\partial_{e}e_{k}\left(i\right)+\varphi\left(i\right)\check{\zeta}_{t}^{t_{0},HtM}\left(i\right)W\partial_{e}v\partial_{e}e_{k}\left(i\right)di\right)$$
$$+\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t}^{t_{0}}\left(i\right)+\varphi\left(i\right)\check{\aleph}_{t}^{t_{0}}\left(i\right)\right)\left[e_{k}(i)-\varsigma\left(i\right)E_{k}\right]di\right)-\delta\int\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t}\left(i\right)b\left(i\right)di\bar{s}_{k}$$
$$-\delta\sum_{u=0}\left(\left(1-\delta\right)\beta\right)^{u}\int\varphi\left(i\right)\check{\aleph}_{t+u}^{t}\left(i\right)b\left(i\right)di\bar{s}_{k}\frac{R-1}{R}+\delta\sum_{u=0}\frac{1}{R^{u}}\int\varphi\left(i\right)b\left(i\right)di\bar{s}_{k}\frac{\hat{R}_{t+u}}{R}$$
$$+\sum_{l,m}\frac{P_{l}P_{k}\partial_{P_{k}}\mathcal{Y}_{l,m}}{A_{m}}Y_{m}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_{l}\frac{WP_{k}\partial_{P_{k}}\mathcal{N}_{l}}{A_{l}}Y_{l}\hat{W}_{t}+\sum_{l}E_{l}\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-Y_{k}\vartheta_{k}\left(\pi_{k,t}-\beta\pi_{k,t+1}\right)$$

As in the previous subsection the goal here is to derive the dynamics of the components of these equation using only a finite number of variables.
First note that we have

$$\mathbb{E}_{\delta,t}\left(\int\varphi\left(i\right)\check{\zeta}_{t}^{t_{0},HtM}\left(i\right)W\partial_{e}v\partial_{e}e_{k}\left(i\right)di\right)=\mathbb{E}_{\delta,t}\left(\int\varphi\left(i\right)\partial_{e}e_{k}\left(i\right)\psi Wn\left(i\right)\left\{\check{\Xi}_{t}-\check{\aleph}_{t}^{t_{0}}\left(i\right)-\frac{G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}dV_{t_{0}}\left(i\right)+\frac{1}{\sigma}e_{t}^{t_{0},HtM}\left(i\right)+\partial_{e}e_{l}\left(i\right)\cdot\hat{P}_{l,t}\right\}di\right)$$
$$=-\mathbb{E}_{\delta,t}\left(\int\varphi\left(i\right)\partial_{e}e_{k}\left(i\right)\psi Wn\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)di\right)$$
$$=-\int\varphi\left(i\right)\partial_{e}e_{k}\left(i\right)\psi Wn\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)di$$

Next, we solve for $\tilde{L}_{k,t}\equiv\mathbb{E}_{\delta,t}\left(\int\left[\left(1-\varphi\left(i\right)\right)\left(\check{\lambda}_{t}^{t_{0}}\left(i\right)-R\check{\lambda}_{t-1}^{t_{0}}\left(i\right)\right)+\check{\zeta}_{t}^{t_{0},u}\left(i\right)\right]\partial_{e}v\partial_{e}e_{k}\left(i\right)\right)$, that we can re-express as:

$$\tilde{L}_{k,t}=\mathbb{E}_{\delta,t}\left(\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)\left\{\frac{G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}-\frac{1}{\sigma\left(i\right)}\hat{e}_{t_{0}}^{t_{0},u}\left(i\right)-\partial_{e}e_{l}\left(i\right)\cdot\hat{P}_{l,t_{0}}+\check{\alpha}_{t_{0}}\left(i\right)-\left(\hat{W}_{t}-\partial_{e}e_{l}\left(i\right)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)-\check{\Xi}_{t}\right\}di\right)$$

Define

$$\begin{aligned}\tilde{\tilde{L}}_{k,t}&=\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)\left\{\frac{G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}-\frac{1}{\sigma\left(i\right)}\hat{e}_{t_{0}}^{t_{0},u}\left(i\right)-\partial_{e}e_{l}\left(i\right)\cdot\hat{P}_{l,t_{0}}+\check{\alpha}_{t_{0}}\left(i\right)-\check{\Xi}_{t}\right\}\right)\\&=\tilde{L}_{k,t}+\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)\left\{\left(\hat{W}_{t}-\partial_{e}e_{l}\left(i\right)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)\right\}di\end{aligned}$$

We have

$$\Delta\tilde{\tilde{L}}_{k,t+1}=-\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)di\Delta\check{\Xi}_{t+1}-\frac{\lambda_{l}}{P_{l}Y_{l}}\sum_{l=1}^{K}\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)\partial_{e}e_{l}\left(i\right)di\Delta\check{\mu}_{l,t+1}+\frac{\delta}{1-\delta}\tilde{\tilde{L}}_{k,t+1}^{0}$$

With

$$\tilde{\tilde{L}}_{k,t}^{0}-\frac{1}{R\left(1-\delta\right)}\tilde{\tilde{L}}_{k,t+1}^{0}+\left(1-\frac{1}{R}\right)\tilde{\tilde{L}}_{k,t}=\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{\sigma e\left(i\right)}{\psi Wn\left(i\right)+\sigma e\left(i\right)}\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)$$

Define

$$\begin{aligned}L_{k,t}&\equiv\tilde{\tilde{L}}_{k,t}-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\tilde{\Lambda}_{t}\\&\quad-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)di,\\L_{k,t}^{0}&\equiv\tilde{\tilde{L}}_{k,t}^{0}-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\tilde{\Lambda}_{t}^{0}.\end{aligned}$$

We have

$$\Delta L_{k,t+1}=\frac{\delta}{1-\delta}L_{k,t+1}^{0}$$

$$\begin{aligned}L_{k,t}^{0}-\frac{1}{R\left(1-\delta\right)}L_{k,t+1}^{0}+\left(1-\frac{1}{R}\right)L_{k,t}&=\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{\sigma e\left(i\right)}{\psi wn\left(i\right)+\sigma e\left(i\right)}\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)di\\&\quad-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)di\end{aligned}$$

The resetting equation becomes

$$\left(\beta\breve{\mu}_{k,t+1}-\left(1+\left(\left(1-\delta\right)\beta\right)\right)\breve{\mu}_{k,t}+\left(1-\delta\right)\breve{\mu}_{k,t-1}\right)=\lambda_{k}\breve{\mu}_{k,t}-\sum_{l=1}^{K}\lambda_{l}\frac{\breve{\mu}_{l,t}}{P_{l}Y_{l}}\frac{P_{k}\mathcal{Y}_{k,l}Y_{l}}{A_{l}}-L_{k,t}+\int\varphi\left(i\right)\partial_{e}e_{k}\psi n\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)di$$
$$+\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)\left\{\left(\hat{W}_{t}-\partial_{e}e_{l}\left(i\right)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)\right\}di-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\tilde{\Lambda}_{t}$$
$$-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)di$$
$$+\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t}^{t_{0}}\left(i\right)+\varphi\left(i\right)\breve{\aleph}_{t}^{t_{0}}\left(i\right)\right)\left[e_{k}(i)-\varsigma\left(i\right)E_{k}\right]di\right)-\delta\int\left(1-\varphi\left(i\right)\right)\breve{\alpha}^{t}\left(i\right)b\left(i\right)di\bar{s}_{k}$$
$$-\delta\sum_{u=0}\left(\left(1-\delta\right)\beta\right)^{u}\int\varphi\left(i\right)\breve{\aleph}_{t+u}^{t}\left(i\right)b\left(i\right)di\bar{s}_{k}\frac{R-1}{R}+\delta\sum_{u=0}\frac{1}{R^{u}}\int\varphi\left(i\right)b\left(i\right)di\bar{s}_{k}\frac{\hat{R}_{t+u}}{R}$$
$$+\sum_{l,m}\frac{P_{l}P_{k}\partial_{P_{k}}\mathcal{Y}_{l,m}}{A_{m}}Y_{m}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_{l}\frac{WP_{k}\partial_{P_{k}}\mathcal{N}_{l}}{A_{l}}Y_{l}\hat{W}_{t}+\sum_{l}E_{l}\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-Y_{k}\vartheta_{k}\left(\pi_{k,t}-\beta\pi_{k,t+1}\right)$$

Next note that we have

$$\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t}^{t_{0}}\left(i\right)+\varphi\left(i\right)\breve{\aleph}_{t}^{t_{0}}\left(i\right)\right)\left[e_{k}(i)-\varsigma\left(i\right)E_{k}\right]di\right)=\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t}^{t_{0}}\left(i\right)+\varphi\left(i\right)\breve{\aleph}_{t}^{t_{0}}\left(i\right)\right)\left[e_{k}(i)-e(i)\bar{s}_{k}\right]di\right)$$
$$+\left(1-\frac{1}{R}\right)\bar{s}_{k}\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t}^{t_{0}}\left(i\right)+\varphi\left(i\right)\breve{\aleph}_{t}^{t_{0}}\left(i\right)\right)b(i)di\right).$$

Define

$$\tilde{A}_{e_{k},t}\equiv\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t}^{t_{0}}\left(i\right)+\varphi\left(i\right)\breve{\aleph}_{t}^{t_{0}}\left(i\right)\right)\left(e_{k}(i)-e(i)\bar{s}_{k}\right)di\right)$$

We have

$$\tilde{A}_{e_{k},t+1}-\left(1-\delta\right)\tilde{A}_{e_{k},t}=\left(1-\delta\right)\hat{R}_{t}\int\left(1-\varphi\left(i\right)\right)\left(e_{k}(i)-e(i)\bar{s}_{k}\right)di+\left(1-\delta\right)\int\varphi\left(i\right)\left(e_{k}(i)-e(i)\bar{s}_{k}\right)di\Delta\breve{\Xi}_{t+1}$$
$$+\left(1-\delta\right)\int\varphi\left(i\right)\left(e_{k}(i)-e(i)\bar{s}_{k}\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\Delta\hat{W}_{t+1}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\Delta\tilde{A}_{l,t+1}+\Delta\hat{P}_{l,t+1}\right)\right)di$$
$$+\left(1-\delta\right)\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\left(e_{k}(i)-e(i)\bar{s}_{k}\right)\Delta\frac{1}{\sigma}\left(\hat{e}_{t+1}^{t_{0},HtM}\left(i\right)-\sum_{l}s_{l}\left(i\right)\hat{P}_{l,t+1}\right)+\Delta\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t+1}\right)$$
$$+\delta\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t+1}^{t+1}+\varphi\left(i\right)\breve{\aleph}_{t+1}^{t+1}\left(i\right)\right)\left(e_{k}(i)-e(i)\bar{s}_{k}\right)di.$$

Define

$$\tilde{A}_{e_{k},t+1}^{0}=\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t+1}^{t+1}\left(i\right)+\varphi\left(i\right)\breve{\aleph}_{t+1}^{t+1}\left(i\right)\right)\left(e_{k}(i)-e(i)\bar{s}_{k}\right)di-\tilde{A}_{e_{k},t+1}$$

We have

$$\tilde{A}_{e_k,t+1}-\tilde{A}_{e_k,t}=\hat{R}_t\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\Delta\check{\Xi}_{t+1}+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\Delta\hat{W}_{t+1}di$$
$$-\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\sum_l\partial_e e_l\left(i\right)\left(\Delta\tilde{A}_{l,t+1}\right)di$$
$$+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{1}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left\{\frac{b\left(i\right)}{R}\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)+Wn(i)\sum\bar{s}_k\Delta\tilde{A}_{k,t+1}-\sum_k e(i)\left(s_k(i)-\bar{s}_k\right)\pi_{k,t+1}\right\}\right)+\frac{\delta}{1-\delta}\tilde{A}^0_{e_k,t+1},$$

$$\tilde{A}^0_{e_k,t}-\frac{1}{\left(1-\delta\right)R}\tilde{A}^0_{e_k,t+1}+\left(1-\frac{1}{R}\right)\tilde{A}_{e_k,t}=-\left(1-\frac{1}{R}\right)\int\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\sum_l\partial_e e_l\left(i\right)\tilde{A}_{l,t}di$$
$$+\int\left(e_k(i)-e(i)\bar{s}_k\right)\left(-\frac{\partial_e vG''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right\}di.$$

Define

$$A_{e_k,t}\equiv\tilde{A}_{e_k,t}+\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}di\tilde{\Lambda}_t,$$
$$A^0_{e_k,t}\equiv\tilde{A}^0_{e_k,t}+\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}di\tilde{\Lambda}^0_t.$$

We have

$$A_{e_k,t+1}-A_{e_k,t}=\hat{R}_t\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di$$
$$-\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\left\{\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l\left(i\right)\Delta\left(\tilde{A}_{l,t+1}+\hat{P}_{l,t+1}\right)\right)\right\}$$
$$+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\Delta\hat{W}_{t+1}di-\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\sum_l\partial_e e_l\left(i\right)\left(\Delta\tilde{A}_{l,t+1}\right)di$$
$$+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{1}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left\{\frac{b\left(i\right)}{R}\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)+Wn(i)\sum\bar{s}_k\Delta\tilde{A}_{k,t+1}-\sum_k e(i)\left(s_k(i)-\bar{s}_k\right)\pi_{k,t+1}\right\}\right)+\frac{\delta}{1-\delta}A^0_{e_k,t+1},$$

$$A^0_{e_k,t}-\frac{1}{\left(1-\delta\right)R}A^0_{e_k,t+1}+\left(1-\frac{1}{R}\right)A_{e_k,t}=-\left(1-\frac{1}{R}\right)\int\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\sum_l\partial_e e_l\left(i\right)\tilde{A}_{l,t}di$$
$$+\int\left(e_k(i)-e(i)\bar{s}_k\right)\left(-\frac{\partial_e vG''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right\}di.$$

Using the evolution of the wage (derived from the output gap Euler)

$$\left(1-\varphi^N\right)\psi\Delta\hat{W}_{t+1}=\left(\left(1-\varphi^N\right)\left(\sigma+\psi\right)-\sigma\left(1-\frac{1}{R}\right)\int\varphi\left(i\right)\frac{b(i)}{RE}\right)\hat{R}_t-\left(1-\varphi^N\right)\bar{s}_k\cdot\Delta\tilde{A}_{l,t+1}$$
$$+\int\varphi\left(i\right)\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)-\frac{e(i)}{E}\sum_k\left(\left(s_k(i)-\bar{s}_k\right)\right)\pi_{k,t+1}\right\}di+\sigma\left(\sum_k-\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\partial_e e_k(i)\pi_{k,t+1}\right)$$

we have

$$A_{e_k,t+1}-A_{e_k,t}-\frac{\delta}{1-\delta}A^0_{e_k,t+1}=-\sum\left(\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)\partial_e e_l\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_l\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\Delta\tilde{A}_{l,t+1}$$
$$+\left(\int\varphi\left(i\right)\frac{\left(\frac{b(i)}{R}\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\left(\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\right)^2}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)$$
$$+\left(\int\varphi\left(i\right)\frac{\left(e_k(i)-e(i)\bar{s}_k\right)Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)Wn(i)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\sum\bar{s}_l\Delta\tilde{A}_{l,t+1}$$
$$-\sum\left(\int\varphi\left(i\right)\frac{\left(e_k(i)-e(i)\bar{s}_k\right)e(i)\left(s_l(i)-\bar{s}_l\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\pi_{k,t+1}.$$

$$A^0_{e_k,t}-\frac{1}{\left(1-\delta\right)R}A^0_{e_k,t+1}+\left(1-\frac{1}{R}\right)A_{e_k,t}$$
$$=\int\left(e_k(i)-e(i)\bar{s}_k\right)\left(-\frac{\partial_e v\left(i\right)G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right\}di$$
$$-\left(1-\frac{1}{R}\right)\int\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\sum_l\partial_e e_l\left(i\right)\tilde{A}_{l,t}di.$$

The resetting equation becomes

$$\left(\beta\check{\mu}_{k,t+1}-\left(1+\left(\left(1-\delta\right)\beta\right)\right)\check{\mu}_{k,t}+\left(1-\delta\right)\check{\mu}_{k,t-1}\right)=\lambda_k\check{\mu}_{k,t}-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{P_lY_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l}-L_{k,t}+\int\varphi\left(i\right)\partial_e e_k\psi n\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\hat{W}_t-\sum_l\partial_e e_l\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)di$$
$$+\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)\left(\hat{W}_t-\partial_e e_l\left(i\right)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)di-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\tilde{\Lambda}_t$$
$$-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_t-\sum_l\partial_e e_l\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)di$$
$$+A_{e_k,t}-\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}di\tilde{\Lambda}_t+\left(1-\frac{1}{R}\right)\bar{s}_k\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t_0}_t\left(i\right)+\varphi\left(i\right)\check{\aleph}^{t_0}_t\left(i\right)\right)b(i)di\right)$$
$$-\delta\int\left(1-\varphi\left(i\right)\right)\check{\alpha}^t\left(i\right)b\left(i\right)di\bar{s}_k-\delta\sum_{u=0}\left(\left(1-\delta\right)\beta\right)^u\int\varphi\left(i\right)\check{\aleph}^t_{t+u}\left(i\right)b\left(i\right)di\bar{s}_k\frac{R-1}{R}+\delta\sum_{u=0}\frac{1}{R^u}\int\varphi\left(i\right)b\left(i\right)di\bar{s}_k\frac{\hat{R}_{t+u}}{R}$$
$$+\sum_{l,m}\frac{P_lP_k\partial_{P_k}\mathcal{Y}_{l,m}}{A_m}Y_m\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_l\frac{WP_k\partial_{P_k}\mathcal{N}_l}{A_l}Y_l\hat{W}_t+\sum_l E_l\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-Y_k\vartheta_k\left(\pi_{k,t}-\beta\pi_{k,t+1}\right)$$

Finally, define

$$\mathcal{A}_{b,t}=\int\left(1-\varphi\left(i\right)\right)\check{\alpha}^t\left(i\right)b\left(i\right)di+\sum_{u=0}\left(\left(1-\delta\right)\beta\right)^u\int\varphi\left(i\right)\check{\aleph}^t_{t+u}\left(i\right)b\left(i\right)di\frac{R-1}{R}-\sum_{u=0}\frac{1}{R^u}\int\varphi\left(i\right)b\left(i\right)di\bar{s}_k\frac{\hat{R}_{t+u}}{R}$$

We have

$$\mathcal{A}_{b,t}-\frac{1}{R}\mathcal{A}_{b,t+1}=-\left(1-\frac{1}{R}\right)\int b\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\sum_{l}\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,t}di$$

$$+\int b\left(i\right)\left(-\frac{\partial_{e}vG''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_{k}\left(s_{k}\left(i\right)-\bar{s}_{k}\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_{k}\bar{s}_{k}\tilde{A}_{k,t+s}\right\}di$$

Optimal Policy Equations: summary

We now collect and slightly simplify the optimal policy equations derived above.[12] We obtain a system of $6+K*6$ equations in the following variables: $\hat{W}_t, Z_t, Z_t^0$, $A_{b,t}, A_{b,t}^0$ and $\mathcal{A}_{b,t}$ and $M_{k,t}, \check{\mu}_{k,t}, L_{k,t}, L_{k,t}^0$ and $A_{e_k,t}, A_{e_k,t}^0$. These replace the interest rate rule. Note that the evolution of $\hat{W}_t$ is given by

$$\begin{aligned}\left(1-\varphi^{N}\right)\psi\Delta\hat{W}_{t+1}=&\left(\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi\right)\hat{R}_{t}-\left(1-\varphi^{N}\right)\bar{s}_{k}\cdot\Delta\tilde{A}_{l,t+1}\\&+\int\varphi\left(i\right)\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)-\frac{e(i)}{E}\sum_{k}\left(\left(s_{k}(i)-\bar{s}_{k}\right)\right)\pi_{k,t+1}\right\}di\\&+\sigma\left(\sum_{k}-\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\partial_{e}e_{k}(i)\pi_{k,t+1}\right).\end{aligned}$$

We renormalize $\check{\mu}_{k,t}\equiv E\check{\mu}_{k,t}$, and define $g(i)=\left(-\frac{\partial_{e}v(i)G''(V(i))\frac{R}{R-1}}{G'(V(i))}+\frac{1}{\sigma e(i)+\psi Wn(i)}\right)E$. Our Labour Market equation equation becomes

$$\begin{aligned}\sum_{k=1}^{K}\Omega_{N,k}\lambda_{k}\check{\mu}_{k,t}=&Z_{t}+\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\frac{1}{R}\left(A_{b,t}-(1-\delta)RA_{b,t-1}\right)\\&-\left(\frac{\int\left(1-\varphi\left(i\right)\right)\frac{Wn(i)}{WN}\psi di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}+\int\varphi\left(i\right)\psi\frac{Wn(i)}{WN}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi n\left(i\right)}di\right)\hat{W}_{t}\\&+\sum_{k}\left(\frac{\int\left(1-\varphi\left(i\right)\right)\frac{Wn(i)}{WN}\psi di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}+\int\varphi\left(i\right)\psi\frac{Wn(i)}{WN}\frac{\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)}{\sigma e\left(i\right)+\psi n\left(i\right)}di\right)\left(\tilde{A}_{k,t}+\hat{P}_{k,t}\right)\\&+\sum_{k,l}\frac{P_{l}W\partial_{W}\mathcal{Y}_{l,k}Y_{k}}{A_{k}E}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_{k}\frac{W\partial_{W}\mathcal{N}_{k}}{A_{k}N}Y_{k}\hat{W}_{t}.\end{aligned}$$

With

$$Z_{t+1}-Z_{t}=\frac{\delta}{1-\delta}Z_{t+1}^{0},$$

$$\begin{aligned}Z_{t-1}^{0}-\frac{1}{\left(1-\delta\right)R}Z_{t}^{0}+\left(1-\frac{1}{R}\right)Z_{t-1}=&\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left(\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}-\frac{\psi Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)di\hat{W}_{t-1}\\&-\sum_{k}\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\partial_{e}e_{k}\left(i\right)\left(\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}-\frac{\psi Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)di\left(\tilde{A}_{k,t-1}+\hat{P}_{k,t-1}\right).\end{aligned}$$

[12] We also derived these equations in a different way, starting from the welfare loss function derived in the next appendix.

And

$$\left(1-\frac{\int\varphi(i)\frac{1}{R}b(i)\,di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}\right)A_{b,t+1}-\left(1-\frac{(1-\delta+R)\int\varphi(i)\frac{1}{R}b(i)\,di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}\right)A_{b,t}-\frac{(1-\delta)R\int\varphi(i)\frac{1}{R}b(i)\,di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}A_{b,t-1}-\frac{\delta}{1-\delta}A^0_{b,t+1}=$$
$$-\sum_l\left(\int\varphi(i)\frac{b(i)}{E}\frac{\sigma e(i)\partial_e e_l(i)}{\sigma e(i)+\psi Wn(i)}di+\frac{\int(1-\varphi(i))\frac{b(i)}{E}di\int(1-\varphi(i))\sigma e(i)\partial_e e_l(i)\,di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}\right)\Delta\tilde{A}_{l,t+1}$$
$$+\left(\int\varphi(i)\frac{(b(i))^2}{ER}\frac{1}{\sigma e(i)+\psi Wn(i)}di+\frac{1}{ER}\frac{\left(\int(1-\varphi(i))b(i)\,di\right)^2}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}\right)\left(\Delta\hat{R}_{t+1}-(R-1)\pi_{cpi,t+1}\right)$$
$$+\left(\int\varphi(i)\frac{\frac{b(i)}{E}Wn(i)}{\sigma e(i)+\psi Wn(i)}di+\frac{\int(1-\varphi(i))\frac{b(i)}{E}di\int(1-\varphi(i))Wn(i)di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}\right)\sum\bar{s}_k\Delta\tilde{A}_{k,t+1}$$
$$-\sum\left(\int\varphi(i)\frac{\frac{b(i)}{E}e(i)(s_k(i)-\bar{s}_k)}{\sigma e(i)+\psi Wn(i)}di+\frac{\int(1-\varphi(i))\frac{b(i)}{E}di\int(1-\varphi(i))e(i)(s_k(i)-\bar{s}_k)\,di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}\right)\pi_{k,t+1}$$

$$\begin{aligned}A^0_{b,t-1}-\frac{1}{(1-\delta)R}A^0_{b,t}+\left(1-\frac{1}{R}\right)A_{b,t-1}&=\left(1-\frac{1}{R}\right)\int\frac{b(i)}{E}g(i)\frac{b(i)}{RE}di\left(\hat{R}_{t-1}-\pi_{cpi,t}\right)\\&-\left(1-\frac{1}{R}\right)\sum_k\int\frac{b(i)}{E}g(i)\frac{e(i)}{E}(s_k(i)-\bar{s}_k)\,di\hat{P}_{k,t-1}\\&+\left(1-\frac{1}{R}\right)\int\frac{b(i)}{E}g(i)\frac{Wn(i)}{WN}di\sum_k\bar{s}_k\tilde{A}_{k,t-1}\\&-\left(1-\frac{1}{R}\right)\sum_k\int\frac{b(i)}{E}\frac{\sigma e(i)\partial_e e_k(i)}{\sigma e(i)+\psi Wn(i)}di\tilde{A}_{k,t-1}.\end{aligned}$$

The Price resetting equation becomes

$$M_{k,t}=\check{\mu}_{k,t}-(1-\delta)\check{\mu}_{k,t-1}$$

$$\beta M_{k,t+1}-M_{k,t}=\lambda_k\check{\mu}_{k,t}-\sum_{l=1}^K\lambda_l\frac{\check{\mu}_{l,t}}{P_lY_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l}+\sum_{l,m}\frac{P_lP_k\partial_{P_k}\mathcal{Y}_{l,m}}{A_mE}Y_m\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_l\frac{P_k\partial_{P_k}\mathcal{N}_l}{A_lN}Y_l\hat{W}_t$$
$$+\sum_l\bar{s}_l\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-\frac{P_kY_k}{E}\vartheta_k\left(\pi_{k,t}-\beta\pi_{k,t+1}\right)-L_{k,t}+\left(\frac{\int(1-\varphi(i))\partial_e e_k(i)\sigma e(i)\,di\int(1-\varphi(i))\psi\frac{Wn(i)}{WN}}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}+\int\varphi(i)\partial_e e_k(i)\frac{\psi Wn(i)}{WN}\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}di\right)\hat{W}_t$$
$$-\sum_l\left(\int\varphi(i)\partial_e e_l(i)\partial_e e_k(i)\psi\frac{Wn(i)}{WN}\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}-\frac{\int(1-\varphi(i))\partial_e e_k(i)\sigma e(i)\,di\int(1-\varphi(i))\sigma\frac{e(i)}{E}\partial_e e_l(i)\,di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}+\int(1-\varphi(i))\sigma\frac{e(i)}{E}\partial_e e_k(i)\partial_e e_l(i)\,di\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)$$
$$+\left(\frac{\int(1-\varphi(i))\partial_e e_k(i)\sigma e(i)\,di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}+\frac{\int\varphi(i)\left(e_k(i)-e(i)\bar{s}_k\right)}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di}\right)\frac{1}{R}\left(A_{b,t}-R(1-\delta)A_{b,t-1}\right)+\left(1-\frac{1}{R}\right)\bar{s}_kA_{b,t}-\delta\bar{s}_k\mathcal{A}_{b,t}+A_{e_k,t}.$$

With $\vartheta_k = \bar{\epsilon}_k \frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)}$ and:

$$\Delta L_{k,t+1} = \frac{\delta}{1-\delta} L^0_{k,t+1},$$

$$L^0_{k,t-1} - \frac{1}{R(1-\delta)} L^0_{k,t} + \left(1 - \frac{1}{R}\right) L_{k,t-1} = \left(1 - \frac{1}{R}\right) \int (1-\varphi(i)) \sigma \frac{e(i)}{E} \left(\frac{\sigma e(i) \partial_e e_k(i)}{\sigma e(i) + \psi W n(i)} - \frac{\int (1-\varphi(i)) \sigma e(i) \partial_e e_k(i) \, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi W n(i)) \, di} \right) di \hat{W}_{t-1}$$
$$- \sum_l \left(1 - \frac{1}{R}\right) \int (1-\varphi(i)) \sigma \frac{e(i)}{E} \partial_e e_l(i) \left(\frac{\sigma e(i) \partial_e e_k(i)}{\sigma e(i) + \psi W n(i)} - \frac{\int (1-\varphi(i)) \sigma e(i) \partial_e e_k(i) \, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi W n(i)) \, di} \right) di \left(\tilde{A}_{l,t-1} + \hat{P}_{l,t-1}\right).$$

$$A_{e_k,t+1} - A_{e_k,t} - \frac{\delta}{1-\delta} A^0_{e_k,t+1} = -\sum \left(\int \varphi(i) \frac{e(i)}{E} (s_k(i) - \bar{s}_k) \frac{\sigma e(i) \partial_e e_l(i)}{\sigma e(i) + \psi W n(i)} di + \frac{\int (1-\varphi(i)) \frac{e(i)}{E} (s_k(i) - \bar{s}_k) \, di \int (1-\varphi(i)) \sigma e(i) \partial_e e_l(i) \, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi W n(i)) \, di} \right) \Delta \tilde{A}_{l,t+1}$$
$$+ \left(\int \varphi(i) \frac{\left(\frac{b(i)}{R}\right) \frac{e(i)}{E} (s_k(i) - \bar{s}_k)}{\sigma e(i) + \psi W n(i)} di + \frac{\int (1-\varphi(i)) b(i) \, di \int (1-\varphi(i)) \frac{e(i)}{E} (s_k(i) - \bar{s}_k) \, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi W n(i)) \, di} \right) \left(\Delta \hat{R}_{t+1} - (R-1) \pi_{cpi,t+1}\right)$$
$$+ \left(\int \varphi(i) \frac{\frac{e(i)}{E} (s_k(i) - \bar{s}_k) W n(i)}{\sigma e(i) + \psi W n(i)} di + \frac{\int (1-\varphi(i)) \frac{e(i)}{E} (s_k(i) - \bar{s}_k) \, di \int (1-\varphi(i)) W n(i) di}{\int (1-\varphi(i)) (\sigma e(i) + \psi W n(i)) \, di} \right) \sum \bar{s}_l \Delta \tilde{A}_{l,t+1}$$
$$- \sum \left(\int \varphi(i) \frac{\frac{e(i)}{E} (s_k(i) - \bar{s}_k) e(i) (s_l(i) - \bar{s}_l)}{\sigma e(i) + \psi W n(i)} di + \frac{\int (1-\varphi(i)) \frac{e(i)}{E} (s_k(i) - \bar{s}_k) \, di \int (1-\varphi(i)) e(i) (s_l(i) - \bar{s}_l)}{\int (1-\varphi(i)) (\sigma e(i) + \psi W n(i)) \, di} \right) \pi_{l,t+1}.$$

$$A^0_{e_k,t-1} - \frac{1}{(1-\delta) R} A^0_{e_k,t} + \left(1 - \frac{1}{R}\right) A_{e_k,t-1} = \left(1 - \frac{1}{R}\right) \int \frac{e(i)}{E} (s_k(i) - \bar{s}_k) g(i) \frac{b(i)}{RE} di \left(\hat{R}_{t-1} - \pi_{cpi,t}\right)$$
$$- \left(1 - \frac{1}{R}\right) \sum_l \int \frac{e(i)}{E} (s_k(i) - \bar{s}_k) g(i) \frac{e(i)}{E} (s_l(i) - \bar{s}_l) di \hat{P}_{l,t-1}$$
$$+ \left(1 - \frac{1}{R}\right) \int \frac{e(i)}{E} (s_k(i) - \bar{s}_k) g(i) \frac{W n(i)}{W N} di \sum_l \bar{s}_l \tilde{A}_{l,t-1}$$
$$- \left(1 - \frac{1}{R}\right) \sum_l \int \frac{e(i) (s_k(i) - \bar{s}_k)}{E} \frac{\sigma e(i) \partial_e e_l(i)}{\sigma e(i) + \psi W n(i)} di \tilde{A}_{l,t-1}.$$

$$\mathcal{A}_{b,t} - \frac{1}{R} \mathcal{A}_{b,t+1} = \left(1 - \frac{1}{R}\right) \int \frac{b(i)}{E} g(i) \frac{b(i)}{RE} di \left(\hat{R}_t - \pi_{cpi,t+1}\right)$$
$$- \left(1 - \frac{1}{R}\right) \sum_k \int \frac{b(i)}{E} g(i) \frac{e(i)}{E} (s_k(i) - \bar{s}_k) di \hat{P}_{k,t}$$
$$+ \left(1 - \frac{1}{R}\right) \int \frac{b(i)}{E} g(i) \frac{W n(i)}{W N} di \sum_k \bar{s}_k \tilde{A}_{k,t}$$
$$- \left(1 - \frac{1}{R}\right) \sum_k \int \frac{b(i)}{E} \frac{\sigma e(i) \partial_e e_k(i)}{\sigma e(i) + \psi W n(i)} di \tilde{A}_{k,t}.$$

F.2 Optimal policy: proof of analytical results in Section 5

In this section, we provide the proofs for section 5.

F.2.1 Proposition 4

Proposition 4 (simplified optimal policy problem). *We provide a slightly more general result than in the main text. We show that under (A.1) and if $b(i) = 0$, optimal policy attempts to jointly stabilize the output gap $\tilde{\mathcal{Y}}_t$ and an inflation index $\pi_t^\theta \equiv \sum_{k=1}^K \frac{\bar{s}_k \vartheta_k}{\vartheta} \pi_{k,t}$ (with $\vartheta = \sum_{k=1}^K \bar{s}_k \vartheta_k$), in the sense that optimal policy can equivalently be derived by solving:*

$$\inf_{\{\tilde{\mathcal{Y}}_t, \pi_t^\theta\}_{t\geq 0}} \mathbb{E}_0 \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\frac{\sigma + \psi}{\sigma \psi} \tilde{\mathcal{Y}}_t^2 + \vartheta \left(\pi_t^\theta \right)^2 \right)$$

$$\text{s.t.} \quad \mathbb{E}_t \pi_{t+1}^\theta - R\pi_t^\theta = -R\lambda \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t - R\lambda \hat{\mathcal{W}}_t^\theta$$

where, denoting $\tilde{P}_{k,t} = \hat{P}_{k,t} + \tilde{A}_{k,t}$, $\hat{\mathcal{W}}_t^\theta = \sum_{k=1}^K \left(\overline{\partial_e e_k} - \frac{\bar{s}_k \vartheta_k}{\vartheta} \right) \tilde{P}_{k,t}$, and $\hat{\mathcal{W}}_t^\theta$ is independent from monetary policy. If ϵ_k are equal across sectors, then π_t^θ is simply the CPI index and $\hat{\mathcal{W}}_t^\theta = \mathcal{N}\mathcal{H}_t$. Under the optimal policy, we have $\tilde{\mathcal{Y}}_0 = -\lambda \vartheta \pi_0^\theta$, and $\tilde{\mathcal{Y}}_t$ partially absorbs the wedge: if $\hat{\mathcal{W}}_t^\theta \geq 0$ at all t then $\tilde{\mathcal{Y}}_t \leq 0$ at all t. In addition, when ϑ goes to infinity, keeping all other parameters fixed, we have $\tilde{\mathcal{Y}}_t = -\frac{\lambda^\theta}{\kappa} \hat{\mathcal{W}}_t^\theta$ and $\pi_t^\theta = 0$: the output gap fully absorbs the wedge. Inversely, when ϑ goes to 0, $\tilde{\mathcal{Y}}_t = 0$: the inflation index fully absorbs the wedge.

Proof. Note that if $b(i) = 0$, we have $A_{b,t} = A_{b,t}^0 = 0$ for all t, and since $e(i) = Wn(i)$, $Z_t = Z_t^0 = 0$ for all t. Defining $\check{\mu}_t \equiv \sum_{k=1}^K \check{\mu}_{k,t}$, we can rewrite the Labour Market equation as:

$$\lambda \check{\mu}_t = -\tilde{\mathcal{Y}}_t.$$

The system of price resetting equations becomes

$$M_{k,t} = \check{\mu}_{k,t} - (1-\delta)\check{\mu}_{k,t-1},$$

$$\begin{aligned} \beta \mathbb{E}_t M_{k,t+1} - M_{k,t} = & -\bar{s}_k \vartheta_k \left(\pi_{k,t} - \beta \mathbb{E}_t \pi_{k,t+1} \right) \\ & + \lambda_k \check{\mu}_{k,t} - \overline{\partial_e e_k} \sum_{l=1}^K \lambda_l \check{\mu}_{l,t} + \overline{\partial_e e_k} \lambda \sum_{l=1}^K \check{\mu}_{l,t} + \overline{\partial_e e_k} \tilde{\mathcal{Y}}_t - L_{k,t} + A_{e_k,t} \\ & + \sum_l \bar{s}_l \bar{\rho}_{l,k} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} \right) - \sigma \sum_l \left(\int \frac{e(i)}{E} \partial_e e_k(i) \partial_e e_l(i) \, di - \overline{\partial_e e_k} \overline{\partial_e e_l} \right) \left(\tilde{A}_{l,t} + \hat{P}_{l,t} \right). \end{aligned}$$

Note that we have:

$$\sum_{k=1}^K L_{k,t} = Z_t,$$

$$\sum_{k=1}^K e_l(i) \rho_{l,k}(i) = \sum_{k=1}^K \bar{s}_l \bar{\rho}_{l,k} = 0,$$

$$\sum_{k=1}^K s_k(i) = \sum_{k=1}^K \bar{s}_k = \sum_{k=1}^K \overline{\partial_e e_k} = \sum_{k=1}^K \partial_e e_k(i) = 1.$$

We therefore have $\sum_{k=1}^K L_{k,t} = \sum_{k=1}^K A_{e_k,t} = \sum_{k=1}^K A_{e_k,t}^0 = 0$. Defining $M_t \equiv \sum_{k=1}^K M_{k,t}$, we have

$$M_t = \check{\mu}_t - (1-\delta)\check{\mu}_{t-1},$$

$$\beta \mathbb{E}_t M_{k,t+1} - M_{k,t} = -\sum_{k=1}^K \bar{s}_k \vartheta_k \left(\pi_{k,t} - \beta \mathbb{E}_t \pi_{k,t+1} \right).$$

Defining;

$$\vartheta \equiv \sum_{k=1}^{K} \bar{s}_k \vartheta_k,$$
$$\pi_t^{\theta} \equiv \sum_{k=1}^{K} \frac{\bar{s}_k \vartheta_k}{\vartheta} \pi_{k,t},$$
$$\hat{\mathcal{W}}_t^{\theta} \equiv \sum_{k=1}^{K} \left(\overline{\partial_e e_k} - \frac{\bar{s}_k \vartheta_k}{\vartheta} \right) \tilde{P}_{k,t},$$

the evolution of the output gap under optimal policy is determined by:

$$\tilde{\mathcal{Y}}_t = -\lambda \check{\mu}_t,$$
$$\check{\mu}_t - (1-\delta)\,\check{\mu}_{t-1} = \vartheta \pi_t^{\theta},$$
$$\mathbb{E}_t \pi_{t+1}^{\theta} - R\pi_t^{\theta} = -R\lambda \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t - R\lambda \hat{\mathcal{W}}_t^{\theta}.$$

Note that we would obtain the same system of equations if the central bank were instead to solve:

$$\inf_{\{\tilde{\mathcal{Y}}_t, \pi_t^{\theta}\}_{t\geq 0}} \quad \mathbb{E}_0 \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\frac{\sigma+\psi}{\sigma\psi} \tilde{\mathcal{Y}}_t^2 + \vartheta \left(\pi_t^{\theta}\right)^2 \right)$$
$$s.t. \quad \mathbb{E}_t \pi_{t+1}^{\theta} - R\pi_t^{\theta} = -R\lambda \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t - R\lambda \hat{\mathcal{W}}_t^{\theta}.$$

In the special case in which $\bar{\epsilon}_k$ are common across sectors, we obtain the problem stated in Proposition 4. Denoting by $\beta^t \check{\check{\mu}}_t$ the Lagrange multiplier on the NKPC, the first-order conditions are:

$$\tilde{\mathcal{Y}}_t = -R\lambda \check{\check{\mu}}_t,$$
$$\vartheta \pi_t^{\theta} = R\check{\check{\mu}}_t - \beta^{-1} \check{\check{\mu}}_{t-1}.$$

Redefining $\check{\mu}_t = R\check{\check{\mu}}_t$ we obtain:

$$\tilde{\mathcal{Y}}_t = -\lambda \check{\mu}_t,$$
$$\check{\mu}_t - (1-\delta)\,\check{\mu}_{t-1} = \vartheta \pi_t^{\theta},$$
$$\mathbb{E}_t \pi_{t+1}^{\theta} - R\pi_t^{\theta} = -R\lambda \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t - R\lambda \hat{\mathcal{W}}_t^{\theta},$$

which is the same system.
Note that under (**A.1**) the wedge $\hat{\mathcal{W}}_t^{\theta}$ evolves independently of monetary policy. The OP system can be rewritten as

$$\mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \left((1-\delta) + R \left(1 + \frac{\sigma\psi}{\sigma+\psi} \vartheta \kappa^2 \right) \right) \tilde{\mathcal{Y}}_t - (1-\delta)\, R \tilde{\mathcal{Y}}_{t-1} = R\lambda^2 \vartheta \hat{\mathcal{W}}_t^{\theta}.$$

Defining

$$\mu_{\pm} \equiv \frac{(1-\delta) + R\left(1 + \frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa^2\right) \pm \sqrt{\left((1-\delta) + R\left(1 + \frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa^2\right)\right)^2 - 4\,(1-\delta)\,R}}{2},$$

and noting that we have $0 < \mu_- < 1 - \delta < R < \mu_+$, we have

$$\tilde{\mathcal{Y}}_t = -\mathbb{E}_t \frac{\lambda^{\theta} \frac{\sigma\psi}{\sigma+\psi} \vartheta \kappa}{(1-\delta)} \sum_{s=0}^{t} \mu_-^{t+1-s} \sum_{u=0}^{+\infty} \mu_+^{-u} \hat{\mathcal{W}}_{s+u}^{\theta}.$$

We directly obtain that if $\hat{\mathcal{W}}_t^{\theta} \gtreqless 0$ for all t then $\tilde{\mathcal{Y}}_t \lesseqgtr 0$ for all t. In addition we have $\lim_{\vartheta\to\infty} \mu_+^{-1} = \lim_{\vartheta\to\infty} \mu_- = 0$ and $\mu_- = (1-\delta) / \left(\frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa\right) + o(1/\vartheta)$, so as ϑ goes to infinity keeping all other parameters fixed, we have

$$\tilde{\mathcal{Y}}_t = -\frac{\lambda}{\kappa}\hat{\mathcal{W}}_t^{\theta}.$$

Inversely when ϑ goes to 0, the output gap goes to 0 and π_t^{θ} fully absorbs the wedge $\hat{\mathcal{W}}_t^{\theta}$.□

F.2.2 Proposition 5

Proposition 5 (signs of responses under OP). *In addition to (**A.1**) and $b(i) = 0$, we now assume that sectoral shocks in k vanish geometrically $\hat{A}_{k,t} = \rho_a^t \hat{A}_{k,0}$. We derive analytical formulas for the evolution of $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t}$ and π_t^{θ} and characterize their sign. First note that for aggregate shocks, we have $\tilde{\mathcal{Y}}_t = \pi_{mcpi,t} = \pi_t^{\theta} = 0$. If $\overline{\partial_e e_k} < \frac{\bar{s}_k\vartheta_k}{\vartheta}$ (note that if ϑ_k are equal across sector the condition simply characterize necessity), following a negative shock in sector k, $\tilde{\mathcal{Y}}_t$ is negative on impact and there t^* such that for $t \geq t^*$, $\tilde{\mathcal{Y}}_t$ is positive. $\pi_{mcpi,t}$ is negative on impact and there t^* such that for $t \geq t^*$, $\pi_{mcpi,t}$ is positive. π_t^{θ} is positive on impact and if δ is small enough there exists t^* such that for $t \geq t^*$, π_t^{θ} is positive. In net present value term, we have $\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t, \sum_{t\geq 0}\frac{1}{R^t}\pi_{mcpi,t} > 0$ and$\sum_{t\geq 0}\frac{1}{R^t}\pi_t^{\theta} < 0$ following a negative shock in k with $\overline{\partial_e e_k} < \frac{\bar{s}_k\vartheta_k}{\vartheta}$.*

Proof. We have that the exogenous wedge is given by:

$$\hat{\mathcal{W}}_t^{\theta} = \sum_{k=1}^{K}\left(\overline{\partial_e e_k} - \frac{\bar{s}_k\vartheta_k}{\vartheta}\right)\left(\tilde{A}_{k,t} + \hat{P}_{k,t}\right).$$

Denoting $P_t^{\Delta} = \sum_{k=1}^{K}\left(\overline{\partial_e e_k} - \frac{\bar{s}_k\vartheta_k}{\vartheta}\right)\hat{P}_{k,t}$, $A_t^{\Delta} = \sum_{k=1}^{K}\left(\overline{\partial_e e_k} - \frac{\bar{s}_k\vartheta_k}{\vartheta}\right)\hat{A}_{k,t}$, The relative price P_t^{Δ} satisfies

$$\mathbb{E}_t\hat{P}_{t+1}^{\Delta} - (1 + R(1+\lambda))\hat{P}_t^{\Delta} + R\hat{P}_t^{\Delta} = R\lambda\rho_a^t\hat{A}_0^{\Delta}$$

Denoting the roots of this polynomial as $\nu_{\pm}$, we have

$$\nu_{\pm} = \frac{1 + R(1+\lambda) \pm \sqrt{(1 + R(1+\lambda))^2 - 4R}}{2},$$

with $0 < \nu_- < 1 < R < \nu_+$. The solution to $\hat{P}_t^{\Delta}$ is given by:

$$P_t^{\Delta} = -\frac{R\lambda}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left(\nu_-^{t+1} - \rho_a^{t+1}\right)\hat{A}_0^{\Delta}.$$

P_t^{Δ} is independent of policy and always has the same sign as $-\hat{A}_{k,0}$. The wedge is then given by:

$$\hat{\mathcal{W}}_t^{\theta} = -\frac{1}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left((R-\nu_-)(1-\nu_-)\nu_-^t - (R-\rho_a)(1-\rho_a)\rho_a^t\right)\hat{A}_0^{\Delta}.$$

Noting that $(R-x)(1-x)$ is positive and decreasing on $[0,1]$, we conclude that the wedge (independently of policy) initially has the same sign as $\hat{A}_0^{\Delta}$ for $t < t^*$ (with t^* the smallest t such that $(R-\rho_a)(1-\rho_a)\rho_a^t > (R-\nu_-)(1-\nu_-)\nu_-^t$ if $\rho_a > \nu_-$, or $(R-\rho_a)(1-\rho_a)\rho_a^t < (R-\nu_-)(1-\nu_-)\nu_-^t$ if $\rho_a < \nu_-$) and has the same sign as $-\hat{A}_{k,0}$ for $t \geq t^*$. Note that $t^* = 1$for transitory shocks, $t^* = \infty$ for permanent shocks.

Plugging this formula in our general expression for the output gap and using the NKPC for the indices π_t^{θ} and $\pi_{mcpi,t}$, and the definition of the nominal interest rate, we obtain:

$$\tilde{\mathcal{Y}}_t = \frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \left\{ \frac{(R-\rho_a)(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \left\{ \rho_a^{t+1} - \mu_-^{t+1} \right\} - \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \left\{ \nu_-^{t+1} - \mu_-^{t+1} \right\} \right\} \hat{A}_0^\Delta,$$

$$\pi_{mcpi,t} = \frac{(R\lambda)^2\vartheta\kappa}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \left\{ \frac{(1-\rho_a)(R-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \left\{ \frac{\rho_a}{R-\rho_a}\rho_a^t - \frac{\mu_-}{R-\mu_-}\mu_-^t \right\} - \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \left\{ \frac{\nu_-}{R-\nu_-}\nu_-^t - \frac{\mu_-}{R-\mu_-}\mu_-^t \right\} \right\} \hat{A}_0^\Delta,$$

$$\pi_t^\theta = -\frac{R\lambda}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \left\{ \frac{(R-\rho_a)(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \left\{ (\rho_a - (1-\delta))\rho_a^t - (\mu_- - (1-\delta))\mu_-^t \right\} - \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \left\{ (\nu_- - (1-\delta))\nu_-^t - (\mu_- - (1-\delta))\mu_-^t \right\} \right\} \hat{A}_0^\Delta,$$

$$\begin{aligned}
\hat{R}_t = & -\frac{1+\psi}{\sigma+\psi}(1-\rho_a)\rho_a^t \sum_k \bar{s}_k A_{k,0} - \frac{\psi}{\sigma+\psi}(1-\rho_a)\rho_a^t \hat{A}_0^\Delta \\
& - \frac{1}{\sigma}\frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \left\{ \frac{(R-\rho_a)(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \left\{ (1-\rho_a)\rho_a^{t+1} - (1-\mu_-)\mu_-^{t+1} \right\} - \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \left\{ (1-\nu_-)\nu_-^{t+1} - (1-\mu_-)\mu_-^{t+1} \right\} \right\} \hat{A}_0^\Delta \\
& + \frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \left\{ \frac{(1-\rho_a)(R-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \left\{ \frac{R\kappa\rho_a}{R-\rho_a}\rho_a^{t+1} - \frac{R\kappa\mu_-}{R-\mu_-}\mu_-^{t+1} \right\} - \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \left\{ \frac{R\kappa\nu_-}{R-\nu_-}\nu_-^{t+1} - \frac{R\kappa\mu_-}{R-\mu_-}\mu_-^{t+1} \right\} \right\} \hat{A}_0^\Delta.
\end{aligned}$$

For aggregate shocks we have $\hat{A}_0^\Delta = 0$, so $\tilde{\mathcal{Y}}_t = \pi_{mcpi,t} = \pi_t^\theta = 0$.

Evaluating these expressions on impact, and simplifying, we obtain:

$$\tilde{\mathcal{Y}}_0 = -\frac{R\lambda^2\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \left(-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1) \right) \hat{A}_0^\Delta,$$

$$\pi_{mcpi,0} = -\frac{(R\lambda)^2\vartheta\kappa}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{R}{R-\mu_-}(\mu_+ - 1)\hat{A}_0^\Delta,$$

$$\pi_0^\theta = -\frac{R\lambda}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \left(-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1) \right) \hat{A}_0^\Delta.$$

Note that since $\rho_a, \nu_- < 1$, $\mu_+ > R$, we have:

$$\begin{aligned}
-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1) &= \mu_+(R + 1 - (\rho_a + \nu_-)) - R + \rho_a\nu_- \\
&\geq R(R + 1 - (\rho_a + \nu_-)) - R + \rho_a\nu_- \\
&= (R - \rho_a)(R - \nu_-) \geq 0
\end{aligned}$$

$\tilde{\mathcal{Y}}_0, \pi_{mcpi,0} \gtrless 0$ and $\pi_0^\theta \lessgtr 0$ if $\hat{A}_0^\Delta \lessgtr 0$. In addition, $\tilde{\mathcal{Y}}_0 = -\lambda\vartheta\pi_0^\theta$. Note in particular that if $\epsilon_k = \epsilon$ across sector, we have $\pi_t^\theta = \pi_{cpi,t}$ and $\hat{A}_t^\Delta = \sum_{k=1}^K \left(\overline{\partial_e e}_k - \bar{s}_k \right) \hat{A}_{k,t}$: $\hat{A}_t^\Delta$ is positive (negative) for negative shocks in necessity (luxury) sectors.

In the medium run, the behavior of $\tilde{\mathcal{Y}}_t, \pi_t^\theta, \pi_{mcpi,t}$ a priori depends on which of the parameters ρ_a, μ_- or ν_- dominates. If $\rho_a > \mu_-, \nu_-$, we have:

$$\tilde{\mathcal{Y}}_t = \frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(R-\rho_a)(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \rho_a^{t+1}\hat{A}_0^\Delta + o\left(\rho_a^t\right),$$

$$\pi_{mcpi,t} = \frac{(R\lambda)^2\vartheta\kappa}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \rho_a^{t+1}\hat{A}_0^\Delta + o\left(\rho_a^t\right),$$

$$\pi_t^\theta = -\frac{R\lambda}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(R-\rho_a)(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} (\rho_a - (1-\delta))\rho_a^t\hat{A}_0^\Delta + o\left(\rho_a^t\right).$$

for t large enough we have $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t} \gtrless 0$ if $\hat{A}_0^\Delta \gtrless 0$. $\pi_t^\theta \gtrless 0$ ($\pi_t^\theta \lessgtr 0$) if $\hat{A}_0^\Delta \gtrless 0$ and $\rho_a < (1-\delta)$ ($\rho_a > (1-\delta)$). Similarly, if $\nu_- > \mu_-, \rho_a$, we have:

$$\tilde{\mathcal{Y}}_t = -\frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)}\nu_-^{t+1}\hat{A}_0^\Delta + o(\nu_-^t),$$
$$\pi_{mcpi,t} = -\frac{(R\lambda)^2\vartheta\kappa}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\frac{(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)}\nu_-^{t+1}\hat{A}_0^\Delta + o(\nu_-^t),$$
$$\pi_t^\theta = \frac{R\lambda}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)}(\nu_- - (1-\delta))\nu_-^t\hat{A}_0^\Delta + o(\nu_-^t).$$

for t large enough we have $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t} \gtrless 0$ if $\hat{A}_0^\Delta \gtrless 0$. $\pi_t^\theta \gtrless 0$ ($\pi_t^\theta \lesseqgtr 0$) if $\hat{A}_0^\Delta \gtrless 0$ and $\nu_- < (1-\delta)$ ($\nu_- > (1-\delta)$). Finally, if $\mu_- > \nu_-, \rho_a$:

$$\tilde{\mathcal{Y}}_t = \frac{R\lambda^2\vartheta}{(\nu_+ - \rho_a)}\frac{\lambda^2\vartheta R(R+\rho_a\nu_-) + \delta(R-\rho_a)(R-\nu_-)}{(\rho_a - \mu_+)(\rho_a - \mu_-)(\nu_- - \mu_+)(\nu_- - \mu_-)}\mu_-^{t+1}\hat{A}_0^\Delta + o(\mu_-^t),$$
$$\pi_{mcpi,t} = \frac{(R\lambda)^2\vartheta\kappa}{(\nu_+ - \rho_a)}\frac{\lambda^2\vartheta R(R+\rho_a\nu_-) + \delta(R-\rho_a)(R-\nu_-)}{(\rho_a - \mu_+)(\rho_a - \mu_-)(\nu_- - \mu_+)(\nu_- - \mu_-)}\frac{1}{R-\mu_-}\mu_-^{t+1}\hat{A}_0^\Delta + o(\mu_-^t),$$
$$\pi_t^\theta = -\frac{R\lambda}{(\nu_+ - \rho_a)}\frac{\lambda^2\vartheta R(R+\rho_a\nu_-) + \delta(R-\rho_a)(R-\nu_-)}{(\rho_a - \mu_+)(\rho_a - \mu_-)(\nu_- - \mu_+)(\nu_- - \mu_-)}(\mu_- - (1-\delta))\mu_-^t\hat{A}_0^\Delta + o(\mu_-^t).$$

Recall that $\mu_- < 1-\delta$ so we have $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t}, \pi_t^\theta \gtrless 0$ if $\hat{A}_0^\Delta \gtrless 0$.

Finally we derive the net present value of $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t}, \pi_t^\theta$ under optimal policy. We have:

$$\mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t = -\frac{(R\lambda)^2\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)}\frac{R(\mu_+ - 1)}{R-\mu_-}\hat{A}_0^\Delta,$$
$$\mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\pi_{mcpi,t} = -\frac{(R\lambda)^2\vartheta\kappa}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)}\frac{R}{(R-\mu_-)^2(R-\rho_a)(R-\nu_-)}\left\{R^2(\mu_+(R-\delta)-R) + \delta R^2(\rho_a+\nu_-) + \left(\mu_-(R-1)+\delta R^2\right)\rho_a\nu_-\right\},$$
$$\mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\pi_t^\theta = \frac{R^2\lambda(R-(1-\delta))}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)}\frac{\mu_+ - 1}{R-\mu_-}\hat{A}_0^\Delta.$$

Note that $\mu_+ > R$, $0 \leq \rho_a, \nu_- \leq 1$ and as $\beta(1-\delta)R = 1$, $R-\delta > 1$ so $R^2(\mu_+(R-\delta)-R) + \delta R^2(\rho_a+\nu_-) + \left(\mu_-(R-1)+\delta R^2\right)\rho_a\nu_- > 0$. We therefore have that $\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t, \sum_{t\geq 0}\frac{1}{R^t}\pi_{mcpi,t} \gtrless 0$ and $\sum_{t\geq 0}\frac{1}{R^t}\pi_t^\theta \lesseqgtr 0$ if $\hat{A}_0^\Delta \lesseqgtr 0$.□

F.2.3 Proposition 6

Proposition 6 (optimal policy versus inflation targeting). *We show that following a negative productivity shock in a necessity sector, optimal policy is relatively loose compared to a strict CPI targeting policy, both initially and in present value terms.*

Proof. Under the assumption that ϵ_k are equal across sector then $\pi_t^\theta = \pi_{cpi,t}$. Using the results of the previous subsection, we have:

$$\begin{aligned}
\tilde{\mathcal{Y}}_0 &= -\frac{R\lambda\lambda\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)}\left(-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1)\right)\hat{A}_0^\Delta, \\
\mathbb{E}_0 \sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t &= -\frac{R^2\lambda\lambda\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)}\frac{R(\mu_+ - 1)}{R - \mu_-}\hat{A}_0^\Delta, \\
\pi_{cpi,0} &= -\frac{R\lambda}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)}\left(-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1)\right)\hat{A}_0^\Delta, \\
\mathbb{E}_0 \sum_{t\geq 0}\frac{1}{R^t}\pi_{cpi,t} &= \frac{R^2\lambda(R-(1-\delta))}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)}\frac{\mu_+ - 1}{R - \mu_-}\hat{A}_0^\Delta.
\end{aligned}$$

Note that using $(\mu_+ - 1)(\mu_+ - R) - R\lambda\vartheta\kappa\mu_+ = 0$ and $-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1) \geq 0$ we have:

$$\begin{aligned}
\tilde{\mathcal{Y}}_0 &= -\frac{\lambda}{\kappa}\frac{1}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(\mu_+ - R)\left(-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1)\right)}{(\mu_+ - \nu_-)\mu_+}\hat{A}_0^\Delta, \\
\left|\tilde{\mathcal{Y}}_0\right| &= \frac{\lambda}{\kappa}\frac{1}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(\mu_+ - R)\left(-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1)\right)}{(\mu_+ - \nu_-)\mu_+}\left|\hat{A}_0^\Delta\right| \\
&< \frac{\lambda}{\kappa}\frac{1}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(\mu_+ - R)\left(-(\rho_a + \nu_-) + (R+1)\right)\mu_+}{(\mu_+ - \nu_-)\mu_+}\left|\hat{A}_0^\Delta\right| \\
&= \frac{\lambda}{\kappa}\frac{R + 1 - (\rho_a + \nu_-)}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(\mu_+ - R)}{(\mu_+ - \nu_-)}\left|\hat{A}_0^\Delta\right| \\
&\leq \frac{\lambda}{\kappa}\frac{R + 1 - (\rho_a + \nu_-)}{(\nu_+ - \rho_a)}\left|\hat{A}_0^\Delta\right|,
\end{aligned}$$

where the last line uses the fact that $\rho_a, \nu_- \leq 1 < R$. Similarly, using $(\mu_- - 1)(\mu_- - R) - R\lambda\vartheta\kappa\mu_- = 0$, we can show that:

$$\begin{aligned}
\mathbb{E}_0 \sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t &= -\frac{\lambda}{\kappa}\frac{R}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(1 - \mu_-)}{(\mu_+ - \nu_-)\mu_-}\hat{A}_0^\Delta, \\
&= -\frac{\lambda}{\kappa}\frac{R}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(1 - \mu_-)}{\left(\frac{1}{\beta} - \nu_-\mu_-\right)}\hat{A}_0^\Delta, \\
\mathbb{E}_0\left|\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t\right| &< \frac{\lambda}{\kappa}\frac{R}{(\nu_+ - \rho_a)}\left|\hat{A}_0^\Delta\right|.
\end{aligned}$$

where the second line uses $\mu_+\mu_- = R(1-\delta) = 1/\beta$ and the last line uses $\frac{1}{\beta} - \nu_-\mu_- \geq 1 - \nu_-\mu_- \geq 1 - \mu_-$ and $\rho_a \leq 1$.
Under strict CPI targeting we have $\pi_{cpi,t} = 0$ at all dates, and

$$\tilde{\mathcal{Y}}_t = -\frac{\lambda}{\kappa}\mathcal{N}\mathcal{H}_t = -\frac{1}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left((R - \rho_a)(1 - \rho_a)\rho_a^t - (R - \nu_-)(1 - \nu_-)\nu_-^t\right)\hat{A}_0^\Delta.$$

So under strict CPI targeting:

$$\begin{aligned}
\tilde{\mathcal{Y}}_0 &= --\frac{\lambda}{\kappa}\frac{1}{(\nu_+ - \rho_a)}\left(R + 1 - (\rho_a + \nu_-)\right)\hat{A}_0^\Delta, \\
\mathbb{E}_0 \sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t &= -\frac{\lambda}{\kappa}\frac{R}{(\nu_+ - \rho_a)}\hat{A}_0^\Delta.
\end{aligned}$$

Denoting with a superscript CPI the variables under CPI targeting, OP the variables under optimal policy we therefore have after a negative shock in a necessity sector:

$$\tilde{\mathcal{Y}}_0^{CPI} < \tilde{\mathcal{Y}}_t^{OP} < 0,$$
$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t^{CPI} < \mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t^{OP} < 0,$$
$$\pi_{cpi,0}^{OP} > \pi_{cpi,0}^{CPI} = 0,$$
$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \pi_{cpi,t}^{OP} > \mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \pi_{cpi,t}^{CPI} = 0.$$

Monetary policy is more accommodative after a negative shock in a necessity sector than strict inflation targeting. After a shock in a luxury sector, we have:

$$\tilde{\mathcal{Y}}_0^{CPI} > \tilde{\mathcal{Y}}_t^{OP} > 0,$$
$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t^{CPI} > \mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t^{OP} > 0,$$
$$\pi_{cpi,0}^{OP} < \pi_{cpi,0}^{CPI} = 0,$$
$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \pi_{cpi,t}^{OP} < \mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \pi_{cpi,t}^{CPI} = 0,$$

i.e. monetary policy is more strict.□

F.3 Welfare loss function

In this appendix, we derive the second order approximation of our social welfare function:

$$\mathcal{W} = (1-\delta)\int G\left(V_{-}(i), i\right) di + \delta \mathbb{E}_0 \sum_{t_0=0}^{\infty} \beta^{t_0} \int G\left(V_{t_0}(i), i\right) di.$$

Recall that the value function of household i born at t_0 is given by:

$$V_{t_0}(i) = \mathbb{E}_{t_0} \sum_{s=0}^{\infty} \left((1-\delta)\beta\right)^s \left\{ (1-\varphi(i)) \left[U_i\left(c^u_{1,t_0+s}(i), \ldots, c^u_{K,t_0+s}(i)\right) - \chi\left(\frac{n^u_{t_0+s}(i)}{\vartheta(i)}\right)\right] + \varphi(i)\left[U_i\left(c^{HtM}_{1,t_0+s}(i), \ldots, c^{HtM}_{K,t_0+s}(i)\right) - \chi\left(\frac{n^{HtM}_{t_0+s}(i)}{\vartheta(i)}\right)\right]\right\},$$

where the quantities $\left\{\boldsymbol{c}^u_{k,t_0+s}(i), \boldsymbol{c}^{HtM}_{1,t_0+s}(i), n^u_{t_0+s}(i), n^{Htm}_{t_0+s}(i)\right\}_{s\geq 0}$ are chosen optimally, as described in the derivation appendix. Applying Roy's identity (and using the fact that in steady state consumption and labor supply is constant across constrained and unconstrained households of the same type i) , the derivative of the value function is given by:

$$dV_{t_0}(i) = \mathbb{E}_{t_0} \sum_{s=0}^{\infty} \left((1-\delta)\beta\right)^s \partial_e v_{t_0+s} \left\{ \frac{b_{t_0+s+1}(i)}{R_{t_0+s}} \hat{R}_{t_0+s} + \hat{W}_{t_0+s} W_{t_0+s} n_{t_0+s}(i) - \sum_{l=1}^{K} c^{t_0}_{l,t_0+s}(i) \int \left(\frac{p_{l,t_0+s}(j)}{P_{l,t_0+s}}\right)^{-\epsilon_l} dp_{l,t_0+s}(j)\, dj + \varsigma(i)\, dDiv_{t_0+s}\right\}$$
$$+ (1-\varphi(i))\, \partial_e v_{t_0} b_{t_0}(i) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0} + \varphi(i)\, \mathbb{E}_{t_0} \sum_{s=0}^{\infty} \left((1-\delta)\beta\right)^s \partial_e v_{t_0+s} \left(1 - \frac{1}{R_{t_0+s}}\right) b_{t_0}(i) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0},$$

with the first order change in dividends given by:

$$dDiv_t = \sum_{k=1}^{K} \int y_k(j) \left(dp_{k,t}(j) - \frac{1}{A_{k,t}} \left(\mathcal{N}_k(\boldsymbol{P}_t, W_t) dW_t + \sum_{l=1}^{K} \mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t) dP_{l,t} - \left(\mathcal{N}_k(\boldsymbol{P}_t, W_t) W_t + \sum_{l=1}^{K} \mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t) P_{l,t} \right) \hat{A}_{k,t} \right) \right)$$
$$+ \sum_{k=1}^{K} \int dy_k(j) \left(p_{k,t}(j) - \frac{1}{A_{k,t}} \left(\mathcal{N}_k(\boldsymbol{P}_t, W_t) W_t + \sum_{l=1}^{K} \mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t) P_{l,t} \right) \right),$$

where the change in demand for variety j is given by:

$$dy_{k,t}(j) = \left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k} \mathbb{E}_{\delta,t} \left(\int -\epsilon_k \left(\frac{dp_{k,t}(j)}{p_{k,t}(j)} - \int \left(\frac{p_{k,t}(j^*)}{P_{k,t}}\right)^{1-\epsilon_k} \frac{dp_{k,t}(j^*)}{p_{k,t}(j^*)} dj^* \right) \left((1-\varphi(i))\, c^{t_0,u}_{k,t}(i) + \varphi(i)\, c^{t_0,HtM}_{k,t} \right) + \left((1-\varphi(i))\, dc^{t_0,u}_{k,t}(i) + \varphi(i)\, dc^{t_0,HtM}_{k,t} \right) di \right)$$
$$+ \left(\frac{p_{k,t}(j)}{P_{k,t}}\right)^{-\epsilon_k} \left(-\epsilon_k \left(\frac{dp_{k,t}(j)}{p_{k,t}(j)} - \int \left(\frac{p_{k,t}(j^*)}{P_{k,t}}\right)^{1-\epsilon_k} \frac{dp_{k,t}(j^*)}{p_{k,t}(j^*)} dj^* \right) \tilde{Y}_{k,t} + d\tilde{Y}_{k,t} \right),$$

and demand for intermediary $d\tilde{Y}_{k,t}$ solves the system

$$d\tilde{Y}_{k,t} = \sum_{l} \left(\partial_W \mathcal{Y}_{k,l}(\boldsymbol{P}_t, W_t) dW_t + \sum_{m=1}^{K} \partial_{P_m} \mathcal{Y}_{k,l}(\boldsymbol{P}_t, W_t) dP_{m,t} \right) Y_{k,t} + \sum_{l} \mathcal{Y}_{k,l}(\boldsymbol{P}_t, W_t) dY_{k,t},$$
$$dY_{k,t} = \int dy_{k,t}(j)\, dj.$$

Around a steady state where prices, consumption, wealth and labor supply are constant (and equal across generation, constrained and unconstrained households of the same type i) and with $p_k(j) = \frac{1}{A}\left(\mathcal{N}_k(\boldsymbol{P},W)W + \sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P},W)P_l\right)$, this simplifies to

$$
\begin{aligned}
dV_{t_0}(i) &= \partial_e v(i)\,\mathbb{E}_{t_0}\sum_{s=0}^{\infty}((1-\delta)\beta)^s\left\{\frac{b(i)}{R}\hat{R}_{t_0+s} + \hat{W}_{t_0+s}Wn(i) - \sum_{l=1}^{K}e_l(i)\hat{P}_{l,t_0+s} + \varsigma(i)\sum_{k=1}^{K}P_kY_k\left(\hat{P}_{k,t_0+s} - \frac{1}{A_kP_k}\left(W\mathcal{N}_k(\boldsymbol{P}_t,W_t)\hat{W}_{t_0+s} + \sum_{l=1}^{K}P_l\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)\hat{P}_{l,t}\right) + \hat{A}_{k,t_0+s}\right)\right\}\\
&\quad + \partial_e v(i)\,b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0},\\
&= \partial_e v(i)\,\mathbb{E}_{t_0}\sum_{s=0}^{\infty}((1-\delta)\beta)^s\left\{\frac{b(i)}{R}\left(\hat{R}_{t_0+s} - \sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right) - \sum_{l=1}^{K}e(i)(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s} + Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}\right\}.
\end{aligned}
$$

Next, we derive the second order approximation of the social welfare function. We use the fact that in steady state, prices, consumption, wealth and labor supply are constant (and equal across generation, constrained and unconstrained households of the same type i) , $G'(V_{t_0}(i),i)\,\partial_e v_{t_0} = 1$ and $p_k(j) = \frac{1}{A}\left(\mathcal{N}_k(\boldsymbol{P},W)W + \sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P},W)P_l\right)$. Using clearing of the bond market, we have:

$$
\begin{aligned}
d^2\mathcal{W} &= (1-\delta)\int G''(i)\,(dV_-(i))^2\,di + \delta\mathbb{E}_0\sum_{t_0=0}^{\infty}\beta^{t_0}\int G''(i)\,(dV_{t_0}(i))^2\,di\\
&\quad - \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{c}_t^{t_0,u} - \sum_k s_k(i)\hat{P}_{k,t}\right) + \sum_k\partial_e e_k(i)\hat{P}_{k,t}\right)\left\{\frac{b(i)}{R}\hat{R}_t - \sum_{l=1}^{K}(e(i)s_l(i) - Wn(i)\bar{s}_l)\hat{P}_{l,t} + Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t} + \mathbb{1}_{t=t_0}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}\right\}di\\
&\quad - \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{c}_t^{t_0,HtM} - \sum_k s_k(i)\hat{P}_{k,t}\right) + \sum_k\partial_e e_k(i)\hat{P}_{k,t}\right)\left\{\frac{b(i)}{R}\hat{R}_t - \sum_{l=1}^{K}(e(i)s_l(i) - Wn(i)\bar{s}_l)\hat{P}_{l,t} + Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t} + \left(1-\frac{1}{R}\right)b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}\right\}di\\
&\quad + \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{dW_tdN_t - \sum_{l=1}^{K}C_{l,t}\int d\left\{\left(\frac{p_{l,t}(j)}{P_{l,t}}\right)^{-\epsilon_l}\right\}dp_{l,t}(j)\,dj + d^2Div_t\right\} + \mathbb{E}_0\sum\beta^t\mathbb{E}_{\delta,t}\left(\int\varphi(i)\frac{dR_t}{R^2}b(i)di\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}\right),
\end{aligned}
$$

with

$$
\int d\left\{\left(\frac{p_{l,t}(j)}{P_{l,t}}\right)^{-\epsilon_l}\right\}dp_{l,t}(j)\,dj = -\epsilon_l\int\left(\frac{p_{l,t}(j)}{P_{l,t}}\right)^{-\epsilon_l}dp_{l,t}(j)\left(\hat{p}_{l,t}(j) - \hat{P}_{l,t}\right),
$$

and

$$
\begin{aligned}
d^2Div_t &= -dW_tdN_t^d - \sum_{k=1}^{K}d\tilde{Y}_{k,t}^d dP_{k,t} - \sum_{k=1}^{K}\tilde{Y}_{k,t}^d d^2P_{k,t} + \sum_{k=1}^{K}\int dy_k(j)\left(dp_{k,t}(j) + P_k\hat{A}_{k,t}\right)dj\\
&\quad + \sum_{k=1}^{K}\frac{Y_k}{A_{k,t}}\left(\mathcal{N}_k(\boldsymbol{P}_t,W_t)dW_t + \sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)dP_{l,t} - \left(\mathcal{N}_k(\boldsymbol{P}_t,W_t)W_t + \sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)P_{l,t}\right)\hat{A}_{k,t}\right)\hat{A}_{k,t}\\
&\quad + \sum_{k=1}^{K}\int dy_k(j)\left(dp_{k,t}(j) - \frac{1}{A_{k,t}}\left(\mathcal{N}_k(\boldsymbol{P}_t,W_t)dW_t + \sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)dP_{l,t}\right) + p_{k,t}(j)\,\hat{A}_{k,t}\right)dj.
\end{aligned}
$$

Simplifying – using market clearing conditions for labor and markets and properties of the steady state – and removing the terms independent of monetary policy, we have:

$$dW_t dN_t - \sum_{l=1}^{K} \int d\left\{ \left(\frac{p_{l,t}(j)}{P_{l,t}} \right)^{-\epsilon_l} \right\} dp_{l,t_0+s}(j)\, dj + d^2 Div_t = -\sum_k P_k dC_{k,t} \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t} \right)$$
$$- \left(\sum_{k,l} \frac{1}{A_l} P_k Y_l \partial_W \mathcal{Y}_{k,l} \hat{W}_t \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t} \right) + \sum_{k,l,m} \frac{1}{A_l} P_k Y_l \hat{P}_{m,t} \partial_{P_m} \mathcal{Y}_{k,l} \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t} \right) \right)$$
$$+ \hat{W}_t \sum_k E_k \tilde{\mathbf{A}}_{k,t} - \sum_{k=1}^{K} P_k Y_k \int \left(\hat{p}_{k,t}(j) - \hat{P}_{k,t} \right) \hat{p}_{k,t}(j)\, dj.$$

Next we have, noting $\hat{p}^*_{k,t}$ the reset price at t

$$\begin{aligned}
\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \int \left(\hat{p}_{k,t}(j) - \hat{P}_{k,t} \right) \hat{p}_{k,t}(j)\, dj &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{1-\theta_k}{1-\beta\theta_k} \left(\hat{p}^*_{k,t} \right)^2 - \left(\hat{P}_{k,t} \right)^2 \right) \\
&= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{(1-\beta\theta_k)(1-\theta_k)} (\pi_{k,t})^2 + 2\frac{1}{(1-\beta\theta_k)} \pi_{k,t} \hat{P}_{k,t-1} + \frac{(1-\theta_k)}{1-\beta\theta_k} \left(\hat{P}_{k,t-1} \right)^2 - \left(\hat{P}_{k,t} \right)^2 \right\} \\
&= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)} (\pi_{k,t})^2 - \frac{\theta_k}{1-\beta\theta_k} \left(\hat{P}_{k,t-1} \right)^2 + \frac{\beta\theta_k}{(1-\beta\theta_k)} \left(\hat{P}_{k,t} \right)^2 \right\} \\
&= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)} (\pi_{k,t})^2 .
\end{aligned}$$

We therefore have

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ dW_t dN_t - \sum_{l=1}^{K} \int d\left\{ \left(\frac{p_{l,t}(j)}{P_{l,t}} \right)^{-\epsilon_l} \right\} dp_{l,t}(j)\, dj + d^2 Div_t \right\} = -\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_k P_k dC_{k,t} \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t} \right)$$
$$- \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\sum_{k,l} \frac{1}{A_l} P_k Y_l W \partial_W \mathcal{Y}_{k,l} \hat{W}_t \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t} \right) + \sum_{k,l,m} \frac{1}{A_l} P_m P_k Y_l \hat{P}_{m,t} \partial_{P_m} \mathcal{Y}_{k,l} \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t} \right) \right)$$
$$+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_k E_k \hat{W}_t \tilde{\mathbf{A}}_{k,t} - P_k Y_k \frac{\theta_k \epsilon_k}{(1-\beta\theta_k)(1-\theta_k)} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\pi_{k,t})^2 .$$

Let us first slightly rewrite the terms coming from substitution in production. We have:

$$- \left(\sum_{k,l} \frac{1}{A_l} P_k Y_l W \partial_W \mathcal{Y}_{k,l} \hat{W}_t \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t} \right) + \sum_{k,l,m} \frac{1}{A_l} P_k P_m Y_l \hat{P}_{m,t} \partial_{P_m} \mathcal{Y}_{k,l} \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t} \right) \right)$$
$$- \sum \frac{1}{A_l} Y_l W^2 \partial_W \mathcal{N}_l \hat{W}_t^{\,2} - \sum_{k,l} \frac{1}{A_l} P_k Y_l W \partial_W \mathcal{Y}_{k,l} \hat{W}_t \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t} \right) - \sum_{k,l} \frac{1}{A_l} W Y_l \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t} \right) P_k \partial_{P_k} \mathcal{N}_l \hat{W}_t - \sum_{k,l,m} \frac{1}{A_l} P_m P_k Y_l \left(\hat{P}_{m,t} + \tilde{\mathbf{A}}_{m,t} \right) \partial_{P_m} \mathcal{Y}_{k,l} \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t} \right),$$

where we used

$$\begin{aligned}
\sum_k P_k \partial_W \mathcal{Y}_{k,l} &= -W \partial_W \mathcal{N}_l, \\
\sum_k P_k \partial_{P_m} \mathcal{Y}_{k,l} &= -W \partial_{P_m} \mathcal{N}_l, \\
\partial_W \mathcal{Y}_{k,l} &= \partial_{P_k} \mathcal{N}_l, \\
\partial_{P_m} \mathcal{Y}_{k,l} &= \partial_{P_k} \mathcal{Y}_{m,l}.
\end{aligned}$$

We define

$$\mathcal{N}\left(W,\boldsymbol{P}\right)\equiv\sum_{l}\frac{1}{A_{l}}Y_{l}\mathcal{N}_{l}\left(W,\boldsymbol{P}\right),$$

$$\mathcal{Y}_{k}\left(W,\boldsymbol{P}\right)=\sum_{l}\frac{1}{A_{l}}Y_{l}\mathcal{Y}_{k,l}\left(W,\boldsymbol{P}\right).$$

We have

$$-\sum\frac{1}{A_{l}}Y_{l}W\partial_{W}\mathcal{N}_{l}\hat{W}_{t}^{2}-\sum_{k,l}\frac{1}{A_{l}}P_{k}Y_{l}W\partial_{W}\mathcal{Y}_{k,l}\hat{W}_{t}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)-\sum_{k,l}\frac{1}{A_{l}}WY_{l}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)P_{k}\partial_{P_{k}}\mathcal{N}_{l}\hat{W}_{t}-\sum_{k,l,m}\frac{1}{A_{l}}P_{m}P_{k}Y_{l}\left(\hat{P}_{m,t}+\tilde{\mathbf{A}}_{m,t}\right)\partial_{P_{m}}\mathcal{Y}_{k,l}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)$$
$$=-W^{2}\partial_{W}\mathcal{N}\hat{W}_{t}^{2}-\sum_{k}P_{k}W\partial_{W}\mathcal{Y}_{k}\hat{W}_{t}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)-\sum_{k}WP_{k}\partial_{P_{k}}\mathcal{N}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)\hat{W}_{t}-\sum_{k,l}P_{l}P_{k}\partial_{P_{l}}\mathcal{Y}_{k}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)\left(\hat{P}_{l,t}+\tilde{\mathbf{A}}_{l,t}\right).$$

Note that we have

$$P_{k}dC_{k,t}=\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)P_{k}dc_{k,t}^{t_{0},u}\left(i\right)+\varphi\left(i\right)P_{k}dc_{k,t}^{t_{0},HtM}\left(i\right)\right)\right),$$

with

$$P_{k}dc_{k,t}^{t_{0}}\left(i\right)=e\left(i\right)\partial_{e}e_{k}\left(i\right)\left(\hat{e}_{t}^{t_{0}}\left(i\right)-\sum_{k}s_{k}\left(i\right)\hat{P}_{k,t}\right)+e_{k}\left(i\right)\sum_{l}\rho_{k,l}\left(i\right)\hat{P}_{l,t}.$$

So rearranging, we obtain:

$$P_{k}dC_{k,t}\left(\hat{W}_{t}-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)=\mathbb{E}_{\delta,t}\int\frac{1}{\sigma}\left(\hat{e}_{t}^{t_{0}}\left(i\right)-\sum_{k}s_{k}\left(i\right)\hat{P}_{k,t}\right)\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)di\left(\hat{W}_{t}-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)$$
$$-\sum_{k}E_{k}\sum_{l}\bar{\rho}_{k,l}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right).$$

So we can rewrite the welfare loss as

$$
\begin{aligned}
d^2\mathcal{W} = (1-\delta)\int G''(i)\,(dV_-(i))^2\,di + \delta\mathbb{E}_0\sum_{t_0=0}^{\infty}\beta^{t_0}\int G''(i)\,(dV_{t_0}(i))^2\,di \\
&- \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}-\sum_k s_k(i)\,\hat{P}_{k,t}\right)+\sum_k\partial_e e_k(i)\,\hat{P}_{k,t}\right)\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e(i)\,s_l(i)-Wn(i)\,\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di \\
&- \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}-\sum_k s_k(i)\,\hat{P}_{k,t}\right)+\sum_k\partial_e e_k(i)\,\hat{P}_{k,t}\right)\left\{\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\,\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di \\
&- \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}-\sum_k s_k(i)\,\hat{P}_{k,t}\right)+\sum_k\partial_e e_k(i)\,\hat{P}_{k,t}\right)\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e(i)\,s_l(i)-Wn(i)\,\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di \\
&- \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}-\sum_k s_k(i)\,\hat{P}_{k,t}\right)+\sum_k\partial_e e_k(i)\,\hat{P}_{k,t}\right)\left\{\left(1-\frac{1}{R}\right)b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\,\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di \\
&+ \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{k,l}^{K}\int\sigma e(i)\,\partial_e e_k(i)\,\partial_e e_l(i)\,di\hat{P}_{l,t}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right) \\
&- \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(-W^2\partial_W\mathcal{N}\hat{W}_t^{\,2}-\sum_k P_kW\partial_W\mathcal{Y}_k\hat{W}_t\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)-\sum_k WP_k\partial_{P_k}\mathcal{N}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)\hat{W}_t-\sum_{k,l}P_lP_k\partial_{P_l}\mathcal{Y}_k\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)\left(\hat{P}_{l,t}+\tilde{\mathbf{A}}_{l,t}\right)\right) \\
&+ \sum_k E_k\sum_l\bar{\rho}_{k,l}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_k E_k\hat{W}_t\tilde{\mathbf{A}}_{k,t}-\sum_k P_kY_k\frac{\theta_k\bar{\epsilon}_k}{(1-\beta\theta_k)(1-\theta_k)}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t(\pi_{k,t})^2 \\
&+ \mathbb{E}_0\sum\beta^t\mathbb{E}_{\delta,t}\left(\frac{dR_t}{R^2}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}\right).
\end{aligned}
$$

Simplification of the terms corresponding to the expenditure of unconstrained households

Here we simplify the second line of the expression. To do so, define

$$
\begin{aligned}
X_t \equiv{}& \frac{1}{\int(1-\varphi(i))\left(\sigma\frac{e(i)}{E}+\frac{Wn(i)}{WN}\psi\right)di}\int(1-\varphi(i))\left\{\frac{Wn(i)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn(i)}{WN}\psi\hat{W}_t+\sum_l\sigma\frac{e(i)}{E}\partial_e e_l(i)\,di\hat{P}_{l,t}\right\}di \\
&-\frac{1}{\int(1-\varphi(i))\left(\sigma\frac{e(i)}{E}+\frac{Wn(i)}{WN}\psi\right)di}\int\varphi(i)\left\{\frac{b(i)}{ER}\left(\hat{R}_t-(R-1)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t}\right\}.
\end{aligned}
$$

X_t is an aggregate variable growing at rate $\hat{R}_t$. Indeed, recall that we have, from the labor market clearing condition:

$$
\begin{aligned}
\sum_l\bar{s}_l\tilde{A}_{l,t}+\psi\left(\hat{W}_t-\sum_l\int\frac{Wn(i)}{WN}\partial_e e_l(i)\,\hat{P}_{l,t}\right) &= \mathbb{E}_{\delta,t}\left(\frac{e(i)}{E}\left(1+\frac{Wn(i)\,\psi}{e(i)\,\sigma}\right)\left((1-\varphi(i))\left(\hat{e}_{t_0+s}^{t_0,u}-\sum_k s_k(i)\,\hat{P}_{k,t_0+s}\right)+\varphi(i)\left(\hat{e}_{t_0+s}^{t_0,HtM}-\sum_k s_k(i)\,\hat{P}_{k,t_0}\right)\right)\right) \\
&\equiv \mathcal{E}_t.
\end{aligned}
$$

And it is direct to show that:

$$\mathbb{E}_t\mathcal{E}_{t+1}-\mathcal{E}_t=\int\left(1-\varphi\left(i\right)\right)\left(\sigma\frac{e\left(i\right)}{E}+\frac{Wn\left(i\right)}{WN}\psi\right)\left(\hat{R}_t-\sum_l\partial_e e_l\left(i\right)\pi_{l,t+1}\right)di$$

$$+\int\varphi\left(i\right)\left\{\frac{b_0}{ER}\Delta\hat{R}_{t+1}+\frac{Wn\left(i\right)}{WN}\psi\Delta\hat{W}_{t+1}-\sum_l\frac{Wn\left(i\right)}{WN}\psi\partial_e e_l\left(i\right)\pi_{l,t+1}-\frac{e\left(i\right)}{E}s_k\left(i\right)\cdot\hat{\pi}_{l,t+1}+\frac{Wn\left(i\right)}{WN}\sum\left\{\bar{s}_k\left(\hat{\pi}_{l,t+1}+\Delta\hat{A}_{l,t+1}\right)\right\}di\right\}.$$

So we have:

$$\mathbb{E}_tX_{t+1}-X_t=\hat{R}_t.$$

Since:

$$\mathbb{E}_t\left(\frac{1}{\sigma}\left(\hat{e}_{t+1}^{t_0,u}-\sum_k s_k\left(i\right)\hat{P}_{k,t+1}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t+1}\right)-\frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}-\sum_k s_k\left(i\right)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t}=\hat{R}_t,$$

we have

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}-\sum_k s_k\left(i\right)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t}\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}-\sum_k s_k\left(i\right)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t}\right)\left\{\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$=-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)$$

$$\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}-\sum_k s_k\left(i\right)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t}-X_t\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^tX_t\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di.$$

Next we have:

$$\mathbb{E}_{t_0}\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{1}{\sigma}\left(\hat{e}_{t_0+s}^{t_0,u}-\sum_k s_k\left(i\right)\hat{P}_{k,t_0+s}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}-X_{t_0+s}\right)$$

$$=\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{E}{\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)}\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+s}\right\}-X_{t_0+s}\right).$$

so

$$
\begin{aligned}
& -\mathbb{E}_0 \sum_{s=0}^{\infty} \frac{1}{R^s} \int (1-\varphi(i)) \left(\frac{1}{\sigma} \left(\hat{e}^{t_0,u}_{t_0+s} - \sum_k s_k(i)\, \hat{P}_{k,t_0+s} \right) + \sum_k \partial_e e_k(i)\, \hat{P}_{k,t_0+s} - X_{t_0+s} \right) \\
& \qquad \cdot \left\{ \frac{b(i)}{R} \hat{R}_{t_0+s} - \sum_{l=1}^{K} \left(e(i)\, s_l(i) - Wn(i)\, \bar{s}_l \right) \hat{P}_{l,t_0+s} + Wn(i) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t_0+s} + \mathbb{1}_{t=t_0} b(i) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0} + \sum_{l=1}^{K} \sigma e(i)\, \partial_e e_k(i) \left(\hat{W}_{t_0+s} - \hat{P}_{k,t_0+s} - 2\tilde{A}_{k,t_0+s} \right) \right\} \\
&= -\left(1-\frac{1}{R}\right) \sum_{s=0}^{\infty} \frac{1}{R^s} \int (1-\varphi(i)) \left(\frac{E}{\sigma e(i) + Wn(i)\,\psi} \left\{ \frac{b(i)}{RE} \left(R_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_l \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l \right) \hat{P}_{l,t_0+s} + \frac{Wn(i)}{WN} \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t_0+s} + \frac{Wn(i)}{WN} \psi \hat{W}_{t_0+s} + \frac{e(i)}{E} \sigma \sum_l \partial_e e_l(i)\, \hat{P}_{l,t_0+} \right.\right. \\
& \qquad \cdot \sum_{s=0}^{\infty} \frac{1}{R^s} \left\{ \frac{b(i)}{R} \left(R_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_l \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l \right) \hat{P}_{l,t_0+s} + Wn(i) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} + \sum_{l=1}^{K} \sigma e(i)\, \partial_e e_k(i) \left(\hat{W}_{t_0+s} - \hat{P}_{k,t_0+s} - 2\tilde{A}_{k,t_0+s} \right) \right\} di \\
&= -\left(1-\frac{1}{R}\right) \int (1-\varphi(i)) \frac{E^2}{\sigma e(i) + Wn(i)\,\psi} \left\{ \sum_{s=0}^{\infty} \frac{1}{R^s} \left\{ \frac{b(i)}{RE} \left(R_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_l \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l \right) \hat{P}_{l,t_0+s} + \frac{Wn(i)}{WN} \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t_0+s} - \frac{e(i)}{E} \sigma \sum_l \partial_e e_l(i)\, \tilde{A}_{k,t_0+s} \right\} \right\}^2 di \\
& \quad - \left(1-\frac{1}{R}\right) \int (1-\varphi(i)) \sum_{s=0}^{\infty} \frac{1}{R^s} \frac{E^2}{\sigma e(i) + Wn(i)\,\psi} \left\{ \frac{Wn(i)}{WN} \psi \hat{W}_{t_0+s} + \frac{e(i)}{E} \sigma \sum_l \partial_e e_l(i)\, \hat{P}_{l,t_0+s} \right\} \sum_{s=0}^{\infty} \frac{1}{R^s} \left\{ \sum_{l=1}^{K} \sigma \frac{e(i)}{E} \partial_e e_k(i) \left(\hat{W}_{t_0+s} - \hat{P}_{k,t_0+s} - 2\tilde{A}_{k,t_0+s} \right) \right\} di \\
& \qquad - \left(1-\frac{1}{R}\right) \int (1-\varphi(i)) \sum_{s=0}^{\infty} \frac{1}{R^s} E \left\{ \frac{b(i)}{RE} \left(R_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_l \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l \right) \hat{P}_{l,t_0+s} + Wn(i) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} \right\} di \sum_{s=0}^{\infty} \frac{1}{R^s} \hat{W}_{t_0+s} \\
& + \left(1-\frac{1}{R}\right) \sum_{s=0}^{\infty} \frac{1}{R^s} X_{t_0+s} \sum_{s=0}^{\infty} \frac{1}{R^s} \int (1-\varphi(i)) \left\{ \frac{b(i)}{R} \left(R_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_l \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l \right) \hat{P}_{l,t_0+s} + Wn(i) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} + \sum_{l=1}^{K} \sigma e(i)\, \partial_e e_k(i) \left(\hat{W}_{t_0+s} - \hat{P}_{k,t_0+s} - 2\tilde{A}_{k,t_0+s} \right) \right\} di
\end{aligned}
$$

First note that we have, up to terms independent from monetary policy:

$$
\begin{aligned}
& -\left(1-\frac{1}{R}\right) \int (1-\varphi(i)) \sum_{s=0}^{\infty} \frac{1}{R^s} \frac{E^2}{\sigma e(i) + Wn(i)\,\psi} \left\{ \frac{Wn(i)}{WN} \psi \hat{W}_{t_0+s} + \frac{e(i)}{E} \sigma \sum_l \partial_e e_l(i)\, \hat{P}_{l,t_0+s} \right\} \sum_{s=0}^{\infty} \frac{1}{R^s} \left\{ \sum_{l=1}^{K} \sigma \frac{e(i)}{E} \partial_e e_k(i) \left(\hat{W}_{t_0+s} - \hat{P}_{k,t_0+s} - 2\tilde{A}_{k,t_0+s} \right) \right\} di \\
&= -\left(1-\frac{1}{R}\right) \int (1-\varphi(i)) \frac{Wn(i)\,\psi\sigma e(i)}{\sigma e(i) + Wn(i)\,\psi} \left(\sum_{s=0}^{\infty} \frac{1}{R^s} \hat{W}_{t_0+s} - \sum_k \partial_e e_k(i) \left(\hat{P}_{k,t_0+s} + \tilde{A}_{k,t_0+s} \right) \right)^2 di \\
& \quad + \left(1-\frac{1}{R}\right) \int (1-\varphi(i))\, \sigma e(i) \left\{ \sum_{s=0}^{\infty} \frac{1}{R^s} \sum_l \partial_e e_l(i) \left(\hat{P}_{l,t_0+s} + \tilde{A}_{l,t_0+s} \right) \right\}^2 di \\
& \qquad - \left(1-\frac{1}{R}\right) \int (1-\varphi(i))\, \sigma e(i) \sum_{s=0}^{\infty} \frac{1}{R^s} \hat{W}_{t_0+s} \sum_{s=0}^{\infty} \frac{1}{R^s} \sum_k \partial_e e_k(i)\, \hat{P}_{k,t_0+s} di.
\end{aligned}
$$

We therefore obtain:

$$
\begin{aligned}
&-\mathbb{E}_0\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}-\sum_k s_k\left(i\right)\hat{P}_{k,t_0+s}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}-X_{t_0+s}\right)\\
&\qquad\cdot\left\{\frac{b\left(i\right)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di\\
&=-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}-\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\tilde{A}_{k,t_0+s}\right\}\right\}^2di\\
&\qquad-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)\psi\sigma e\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)^2di\\
&\qquad+\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\sum_l\partial_e e_l\left(i\right)\left(\hat{P}_{l,t_0+s}+\tilde{A}_{l,t_0+s}\right)\right\}^2di\\
&\qquad-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum\frac{1}{R^s}\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}di\\
&\qquad-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sum_{s=0}^{\infty}\frac{1}{R^s}E\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\\
&+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di
\end{aligned}
$$

Next we have:

$$\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_l\bar{s}_l\hat{P}_{l,t_0}$$

$$=\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\bar{\psi}\hat{W}_{t_0+s}+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l di\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l di\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{k,t_0+s}\right\}di$$

$$-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di$$

$$+\left(1-\frac{1}{R}\right)\sum\frac{1}{R^s}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_l\bar{s}_l\hat{P}_{l,t_0},$$

where $\varphi^E \equiv \int \varphi(i)\frac{e(i)}{E}di$,$\varphi^N \equiv \int \varphi(i)\frac{Wn(i)}{WN}di$. Let us first rewrite the first term of this expression. We have:

$$\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l \bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum\frac{1}{R^s}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)di\right)^2$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}di+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma\frac{e(i)}{E}+\frac{Wn(i)}{WN}\psi\right)di}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di.$$

Next, the second and third terms can be rewritten:

$$\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}\right\}di$$

$$-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}di\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}di$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l di\hat{P}_{l,t_0+s}di\sum\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}\right\}di$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}\right\}di$$

$$+2\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di\sum\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t_0+s}di.$$

Using:

$$\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di=-\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di,$$

and the fact that in steady state:

$$\int b\left(i\right)di=\int\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)di=0,$$

we have:

$$\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}$$

$$=\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_{k}\partial_e e_k\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum\frac{1}{R^s}\sum_{k}\partial_e e_k\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_{k}\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)di\right)^2$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_{k}\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}di$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l}\bar{s}_l\tilde{A}_{l,t_0+s}\right\}di$$

$$+\left(1-\frac{1}{R}\right)\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}-\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t_0+s}\right\}di\right)^2$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_{l}\bar{s}_l\hat{P}_{l,t_0}.$$

Putting everything together, we have:

$$-\mathbb{E}_0\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}\left(i\right)-\sum_k s_k\left(i\right)\hat{P}_{k,t_0+s}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}-X_{t_0+s}\right)$$

$$\cdot\left\{\frac{b\left(i\right)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}-\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\tilde{A}_{k,t_0+s}\right\}\right\}^2di$$

$$-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)\psi\sigma e\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)^2di$$

$$+\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2di$$

$$-\left(1-\frac{1}{R}\right)\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)di\right)^2$$

$$+\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^E\right)\sigma}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2di$$

$$-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_l\bar{s}_l\hat{P}_{l,t_0}di\sum\frac{1}{R^s}\hat{W}_{t_0+s}+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_l\bar{s}_l\hat{P}_{l,t_0}$$

$$+\frac{1-\frac{1}{R}}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}-\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t_0+s}\right\}di\right)^2.$$

Finally, we rewrite some coefficients, we have:

$$\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)\psi\sigma e\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}di-\frac{\int\left(1-\varphi\left(i\right)\right)\psi Wn\left(i\right)di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}$$

$$=\frac{\sigma\psi}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\left\{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)\frac{Wn\left(i\right)\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}{\sigma e\left(i\right)+Wn\left(i\right)\psi}di-\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)di\int\left(1-\varphi\left(i\right)\right)e\left(i\right)\frac{\sigma e\left(i\right)+Wn\left(i\right)\psi}{\sigma e\left(i\right)+Wn\left(i\right)\psi}di\right\}$$

$$=\frac{\sigma\psi}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\left\{\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\frac{Wn\left(i\right)\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di-e\left(i\right)\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)di}{\sigma e\left(i\right)+Wn\left(i\right)\psi}di\right\}$$

$$=\left(1-\frac{1}{R}\right)\frac{\sigma\int\left(1-\varphi\left(i\right)\right)\psi Wn\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\left\{\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\frac{\frac{Wn(i)}{\int(1-\varphi(i))\psi Wn(i)di}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di-b\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}di\right\}$$

$$=-\frac{\int\left(1-\varphi\left(i\right)\right)\psi\frac{Wn(i)}{WN}di}{\left(1+\frac{\int(1-\varphi(i))\left(\frac{Wn(i)}{WN}\psi\right)di}{\int(1-\varphi(i))\frac{\sigma e(i)}{E}di}\right)}\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di}-\frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}.$$

Define the variance covariance matrix of marginal propensities to spend as:

$$\mathcal{E}_{k,l} = \int \frac{(1-\varphi(i))\, e(i)}{\int (1-\varphi(i))\, e(i)\, di} \partial_e e_l(i)\, \partial_e e_k(i)\, di - \frac{\int (1-\varphi(i))\, e(i)\, \partial_e e_l(i)\, di \int (1-\varphi(i))\, e(i)\, \partial_e e_l(i)\, di}{\left(\int (1-\varphi(i))\, e(i)\, di\right)^2}.$$

So we have:

$$\begin{aligned}
&-\mathbb{E}_0 \sum_{s=0}^{\infty} \frac{1}{R^s} \int (1-\varphi(i)) \left(\frac{1}{\sigma} \left(\hat{e}^{t_0,u}_{t_0+s}(i) - \sum_k s_k(i)\, \hat{P}_{k,t_0+s} \right) + \sum_k \partial e_k(i)\, \hat{P}_{k,t_0+s} - X_{t_0+s} \right) \\
&\qquad \cdot \left\{ \frac{b(i)}{R} \hat{R}_{t_0+s} - \sum_{l=1}^{K} \left(e(i)\, s_l(i) - Wn(i)\, \bar{s}_l\right) \hat{P}_{l,t_0+s} + Wn(i) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t_0+s} + \mathbb{1}_{t=t_0} b(i) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0} + \sum_{l=1}^{K} \sigma e(i)\, \partial_e e_k(i) \left(\hat{W}_{t_0+s} - \hat{P}_{k,t_0+s} - 2\tilde{A}_{k,t_0+s}\right) \right\} \\
&= -\left(1-\frac{1}{R}\right) \int (1-\varphi(i)) \frac{E^2}{\sigma e(i) + Wn(i)\, \psi} \left\{ \sum_{s=0}^{\infty} \frac{1}{R^s} \left\{ \frac{b(i)}{RE} \left(R_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_l \frac{e(i)}{E} (s_l - \bar{s}_l)\, \hat{P}_{l,t_0+s} + \frac{Wn(i)}{WN} \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t_0+s} - \frac{e(i)}{E} \sigma \sum_l \partial_e e_l(i)\, \tilde{A}_{k,t_0+s} \right\} \right\}^2 di \\
&\quad + \left(1-\frac{1}{R}\right) E \frac{\int (1-\varphi(i))\, \psi \frac{Wn(i)}{WN} di}{\left(1 + \frac{\int (1-\varphi(i)) \left(\frac{Wn(i)}{WN}\psi\right) di}{\int (1-\varphi(i)) \frac{\sigma e(i)}{E} di}\right)} \int \left(1-\frac{1}{R}\right) \frac{\frac{(1-\varphi(i)) b(i)}{\int (1-\varphi(i)) e(i) di} - \frac{(1-\varphi(i)) Wn(i)}{\int (1-\varphi(i)) Wn(i) di} \frac{\int (1-\varphi(i)) b(i) di}{\int (1-\varphi(i)) e(i) di}}{1 + \frac{Wn(i)\psi}{\sigma e(i)}} \left\{ \sum_{s=0}^{\infty} \frac{1}{R^s} \left(\hat{W}_{t_0+s} - \sum_k \partial_e e_k(i) \left(\hat{P}_{k,t_0+s} + \tilde{A}_{k,t_0+s}\right) \right) \right\}^2 di \\
&\quad + \left(1-\frac{1}{R}\right) \frac{\left((1-\varphi^E)\, \sigma\right)^2}{\int (1-\varphi(i)) \left(\sigma \frac{e(i)}{E} + \frac{Wn(i)}{WN}\psi\right) di} \sum_{k,l} \mathcal{E}_{k,l} \left(\sum_{s=0}^{\infty} \frac{1}{R^s} \left(\hat{P}_{k,t_0+s} + \tilde{A}_{k,t_0+s}\right) \right) \left(\sum_{s=0}^{\infty} \frac{1}{R^s} \left(\hat{P}_{l,t_0+s} + \tilde{A}_{l,t_0+s}\right) \right) \\
&\quad - \left(1-\frac{1}{R}\right) \int (1-\varphi(i))\, b(i)\, di \sum_l \bar{s}_l \hat{P}_{l,t_0} di \sum_{s=0}^{\infty} \frac{1}{R^s} \hat{W}_{t_0+s} + \left(1-\frac{1}{R}\right) \sum_{s=0}^{\infty} \frac{1}{R^s} X_{t_0+s} \int (1-\varphi(i))\, b(i)\, di \sum_l \bar{s}_l \hat{P}_{l,t_0} \\
&\quad + \frac{1-\frac{1}{R}}{(1-\varphi^E)\, \sigma + (1-\varphi^N)\, \psi} \left(\sum_{s=0}^{\infty} \frac{1}{R^s} \int (1-\varphi(i)) \left\{ \frac{b(i)}{R} \left(R_{t_0+s} - (R-1) \sum_l \bar{s}_l \hat{P}_{l,t_0+s} \right) - \sum_l \frac{e(i)}{E} (s_l - \bar{s}_l)\, \hat{P}_{l,t_0+s} + Wn(i) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} - \sum_{l=1}^{K} \sigma e(i)\, \partial_e e_k(i)\, \tilde{A}_{k,t_0+s} \right\} di \right)^2.
\end{aligned}$$

Simplification of the terms corresponding to the expenditure of HtM households

Here we simplify the third line of the of the social welfare function. Recall that we have:

$$\begin{aligned}
&\frac{1}{\sigma} \left(\hat{e}^{t_0,HtM}_t - \sum_k s_k(i)\, \hat{P}_{k,t} \right) + \sum_k \partial_e e_k(i)\, \hat{P}_{k,t} \\
&= \frac{E}{(\sigma e(i) + Wn(i)\, \psi)} \left\{ \frac{b(i)}{RE} R_t + \frac{Wn(i)}{WN} \psi \sum_{l=1}^{K} \left(\hat{W}_t - \partial_e e_k(i)\, \hat{P}_{l,t}\right) - \sum_l \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) \hat{P}_{l,t} + \frac{Wn(i)}{WN} \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} + \left(1-\frac{1}{R}\right) b(i) \sum_l \bar{s}_l \left(\hat{P}_{l,t_0} - \hat{P}_{l,t}\right) \right\} + \sum_k \partial_e e_k(i)\, \hat{P}_{k,t} \\
&= \frac{E}{(\sigma e(i) + Wn(i)\, \psi)} \left\{ \frac{b(i)}{RE} R_t + \frac{Wn(i)}{WN} \psi \hat{W}_t + \frac{\sigma e(i)}{E} \sum_{l=1}^{K} \partial_e e_k(i)\, \hat{P}_{l,t} - \sum_l \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) \hat{P}_{l,t} + \frac{Wn(i)}{WN} \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} + \left(1-\frac{1}{R}\right) \frac{b(i)}{E} \sum_l \bar{s}_l \left(\hat{P}_{l,t_0} - \hat{P}_{l,t}\right) \right\}.
\end{aligned}$$

We then get

$$-\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\varphi\left(i\right)\left(\frac{1}{\sigma}\left(\hat{e}_{t_0+s}^{t_0,HtM}-\sum_{k}s_{k}\left(i\right)\hat{P}_{k,t_0+s}\right)+\sum_{k}\partial_{e}e_{k}\left(i\right)\hat{P}_{k,t_0+s}\right)$$

$$\cdot\left\{\frac{b\left(i\right)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\left(e\left(i\right)s_{l}\left(i\right)-Wn\left(i\right)\bar{s}_{l}\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_{l}\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)b\left(i\right)\sum_{l=1}^{K}\bar{s}_{l}\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\varphi\left(i\right)\frac{E^{2}}{\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)}\left\{\frac{b\left(i\right)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_{l}\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_{l}\bar{s}_{l}\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)-\frac{\sigma e\left(i\right)}{E}\sum_{l=1}^{K}\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\tilde{A}_{k,t_0+s}\right\}^{2}$$

$$-\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\varphi\left(i\right)\frac{E^{2}}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t_0+s}\right\}\left\{\sum_{l=1}^{K}\sigma\frac{e\left(i\right)}{E}\partial_{e}e_{k}\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$-\sum_{s=0}^{\infty}\hat{W}_{t_0+s}\frac{1}{R^{s}}\int\varphi\left(i\right)E\left\{\frac{b\left(i\right)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_{l}\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_{l}\bar{s}_{l}\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)\right\}.$$

First note that we have, up to terms independent from monetary policy:

$$-\mathbb{E}_{0}\sum_{s\geq0}\frac{1}{R^{s}}\int\varphi\left(i\right)\frac{E^{2}}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t_0+s}\right\}\left\{\sum_{l=1}^{K}\sigma\frac{e\left(i\right)}{E}\partial_{e}e_{k}\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\mathbb{E}_{0}\sum_{s\geq0}\frac{1}{R^{s}}\int\varphi\left(i\right)\frac{\sigma\psi e\left(i\right)Wn\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\hat{W}_{t_0+s}-\sum_{k}\partial_{e}e_{k}\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^{2}di$$

$$+\mathbb{E}_{0}\sum_{s\geq0}\frac{1}{R^{s}}\int\varphi\left(i\right)\sigma e\left(i\right)\left\{\sum_{l,k}\partial_{e}e_{l}\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^{2}di-\mathbb{E}_{0}\sum_{s\geq0}\frac{1}{R^{s}}\int\varphi\left(i\right)\sigma e\left(i\right)\hat{W}_{t_0+s}\sum_{k}\partial_{e}e_{k}\left(i\right)\hat{P}_{k,t_0+s}di.$$

So we have:

$$-\mathbb{E}_{0}\sum_{t=0}^{\infty}\beta^{t}\mathbb{E}_{\delta,t}\int\varphi\left(i\right)\left(\frac{1}{\sigma}\left(\hat{e}_{t}^{t_0,HtM}-\sum_{k}s_{k}\left(i\right)\hat{P}_{k,t}\right)+\sum_{k}\partial_{e}e_{k}\left(i\right)\hat{P}_{k,t}\right)$$

$$\cdot\left\{\frac{b\left(i\right)}{R}\hat{R}_{t}-\sum_{l=1}^{K}\left(e\left(i\right)s_{l}\left(i\right)-Wn\left(i\right)\bar{s}_{l}\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_{l}\tilde{A}_{l,t}+\left(1-\frac{1}{R}\right)b\left(i\right)\sum_{l=1}^{K}\bar{s}_{l}\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\left(\hat{W}_{t}-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$=-\mathbb{E}_{0}\sum_{t=0}^{\infty}\beta^{t}\mathbb{E}_{\delta,t}\frac{E^{2}}{\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)}\left\{\frac{b\left(i\right)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_{l}\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_{l}\bar{s}_{l}\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)-\frac{\sigma e\left(i\right)}{E}\sum_{l=1}^{K}\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\tilde{A}_{k,t_0+s}\right\}^{2}$$

$$-\mathbb{E}_{0}\sum\beta^{t}\int\varphi\left(i\right)\frac{\sigma\psi e\left(i\right)Wn\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\hat{W}_{t}-\sum_{k}\partial_{e}e_{k}\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^{2}di+\sum\beta^{t}\int\varphi\left(i\right)\sigma e\left(i\right)\left\{\sum_{l,k}\partial_{e}e_{l}\left(i\right)\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)\right\}^{2}di-\sum\beta^{t}\int\varphi\left(i\right)\sigma e\left(i\right)\hat{W}_{t}\sum_{k}\partial_{e}e_{k}\left(i\right)\hat{P}_{k,t}di$$

$$-\mathbb{E}_{0}\sum_{t=0}^{\infty}\beta^{t}\mathbb{E}_{\delta,t}\hat{W}_{t}\int\varphi\left(i\right)E\left\{\frac{b\left(i\right)}{RE}\hat{R}_{t}-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)\hat{P}_{l,t}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_{l}\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_{l}\bar{s}_{l}\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)\right\}.$$

Next, we gather all the terms of the form $\mathbb{E}_{0}\sum_{t=0}^{\infty}\beta^{t}x_{t}z_{t}$, that is:

$$
\begin{aligned}
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}-\sum_k s_k(i)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k(i)\hat{P}_{k,t}\right)\\
&\qquad\cdot\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^K\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t}+\left(1-\frac{1}{R}\right)b(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^K\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\
&\qquad+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{k,l}^K\int\sigma e(i)\partial_e e_k(i)\partial_e e_l(i)\,di\hat{P}_{l,t}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_k E_k\hat{W}_t\tilde{\mathbf{A}}_{k,t}\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^tX_t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^K\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^K\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di.
\end{aligned}
$$

Following the same step as above, we rewrite the third line as:

$$
\begin{aligned}
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^tX_t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^K\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^K\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\
&=-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left\{\int(1-\varphi(i))\frac{Wn(i)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn(i)}{WN}\bar{\psi}\hat{W}_t+\sum_l\sigma\frac{e(i)}{E}\partial_e e_l(i)\,di\hat{P}_{l,t}di\right\}\int(1-\varphi(i))\sum_{l=1}^K\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)di\\
&\qquad-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left\{\int(1-\varphi(i))\frac{Wn(i)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn(i)}{WN}\bar{\psi}\hat{W}_t+\sum_l\sigma\frac{e(i)}{E}\partial_e e_l(i)\,di\hat{P}_{l,t}di\right\}\\
&\qquad\cdot\int(1-\varphi(i))\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^K\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t}\right\}di\\
&+\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi(i)\left\{\frac{b(i)}{ER}\left(\hat{R}_t-(R-1)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t}di\right\}\int(1-\varphi(i))\left\{\sum_{l=1}^K\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\
&\qquad+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi(i)\left\{\frac{b(i)}{ER}\left(\hat{R}_t-(R-1)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t}di\right\}\\
&\qquad\cdot\int(1-\varphi(i))\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^K\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t}\right\}di-\delta\mathbb{E}_0\sum_{t=0}^{\infty}\beta^tX_t\int(1-\varphi(i))b(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t}di.
\end{aligned}
$$

First, we have

$$
\begin{aligned}
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left\{\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_t+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}di\right\}\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)di\\
&=-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\hat{W}_t-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2di\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2di\\
&+\sum_{t=0}^{\infty}\beta^t E\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)di\right)^2\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\sum_k\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)di\hat{W}_t\hat{P}_{k,t}\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}di\sum_l\bar{s}_l\tilde{A}_{l,t}di\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)di
\end{aligned}
$$

Next,

$$
\begin{aligned}
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_t+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}\right\}di\\
&\cdot\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}\right\}di\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\
&=\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\hat{W}_t\int\varphi\left(i\right)E\left\{\frac{b\left(i\right)}{RE}\hat{R}_t-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}-\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\hat{P}_{l,t}\right\}di\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\hat{W}_t\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}di\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left\{\int\left(1-\varphi\left(i\right)\right)\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}di\right\}\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}di\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}di\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}\right\}di\\
&-2\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}\right\}di\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t}di
\end{aligned}
$$

Gathering the terms, we obtain:

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}-\sum_k s_k(i)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k(i)\hat{P}_{k,t}\right)$$

$$\cdot\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^{K}(e(i)s_l(i)-Wn(i)\bar{s}_l)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\left(1-\frac{1}{R}\right)b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{k,l}^{K}\int\sigma e(i)\partial_e e_k(i)\partial e_l(i)di\hat{P}_{l,t}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_k E_k\hat{W}_t\tilde{\mathbf{A}}_{k,t}$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t X_t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^{K}(e(i)s_l(i)-Wn(i)\bar{s}_l)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$=-\sum_{s=0}^{\infty}\frac{1}{R^s}\int\varphi(i)\frac{E^2}{(\sigma e(i)+Wn(i)\psi)}\left\{\frac{b(i)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)-\frac{\sigma e(i)}{E}\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\tilde{A}_{k,t_0+s}\right\}^2$$

$$-\mathbb{E}_0\sum\beta^t\int\varphi(i)\frac{\sigma\psi e(i)Wn(i)}{\sigma e(i)+Wn(i)\psi}\left\{\hat{W}_t-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\int(1-\varphi(i))\sigma\frac{e(i)}{E}\left\{\hat{W}_t-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\frac{\left((1-\varphi^E)\sigma\right)^2}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\sum_{k,l}\mathcal{E}_{k,l}\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-\delta\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(1-\frac{1}{R}\right)\int(1-\varphi(i))b(i)di\sum_l\bar{s}_l\hat{P}_{l,t}di\sum_{s\geq 0}\frac{1}{R^s}\hat{W}_{t+s}$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\left(\int(1-\varphi(i))\left\{\frac{b(i)}{ER}\left(\hat{R}_t-(R-1)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}-\sigma\frac{e(i)}{E}\sum_{l=1}^{K}\partial_e e_k(i)\tilde{A}_{k,t}di\right\}\right)^2$$

$$-\delta\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t X_t\int(1-\varphi(i))b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t}di.$$

Social Welfare Function

Note that we have, as $\mathbb{E}_t X_{t+1}-X_t=\hat{R}_t$

$$\mathbb{E}_0 X_t\int(1-\varphi(i))b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t}di+\mathbb{E}_0\left(1-\frac{1}{R}\right)\sum\frac{1}{R^s}X_{t+s}\int(1-\varphi(i))b(i)di\sum_l\bar{s}_l\hat{P}_{l,t}=\sum\frac{\hat{R}_s}{R^{s+1}}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}.$$

In addition, for any variable x_t, z_t, we have

$$\sum_{t=0}^{\infty}\frac{1}{R^t}\left(x_{t_0+t}+\left(1-\frac{1}{R}\right)z_{t_0}\right)^2-\left(1-\frac{1}{R}\right)\left(\sum_{t=0}^{\infty}\frac{1}{R^t}x_{t_0+t}+z_{t_0}\right)^2=\mathbb{E}_0\sum_{t=0}^{\infty}\frac{1}{R^t}\left(x_{t_0+t}\right)^2-\mathbb{E}_0\left(1-\frac{1}{R}\right)\left(\sum_{t=0}^{\infty}\frac{1}{R^t}x_{t_0+t}\right)^2.$$

Finally, we denote for any variable Z_{t_0},

$$\mathbb{E}_\beta\left(Z_{t_0}\right)\equiv\mathbb{E}_0(1-\beta)\left(\frac{1}{1-\frac{1}{R}}(1-\delta)Z_-+\delta\frac{1}{1-\frac{1}{R}}\sum_{t_0}\beta^{t_0}Z_{t_0}\right)$$

the social average over the population of the variable Z_{t_0}(Note that if $Z_{t_0}$is constant, we have $\mathbb{E}_\beta\left(Z_{t_0}\right)=Z$). Using this facts and the previous derivations, we can write – after normalization – the social welfare function as:

$$\mathcal{L} = (1-\beta)\sum\beta^s \frac{\left(1-\varphi^N\right)\psi}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)} \left\{ \underbrace{\int \frac{\left(1-\varphi\left(i\right)\right)e\left(i\right)}{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di}\left(\hat{W}_s - \sum_l \partial_e e_l\left(i\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^2 di}_{\text{Labor Distortions with average wealth effect}} \right\}$$

$$-(1-\beta)\sum\beta^s \frac{\left(1-\varphi^N\right)\psi}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)} \underbrace{\mathbb{E}_\beta\left(\left\{\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di} - \frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left(\left(1-\tilde\beta\right)\sum_{u\ge t_0}\tilde\beta^{u-t_0}\left(\hat W_u - \sum_l \partial_e e_l\left(i\right)\left(\hat P_{l,u}+\tilde A_{l,u}\right)\right)\right)^2\right\}\right)}_{\text{Correction for the dispersion in wealth effects}}$$

$$+(1-\beta)\sum\beta^s \underbrace{\int \varphi\left(i\right)\frac{Wn\left(i\right)}{WN}\frac{\psi}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left(\hat W_s - \sum_l \partial_e e_l\left(i\right)\left(\hat P_{l,s}+\tilde A_{l,s}\right)\right)^2 di}_{\text{Labor distortion HtM}}$$

$$+(1-\beta)\sum\beta^s\frac{\left(1-\varphi^E\right)\sigma}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)}\underbrace{\sum_{k,l}\mathcal{E}_{k,l}\left\{\left(\hat P_{k,s}+\tilde A_{k,s}\right)\left(\hat P_{l,s}+\tilde A_{l,s}\right) - \mathbb{E}_\beta\left(\left(1-\tilde\beta\right)\left(\sum_{s\ge t_0}\tilde\beta^{s-t_0}\left(\hat P_{k,s}+\tilde A_{k,s}\right)\right)\left(\left(1-\tilde\beta\right)\sum_{s\ge t_0}\tilde\beta^{s-t_0}\left(\hat P_{l,s}+\tilde A_{l,s}\right)\right)\right)\right\}}_{\text{Intertemporal misallocation of expenditures}}$$

$$-(1-\beta)\sum\beta^s\underbrace{\sum_k \bar s_k\sum_l \bar\rho_{k,l}\left(\hat P_{k,s}+\tilde A_{k,s}\right)\left(\hat P_{l,s}+\tilde A_{l,s}\right)}_{\text{Intratemporal misallocation of expenditures}} + (1-\beta)\sum\beta^s\underbrace{\left[\bar s_1,...,\bar s_K\right]\left(1-\Omega\right)^{-1}\left[\vartheta_1\left(\pi_{1,s}\right)^2,...,\vartheta_K\left(\pi_{K,s}\right)^2\right]'}_{\text{Within Sector misallocation}} + (1-\beta)\sum\beta^s\underbrace{\left(\hat W_s,\hat P_s+\tilde{\mathbf{A}}_s\right)^T\mathcal{O}\left(\hat W_s,\hat P_s+\tilde{\mathbf{A}}_s\right)}_{\text{Input misallocation}}$$

$$+\underbrace{\mathbb{E}_\beta\left(\int g\left(i\right)\left(\nu_{t_0}(i)+\left(1-\tilde\beta\right)\sum_{s\ge t_0}\tilde\beta^{s-t_0}\sum_l\frac{Wn\left(i\right)}{WN}\bar s_l\tilde A_{l,s}\right)^2 di\right)}_{\text{Redistribution}} - \underbrace{2\mathbb{E}_\beta\left(\left(1-\tilde\beta\right)\sum_{s\ge t_0}\int\left(\nu_{t_0}(i)\tilde\beta^{s-t_0}\sum_l\frac{\left(1-\varphi\left(i\right)\right)\partial_e e_l\left(i\right)}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\tilde A_{l,s}\right)di\right)}_{\text{"Bang for Buck non HtM”}}$$

$$+\underbrace{\mathbb{E}_\beta\left(\left(1-\tilde\beta\right)\sum\tilde\beta^{s-t_0}\int\varphi\left(i\right)\frac{E}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\aleph_s^{t_0}(i)+\sum_l\frac{Wn\left(i\right)}{WN}\bar s_l\tilde A_{l,s}\right\}^2 - \int\varphi\left(i\right)\frac{E}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\nu_{t_0}(i)+\left(1-\tilde\beta\right)\sum_{s\ge t_0}\tilde\beta^{s-t_0}\frac{Wn\left(i\right)}{WN}\bar s_l\tilde A_{l,s}\right\}^2\right)}_{\text{Non Smoothing compensation for HtM}}$$

$$-\underbrace{2\mathbb{E}_\beta\left(\left(1-\tilde\beta\right)\sum_{s\ge t_0}\tilde\beta^{s-t_0}\int\aleph_s^{t_0}(i)\sum_l\frac{\varphi\left(i\right)\partial_e e_l\left(i\right)}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\tilde A_{l,s}di\right)}_{\text{"Bang for Buck HtM”}}$$

$$+\frac{E}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\mathbb{E}_\beta\underbrace{\left\{\left(1-\tilde\beta\right)\sum_{s\ge t_0}\tilde\beta^{s-t_0}\left(\int\left(1-\varphi\left(i\right)\right)\left\{\aleph_s^{t_0}(i)+\sum_l\frac{Wn\left(i\right)}{WN}\bar s_l\tilde A_{l,s}-\sum_l\frac{\sigma e\left(i\right)}{E}\partial_e e_l\left(i\right)\tilde A_{l,s}\right\}\right)^2\right\}}_{\text{Smoothing Premium for non HtM}}$$

$$-\frac{E}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\mathbb{E}_\beta\underbrace{\left\{\left(\int\left(1-\varphi\left(i\right)\right)\left\{\nu_{t_0}(i)+\left(1-\tilde\beta\right)\sum_{s\ge t_0}\tilde\beta^{s-t_0}\sum_l\left(\frac{Wn\left(i\right)}{WN}\bar s_l\tilde A_{l,s}-\frac{\sigma e\left(i\right)}{E}\partial_e e_l\left(i\right)\tilde A_{l,s}\right)\right\}di\right)^2\right\}}_{\text{Smoothing Premium for non HtM}}$$

where $\mathcal{O}$ denotes the aggregate input substitution matrix which is of dimension $(K+1)\times(K+1)$, and given by $\mathcal{O}=\begin{bmatrix} -\frac{W\partial_W\mathcal{N}}{N} & -\left(\frac{P_k\partial_{P_k}\mathcal{N}}{N}\right)^T \\ -\frac{P_k\partial_{P_k}\mathcal{N}}{N} & -\frac{P_k\mathcal{Y}_k}{E}\frac{P_l\partial_{P_l}\mathcal{Y}_k}{\mathcal{Y}_k}\end{bmatrix}$.

The last five lines constitute the redistributive motive. The first term ("Redistribution") is standard and is 0 when households are compensated for their loss of income. The second term("Bang for Buck") is an adjustment taking into account that it is relatively more efficient to compensate households who consume more in relatively more productive sectors. The last four terms are adjustments taking into account that HtM households cannot smooth their consumption.
With

$$\mathcal{E}_{k,l}=\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\partial_e e_l(i)\,\partial_e e_k(i)\,di-\frac{\int(1-\varphi(i))\,e(i)\,\partial_e e_k(i)\,di\int(1-\varphi(i))\,e(i)\,\partial_e e_l(i)\,di}{\left(\int(1-\varphi(i))\,e(i)\,di\right)^2},$$

$$\nu_{t_0}(i)=(1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\frac{b(i)}{ER}\left(\hat{R}_s-\sum_{l=1}^{K}\bar{s}_l\pi_{l,s+1}\right)-\sum_{l=1}^{K}\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,s}\right),$$

$$\nu_-(i)=(1-\tilde{\beta})\sum_{s\geq 0}\tilde{\beta}^{s}\left(\frac{b(i)}{ER}\left(\hat{R}_s-(R-1)\sum_l\bar{s}_l\hat{P}_{l,s}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,s}\right),$$

$$\aleph_s^{t_0}(i)=\left(\frac{b(i)}{ER}\left(\hat{R}_s-(R-1)\sum_l\bar{s}_l\hat{P}_{l,s}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,s}\right)+\sum_l\frac{R-1}{R}\frac{b(i)}{E}\bar{s}_l\hat{P}_{l,t_0},$$

$$\aleph_-(i)=\left(\frac{b(i)}{ER}\left(\hat{R}_s-(R-1)\sum_l\bar{s}_l\hat{P}_{l,s}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,s}\right),$$

$$g(i)=-\frac{G''(V(i))\,v'(e(i))\,E}{(1-\tilde{\beta})\,G'(V(i))}+\frac{E}{\sigma e(i)+Wn(i)\,\psi}.$$

Without HtM households and IO, the loss function simplifies to:

$$\mathcal{L}=(1-\beta)\sum\beta^s\frac{\psi}{1+\frac{\psi}{\sigma}}\left\{\underbrace{\int\frac{e(i)}{E}\left(\hat{W}_s-\sum_l\partial_e e_l(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^2 di}_{\text{Labor Distortions with average wealth effect}}-\underbrace{\mathbb{E}_\beta\left(\left\{\int\left(1-\frac{1}{R}\right)\frac{\frac{b(i)}{E}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left((1-\tilde{\beta})\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\hat{W}_u-\sum_l\partial_e e_l(i)\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)^2\right\}\right)}_{\text{Correctionfor the dispersion in wealth effects}}\right\}$$

$$(1-\beta)\sum\beta^s\frac{\sigma}{1+\frac{\psi}{\sigma}}\underbrace{\sum_{k,l}\mathcal{E}_{k,l}\left\{\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_\beta\left((1-\tilde{\beta})\left(\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left((1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)\right\}}_{\text{Intertemporal misallocation of expenditures}}$$

$$-(1-\beta)\sum\beta^s\sum_k\bar{s}_k\sum_l\bar{\rho}_{k,l}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)+(1-\beta)\sum\beta^s\sum\bar{s}_k\vartheta_k\left(\pi_{k,s}\right)^2$$

$$+\underbrace{\mathbb{E}_\beta\left(\int g(i)\left(\nu_{t_0}(i)+(1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\sum_l\frac{Wn(i)}{WN}\bar{s}_l\tilde{A}_{l,s}\right)^2 di\right)}_{\text{Redistribution}}-\underbrace{2\mathbb{E}_\beta\left((1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\int\left(\nu_{t_0}(i)\sum_l\frac{\partial_e e_l(i)}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\tilde{A}_{l,s}\right)di\right)}_{\text{"Bang for Buck"}}$$

Defining $\tilde{P}_{l,s} = \hat{P}_{l,s} + \tilde{A}_{l,s}$, the markup in k, and using $\tilde{\mathcal{Y}}_t = \frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t - \sum_l \overline{\partial_e e_l}\left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right)\right)$ we can re-express the loss function directly in terms of the output gap :

$$\mathcal{L} = (1-\beta)\sum \beta^s \frac{\psi}{1+\frac{\psi}{\sigma}}\left\{\left(\frac{1}{\sigma}+\frac{1}{\psi}\right)^2 \underbrace{\tilde{\mathcal{Y}}_s^2} + \mathcal{E}_{k,l}\tilde{P}_{k,s}\tilde{P}_{l,s} - \mathbb{E}_\beta\left(\left\{\int\left(1-\frac{1}{R}\right)\frac{\frac{b(i)}{E}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left((1-\tilde{\beta})\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\frac{1}{\sigma}+\frac{1}{\psi}\right)\tilde{\mathcal{Y}}_u - \sum_l\left(\partial_e e_l\,(i) - \overline{\partial_e e_l}\right)\tilde{P}_{l,s}\right)^2\right\}\right)\right\}$$

$$(1-\beta)\sum \beta^s \frac{\sigma}{1+\frac{\psi}{\sigma}}\sum_{k,l}\mathcal{E}_{k,l}\left\{\tilde{P}_{k,s}\tilde{P}_{l,s} - \mathbb{E}_\beta\left(\left((1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\tilde{P}_{k,s}\right)\left((1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\tilde{P}_{l,s}\right)\right)\right\}$$

$$-(1-\beta)\sum\beta^s\sum_k \bar{s}_k\sum_l\bar{\rho}_{k,l}\tilde{P}_{k,s}\tilde{P}_{l,s} + (1-\beta)\sum\beta^s\sum\bar{s}_k\vartheta_k\left(\pi_{k,s}\right)^2$$

$$+\mathbb{E}_\beta\left(\int g\,(i)\left(\nu_{t_0}(i) + (1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\sum_l\frac{Wn\,(i)}{WN}\bar{s}_l\tilde{A}_{l,s}\right)^2 di\right) - 2\mathbb{E}_\beta\left((1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\int\left(\nu_{t_0}(i)\sum_l\frac{\partial_e e_l\,(i)}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\tilde{A}_{l,s}\right)di\right)$$

System of Equations without redistributive motive:

We now derive the system of optimal equation when the central bank ignores redistributive motives (last three lines of the loss function for the full model). It is possible to show that if CB had access to lump sum transfers (that depends on time, the birth cohort of a household and whether or not the household is Hand-to-Mouth) then at the optimum, these terms of the loss function would become independent from monetary policy. Defining:

$$\mathcal{L}^{nd} \equiv (1-\beta)\sum\beta^s\frac{\left(1-\varphi^N\right)\psi}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)}\left\{\underbrace{\int\frac{(1-\varphi\,(i))\,e\,(i)}{\int(1-\varphi\,(i))\,e\,(i)\,di}\left(\hat{W}_s - \sum_l\partial_e e_l\,(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^2 di}_{\text{Labor Distortions with average wealth effect}}\right\}$$

$$-(1-\beta)\sum\beta^s\frac{\left(1-\varphi^N\right)\psi}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)}\underbrace{\mathbb{E}_\beta\left(\left\{\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di} - \frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left((1-\tilde{\beta})\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\hat{W}_u - \sum_l\partial_e e_l\,(i)\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)\right)^2\right\}\right)}_{\text{Correctionfor the dispersion in wealth effects}}$$

$$+(1-\beta)\sum\beta^s\underbrace{\int\varphi\,(i)\frac{Wn\,(i)}{WN}\frac{\psi}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left(\hat{W}_s - \sum_l\partial_e e_l\,(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^2 di}_{\text{Labor distortion HtM}}$$

$$(1-\beta)\sum\beta^s\frac{\left(1-\varphi^E\right)\sigma}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)}\underbrace{\sum_{k,l}\mathcal{E}_{k,l}\left\{\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right) - \mathbb{E}_\beta\left((1-\tilde{\beta})\left(\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left((1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)\right\}}_{\text{Intertemporal misallocation of expenditures}}$$

$$-(1-\beta)\sum\beta^s\sum_k\bar{s}_k\sum_l\bar{\rho}_{k,l}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right) + (1-\beta)\sum\beta^s\sum\bar{s}_k\vartheta_k\left(\pi_{k,s}\right)^2$$

$$-(1-\beta)\sum\beta^s\left\{\frac{W\partial_W\mathcal{N}}{N}\hat{W}_s^{\,2} + \sum_k\frac{P_k\mathcal{Y}_k}{E}\frac{W\partial_W\mathcal{Y}_k}{\mathcal{Y}_k}\hat{W}_s\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right) + \sum_k\frac{P_k\partial_{P_k}\mathcal{N}}{N}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)\hat{W}_s + \sum_{k,l}\frac{P_k\mathcal{Y}_k}{E}\frac{P_l\partial_{P_l}\mathcal{Y}_k}{\mathcal{Y}_k}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{\mathbf{A}}_{l,s}\right)\right\}$$

The central bank solves

$$\inf_{\{\hat{W}_t,\hat{\boldsymbol{P}}_t\}_{t\geq 0}} \mathcal{L}^{nd}$$

under the constraints

$$\pi_{k,t} = \lambda_k\left(\left(\frac{1}{\sigma}+\frac{1}{\psi}\right)\tilde{\mathcal{Y}}_t + \Omega_{N,k}\sum_l \overline{\partial_e e_l}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}-\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right) + \sum_l \Omega_{k,l}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}-\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right)\right) + \tilde{\beta}\pi_{k,t+1},$$

$$\hat{P}_{k,t} = \hat{P}_{k,t-1}+\pi_{k,t},$$

$$\left(1-\varphi^N\right)\left(\mathbb{E}_t\tilde{\mathcal{Y}}_{t+1}-\tilde{\mathcal{Y}}_t\right) = \left(1-\varphi^N\right)\sigma\mathbb{E}_t\left(\hat{R}_t-\sum_k \overline{\partial_e e_k}\pi_{k,t+1}-r_t^*\right)+\mathbb{E}_t\int\frac{\sigma\varphi(i)}{\sigma+\psi}\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1}-\sigma(R-1)\hat{R}_t\right)-\frac{e(i)}{E}\sum_k\left((s_k(i)-\bar{s}_k)-\sigma(\partial_e e_k(i)-\overline{\partial_e e_k})\right)\pi_{k,t+1}\right\}di$$
$$-\mathbb{E}_t\int\frac{\sigma\varphi(i)}{\sigma+\psi}\left\{\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\left(\pi_{cpi,t+1}-\sigma\pi_{mcpi,t+1}\right)\right\}di,$$

$$\tilde{\mathcal{Y}}_t = \frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t-\sum_k\overline{\partial_e e_k}\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right).$$

Denoting by $\check{\mu}_{k,t}$ the lagrange multiplier on the NKPC of sector k and $M_{k,t}$ the lagrange multiplier on the price evolution equation and taking derivatives with respect to the wage at t directly gives:

$$\sum_{k=1}^K\Omega_{N,k}\lambda_k\check{\mu}_{k,t} = Z_t+\sum_{k,l}\frac{P_l W\partial_W\mathcal{Y}_{l,k}Y_k}{A_k E}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_k\frac{W\partial_W\mathcal{N}_k}{A_k N}Y_k\hat{W}_t$$
$$-\left(\frac{\int(1-\varphi(i))\frac{Wn(i)}{WN}\psi di\int(1-\varphi(i))\sigma e(i)di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di}+\int\varphi(i)\psi\frac{Wn(i)}{WN}\frac{\sigma e(i)}{\sigma e(i)+\psi n(i)}di\right)\hat{W}_t$$
$$+\sum_k\left(\frac{\int(1-\varphi(i))\frac{Wn(i)}{WN}\psi di\int(1-\varphi(i))\sigma e(i)\partial_e e_k(i)di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di}+\int\varphi(i)\psi\frac{Wn(i)}{WN}\frac{\sigma e(i)\partial_e e_k(i)}{\sigma e(i)+\psi n(i)}di\right)\left(\tilde{A}_{k,t}+\hat{P}_{k,t}\right),$$

With

$$Z_{t+1}-Z_t = \frac{\delta}{1-\delta}Z^0_{t+1},$$

$$Z^0_{t-1}-\frac{1}{(1-\delta)R}Z^0_t+\left(1-\frac{1}{R}\right)Z_{t-1} = \left(1-\frac{1}{R}\right)\int(1-\varphi(i))\sigma\frac{e(i)}{E}\left(\frac{\int(1-\varphi(i))Wn(i)\psi di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di}-\frac{\psi Wn(i)}{\sigma e(i)+\psi Wn(i)}\right)di\hat{W}_{t-1}$$
$$-\sum_k\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\sigma\frac{e(i)}{E}\partial_e e_k(i)\left(\frac{\int(1-\varphi(i))Wn(i)\psi di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di}-\frac{\psi Wn(i)}{\sigma e(i)+\psi Wn(i)}\right)di\left(\tilde{A}_{k,t-1}+\hat{P}_{k,t-1}\right).$$

Taking derivatives with respect to the price of k at t gives:

$$M_{k,t} = \check{\mu}_{k,t}-(1-\delta)\check{\mu}_{k,t-1}$$

$$
\begin{aligned}
\beta M_{k,t+1} - M_{k,t} = \lambda_k \check{\mu}_{k,t} - \sum_{l=1}^{K} \lambda_l \frac{\check{\mu}_{l,t}}{P_l Y_l} \frac{P_k \mathcal{Y}_{k,l} Y_l}{A_l} + \sum_{l,m} \frac{P_l P_k \partial_{P_k} \mathcal{Y}_{l,m}}{A_m E} Y_m \left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_l \frac{P_k \partial_{P_k} \mathcal{N}_l}{A_l N} Y_l \hat{W}_t \\
+ \sum_l \bar{s}_l \bar{\rho}_{l,k} \left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right) - \frac{P_k Y_k}{E} \vartheta_k \left(\pi_{k,t} - \beta \pi_{k,t+1}\right) - L_{k,t} \\
+ \left(\frac{\int \left(1-\varphi\left(i\right)\right) \partial_e e_k\left(i\right) \sigma e\left(i\right) di \int \left(1-\varphi\left(i\right)\right) \psi \frac{Wn(i)}{WN}}{\int \left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right) + \psi W n\left(i\right)\right) di} + \int \varphi\left(i\right) \partial_e e_k\left(i\right) \frac{\psi W n\left(i\right)}{WN} \frac{\sigma e\left(i\right)}{\sigma e\left(i\right) + \psi W n\left(i\right)} di \right) \hat{W}_t \\
- \sum_l \left(\int \varphi\left(i\right) \partial_e e_l\left(i\right) \partial_e e_k\left(i\right) \psi \frac{W n\left(i\right)}{WN} \frac{\sigma e\left(i\right)}{\sigma e\left(i\right) + \psi W n\left(i\right)} - \frac{\int \left(1-\varphi\left(i\right)\right) \partial_e e_k\left(i\right) \sigma e\left(i\right) di \int \left(1-\varphi\left(i\right)\right) \sigma \frac{e(i)}{E} \partial_e e_l di}{\int \left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right) + \psi W n\left(i\right)\right) di} + \int \left(1-\varphi\left(i\right)\right) \sigma \frac{e\left(i\right)}{E} \partial_e e_k\left(i\right) \partial_e e_l\left(i\right) di \right) \left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)
\end{aligned}
$$

With

$$
\Delta L_{k,t+1} = \frac{\delta}{1-\delta} L^0_{k,t+1},
$$

$$
\begin{aligned}
L^0_{k,t-1} - \frac{1}{R\left(1-\delta\right)} L^0_{k,t} + \left(1 - \frac{1}{R}\right) L_{k,t-1} = \left(1 - \frac{1}{R}\right) \int \left(1-\varphi\left(i\right)\right) \sigma \frac{e\left(i\right)}{E} \left(\frac{\sigma e\left(i\right) \partial_e e_k\left(i\right)}{\sigma e\left(i\right) + \psi W n\left(i\right)} - \frac{\int \left(1-\varphi\left(i\right)\right) \sigma e\left(i\right) \partial_e e_k\left(i\right) di}{\int \left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right) + \psi W n\left(i\right)\right) di} \right) di \hat{W}_{t-1} \\
- \sum_l \left(1 - \frac{1}{R}\right) \int \left(1-\varphi\left(i\right)\right) \sigma \frac{e\left(i\right)}{E} \partial_e e_l\left(i\right) \left(\frac{\sigma e\left(i\right) \partial_e e_k\left(i\right)}{\sigma e\left(i\right) + \psi W n\left(i\right)} - \frac{\int \left(1-\varphi\left(i\right)\right) \sigma e\left(i\right) \partial_e e_k\left(i\right) di}{\int \left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right) + \psi W n\left(i\right)\right) di} \right) di \left(\tilde{A}_{l,t-1} + \hat{P}_{l,t-1}\right).
\end{aligned}
$$

Efficiency of the Steady State

For first order change in prices, the change in welfare is

$$
d\mathcal{W} = \mathbb{E}_\beta \int G'\left(V(i)\right) \partial_e v\left(i\right) \mathbb{E}_{t_0} \sum_{s=0}^{\infty} \left(\left(1-\delta\right)\beta\right)^s \left\{ \left(\frac{b\left(i\right)}{ER} \left(\hat{R}_s - \left(R-1\right) \sum_l \bar{s}_l \hat{P}_{l,s} \right) - \sum_l \frac{e\left(i\right)}{E} \left(s_l - \bar{s}_l\right) \hat{P}_{l,s} \right) + \sum_l \frac{R-1}{R} \frac{b\left(i\right)}{E} \bar{s}_l \hat{P}_{l,t_0} \right\} di.
$$

Using the fact that $G'\left(V(i)\right) \partial_e v\left(i\right) = 1$ and that the market for bonds clear, we have that

$$
d\mathcal{W} = 0,
$$

for any change in prices (subvariety prices, wage and interest rate). Therefore, there is no change in monetary policy that can improve upon the steady state. It is also direct to verify that there are no taxes at any date t, financed by a lump sum at t than can improve the steady state. To see this, consider for example a wage subsidy in sector k at t subsidized by an arbitrary lump sum. The total impact on welfare is given by

$$
\beta^t \mathbb{E}_0 \int G'\left(V(i)\right) \partial_e v\left(i\right) \left\{ \varsigma\left(i\right) \frac{Y_k}{A_k} \mathcal{N}_k(\boldsymbol{P}_t, W_t) dW_{k,t} - d\tau_t\left(i\right) \right\} di,
$$

with

$$
\int \frac{Y_k}{A_{k,t}} \mathcal{N}_k(\boldsymbol{P}_t, W_t) dW_{k,t} di = \int d\tau_t\left(i\right) di.
$$

Using again $G'\left(V(i)\right) \partial_e v\left(i\right) = 1$ and $\int \varsigma\left(i\right) di = 1$, this subsidy has no impact on welfare. Similarly, wage subsidy to workers, input taxes and commodity taxes have no impact on welfare. The economy is therefore constrained efficient: the only inefficiency is the uninsured death risk which could be corrected through an annuity market but neither through monetary policy or through taxes (if the government cannot have debt).

We now verify that the second order approximation of the Welfare Function is always negative. This will imply that a steady state with 0 inflation and output gap is a local maximum. First, note that a direct application of Cauchy-Schwartz gives that for any variable x_t and any weight $0 < \alpha < 1$,

$$
\mathbb{E}_0 \left(\sum_{s \geq 0} \alpha^s x_{t+s} \right)^2 \leq \mathbb{E}_0 \frac{1}{\left(1-\alpha\right)} \sum_{s \geq 0} \alpha^s \left(x_{t+s}\right)^2.
$$

Given the definition of $\mathbb{E}_\beta$ we have that $\sum_{s=0}^{\infty} \beta^s X_s = \mathbb{E}_\beta \left[\sum_{s \geq t_0} \tilde{\beta}^{s-t_0} X_s \right]$ so we can rewrite the average labour distortion term (which we denote by L_1) as

$$(1-\beta)\sum\beta^{s}\left\{\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\left(\hat{W}_{s}-\sum_{l}\partial_{e}e_{l}(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^{2}di\right\}=(1-\tilde{\beta})\,\mathbb{E}_{\beta}\left(\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\left(\hat{W}_{s}-\sum_{l}\partial_{e}e_{l}(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^{2}di\right).$$

We can re-write the term that appears in the dispersion of wealth effect as

$$\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di}-\frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}=\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}-(1-\varphi(i))\frac{e(i)\,Wn(i)\,/E}{\sigma e(i)+Wn(i)\,\psi}\left(\frac{\sigma}{(1-\varphi^{N})}+\frac{\psi}{(1-\varphi^{E})}\right),$$

where the second term is always weakly negative and hence

$$L_{2}\equiv\mathbb{E}_{\beta}\left(\left\{\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di}-\frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left((1-\tilde{\beta})\sum_{u\geq t_{0}}\tilde{\beta}^{u-t_{0}}\left(\hat{W}_{u}-\sum_{l}\partial_{e}e_{l}(i)\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)\right)^{2}di\right\}\right)\leq$$

$$\mathbb{E}_{\beta}\left(\left\{\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\left((1-\tilde{\beta})\sum_{u\geq t_{0}}\tilde{\beta}^{u-t_{0}}\left(\hat{W}_{u}-\sum_{l}\partial_{e}e_{l}(i)\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)\right)^{2}di\right\}\right).$$

Putting these results together we have that

$$L_{1}+L_{2}\geq(1-\tilde{\beta})\,\mathbb{E}_{\beta}\left(\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{W}_{s}-\sum_{l}\partial_{e}e_{l}(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^{2}di\right)-\mathbb{E}_{\beta}\left(\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\left((1-\tilde{\beta})\sum_{u\geq t_{0}}\tilde{\beta}^{u-t_{0}}\left(\hat{W}_{u}-\sum_{l}\partial_{e}e_{l}(i)\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)\right)\right.$$

$$\geq 0.$$

where the last line follows from our preliminary Cauchy-Schwartz result.
Next consider the term

$$(1-\beta)\sum\beta^{s}\left\{\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_{\beta}\left((1-\tilde{\beta})\left(\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left((1-\tilde{\beta})\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)\right\}$$

$$=(1-\tilde{\beta})\,\mathbb{E}_{\beta}\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_{\beta}\left((1-\tilde{\beta})\left(\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left((1-\tilde{\beta})\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)$$

$$=(1-\tilde{\beta})\,\mathbb{E}_{\beta}\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}-(1-\tilde{\beta})\left(\sum_{u\geq t_{0}}\tilde{\beta}^{u-t_{0}}\left(\hat{P}_{k,u}+\tilde{A}_{k,s}\right)\right)\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}-(1-\tilde{\beta})\sum_{s\geq t_{0}}\tilde{\beta}^{u-t_{0}}\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)$$

Since $\mathcal{E}$ is a variance-covariance matrix, it is positive semi-definite, therefore:

$$(1-\beta)\sum\beta^{s}\frac{\left(1-\varphi^{E}\right)\sigma}{\left(1+\frac{(1-\varphi^{N})\psi}{(1-\varphi^{E})\sigma}\right)}\sum_{k,l}\mathcal{E}_{k,l}\left\{\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_{\beta}\left((1-\tilde{\beta})\left(\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left((1-\tilde{\beta})\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)\right\}\geq 0$$

Next, we have:

$$-(1-\beta)\sum\beta^{s}\sum_{k}\bar{s}_{k}\sum_{l}\complement_{k,l}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\geq 0,$$

$$-(1-\beta)\sum\beta^{s}\left\{\frac{W\partial_{W}\mathcal{N}}{N}\hat{W}_{s}^{2}+\sum_{k}\frac{P_{k}\mathcal{Y}_{k}}{E}\frac{W\partial_{W}\mathcal{Y}_{k}}{\mathcal{Y}_{k}}\hat{W}_{s}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)+\sum_{k}\frac{P_{k}\partial_{P_{k}}\mathcal{N}}{N}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)\hat{W}_{s}+\sum_{k,l}\frac{P_{k}\mathcal{Y}_{k}}{E}\frac{P_{l}\partial_{P_{l}}\mathcal{Y}_{k}}{\mathcal{Y}_{k}}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{\mathbf{A}}_{l,s}\right)\right\}\geq 0,$$

since both the substitution and the transformation matrix are negative semi-definite.

Finally, we can rewrite the redistribution terms as

$$+\mathbb{E}_{\beta}\left(\int-\frac{G''\left(V\left(i\right)\right)v'\left(e\left(i\right)\right)E}{\left(1-\tilde{\beta}\right)G'\left(V\left(i\right)\right)}\left(\nu_{t_0}\left(i\right)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\sum_{l}\frac{e\left(i\right)}{E}s_l\left(i\right)\tilde{A}_{l,s}\right)^2 di\right)$$

$$+\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\int\left(1-\varphi\left(i\right)\right)\frac{E}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left(\nu_{t_0}\left(i\right)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\sum_{l}\frac{e\left(i\right)}{E}s_l\left(i\right)\tilde{A}_{l,s}-\partial_e e_l\left(i\right)\tilde{A}_{l,s}\right)\right)di\right)$$

$$+\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\sum\tilde{\beta}^{s-t_0}\int\varphi\left(i\right)\frac{E}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\aleph_s^{t_0}\left(i\right)+\sum_{l}\frac{e\left(i\right)}{E}s_l\left(i\right)\tilde{A}_{l,s}-\partial_e e_l\left(i\right)\tilde{A}_{l,s}\right\}^2\right)$$

$$+\frac{E}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\mathbb{E}_{\beta}\left\{\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\int\left(1-\varphi\left(i\right)\right)\left\{\aleph_s^{t_0}\left(i\right)+\sum_{l}\frac{e\left(i\right)}{E}s_l\left(i\right)\tilde{A}_{l,s}-\sum_{l}\frac{\sigma e\left(i\right)}{E}\partial_e e_l\left(i\right)\tilde{A}_{l,s}\right\}\right)^2\right\}$$

$$-\frac{E}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\mathbb{E}_{\beta}\left\{\left(\int\left(1-\varphi\left(i\right)\right)\left\{\nu_{t_0}\left(i\right)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\sum_{l}\left(\frac{e\left(i\right)}{E}s_l\left(i\right)\tilde{A}_{l,s}-\frac{\sigma e\left(i\right)}{E}\partial_e e_l\left(i\right)\tilde{A}_{l,s}\right)\right\}di\right)^2\right\}.$$

Note that

$$\left(1-\tilde{\beta}\right)\sum_{s=t_0}^{\infty}\tilde{\beta}^{s-t_0}\aleph_s^{t_0}\left(i\right)=\nu_{t_0}\left(i\right),$$

So

$$\mathbb{E}_{\beta}\left\{\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\int\left(1-\varphi\left(i\right)\right)\left\{\aleph_s^{t_0}\left(i\right)+\sum_{l}\frac{e\left(i\right)}{E}s_l\left(i\right)\tilde{A}_{l,s}-\sum_{l}\frac{\sigma e\left(i\right)}{E}\partial_e e_l\left(i\right)\tilde{A}_{l,s}\right\}\right)^2\right\}$$

$$-\mathbb{E}_{\beta}\left\{\left(\int\left(1-\varphi\left(i\right)\right)\left\{\nu_{t_0}\left(i\right)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\sum_{l}\left(\frac{e\left(i\right)}{E}s_l\left(i\right)\tilde{A}_{l,s}-\frac{\sigma e\left(i\right)}{E}\partial_e e_l\left(i\right)\tilde{A}_{l,s}\right)\right\}di\right)^2\right\}\geq 0.$$

Therefore, all terms in the social welfare function are positive: there are no second order deviations in prices, wage and interest rates improving on the steady state.