Online Appendix for

"Liquidity Reallocation and Run Resolution"

Haelim Anderson Kinda Hachem Simpson Zhang

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Online Appendix A – Proofs

Proof of Lemma 1

If $c_i + x\widetilde{z} < \omega_i$, then bank i has to be liquidated at $t = 1$, i.e., $\ell_i = \widetilde{z}$, and does not survive to $t = 2$. If $c_i + x\widetilde{z} \geq \omega_i$, then the Lagrangian of bank i's problem is

$$\begin{aligned}\mathcal{L} &= c_i + x\ell_i - z_i - \omega_i + (1 + r_z)(\widetilde{z} - \ell_i + z_i) - (1 + r)(1 - \omega_i) - \frac{1}{2}\zeta z_i^2 \\ &\quad + \lambda_i [c_i + x\ell_i - z_i - \omega_i]\end{aligned}$$

where $\lambda_i \geq 0$ is the Lagrange multiplier on the bank's constraint at $t = 1$.

The first order conditions (F.O.C.s) for ℓ_i and z_i are governed by

$$\frac{\partial \mathcal{L}}{\partial \ell_i} = x(1 + \lambda_i) - (1 + r_z)$$

$$\frac{\partial \mathcal{L}}{\partial z_i} = r_z - \lambda_i - \zeta z_i$$

where we do not immediately set the derivatives to zero since ℓ_i and z_i are not assumed to be interior.

There are several cases to consider:

- If $\lambda_i = 0$, then (i) $\frac{\partial \mathcal{L}}{\partial \ell_i} < 0$ and hence $\ell_i = 0$ and (ii) setting $\frac{\partial \mathcal{L}}{\partial z_i} = 0$ delivers $z_i = \frac{r_z}{\zeta}$. Confirming $\lambda_i = 0$ requires that the bank's constraint is satisfied at these solutions, i.e., $c_i \geq \omega_i + \frac{r_z}{\zeta}$.
- If $\lambda_i \in (0, r_z)$, then (i) $\frac{\partial \mathcal{L}}{\partial \ell_i} < 0$ and again $\ell_i = 0$ and (ii) setting $\frac{\partial \mathcal{L}}{\partial z_i} = 0$ delivers $z_i = \frac{r_z - \lambda_i}{\zeta}$. Moreover, with $\lambda_i > 0$, the bank's constraint now binds, so $z_i = c_i - \omega_i$ and hence $\lambda_i = r_z - \zeta(c_i - \omega_i)$. Confirming $\lambda_i \in (0, r_z)$ therefore requires $c_i \in \left(\omega_i, \omega_i + \frac{r_z}{\zeta}\right)$.
- If $\lambda_i \in \left[r_z, \frac{1+r_z}{x} - 1\right)$, then (i) $\frac{\partial \mathcal{L}}{\partial \ell_i} < 0$ and again $\ell_i = 0$ and (ii) $\frac{\partial \mathcal{L}}{\partial z_i} < 0$ for any $z_i > 0$, hence $z_i = 0$. The bank's binding constraint is now $c_i = \omega_i$.
- If $\lambda_i = \frac{1+r_z}{x} - 1$, then (i) $\frac{\partial \mathcal{L}}{\partial z_i} < 0$ and hence $z_i = 0$ and (ii) the bank's binding constraint now delivers $\ell_i = \frac{\omega_i - c_i}{x}$. This is only valid if $\ell_i \in (0, \widetilde{z})$, or equivalently $c_i \in (\omega_i - x\widetilde{z}, \omega_i)$.
- If $\lambda_i > \frac{1+r_z}{x} - 1$, then (i) $\frac{\partial \mathcal{L}}{\partial z_i} < 0$ and again $z_i = 0$ and (ii) $\frac{\partial \mathcal{L}}{\partial \ell_i} > 0$ and hence $\ell_i = \widetilde{z}$. The bank's binding constraint is now $c_i = \omega_i - x\widetilde{z}$.

We can summarize these cases as follows. If $c_i \geq \omega_i + \frac{r_z}{\zeta}$, then $\ell_i = 0$ and $z_i = \frac{r_z}{\zeta}$. Therefore,

$$V_i(\omega_i) = c_i + (1 + r_z)\widetilde{z} - (1 + r) + r\omega_i + \frac{r_z^2}{2\zeta} \geq 0$$

Next consider $c_i \in \left[\omega_i, \omega_i + \frac{r_z}{\zeta}\right)$, in which case $\ell_i = 0$ and $z_i = c_i - \omega_i$. Therefore,

$$V_i(\omega_i) = (1 + r_z)(\widetilde{z} + c_i - \omega_i) - (1 + r)(1 - \omega_i) - \frac{\zeta}{2}(c_i - \omega_i)^2$$

and $V_i(\omega_i) \geq 0$ reduces to

$$\frac{\zeta}{2}(c_i - \omega_i)^2 - (1 + r_z)(c_i - \omega_i) - (1 + r_z)\widetilde{z} + (1 + r)(1 - \omega_i) \leq 0$$

The associated quadratic has roots

$$c_i = \omega_i + \frac{1 + r_z}{\zeta} \pm \frac{1 + r_z}{\zeta}\sqrt{1 + 2\zeta\frac{(1 + r_z)\widetilde{z} - (1 + r)(1 - \omega_i)}{(1 + r_z)^2}}$$

so we need c_i between the smaller and larger roots, which is satisfied by any $c_i \in \left[\omega_i, \omega_i + \frac{r_z}{\zeta}\right)$. Note here that the term in the square root is greater than 1 since

$$(1 + r_z)\widetilde{z} - (1 + r)(1 - \omega_i) \geq (1 + r_z)\widetilde{z} - (1 + r)(1 - \rho) = (1 + r)\left(\rho + \frac{1 + r_z}{1 + r}\widetilde{z} - 1\right) \geq 0$$

where the inequality follows from the ex ante solvency assumption. Accordingly, we have established $V_i(\omega_i) \geq 0$ for any $c_i \geq \omega_i$. Finally, if $c_i \in [\omega_i - x\widetilde{z}, \omega_i)$, then $z_i = 0$ and $\ell_i = \frac{\omega_i - c_i}{x}$. Therefore,

$$V_i(\omega_i) = (1 + r_z)\left(\widetilde{z} - \frac{\omega_i - c_i}{x}\right) - (1 + r)(1 - \omega_i)$$

and $V_i(\omega_i) \geq 0$ reduces to

$$c_i + x\widetilde{z} \geq \omega_i + \frac{(1 + r)(1 - \omega_i)}{\frac{1 + r_z}{x}} \equiv \widetilde{A}(\omega_i)$$

Thus, $c_i \in [\omega_i - x\widetilde{z}, \omega_i)$ delivers $V_i(\omega_i) \geq 0$ if and only if $c_i \geq \widetilde{A}(\omega_i) - x\widetilde{z}$.

We now need to determine where $\widetilde{A}(\omega_i) - x\widetilde{z}$ lies relative to the interval $[\omega_i - x\widetilde{z}, \omega_i)$. Since $\widetilde{A}(\omega_i) \geq \omega_i$, the only cases to consider pertain to the upper bound of the interval. If $\widetilde{A}(\omega_i) - x\widetilde{z} < \omega_i$, then $V_i(\omega_i) \geq 0$ for the subset $c_i \in \left[\widetilde{A}(\omega_i) - x\widetilde{z}, \omega_i\right)$, which establishes $V_i(\omega_i) \geq 0$ if and only if $c_i \geq \widetilde{A}(\omega_i) - x\widetilde{z}$. If instead $\widetilde{A}(\omega_i) - x\widetilde{z} \geq \omega_i$, then $V_i(\omega_i) \geq 0$ if and only if $c_i \geq \omega_i$. Notice that we have now established $V_i(\omega_i) \geq 0$ if and only if $c_i \geq \min\left\{\widetilde{A}(\omega_i) - x\widetilde{z}, \omega_i\right\}$, where this lower bound on c_i is non-decreasing in ω_i.

We conclude with a sufficient condition for the aforementioned lower bound on c_i to always equal $\widetilde{A}(\omega_i) - x\widetilde{z}$ so that $A(\omega_i) = \widetilde{A}(\omega_i)$ for all $\omega_i \in [\rho, 1]$ as in the second part of the statement of the lemma. Rewrite $\widetilde{A}(\omega_i) - x\widetilde{z} \leq \omega_i$ as $\omega_i + \frac{(1+r_z)\widetilde{z}}{1+r} \geq 1$ and recall $\omega_i \geq \rho$ along with the ex ante solvency assumption. Then $c_{\min} \leq \rho$ constitutes a sufficient condition for $\widetilde{A}(\omega_i) - x\widetilde{z} \leq \omega_i$. ■

Proof of Proposition 1

Let $\lambda_i \geq 0$ denote the Lagrange multiplier on the constraint in bank i's optimization problem in Eq. (3). The Lagrangian is then

$$\begin{aligned}\mathcal{L} \;=\;& c_i + x\ell_i - z_i + \Delta_i - \omega_i - \frac{1}{2}\zeta z_i^2 \\ & + (1+r_z)(\widetilde{z} - \ell_i + z_i) - (1+r_b)\Delta_i - (1+r)(1-\omega_i) \\ & + \lambda_i [c_i + x\ell_i - z_i + \Delta_i - \omega_i]\end{aligned}$$

F.O.C. wrt ℓ_i:

$$\frac{\partial \mathcal{L}}{\partial \ell_i} = x(1+\lambda_i) - (1+r_z)$$

F.O.C. wrt z_i:

$$\frac{\partial \mathcal{L}}{\partial z_i} = r_z - \lambda_i - \zeta z_i$$

F.O.C. wrt Δ_i:

$$\frac{\partial \mathcal{L}}{\partial \Delta_i} = \lambda_i - r_b$$

Consider the various cases for ℓ_i:

1. If $\ell_i \in (0, \widetilde{z})$, then $\lambda_i = \frac{1+r_z}{x} - 1 > 0$, where the inequality follows from the assumptions of $r_z > 0$ and $x \in (0,1)$ in the main text. Substituting λ_i into the F.O.C. for z_i:

$$\frac{\partial \mathcal{L}}{\partial z_i} = -(1+r_z)\left(\frac{1}{x} - 1\right) - \zeta z_i < 0$$

   Therefore $z_i = 0$. Now use $\lambda_i > 0$ to conclude that the constraint is binding and hence

$$\Delta_i = \omega_i - c_i - x\ell_i \in (\omega_i - c_i - x\widetilde{z}, \omega_i - c_i) \tag{A.1}$$

   Next, substitute λ_i into the F.O.C. for Δ_i:

$$\frac{\partial \mathcal{L}}{\partial \Delta_i} = \frac{1+r_z}{x} - (1+r_b)$$

   Notice that (A.1) requires Δ_i interior, so $r_b = \frac{1+r_z}{x} - 1$.

2. If $\ell_i = \widetilde{z}$, then $\lambda_i > \frac{1+r_z}{x} - 1 > 0$ and again $z_i = 0$ but now with $\Delta_i = \omega_i - c_i - x\widetilde{z}$. This implies Δ_i interior, which requires $\lambda_i = r_b$. Thus, $r_b > \frac{1+r_z}{x} - 1$.

3. If $\ell_i = 0$, then $\lambda_i < \frac{1+r_z}{x} - 1$.

   (a) If $\lambda_i > r_z$, then $z_i = 0$. Moreover, $\lambda_i > 0$ now implies $\Delta_i = \omega_i - c_i$, which again requires $\lambda_i = r_b$. Thus, $r_b \in \left(r_z, \frac{1+r_z}{x} - 1\right)$.

   (b) If $\lambda_i \in (0, r_z]$, then

$$\frac{\partial \mathcal{L}}{\partial z_i} = 0 \longrightarrow z_i = \frac{r_z - \lambda_i}{\zeta}$$

Moreover, $\lambda_i > 0$ implies a binding constraint, i.e.,

$$c_i - \frac{r_z - \lambda_i}{\zeta} + \Delta_i = \omega_i \longrightarrow \lambda_i = r_z + \zeta\left(\omega_i - c_i - \Delta_i\right)$$

Thus,

$$\frac{\partial \mathcal{L}}{\partial \Delta_i} = r_z - r_b + \zeta\left(\omega_i - c_i - \Delta_i\right)$$

Note that $\frac{\partial \mathcal{L}}{\partial \Delta_i} \neq 0$ will imply a contradiction (e.g., $\frac{\partial \mathcal{L}}{\partial \Delta_i} > 0$ implies that Δ_i is set as high as possible, but setting Δ_i as high as possible implies $\frac{\partial \mathcal{L}}{\partial \Delta_i} < 0$). Accordingly, $\frac{\partial \mathcal{L}}{\partial \Delta_i} = 0$ and hence

$$\Delta_i = \omega_i - c_i + \frac{r_z - r_b}{\zeta}$$

We can then back out

$$\lambda_i = r_b$$

so $r_b \in (0, r_z]$.

(c) If $\lambda_i = 0$, then

$$\frac{\partial \mathcal{L}}{\partial \ell_i} = x - (1 + r_z) < 0 \longrightarrow \ell_i = 0$$

$$\frac{\partial \mathcal{L}}{\partial z_i} = r_z - \zeta z_i \longrightarrow z_i = \frac{r_z}{\zeta}$$

$$\frac{\partial \mathcal{L}}{\partial \Delta_i} = -r_b$$

where the constraint requires

$$\Delta_i \geq \omega_i - c_i + \frac{r_z}{\zeta}$$

If $r_b > 0$, then bank i wants to set Δ_i as low as possible, which means that its constraint cannot be slack. Therefore, $r_b = 0$. But $r_b = 0$ means that Δ_i cancels out of the bank's objective function, so it is without loss of generality for the bank to set Δ_i so that its constraint holds with equality. This bullet then extends the previous one to cover $r_b = 0$.

Putting things together:

1. If $r_b \leq r_z$, then

$$\ell_i = 0$$

$$z_i = \frac{r_z - r_b}{\zeta}$$

$$\Delta_i = \omega_i - c_i + \frac{r_z - r_b}{\zeta}$$

$$V_i\left(\omega_i; r_b\right) = \left(r_z - r_b\right)\widetilde{z} + \left(1 + r_b\right)\left(c_i + \widetilde{z} - 1\right) + \frac{\left(r_z - r_b\right)^2}{2\zeta} + \left(r_b - r\right)\left(1 - \omega_i\right)$$

If $r_b \geq r$, then V_i is (weakly) decreasing in ω_i. In the worst-case scenario of $\omega_i = 1$,

$$V_i(1; r_b) = (r_z - r_b)\widetilde{z} + (1 + r_b)(c_i + \widetilde{z} - 1) + \frac{(r_z - r_b)^2}{2\zeta} > 0$$

so the bank is run-proof and thus $\omega_i = \rho$, i.e., only depositors that really need to withdraw at $t = 1$ do. If instead $r_b < r$, then V_i is increasing in ω_i, i.e., it is less costly for the bank to cover a run by borrowing on the interbank market and repaying the interest rate r_b than it is to pay patient depositors the interest rate r if they do not run. Note that

$$V_i(\rho; r_b) = (1 + r_b) c_i + (1 + r_z)\widetilde{z} - (1 + r) + \frac{(r_z - r_b)^2}{2\zeta} + (r - r_b)\rho > 0$$

where the inequality follows from $q_i \equiv c_i + \widetilde{z} - 1$ and $q_i + r_z\widetilde{z} - r \geq 0$ for all i. Thus, $\omega_i = \rho$ regardless of the sign of $r_b - r$. This is true for all banks so the interbank market clearing condition pins down

$$r_b = \max\left\{r_z - \zeta\left(\int_0^1 c_i di - \rho\right), 0\right\}$$

The case $r_b \leq r_z$ is thus valid if and only if $\int_0^1 c_i di \geq \rho$ (i.e., aggregate cash holdings are sufficient to cover all depositors that really need to withdraw at $t = 1$).

2. If $r_b \in \left(r_z, \frac{1+r_z}{x} - 1\right)$, then

$$\ell_i = 0$$

$$z_i = 0$$

$$\Delta_i = \omega_i - c_i$$

$$V_i(\omega_i; r_b) = (r_z - r_b)\widetilde{z} + (1 + r_b)(c_i + \widetilde{z} - 1) + (r_b - r)(1 - \omega_i)$$

In the worst-case scenario of $\omega_i = 1$,

$$V_i(1; r_b) = (r_z - r_b)\widetilde{z} + (1 + r_b)(c_i + \widetilde{z} - 1)$$

The bank is run-proof if and only if $c_i \geq \overline{c}(r_b)$, where $\overline{c}(\cdot)$ is as defined in Eq. (4). If $c_i \geq \overline{c}(r_b)$, then $\omega_i = \rho$ with ℓ_i, z_i, and Δ_i as above. If $c_i < \overline{c}(r_b)$, then the bank experiences a run at $t = 1$ (i.e., $\omega_i = 1$) and fails with $\ell_i = 1$, $z_i = 0$, and $\Delta_i = 0$. The interbank market clearing condition is given by Eq. (5). The equilibrium value of r_b is pinned down by Eq. (5), with $\overline{c}(\cdot)$ as defined in Eq. (4). Now we need to confirm that the equilibrium value of r_b satisfies $r_b \in \left(r_z, \frac{1+r_z}{x} - 1\right)$. Define

$$H(r_b) \equiv \int_{\left\{i | c_i \geq 1 - \frac{(1+r_z)\widetilde{z}}{1+r_b^*}\right\}} (\rho - c_i)\, di$$

With a continuous distribution of c_i, there exists a unique solution $r_b^* \in \left(r_z, \frac{1+r_z}{x} - 1\right)$ to

$H\left(r_b^*\right) = 0$ if and only if $H\left(r_z\right) > 0$ and $H\left(\frac{1+r_z}{x} - 1\right) < 0$. These conditions are equivalent to $\int_0^1 c_i di < \rho$ and $\int_{\{i|c_i \geq 1-x\widetilde{z}\}} \left(\rho - c_i\right) di < 0$, where the latter can be rewritten as

$$\frac{1}{1 - F\left(1 - x\widetilde{z}\right)} \int_{c_i \geq 1 - x\widetilde{z}} c_i dF\left(c_i\right) > \rho$$

The left-hand side is decreasing in x, so it follows from $c_{\max} > \rho$ that there exists an $\widehat{x} \in (0, 1)$ such that this inequality is satisfied if and only if $x \in (0, \widehat{x})$.

3. If $r_b = \frac{1+r_z}{x} - 1$, then

$$z_i = 0$$

$$\Delta_i = \omega_i - c_i - x\ell_i \in \left(\omega_i - c_i - x\widetilde{z}, \omega_i - c_i\right)$$

$$\ell_i \in \left(0, \widetilde{z}\right)$$

$$V_i\left(\omega_i; r_b\right) = \left(1 + r_z\right)\left(\widetilde{z} - \frac{\omega_i - c_i}{x}\right) - \left(1 + r\right)\left(1 - \omega_i\right)$$

In the worst-case scenario of $\omega_i = 1$,

$$V_i\left(1; r_b\right) = \left(1 + r_z\right)\left(\widetilde{z} - \frac{1 - c_i}{x}\right)$$

The bank is run-proof if and only if $c_i \geq 1 - x\widetilde{z}$, in which case all run-proof banks have $\omega_i = \rho$. Interbank market clearing then pins down aggregate liquidations among run-proof banks as

$$\int_{\{i|c_i \geq 1-x\widetilde{z}\}} \left(\rho - c_i - x\ell_i\right) di = 0 \tag{A.2}$$

Eq. (A.2) has a non-negative solution for aggregate liquidations if and only if $\int_{\{i|c_i \geq 1-x\widetilde{z}\}} \left(\rho - c_i\right) di \geq 0$, which from the previous bullet means $\int_0^1 c_i di < \rho$ and $x \in [\widehat{x}, 1)$. It is also straightforward to confirm that aggregate liquidations do not exceed the maximum feasible amount; $\rho - c_i - x\widetilde{z} < 0$ for all $c_i \geq 1 - x\widetilde{z}$ so Eq. (A.2) could not hold with equality.

4. If $r_b > \frac{1+r_z}{x} - 1$, then

$$\ell_i = \widetilde{z}$$

$$z_i = 0$$

$$\Delta_i = \omega_i - c_i - x\widetilde{z}$$

$$V_i\left(\omega_i; r_b\right) = \left(1 + r_b\right)\left(c_i + x\widetilde{z} - 1\right) + \left(r_b - r\right)\left(1 - \omega_i\right)$$

In the worst-case scenario of $\omega_i = 1$,

$$V_i\left(1; r_b\right) = \left(1 + r_b\right)\left(c_i + x\widetilde{z} - 1\right)$$

The bank is run-proof if and only if $c_i + x\widetilde{z} \geq 1$, in which case all run-proof banks have

$\Delta_i = \rho - (c_i + x\widetilde{z}) < 0$, i.e., all run-proof banks are net lenders on the interbank market. This violates the market clearing condition unless the measure of run-proof banks is zero, i.e., unless $c_{\max} + x\widetilde{z} < 1$. Therefore, with $c_{\max} + x\widetilde{z} > 1$, we can rule out $r_b > \frac{1+r_z}{x} - 1$.

Recall that in autarky the condition for bank i to be run-proof is $c_i \geq 1 - x\widetilde{z}$. Note from the definition of $\overline{c}(\cdot)$ in Eq. (4) that $\overline{c}\left(\frac{1+r_z}{x} - 1\right) = 1 - x\widetilde{z}$. Also recall that we have just shown $r_b^* < \frac{1+r_z}{x} - 1$, and thus $\overline{c}(r_b^*) < 1 - x\widetilde{z}$, when $\int_0^1 c_i di \geq \rho$ or $x \in (0, \widehat{x})$. In words, there are strictly fewer bank runs in the model with interbank trade than in autarky if $\int_0^1 c_i di \geq \rho$ or $x \in (0, \widehat{x})$. If instead $\int_0^1 c_i di < \rho$ and $x \in [\widehat{x}, 1)$, then the measure of bank runs is the same. ■

Proof of Proposition 2

This proof lays out the second-best problem assuming no reallocation from failing banks. For brevity, set $\zeta \to \infty$ as noted at the beginning of Section 3.1 so that $z_i = 0$, leaving only choices of ℓ_i and Δ_i.

Planner's Solution Abstracting from Depositor Behavior We first construct the objective function of a planner who chooses a cash threshold $\widehat{c} \geq c_{\min}$ for run-proofness without access to the resources of run-prone (failing) banks.

Recall $\widetilde{c} \equiv \int_0^1 c_i di$ and let ℓ denote the average liquidation among run-proof banks. Welfare is then

$$
\begin{aligned}
\mathcal{W} &= \widetilde{c} + x\widetilde{z}F(\widehat{c}) + [(1+r_z)(\widetilde{z} - \ell) + x\ell][1 - F(\widehat{c})] \\
&= \widetilde{c} + x\widetilde{z}F(\widehat{c}) + (1+r_z)\widetilde{z}[1 - F(\widehat{c})] - (1 + r_z - x)\underbrace{\ell[1 - F(\widehat{c})]}_{\equiv L}
\end{aligned}
$$

where L is aggregate liquidations.

The resource constraint of each run-proof bank i at $t = 1$ implies

$$\Delta_i = \rho - c_i - x\ell_i$$

Thus, market clearing (or equivalently the aggregate resource constraint) is

$$\int_{\widehat{c}}^{c_{\max}} (\rho - c_i - x\ell_i)\, dF(c_i) = 0$$

Notice that we can express the market clearing condition as

$$\int_{\widehat{c}}^{c_{\max}} (\rho - c_i)\, dF(c_i) = xL$$

which allows us to substitute L of the planner's objective function and write

$$\begin{aligned}\mathcal{W} &= \widetilde{c}+x\widetilde{z}F(\widehat{c})+(1+r_z)\,\widetilde{z}\,[1-F(\widehat{c})]-\left(\frac{1+r_z}{x}-1\right)\int_{\widehat{c}}^{c_{\max}}(\rho-c_i)\,dF(c_i)\\ &= \widetilde{c}+x\widetilde{z}+\left(\frac{1+r_z}{x}-1\right)\int_{\widehat{c}}^{c_{\max}}(c_i+x\widetilde{z}-\rho)\,dF(c_i)\end{aligned}$$

Now consider the constraints faced by the planner choosing $\widehat{c}\geq c_{\min}$. One constraint is that aggregate liquidations cannot be negative, i.e., $L\geq 0$, or equivalently

$$\int_{\widehat{c}}^{c_{\max}}(\rho-c_i)\,dF(c_i)\geq 0$$

Recall that $\overline{c}$ (the decentralized solution) is pinned down by

$$\int_{\overline{c}}^{c_{\max}}(\rho-c_i)\,dF(c_i)=0$$

Thus, the constraint $L\geq 0$ reduces to

$$\int_{\widehat{c}}^{\overline{c}}(\rho-c_i)\,dF(c_i)\geq 0$$

which means the planner chooses $\widehat{c}\leq\overline{c}$. Another constraint is that aggregate liquidations cannot exceed the total liquidation value of run-proof banks, i.e., $L\leq\widetilde{z}\,[1-F(\widehat{c})]$, but this will never bind and can therefore be ignored.[1]

If the planner were to choose $\widehat{c}$ without any other constraints, then the choice would be governed by

$$\eta_0-\eta_1=\left(\frac{1+r_z}{x}-1\right)(\widehat{c}+x\widetilde{z}-\rho)\,f(\widehat{c})$$

where $\eta_0,\eta_1\geq 0$ are Lagrange multipliers on the constraints $\widehat{c}\geq c_{\min}$ and $\widehat{c}\leq\overline{c}$ respectively. Complementary slackness conditions then imply

$$\widehat{c}=\begin{cases}\overline{c} & if \quad x\widetilde{z}\leq\rho-\overline{c}\\ \rho-x\widetilde{z} & if \quad x\widetilde{z}\in(\rho-\overline{c},\rho-c_{\min})\\ c_{\min} & if \quad x\widetilde{z}\geq\rho-c_{\min}\end{cases}\tag{A.3}$$

Next, we turn to implementation of this solution as a decentralized equilibrium. Specifically, if there exist taxes and transfers that achieve the planner's unconstrained choice of $\widehat{c}$, then the choice also satisfies the price-dependent run-proofness constraints introduced by depositor behavior.

[1] To see this formally, use the expression for L from market clearing to write $L\leq\widetilde{z}\,[1-F(\widehat{c})]$ as

$$\int_{\widehat{c}}^{c_{\max}}(c_i+x\widetilde{z}-\rho)\,dF(c_i)\geq 0$$

The left-hand side of this expression is the endogenous part of the planner's objective function, i.e., the planner seeks to maximize it. Moreover, the left-hand side evaluated at $\widehat{c}=\overline{c}$ is strictly positive, i.e., $x\widetilde{z}\,[1-F(\overline{c})]>0$. Thus, the constraint will never hold with equality.

Clearly, this implementation is only relevant for $x\widetilde{z} > \rho - \bar{c}$.

Implementation of Planner's Solution Respecting Depositor Behavior Consider a flat tax rate $\tau > 0$ on interbank borrowing that is rebated (to run-proof banks) through a lump-sum transfer $T > 0$. The tax rate faced by bank i is

$$\tau_i = \begin{cases} 0 & if \quad \Delta_i \leq 0 \\ \tau & if \quad \Delta_i > 0 \end{cases}$$

and implements

$$(1+\tau)(1+r_b) \equiv \frac{1+r_z}{x} \tag{A.4}$$

where $\tau > 0$. Clearly, $\frac{dr_b}{d\tau} < 0$.

The optimality conditions for any bank that chooses $\Delta_i \leq 0$ are as in the baseline model, i.e.,

$$\frac{\partial \mathcal{L}}{\partial \ell_i} = x(1+\lambda_i) - (1+r_z)$$

$$\frac{\partial \mathcal{L}}{\partial \Delta_i} = \lambda_i - r_b$$

Setting $\frac{\partial \mathcal{L}}{\partial \Delta_i} = 0$ delivers $\lambda_i = r_b$, so (i) $\frac{\partial \mathcal{L}}{\partial \ell_i} \propto 1 + r_b - \frac{1+r_z}{x} < 0$ by Eq. (A.4) and hence $\ell_i = 0$ and (ii) $\Delta_i = \omega_i - c_i$.

The banks that choose $\Delta_i > 0$ will be banks with $c_i < \rho$. This can be seen by adding the constraint $\Delta_i \geq 0$ with Lagrange multiplier $\mu_i \geq 0$ to the optimization problem in the proof of Proposition 1 and evaluating the complementary slackness condition. Specifically, with $\zeta \to \infty$, the bank Lagrangian is

$$\begin{aligned} \mathcal{L} \quad = \quad & c_i + x\ell_i + \Delta_i - \omega_i \\ & + (1+r_z)(\widetilde{z} - \ell_i) - (1+\tau)(1+r_b)\Delta_i - (1+r)(1-\omega_i) \\ & + \lambda_i [c_i + x\ell_i + \Delta_i - \omega_i] + \mu_i \Delta_i + T \end{aligned}$$

with optimality conditions

$$\frac{\partial \mathcal{L}}{\partial \ell_i} = x(1+\lambda_i) - (1+r_z)$$

$$\frac{\partial \mathcal{L}}{\partial \Delta_i} = 1 + \lambda_i - (1+\tau)(1+r_b) + \mu_i$$

Setting $\frac{\partial \mathcal{L}}{\partial \Delta_i} = 0$ delivers $\lambda_i = r_b + \tau(1+r_b) - \mu_i$. If $\mu_i = 0$, then $\lambda_i > 0$ and hence $\Delta_i = \omega_i - c_i - x\ell_i$. Moreover, $\frac{\partial \mathcal{L}}{\partial \ell_i} \propto (1+\tau)(1+r_b) - \frac{1+r_z}{x} = 0$. By Eq. (A.4), we get $\ell_i \in [0, \widetilde{z}]$ interior. If $\mu_i > 0$, then $\Delta_i = 0$ and $x\ell_i \geq \omega_i - c_i$ (this need not hold with equality because it is possible to have $\lambda_i = 0$). Moreover, $\frac{\partial \mathcal{L}}{\partial \ell_i} \propto (1+\tau)(1+r_b) - \frac{1+r_z}{x} - \mu_i < 0$ so $\ell_i = 0$ and hence $c_i \geq \omega_i$. In equilibrium, each run-proof bank faces $\omega_i = \rho$, so it is banks with $c_i \geq \rho$ that are constrained, i.e., these banks want to lend. The value of an unconstrained (borrowing) bank is

$$\begin{aligned} V_i(\omega_i; r_b) &= (1+r_z)\widetilde{z} + (1+\tau)(1+r_b)c_i + \underbrace{\left((1+\tau)(1+r_b) - \frac{1+r_z}{x}\right)}_{=0} x\ell_i \\ &\quad - (1+r) - \underbrace{((1+\tau)(1+r_b) - (1+r))}_{=\frac{1+r_z}{x}-(1+r)>0}\omega_i + T \end{aligned}$$

so the run-proofness condition of $V_i(1; r_b) \geq 0$ is

$$c_i \geq 1 - \frac{(1+r_z)\widetilde{z} + T}{(1+\tau)(1+r_b)}$$

which can be rewritten as

$$c_i \geq 1 - x\widetilde{z} - \frac{T}{\frac{1+r_z}{x}}$$

using Eq. (A.4). Thus,

$$\widehat{c} = 1 - x\widetilde{z} - \frac{T}{\frac{1+r_z}{x}} \tag{A.5}$$

The market-clearing condition is

$$\int_{\widehat{c}}^{\rho} (\rho - c_i - x\ell_i)\, dF(c_i) = \int_{\rho}^{c_{\max}} (c_i - \rho)\, dF(c_i)$$

while the budget constraint for the tax/transfer scheme requires

$$\tau \int_{\widehat{c}}^{\rho} (\rho - c_i - x\ell_i)\, dF(c_i) = T[1 - F(\widehat{c})]$$

Satisfying both market clearing and the budget constraint requires

$$\int_{\rho}^{c_{\max}} (c_i - \rho)\, dF(c_i) = \frac{T}{\tau}[1 - F(\widehat{c})] \tag{A.6}$$

Notice that Eqs. (A.5) and (A.6) define a system in terms of τ and T for a given $\widehat{c}$. This system can thus be solved to find the τ and T that implement Eq. (A.3) when $x\widetilde{z} > \rho - \overline{c}$. Accordingly, the second-best for a planner that cannot reallocate away from failing banks is given by Eq. (A.3) and implemented by an interbank borrowing tax rate

$$\tau = \begin{cases} \frac{\frac{1+r_z}{x}(1-\rho)[1-F(\rho - x\widetilde{z})]}{\int_{\rho}^{c_{\max}}(c_i-\rho)dF(c_i)} & if \quad x\widetilde{z} \in (\rho - \overline{c}, \rho - c_{\min}) \\ \frac{\frac{1+r_z}{x}(1-x\widetilde{z}-c_{\min})}{\int_{\rho}^{c_{\max}}(c_i-\rho)dF(c_i)} & if \quad x\widetilde{z} \geq \rho - c_{\min} \end{cases}$$

and a lump-sum transfer

$$T = \begin{cases} \frac{1+r_z}{x}(1-\rho) & if \quad x\widetilde{z} \in (\rho - \overline{c}, \rho - c_{\min}) \\ \frac{1+r_z}{x}(1 - x\widetilde{z} - c_{\min}) & if \quad x\widetilde{z} \geq \rho - c_{\min} \end{cases}$$

to run-proof banks. ■

Proof of Lemma 2

With $\mathcal{W}_b^{(2)}$ and $\mathcal{W}_{RP}^{(2)}$ as per Eqs. (7) and (8) respectively, showing $\mathcal{W}_b^{(2)} < \mathcal{W}_{RP}^{(2)}$ simplifies to showing

$$\int_{\{i|c_i \geq \bar{c}(r_b^*)\}} di < \frac{\int_0^1 c_i di}{\rho}$$

Rewrite this inequality as

$$\int_{\{i|c_i \geq \bar{c}(r_b^*)\}} \rho di < \int_0^1 c_i di$$

Next, use Eq. (5) to substitute out the left-hand side and get

$$\int_{\{i|c_i \geq \bar{c}(r_b^*)\}} c_i di < \int_0^1 c_i di$$

which is true.

To show that this result is not driven by linearity in the welfare function, suppose depositors have concave utility over payoffs, $u(\cdot)$. Then the welfare functions to be compared are

$$\widetilde{\mathcal{W}}_b^{(2)} = \int_{c_i < \bar{c}(r_b^*)} u(c_i + x\widetilde{z})\, dF(c_i) + \int_{c_i \geq \bar{c}(r_b^*)} [\rho u(1) + (1-\rho) u(1+r)]\, dF(c_i)$$

and

$$\widetilde{\mathcal{W}}_{RP}^{(2)} = u(x\widetilde{z}) \left(1 - \frac{\int_0^1 c_i di}{\rho}\right) + [\rho u(1) + (1-\rho) u(1+r)] \frac{\int_0^1 c_i di}{\rho}$$

We define these welfare functions over only depositors since banks always have higher welfare when the measure of run-proof banks is maximized. With $u(\cdot) = \log(\cdot)$ and a uniform cash distribution, $\widetilde{\mathcal{W}}_b^{(2)} < \widetilde{\mathcal{W}}_{RP}^{(2)}$ if and only if

$$\begin{aligned} &\left(\frac{c_{\min} + c_{\max}}{2\rho} - 1 + \frac{2\rho - c_{\max}}{c_{\max} - c_{\min}}\right)(1-\rho)\log(1+r) + \left(1 - \frac{c_{\min} + c_{\max}}{2\rho}\right)\log(x\widetilde{z}) \\ > \;& \frac{(2\rho - c_{\max} + x\widetilde{z})\log(2\rho - c_{\max} + x\widetilde{z}) - (c_{\min} + x\widetilde{z})\log(c_{\min} + x\widetilde{z}) + c_{\min} + c_{\max} - 2\rho}{c_{\max} - c_{\min}} \end{aligned}$$

where we have used Eq. (5) to substitute out $\bar{c}(r_b^*)$ in $\widetilde{\mathcal{W}}_b^{(2)}$. The set of parameters for which this inequality is satisfied is non-empty. For example, it is satisfied for the parameters in Section 5.2.1. ■

Proof of Proposition 3

Recall from the proof of Proposition 2 the objective function

$$\mathcal{W} = \widetilde{c} + x\widetilde{z}F(\widehat{c}) + (1+r_z)\widetilde{z}[1 - F(\widehat{c})] - (1 + r_z - x)L$$

The market clearing (aggregate resource) equation is now

$$\int_{\widehat{c}}^{c_{\max}} (\rho - c_i)\, dF(c_i) = xL + \underbrace{\int_{c_{\min}}^{\widehat{c}} c_i dF(c_i)}_{\text{from reserve pooling}}$$

which can be rewritten as

$$L = \frac{\rho\left[1 - F(\widehat{c})\right] - \widetilde{c}}{x}$$

to isolate aggregate liquidations L. Substitute this expression for L into the objective $\mathcal{W}$ and rearrange to get

$$\mathcal{W} = \widetilde{c} + x\widetilde{z} + \left(\frac{1 + r_z}{x} - 1\right)\left[\left(x\widetilde{z} - \rho\right)\left[1 - F(\widehat{c})\right] + \widetilde{c}\right]$$

Thus,

$$\frac{\partial \mathcal{W}}{\partial \widehat{c}} = \left(\frac{1 + r_z}{x} - 1\right)\left(\rho - x\widetilde{z}\right) f(\widehat{c})$$

If $x\widetilde{z} < \rho$, then the planner will set $\widehat{c}$ as high as possible, specifically $\widehat{c}$ such that $L = 0$. This implies $\widehat{c}$ such that $1 - F(\widehat{c}) = \frac{\widetilde{c}}{\rho}$, so the planner only does reserve pooling here. If instead $x\widetilde{z} > \rho$, then the planner will want to set $\widehat{c} = c_{\min}$, which is implemented with the borrowing tax and transfer scheme $\{\tau, T\}$, specifically

$$\tau = \frac{\frac{1+r_z}{x}\left(1 - x\widetilde{z} - c_{\min}\right)}{\int_{\rho}^{c_{\max}} (c_i - \rho)\, dF(c_i)}$$

$$T = \frac{1 + r_z}{x}\left(1 - x\widetilde{z} - c_{\min}\right)$$

There are no failing banks to pull liquidity away from, so the planner only does the borrowing tax here.

The planner's preference for reserve pooling at low x is not driven by linear utility. To show this formally, take the extreme case of $x \to 0$ where liquidations are impossible so the borrowing tax is not an option. Depositor welfare is

$$\mathcal{D} = \int_{c_{\min}}^{\widehat{c}} u(c_i - \varepsilon)\, dF(c_i) + \left[\rho u(1) + (1 - \rho) u(1 + r)\right]\left[1 - F(\widehat{c})\right]$$

where $\widehat{c}$ solves

$$\int_{\widehat{c}}^{c_{\max}} (\rho - c_i)\, dF(c_i) = \varepsilon F(\widehat{c})$$

and $\varepsilon > 0$ captures a perturbation into reserve pooling. Note

$$\frac{d\mathcal{D}}{d\varepsilon} = -\left[\rho u(1) + (1 - \rho) u(1 + r) - u(\widehat{c} - \varepsilon)\right] f(\widehat{c}) \frac{d\widehat{c}}{d\varepsilon} - \int_{c_{\min}}^{\widehat{c}} u'(c_i - \varepsilon)\, dF(c_i)$$

where

$$\frac{d\widehat{c}}{d\varepsilon} = -\frac{1}{\rho - \widehat{c} + \varepsilon} \frac{F(\widehat{c})}{f(\widehat{c})}$$

Therefore,

$$\frac{d\mathcal{D}}{d\varepsilon} \overset{sign}{=} \frac{\rho u(1) + (1-\rho) u(1+r) - u(\widehat{c}-\varepsilon)}{\rho - \widehat{c} + \varepsilon} - \frac{1}{F(\widehat{c})} \int_{c_{\min}}^{\widehat{c}} u'(c_i - \varepsilon) \, dF(c_i)$$

The condition for $\left.\frac{d\mathcal{D}}{d\varepsilon}\right|_{\varepsilon=0} > 0$ is

$$\frac{\rho u(1) + (1-\rho) u(1+r) - u(\widehat{c}_0)}{\rho - \widehat{c}_0} > \frac{1}{F(\widehat{c}_0)} \int_{c_{\min}}^{\widehat{c}_0} u'(c_i) \, dF(c_i)$$

where $\overline{c}_0$ solves $\int_{\overline{c}_0}^{c_{\max}} (\rho - c_i) \, dF(c_i) = 0$. With a uniform cash distribution and log utility, this condition reduces to

$$(2\rho - c_{\max})(1-\rho)\log(1+r) > \rho \log(2\rho - c_{\max}) - (c_{\max} - \rho)\log(c_{\min})$$

The set of parameters for which this inequality is satisfied is non-empty. For example, it is satisfied for the parameters in Section 5.2.1. ■

Proof of Lemma 3

From the proof of Proposition 1, a most-pessimistic equilibrium when $\int_0^1 c_i di < \rho$ and $x \in (0, \widehat{x})$ involves $\Delta_i = \omega_i - c_i$ and the set of run-proof banks is $S \equiv \{i | c_i \geq \overline{c}(r_b)\}$. Denote by $\widetilde{C}$ the cash from failing banks that gets moved to the run-proof banks when runs are within the banking system.

Suppose bank $i \in S$ gets a fraction $\pi_i \in [0,1]$ of $\widetilde{C}$ that is immediately available to be redeployed, where $\int_{\{i|c_i \geq \overline{c}(r_b)\}} \pi_i di \equiv \Pi \in [0,1]$. Then the new cash level of a run-proof bank i is $c_i + \pi_i \widetilde{C}$ and its new net interbank position is $\Delta_i^{new} = \omega_i - \left(c_i + \pi_i \widetilde{C}\right)$, where $\omega_i = \rho$ for run-proof banks. Market clearing thus becomes

$$\int_{\{i|c_i \geq \overline{c}(r_b)\}} \Delta_i^{new} di = 0$$

$\iff$

$$\int_{\{i|c_i \geq \overline{c}(r_b)\}} \left(\rho - \left(c_i + \pi_i \widetilde{C}\right)\right) di = 0$$

$\iff$

$$\int_{\{i|c_i \geq \overline{c}(r_b)\}} (\rho - c_i) \, di = \Pi \widetilde{C} \tag{A.7}$$

$\iff$

$$|S| = \frac{\int_0^1 c_i di}{\rho} + \frac{\Pi \widetilde{C} - \int_{\{i|c_i < \overline{c}(r_b)\}} c_i di}{\rho}$$

so $|S|$ is increasing in Π.

Suppose $\Pi = 1$, i.e., all the cash from failing banks that gets moved to the run-proof banks is immediately redeployed. Then the maximum level of welfare $\mathcal{W}_{\max}^{(2)}$ is achieved if and only if $\widetilde{C} = \int_{\{i|c_i < \overline{c}(r_b)\}} c_i di$, i.e., all of the cash from failing banks moves to the run-proof banks via withdrawals by patient depositors and none moves out of the system via withdrawals by impatient depositors. This cannot be true when patient and impatient depositors have similar arrival rates at the bank.[2]

As long as $\Pi\widetilde{C} > 0$, Eq. (A.7) will pin down a lower r_b than the solution r_b^* to Eq. (5). Since $\overline{c}(\cdot)$ is the same increasing function of r_b in both equations, Eq. (A.7) also implies a lower measure of failed banks than Eq. (5).

Therefore, relaxing the assumption that all withdrawals are cash-outs from the system leads to an improvement in the decentralized equilibrium but not to the point that a welfare improvement from reserve pooling (if feasible) is eliminated. ■

Proof of Proposition 4

Consider the model with a decentralized interbank market.

If $r_b \in \left(r_z, \frac{1+r_z}{x} - 1\right)$, then

$$V_i(\omega_i; r_b) = (r_z - r_b)\widetilde{z} + (1 + r_b)(c_i + \widetilde{z} - 1) + (r_b - r)(1 - \omega_i)$$

from the proof of Proposition 1. This is the maximized value of bank i at $t = 2$ if it honors withdrawals ω_i at $t = 1$.

With information suppression, $\omega_i = \phi$ for all $i \in [0, 1]$, where $\phi \in [\rho, 1]$ is to be determined in equilibrium. If $\phi \in (\rho, 1)$, then patient depositors are playing a mixed strategy where they withdraw at $t = 1$ with probability $\frac{\phi-\rho}{1-\rho} \in (0, 1)$.

Bank i does not default at $t = 2$ if and only if $V_i(\phi; r_b) \geq 0$, or equivalently,

$$c_i \geq \overline{\overline{c}}(r_b) \equiv \overline{c}(r_b) - \frac{(r_b - r)(1 - \phi)}{1 + r_b} \tag{A.8}$$

where $\overline{c}(\cdot)$ is as defined in Eq. (4). We can see from Eq. (A.8) that $\overline{\overline{c}}(r_b) \leq \overline{c}(r_b)$ for the same interest rate r_b, with strict inequality if and only if $\phi < 1$.

Let r_b^a denote the equilibrium interbank rate with information suppression. Market clearing pins down r_b^a as the solution to

$$\int_{\left\{i | c_i \geq \overline{\overline{c}}\left(r_b^a\right)\right\}} (\phi - c_i)\, di = 0 \tag{A.9}$$

Suppose there exists an equilibrium with $\phi = \rho$. Comparing Eqs. (5) and (A.9), it must be the case that $\overline{\overline{c}}(r_b^a) = \overline{c}(r_b^*)$, where r_b^* is the equilibrium interbank rate without information suppression. Comparing Eqs. (4) and (A.8), we then conclude $\overline{c}(r_b^a) > \overline{c}(r_b^*)$ and $r_b^a > r_b^*$.

Next, define

$$c_\phi \equiv \phi - x\widetilde{z}$$

[2] The liquidation of failing banks happens after the interbank market clears, so while the cash available to split between patient and impatient depositors is ultimately $\int_{\left\{i | c_i < \overline{c}\left(r_b^*\right)\right\}} (c_i + x\widetilde{z})\, di$, only $\int_{\left\{i | c_i < \overline{c}\left(r_b^*\right)\right\}} c_i di$ is available at the time that transfers by patient depositors to other banks can be redeployed. In practice, a receiver will be appointed to liquidate the assets of a failing bank, which is what delays the transfer of liquidation proceeds.

A sufficient condition for $c_\phi < \bar{\bar{c}}\left(r_b^a\right)$ is

$$\phi < 1 - \frac{(1-x)\left(1+r_z\right)\widetilde{z}}{1+r}$$

Suppose this sufficient condition is satisfied. Then, banks with $c_i \in \left(c_\phi, \bar{\bar{c}}\left(r_b^a\right)\right)$ only have to liquidate $\ell_i = \frac{\phi - c_i}{x} < \widetilde{z}$ to satisfy depositor withdrawals at $t = 1$. Welfare is then

$$\begin{aligned}
\mathcal{W}_a^{(2)} &= \widetilde{c} + x\widetilde{z} + \left[\left(1+r_z\right)\widetilde{z} - x\widetilde{z}\right]\int_{\left\{i \mid c_i \geq \bar{\bar{c}}\left(r_b^a\right)\right\}} di \\
&\quad + \underbrace{\left(\frac{1+r_z}{x} - 1\right)}_{\text{positive}} \underbrace{\int_{\left\{i \mid c_i \in \left(c_\phi, \bar{\bar{c}}\left(r_b^a\right)\right)\right\}} \left(c_i + x\widetilde{z} - \phi\right) di}_{\text{positive by definition of } c_\phi}
\end{aligned}$$

If there exists an equilibrium with $\phi = \rho$, then $\mathcal{W}_a^{(2)}$ exceeds $\mathcal{W}_b^{(2)}$ as defined in Eq. (7) if $\rho < 1 - \frac{(1-x)(1+r_z)\widetilde{z}}{1+r}$. ■

Proof of Lemma 4

Bank i is solvent at $t = 2$ if and only if $c_i \geq \bar{\bar{c}}$, where $\bar{\bar{c}}$ is to be determined in an information suppression equilibrium. Assessing his bank under the assumption that everyone withdraws from it at $t = 1$, the expected value of a patient depositor at bank i if he waits is

$$E_{wait} = (1+r)\int_{\left\{i \mid c_i \geq \bar{\bar{c}}\right\}} di$$

while his expected value if he runs is

$$E_{run} = \int_{\left\{i \mid c_i < \bar{\bar{c}}\right\}} \left(c_i + x\widetilde{z}\right) di + \int_{\left\{i \mid c_i \geq \bar{\bar{c}}\right\}} di$$

Therefore, he waits if and only if $E_{wait} \geq E_{run}$, i.e.,

$$r\int_{\left\{i \mid c_i \geq \bar{\bar{c}}\right\}} di \geq \int_{\left\{i \mid c_i < \bar{\bar{c}}\right\}} \left(c_i + x\widetilde{z}\right) di$$

or equivalently

$$r\left(c_{\max} - \bar{\bar{c}}\right) \geq \left(\bar{\bar{c}} - c_{\min}\right)\left(\frac{\bar{\bar{c}} + c_{\min}}{2} + x\widetilde{z}\right) \tag{A.10}$$

when $c_i \sim U\left[c_{\min}, c_{\max}\right]$.

The analysis that follows posits an equilibrium where each bank only experiences withdrawals ρ at $t = 1$ under information suppression then checks if patient depositors at any one bank have an incentive to deviate when assessing expected solvency under the assumption that everyone withdraws from this one bank. If yes, then the posited no-run equilibrium does not exist. If no, then it exists but may not be unique, which we return to at the end of the proof.

In any equilibrium where patient depositors do not run, $\bar{\bar{c}}$ is pinned down by

$$\int_{\{i|c_i \geq \bar{\bar{c}}\}} (c_i - \rho)\, di = 0$$

or equivalently

$$\left(c_{\max} - \bar{\bar{c}}\right)\left(\rho - \frac{c_{\max} + \bar{\bar{c}}}{2}\right) = 0 \tag{A.11}$$

with the uniform distribution. We are assuming that information suppression is only with the public; banks can observe everyone's cash positions so that the interbank market functions as in the benchmark model. Thus, interbank participants do not liquidate any assets when $x \in (0, \widehat{x})$. Recall that we are also assuming $\int_0^1 c_i di < \rho$, so market clearing holds with equality.[3] If $\bar{\bar{c}} < c_{\max}$, then Eq. (A.11) pins down

$$\bar{\bar{c}} = 2\rho - c_{\max}$$

and satisfies $\bar{\bar{c}} < c_{\max}$ if and only if $c_{\max} > \rho$, which we have imposed. Condition (A.10) for no runs by patient depositors then becomes

$$c_{\max}^2 - 2\left(2r + 2\rho + x\widetilde{z}\right)c_{\max} + 4r\rho + \left(2\rho - c_{\min}\right)\left(2\rho + 2x\widetilde{z} + c_{\min}\right) \leq 0$$

The roots of the associated quadratic are

$$c_{\max} = 2r + 2\rho + x\widetilde{z} \pm \sqrt{4r\left(r + \rho + x\widetilde{z}\right) + \left(c_{\min} + x\widetilde{z}\right)^2}$$

Thus, we need

$$c_{\max} \geq 2r + 2\rho + x\widetilde{z} - \sqrt{4r\left(r + \rho + x\widetilde{z}\right) + \left(c_{\min} + x\widetilde{z}\right)^2}$$

Note that $c_{\max}$ less than or equal to the larger root is guaranteed by $\int_0^1 c_i di < \rho$.

We have now shown that there exists a no-run equilibrium under information suppression if $c_{\max}$ is sufficiently high. This need not be the only equilibrium, i.e., the existence of a no-run equilibrium does not mean that information suppression eliminates runs. Suppose we start from an equilibrium where each bank experiences a run then check if patient depositors at any one bank have an incentive to deviate when assessing expected solvency under the assumption that everyone withdraws from this one bank. If yes, then we can claim uniqueness of the no-run equilibrium (at least among pure strategy equilibria). In any equilibrium where patient depositors run, $\bar{\bar{c}} = 1 - x\widetilde{z}$. The no-run condition in (A.10) is then

$$c_{\max} \geq 1 - x\widetilde{z} + \frac{1 - \left(c_{\min} + x\widetilde{z}\right)^2}{2r}$$

This is clearly violated as $r \to 0$, so for sufficiently low r, there is always an equilibrium where all banks experience runs under information suppression even if we assume $c_{\max} + x\widetilde{z} > 1$ to eliminate

[3] If $\int_0^1 c_i di > \rho$, then $\bar{\bar{c}} = c_{\min}$, i.e., condition (A.10) is trivially satisfied and there exists an equilibrium without runs.

such an equilibrium without information suppression. ■

Proof of Proposition 5

Let $\lambda_i \geq 0$ denote the Lagrange multiplier on the first constraint in bank i's optimization problem in Eq. (9). Simplify the second constraint to $k_i \in [0, k_i^{\max}]$, where $k_i^{\max} \leq \min\left\{\nu_i, \widehat{k}_i + \overline{k}\right\}$, with multipliers $\mu_i^0 \geq 0$, and $\mu_i^1 \geq 0$. The Lagrangian is then

$$\begin{aligned}\mathcal{L} &= \widetilde{c} - (\nu_i - k_i) + \left(\overline{\nu} - \overline{k}\right) + x\ell_i - z_i + \Delta_i - \omega_i - \frac{1}{2}\zeta z_i^2 \\ &\quad + (1 + r_z)(\widetilde{z} - \ell_i + z_i) - (1 + r_b)\Delta_i - (1 + r)(1 - \omega_i) - (1 + r_k)\left(k_i - \overline{k}\right) \\ &\quad + \lambda_i\left[\widetilde{c} - (\nu_i - k_i) + \left(\overline{\nu} - \overline{k}\right) + x\ell_i - z_i + \Delta_i - \omega_i\right] + \mu_i^0 k_i + \mu_i^1\left[k_i^{\max} - k_i\right]\end{aligned}$$

F.O.C. wrt ℓ_i:

$$\frac{\partial\mathcal{L}}{\partial\ell_i} = x(1 + \lambda_i) - (1 + r_z)$$

F.O.C. wrt z_i:

$$\frac{\partial\mathcal{L}}{\partial z_i} = r_z - \lambda_i - \zeta z_i$$

F.O.C. wrt Δ_i:

$$\frac{\partial\mathcal{L}}{\partial\Delta_i} = \lambda_i - r_b$$

F.O.C. wrt k_i:

$$\frac{\partial\mathcal{L}}{\partial k_i} = \mu_i^0 - \mu_i^1 + \lambda_i - r_k$$

We restrict attention to combinations of r_k and $\left\{\widehat{k}_i\right\}_{i\in[0,1]}$ that implement $\Delta_i = 0$ for all i (i.e., all interbank trade is conducted through loan certificates) in equilibrium. We then show that there exists such a combination where welfare is higher than in the decentralized equilibrium.

Notice that $\Delta_i = 0$ satisfies the F.O.C. for $\Delta_i \in \mathbb{R}$ if (and only if) $\lambda_i = r_b$. The remaining conditions are:

$$\frac{\partial\mathcal{L}}{\partial\ell_i} \overset{sign}{=} r_b - \left(\frac{1 + r_z}{x} - 1\right)$$

$$\frac{\partial\mathcal{L}}{\partial z_i} \overset{sign}{=} \frac{r_z - r_b}{\zeta} - z_i$$

$$\frac{\partial\mathcal{L}}{\partial k_i} = \mu_i^0 - \mu_i^1 + r_b - r_k$$

In the equilibrium we are considering, no one transacts at the interest rate r_b. However, a value of r_b must still be specified in case a bank were to deviate and use the interbank market off equilibrium, e.g., in the evaluation of run-proofness. Specifying a latent value of r_b effectively selects an equilibrium from a continuum of possible equilibria.[4]

[4] In the absence of an interbank market, the latent r_b could come from a discount window or any form of lender of last resort facility, including one set up by a fiscal rather than monetary authority. Since no one transacts at r_b, the facility is not used in equilibrium and hence its presence does not increase the amount of liquidity in the system, preserving the assumption of $\widetilde{c} < \rho$.

We set the latent interbank rate at $r_b = r_k$ and conjecture $r_k \in \left(r_z, \frac{1+r_z}{x} - 1\right)$, in which case $\ell_i = 0$ and $z_i = 0$. Moreover, $\widetilde{c} - (\nu_i - k_i) + \left(\overline{\nu} - \overline{k}\right) + \Delta_i = \omega_i$ by complementary slackness with $\lambda_i = r_b > 0$. Bank i's maximized value is then

$$\widetilde{V}_i\left(\omega_i; r_k, r_k\right) = V_i\left(\omega_i; r_k\right) = \left(1 + r_z\right)\widetilde{z} + \left(1 + r_k\right)\left(\widetilde{c} - \nu_i + \overline{\nu}\right) - \left(1 + r\right) - \left(r_k - r\right)\omega_i$$

where we recall

$$c_i \equiv \widetilde{c} - \nu_i + \overline{\nu} \tag{A.12}$$

The bank is run-proof if and only if $V_i\left(1; r_k\right) \geq 0$, or equivalently $c_i \geq \overline{c}\left(r_k\right)$, where $\overline{c}\left(\cdot\right)$ is as defined in Eq. (4). Thus, we focus on $\Delta_i = 0$ and

$$\widetilde{c} - \left(\nu_i - k_i\right) + \left(\overline{\nu} - \overline{k}\right) = \rho \tag{A.13}$$

for all i such that $c_i \geq \overline{c}\left(r_k\right)$. Banks that are not run-proof fail at $t = 1$ and are precluded from interbank borrowing, i.e., they also have $\Delta_i = 0$.

With $r_b = r_k$, bank i is indifferent between any $k_i \in [0, k_i^{\max}]$ so we proceed with $k_i = k_i^{\max}$.[5] Consider $\widehat{k}_i = 0$ if $c_i < \overline{c}\left(r_k\right)$ and

$$k_i^{\max} = \begin{cases} \max\left\{\overline{k} - c_i, 0\right\} & if \quad c_i < \overline{c}\left(r_k\right) \\ \min\left\{\nu_i, \widehat{k}_i + \overline{k}\right\} & if \quad c_i \geq \overline{c}\left(r_k\right) \end{cases}$$

where we note

$$\overline{k} - c_i \equiv \nu_i - \left(\widetilde{c} + \left(\overline{\nu} - \overline{k}\right)\right)$$

from Eq. (A.12). In words, any bank i with $c_i < \overline{c}\left(r_k\right)$ receives no initial allocation of loan certificates and must use its available cash, $\widetilde{c} + \left(\overline{\nu} - \overline{k}\right)$, to cover its payment obligations ν_i before it is allowed to use any recirculated loan certificates $\overline{k}$. This corresponds to the recirculation policy in Eq. (11). Then,

$$\overline{k} \equiv \int_0^1 k_i di = \int_{\{i|c_i<\overline{c}(r_k)\}} \max\left\{\overline{k} - c_i, 0\right\} di + \int_{\{i|c_i\geq\overline{c}(r_k)\}} \min\left\{\nu_i, \widehat{k}_i + \overline{k}\right\} di \tag{A.14}$$

where Eq. (A.13) implies

$$\min\left\{\nu_i, \widehat{k}_i + \overline{k}\right\} = \rho + \overline{k} - c_i \tag{A.15}$$

for all i such that $c_i \geq \overline{c}\left(r_k\right)$. Substituting Eq. (A.15) into (A.14) delivers

$$\int_{\{i|c_i\geq\overline{c}(r_k)\}} \left(\rho - c_i\right) di = \int_{\{i|c_i<\overline{c}(r_k)\}} \min\left\{\overline{k}, c_i\right\} di \tag{A.16}$$

[5] Alternatively, we could have set $r_b = r_k + \epsilon$, with $\epsilon > 0$ arbitrarily small, in which case the cutoff for run-proofness would be arbitrarily close to $\overline{c}\left(r_k\right)$ and the F.O.C. for k_i would deliver $k_i = k_i^{\max}$ without indifference. Setting instead $r_b = r_k - \epsilon$ would deliver $k_i = 0$ for all i, i.e., the environment in Proposition 1 with $r_b = r_b^*$, and will be ruled out with $r_k < r_b^*$ as derived below.

which pins down r_k conditional on the initial allocations $\widehat{k}_i$ for $c_i \geq \overline{c}(r_k)$.

Social welfare takes the same form as Eq. (7), but with r_k in place of r_b^*, where r_b^* solves (5). The right-hand side of Eq. (A.16) is strictly positive if loan certificates are issued, i.e., if $\overline{k} > 0$, in which case Eqs. (5) and (A.16) imply

$$\int_{\{i|c_i \geq \overline{c}(r_k)\}} (\rho - c_i)\, di > \int_{\{i|c_i \geq \overline{c}(r_b^*)\}} (\rho - c_i)\, di \tag{A.17}$$

Recall $\overline{c}'(\cdot) > 0$ from Eq. (4). Then, $r_k < r_b^*$ from (A.17) and it follows immediately from Eq. (7) that welfare is higher than in the decentralized equilibrium. Also recall from Proposition 1 that $r_b^* < \frac{1+r_z}{x} - 1$. Thus, Eq. (A.16) delivers $r_k < \frac{1+r_z}{x} - 1$, verifying our initial conjecture about r_k. Verification of $r_k > r_z$ follows trivially from $\int_0^1 c_i di < \rho$.

It only remains to find initial allocations $\widehat{k}_i$ for $c_i \geq \overline{c}(r_k)$ such that $\overline{k} > 0$. Use Eq. (A.12) to rewrite Eq. (A.15) as

$$\min\left\{\nu_i, \widehat{k}_i + \overline{k}\right\} = \left(\rho + \overline{k} - \overline{\nu} - \widetilde{c}\right) + \nu_i \tag{A.18}$$

The planner can satisfy Eq. (A.18) for all $c_i \geq \overline{c}(r_k)$ by setting an initial allocation of loan certificates

$$\widehat{k}_i = \begin{cases} 0 & if \quad c_i < \overline{c}(r_k) \\ \nu_i & if \quad c_i \geq \overline{c}(r_k) \end{cases} \tag{A.19}$$

and an interest rate r_k such that

$$\overline{k} = \overline{\nu} + \widetilde{c} - \rho \tag{A.20}$$

Notice that $\overline{k} > 0$ will require $\overline{\nu} > \rho - \widetilde{c}$. Substituting Eqs. (A.19) and (A.20) into the definition of $\overline{k}$ in Eq. (A.14) delivers

$$\int_{\{i|c_i < \overline{c}(r_k)\}} \min\{\nu_i, \rho\}\, di = \rho - \widetilde{c} \tag{A.21}$$

The left-hand side of Eq. (A.21) is positive, strictly increasing in r_k, and ranges from 0 to at most $\overline{\nu}$. Therefore, with $\overline{\nu} > \rho - \widetilde{c} > 0$, there is a unique solution for r_k. ■

Proof of Lemma 5

If low cash banks use recirculated loan certificates before cash, then $k_i = \min\left\{\nu_i, \overline{k}\right\}$ for $c_i < \overline{c}(r_k)$. The expression for $\overline{k}$, previously given by Eq. (A.14), is now

$$\overline{k} \equiv \int_0^1 k_i di = \int_{\{i|c_i < \overline{c}(r_k)\}} \min\left\{\nu_i, \overline{k}\right\} di + \int_{\{i|c_i \geq \overline{c}(r_k)\}} \min\left\{\nu_i, \widehat{k}_i + \overline{k}\right\} di \tag{A.22}$$

With the allocation in Eq. (A.19), $\widehat{k}_i = \nu_i$ for $c_i \geq \overline{c}(r_k)$. Therefore, Eq. (A.22) becomes

$$\overline{k} = \int_{\{i|c_i < \overline{c}(r_k)\}} \min\left\{\nu_i, \overline{k}\right\} di + \int_{\{i|c_i \geq \overline{c}(r_k)\}} \nu_i di$$

Using Eq. (A.20) to substitute out $\overline{k}$ gives

$$\overline{\nu}+\widetilde{c}-\rho=\int_{\{i|c_i<\overline{c}(r_k)\}}\min\left\{\nu_i,\overline{\nu}+\widetilde{c}-\rho\right\}di+\int_{\{i|c_i\geq\overline{c}(r_k)\}}\nu_i di$$

Then using Eq. (A.12) to substitute out ν_i gives

$$\overline{\nu}+\widetilde{c}-\rho=\int_{\{i|c_i<\overline{c}(r_k)\}}\min\left\{\overline{\nu}+\widetilde{c}-c_i,\overline{\nu}+\widetilde{c}-\rho\right\}di+\int_{\{i|c_i\geq\overline{c}(r_k)\}}\left(\overline{\nu}+\widetilde{c}-c_i\right)di$$

Rewrite as

$$\overline{\nu}+\widetilde{c}-\rho=\int_{\{i|c_i<\overline{c}(r_k),c_i<\rho\}}\left(\overline{\nu}+\widetilde{c}-\rho\right)di+\int_{\{i|c_i<\overline{c}(r_k),c_i\geq\rho\}}\left(\overline{\nu}+\widetilde{c}-c_i\right)di+\int_{\{i|c_i\geq\overline{c}(r_k)\}}\left(\overline{\nu}+\widetilde{c}-c_i\right)di$$

$$\iff$$

$$\overline{\nu}+\widetilde{c}-\rho=\int_{\{i|c_i<\overline{c}(r_k)\}}\left(\overline{\nu}+\widetilde{c}-\rho\right)di+\int_{\{i|c_i<\overline{c}(r_k),c_i\geq\rho\}}\left(\rho-c_i\right)di+\int_{\{i|c_i\geq\overline{c}(r_k)\}}\left(\overline{\nu}+\widetilde{c}-c_i\right)di$$

$$\iff$$

$$\int_{\{i|c_i<\overline{c}(r_k),c_i\geq\rho\}}\left(\rho-c_i\right)di+\int_{\{i|c_i\geq\overline{c}(r_k)\}}\left(\rho-c_i\right)di=0 \tag{A.23}$$

If $\{i|c_i<\overline{c}(r_k),c_i\geq\rho\}\neq\emptyset$, then the first term in Eq. (A.23) is negative. The second term is then also negative since it integrates over higher values of c_i. But then Eq. (A.23) cannot hold, so it follows that $\{i|c_i<\overline{c}(r_k),c_i\geq\rho\}=\emptyset$. Accordingly, Eq. (A.23) simplifies to

$$\int_{\{i|c_i\geq\overline{c}(r_k)\}}\left(\rho-c_i\right)di=0$$

which is the same as Eq. (5), i.e., the equation that pins down r_b^*. ■

Proof of Lemma 6

Consider the parameter space where $z_i=\ell_i=0$, i.e., no liquidations or additional investments made. In the decentralized equilibrium, this means considering parameters that deliver an equilibrium interbank rate of $r_b^*\in\left(r_z,\frac{1+r_z}{x}-1\right)$, i.e., the second bullet in Proposition 1 which motivated corrective intervention.

The value of a bank with cash $c_i\geq\overline{c}\left(r_b^*\right)$ in the decentralized equilibrium is

$$V_i\left(\rho;r_b^*\right)=\left(1+r_z\right)\widetilde{z}+\left(1+r_b^*\right)c_i-\left(1+r\right)-\left(r_b^*-r\right)\rho$$

while banks with $c_i<\overline{c}\left(r_b^*\right)$ fail and have a value of zero. In the allocation with loan certificates, banks with $c_i\geq\overline{c}\left(r_k^*\right)$ get $V_i\left(\rho;r_k^*\right)$ while banks with $c_i<\overline{c}\left(r_k^*\right)$ fail and have a value of zero.

If banks have to choose whether or not to participate in loan certificates before knowing their realization of c_i, then the condition for banks to prefer to participate in loan certificates is

$$\int_{\{i|c_i \geq \overline{c}(r_k^*)\}} V_i\left(\rho; r_k^*\right) di > \int_{\{i|c_i \geq \overline{c}(r_b^*)\}} V_i\left(\rho; r_b^*\right) di$$

or equivalently

$$\int_{\{i|c_i \in (\overline{c}(r_k^*), \overline{c}(r_b^*))\}} V_i\left(\rho; r_k^*\right) di > \left(r_b^* - r_k^*\right) \underbrace{\int_{\{i|c_i \geq \overline{c}(r_b^*)\}} \left(c_i - \rho\right) di}_{=0 \text{ by market clearing}}$$

That is, the ex ante participation constraint for loan certificates simplifies to

$$\int_{\{i|c_i \in (\overline{c}(r_k^*), \overline{c}(r_b^*))\}} V_i\left(\rho; r_k^*\right) di > 0$$

which is true since $V_i\left(\rho; r_k^*\right) > 0$ for all $c_i > \overline{c}\left(r_k^*\right)$. ■

Proof of Proposition 6

Recall Eq. (A.21), which pins down r_k under the loan certificate arrangement in Proposition 5:[6]

$$\int_{\{i|c_i < \overline{c}(r_k)\}} \min\left\{\nu_i, \rho\right\} di = \rho - \widetilde{c}$$

Suppose the parameters are such that the solution to Eq. (A.21) satisfies $\overline{c}\left(r_k\right) \leq \overline{\nu} + \widetilde{c} - \rho$. Then, using Eq. (A.12), the solution must also satisfy $\overline{c}\left(r_k\right) \leq c_i + \nu_i - \rho$ for all i. This implies $\nu_i > \rho$ for any i with $c_i < \overline{c}\left(r_k\right)$, simplifying Eq. (A.21) to

$$\int_{\{i|c_i \geq \overline{c}(r_k)\}} di = \frac{\widetilde{c}}{\rho}$$

Welfare with loan certificates (given by Eq. (7) with r_k in place of r_b^*) then simplifies to

$$\widetilde{c} + x\widetilde{z} + \left[\left(1 + r_z\right)\widetilde{z} - x\widetilde{z}\right]\frac{\widetilde{c}}{\rho}$$

which is exactly the welfare in Eq. (8).

Using Eq. (A.20), $\overline{c}\left(r_k\right) \leq \overline{\nu} + \widetilde{c} - \rho$ can also be expressed as $\overline{c}\left(r_k\right) \leq \overline{k}$. Any bank i with $c_i < \overline{c}\left(r_k\right)$ thus has

$$\nu_i - \left(\widetilde{c} + \left(\overline{\nu} - \overline{k}\right)\right) \equiv \overline{k} - c_i > 0$$

which is to say it uses all of its cash and then some recirculated loan certificates to cover its payment obligations. We call this the "unconstrained case" because payment obligations are such that all of the cash holdings of failing banks can be transferred towards run-proof banks.

[6] Recall that Eq. (A.21) is equivalent to Eq. (12) with c_i as per Eq. (A.12) and $\overline{k}$ as per Eq. (A.20).

If instead $\overline{c}(r_k) > \overline{\nu} + \widetilde{c} - \rho$, then $\overline{c}(r_k) > \overline{k}$ and any bank i with $c_i \in \left(\overline{k}, \overline{c}(r_k)\right)$ has

$$\nu_i - \left(\widetilde{c} + \left(\overline{\nu} - \overline{k}\right)\right) \equiv \overline{k} - c_i < 0$$

which is to say it covers all of its payment obligations with cash and still has some cash left over, i.e., it does not use any recirculated loan certificates. We call this the "constrained case" because payment obligations are such that not all of the cash holdings of failing banks can be transferred towards run-proof banks.

Both sides of Eq. (A.21) are increasing in ρ, with the right-hand side increasing one-for-one and the left-hand side increasing less than one-for-one. The left-hand side is also increasing in r_k, thus Eq. (A.21) defines r_k increasing in ρ. The unconstrained case, $\overline{c}(r_k) \leq \overline{\nu} + \widetilde{c} - \rho$, will therefore require ρ not too high.

Alternatively, consider an increase in payment obligations from ν_i to $\nu_i + \varepsilon$ at all banks i, where $\varepsilon > 0$. From Eq. (A.12), c_i is unchanged. The left-hand side of Eq. (A.21) is then increasing in ε, which means the solution r_k is decreasing in ε. The condition for the unconstrained case, $\overline{c}(r_k) \leq \overline{\nu} + \widetilde{c} - \rho$, is therefore easier to satisfy for higher ε. ■

Proof of Lemma 7

Start from the loan certificate allocation in Proposition 5, where the cutoff cash level is pinned down by Eq. (12). We obtain a closed-form solution for this cutoff, denoted here by $\overline{\overline{c}}_l$, using a uniform cash distribution as in Lemma 4. If $\overline{\overline{c}}_l \leq \overline{k}$, then Eq. (12) is

$$\int_{\overline{\overline{c}}_l}^{c_{\max}} (\rho - c_i)\, dF(c_i) = \int_{c_{\min}}^{\overline{\overline{c}}_l} c_i dF(c_i)$$

which reduces to

$$\rho\left(1 - F\left(\overline{\overline{c}}_l\right)\right) = \widetilde{c}$$

and hence

$$\overline{\overline{c}}_l = c_{\max} - \frac{c_{\max}^2 - c_{\min}^2}{2\rho}$$

with the uniform distribution. Confirming $\overline{\overline{c}}_l \leq \overline{k}$ then requires $c_{\max} \geq \sqrt{2\rho^2 + c_{\min}^2}$. Recall from the proof of Lemma 4 that the cutoff without loan certificates was $\overline{\overline{c}} = 2\rho - c_{\max}$. It is straightforward to show $\overline{\overline{c}}_l < \overline{\overline{c}}$ when $\widetilde{c} < \rho$.

A depositor will get 1 from running and $1 + r$ from waiting for any bank with $c_i \geq \overline{\overline{c}}_l$. For $c_i < \overline{\overline{c}}_l$, the bank has available resources $c_i + k_i - \overline{k} + x\ell_i$ at $t = 1$. Under the loan certificate allocation in Proposition 5, $k_i = \max\left\{\overline{k} - c_i, 0\right\}$. Therefore, any bank with $c_i < \overline{\overline{c}}_l$ has available resources $\max\left\{c_i - \overline{k}, 0\right\} + x\ell_i$ at $t = 1$. When $x \in (0, \widehat{x})$, we have $\overline{\overline{c}} + x\widetilde{z} < 1$ and thus $\overline{\overline{c}}_l + x\widetilde{z} < 1$. It then follows that a depositor will get $c_i + k_i - \overline{k} + x\widetilde{z}$, or equivalently $c_i + x\widetilde{z} - \min\left\{c_i, \overline{k}\right\}$, from running and 0 from waiting for any bank with $c_i < \overline{\overline{c}}_l$. The expected value to a patient depositor of running is therefore

$$E_{run} = \int_{c_{\min}}^{\bar{\bar{c}}_l} \left[c_i + x\widetilde{z} - \min\left\{c_i, \overline{k}\right\}\right] dF\left(c_i\right) + \int_{\bar{\bar{c}}_l}^{c_{\max}} dF\left(c_i\right)$$

while the expected value of waiting is

$$E_{wait} = \int_{\bar{\bar{c}}_l}^{c_{\max}} (1+r)\, dF\left(c_i\right)$$

The depositor waits if and only if $E_{wait} \geq E_{run}$, i.e.,

$$r\left(c_{\max} - \bar{\bar{c}}_l\right) \geq \left(\bar{\bar{c}}_l - c_{\min}\right)\left(\frac{\bar{\bar{c}}_l + c_{\min}}{2} + x\widetilde{z}\right) - \int_{c_{\min}}^{\bar{\bar{c}}_l} \min\left\{c_i, \overline{k}\right\} dc_i \tag{A.24}$$

This expression differs from condition (A.10) in the proof of Lemma 4 by the last term. Note

$$\int_{c_{\min}}^{\bar{\bar{c}}_l} \min\left\{c_i, \overline{k}\right\} dc_i = \frac{1}{2}\left(c_{\max} - \bar{\bar{c}}_l\right)\left(2\rho - c_{\max} - \bar{\bar{c}}_l\right)$$

from Eq. (12) with uniform distribution. We can then rewrite condition (A.24) as

$$Q\left(\bar{\bar{c}}_l\right) \equiv r\left(c_{\max} - \bar{\bar{c}}_l\right) - \left(\bar{\bar{c}}_l - c_{\min}\right)\left(\frac{\bar{\bar{c}}_l + c_{\min}}{2} + x\widetilde{z}\right) + \frac{1}{2}\left(c_{\max} - \bar{\bar{c}}_l\right)\left(2\rho - c_{\max} - \bar{\bar{c}}_l\right) \geq 0$$

where

$$Q'\left(\bar{\bar{c}}_l\right) = -\left(\rho + x\widetilde{z} + r\right) < 0$$

Thus, $Q\left(\bar{\bar{c}}_l\right) > Q\left(\bar{\bar{c}}\right)$ follows from $\bar{\bar{c}}_l < \bar{\bar{c}}$. In other words, the no-run condition under information suppression is easier to satisfy with loan certificates.[7] ■

Proof of Lemma 8

We first rewrite $x(L) = x_0 e^{-\lambda L}$ as

$$\widetilde{L}(x) = \frac{1}{\lambda} \ln\left(\frac{x_0}{x}\right)$$

where $\widetilde{L}(0) = \infty$ and $\widetilde{L}(x_0) = 0$. If $x_0 \leq \widehat{x}$, then the intermediate value theorem implies existence of an $x^* \in (0, \widehat{x})$ such that $\widetilde{L}(x^*) = \widetilde{z}F(\overline{c})$. Since $L(x) = \widetilde{z}F(\overline{c})$ for any $x \in (0, \widehat{x})$, it follows that there exists a solution $x^* \in (0, \widehat{x})$ such that $\widetilde{L}(x^*) = L(x^*)$. That this solution is unique follows from $\widetilde{L}'(x) < 0$.

The direct effect of loan certificates is on the function $L(\cdot)$. Specifically, this policy lowers $\overline{c}$,

[7] We also include here a note on the run equilibrium under information suppression, which was shown to exist for sufficiently low r in the proof of Lemma 4. The maximum measure of run-proof banks is $\frac{\widetilde{c}}{1-x\widetilde{z}}$ if all depositors run on all banks. When loan certificates reallocate all the cash away from banks with $c_i < \bar{\bar{c}}_l$, the condition $E_{wait} \geq E_{run}$ can be written as $r\left[1 - F\left(\bar{\bar{c}}_l\right)\right] \geq x\widetilde{z}F\left(\bar{\bar{c}}_l\right)$, or equivalently $1 - F\left(\bar{\bar{c}}_l\right) \geq \frac{x\widetilde{z}}{r+x\widetilde{z}}$. A necessary condition to eliminate the run equilibrium under information suppression is therefore $\frac{\widetilde{c}}{1-x\widetilde{z}} \geq \frac{x\widetilde{z}}{r+x\widetilde{z}}$, or equivalently $\widetilde{c} \geq \frac{(1-x\widetilde{z})x\widetilde{z}}{r+x\widetilde{z}}$, which is not generically true. Thus, loan certificates do not necessarily eliminate the run equilibrium under information suppression.

so the flat part of the $L(\cdot)$ function is pushed down and delivers a new solution $x^{**} \in (x^*, \widehat{x})$ such that $\widetilde{L}(x^{**}) = L(x^{**})$. Recall that welfare is

$$\mathcal{W} = \widetilde{c} + x\widetilde{z} + [(1+r_z)\widetilde{z} - x\widetilde{z}][1 - F(\overline{c})]$$

for a given cutoff $\overline{c}$. Rewrite as

$$\mathcal{W} = \widetilde{c} + x\widetilde{z}F(\overline{c}) + (1+r_z)\widetilde{z}[1 - F(\overline{c})]$$

Loan certificates now have two positive effects on welfare. First, and as before, they decrease $F(\overline{c})$. Second, they also increase x from x^* to x^{**}. ■

Proof of Lemma 9

Return to the four cases for r_b in the proof of Proposition 1.

If $r_b \leq r_z$, then we have already shown that all banks are run-proof for $\overline{\omega} = 1$ when V_i is weakly decreasing in ω_i (which occurs if $r_b \geq r$) and that all banks are run-proof for $\overline{\omega} = \rho$ when V_i is increasing in ω_i (which occurs if $r_b < r$). Hence, they are all run-proof for any $\overline{\omega} \in [\rho, 1]$ so nothing changes here.

If $r_b \in \left(r_z, \frac{1+r_z}{x} - 1\right)$, then the condition for bank i to be run-proof is

$$V_i(\overline{\omega}; r_b) = (r_z - r_b)\widetilde{z} + (1+r_b)(c_i + \widetilde{z} - 1) + (r_b - r)(1 - \overline{\omega}) \overset{?}{\geq} 0$$

or equivalently

$$c_i \overset{?}{\geq} 1 - \frac{(1+r_z)\widetilde{z}}{1+r_b} - \left(1 - \frac{1+r}{1+r_b}\right)(1 - \overline{\omega}) \equiv \overline{c}(r_b; \overline{\omega})$$

Naturally, $\overline{c}(r_b)$ in Eq. (4) is equivalent to $\overline{c}(r_b; 1)$. Market clearing is still given by Eq. (5), but with $\overline{c}(r_b; \overline{\omega})$ instead of $\overline{c}(r_b)$ in the limits of integration, i.e.,

$$\int_{\{i | c_i \geq \overline{c}(r_b^*; \overline{\omega})\}} (\rho - c_i)\, di = 0 \tag{A.25}$$

Denote by $r_b^*(\overline{\omega})$ the solution to Eq. (A.25). We now need conditions such that $r_b^*(\overline{\omega}) \in \left(r_z, \frac{1+r_z}{x} - 1\right)$. Define

$$\widehat{H}(r_b; \overline{\omega}) \equiv \int_{\{i | c_i \geq \overline{c}(r_b; \overline{\omega})\}} (\rho - c_i)\, di$$

The conditions are again $\widehat{H}(r_z; \overline{\omega}) > 0$ and $\widehat{H}\left(\frac{1+r_z}{x} - 1; \overline{\omega}\right) < 0$. The condition for $\widehat{H}(r_z; \overline{\omega}) > 0$ reduces to $\int_0^1 c_i di < \rho$, as before. For $\widehat{H}\left(\frac{1+r_z}{x} - 1; \overline{\omega}\right) < 0$, note

$$\overline{c}\left(\frac{1+r_z}{x} - 1; \overline{\omega}\right) = 1 - x\widetilde{z} - \left(1 - \frac{x(1+r)}{1+r_z}\right)(1 - \overline{\omega})$$

where

$$\begin{aligned}
\frac{\partial \overline{c}\left(\frac{1+r_z}{x}-1;\overline{\omega}\right)}{\partial x} &= \frac{(1+r)(1-\overline{\omega})}{1+r_z}-\widetilde{z} \\
&\leq \frac{(1+r)(1-\rho)}{1+r_z}-\widetilde{z} \overset{sign}{=} \underbrace{\int_0^1 c_i di-(1+r)\rho}_{<0 \text{ from } \int_0^1 c_i di<\rho}-\underbrace{\left[\int_0^1 c_i di+(1+r_z)\widetilde{z}-(1+r)\right]}_{>0 \text{ from } q_i+r_z\widetilde{z}-r\geq 0 \text{ for all } i}<0
\end{aligned}$$

Thus, there exists an $\widehat{x}(\overline{\omega}) \in (0,1)$ such that $\widehat{H}\left(\frac{1+r_z}{x}-1;\overline{\omega}\right)<0$ if and only if $x \in (0,\widehat{x}(\overline{\omega}))$, similar to before.

The properties of $\overline{c}(r_b;\overline{\omega})$ are as follows:

$$\frac{\partial \overline{c}(r_b;\overline{\omega})}{\partial \overline{\omega}}=\frac{r_b-r}{1+r_b}>0$$

$$\frac{\partial \overline{c}(r_b;\overline{\omega})}{\partial r_b}=\frac{(1+r_z)\widetilde{z}-(1+r)(1-\overline{\omega})}{(1+r_b)^2}>0$$

$$\frac{\partial^2 \overline{c}(r_b;\overline{\omega})}{\partial \overline{\omega}\partial r_b}=\frac{1+r}{(1+r_b)^2}>0$$

Importantly, there is a unique cutoff $\overline{c}(\cdot)$ that satisfies Eq. (A.25). Thus, $\overline{c}(r_b^*(\overline{\omega});\overline{\omega})=\overline{c}(r_b^*(\overline{\omega}');\overline{\omega}')$ for any $\overline{\omega}' \neq \overline{\omega}$, where $\overline{\omega},\overline{\omega}' \in [\rho,1]$. It then follows from $\frac{\partial \overline{c}(r_b;\overline{\omega})}{\partial \overline{\omega}}>0$ and $\frac{\partial \overline{c}(r_b;\overline{\omega})}{\partial r_b}>0$ that $\frac{\partial r_b^*(\overline{\omega})}{\partial \overline{\omega}}<0$. It also follows from $\frac{\partial \overline{c}(r_b;\overline{\omega})}{\partial \overline{\omega}}>0$ that $\frac{\partial \widehat{x}(\overline{\omega})}{\partial \overline{\omega}}>0$.

If $r_b=\frac{1+r_z}{x}-1$, then interbank market clearing pins down aggregate liquidations among run-proof banks as

$$\int_{\left\{i|c_i\geq 1-x\widetilde{z}-\left(1-\frac{x(1+r)}{1+r_z}\right)(1-\overline{\omega})\right\}}(\rho-c_i-x\ell_i)\,di=0$$

which has a non-negative solution for aggregate liquidations if and only if $\int_0^1 c_i di<\rho$ and $x \in [\widehat{x}(\overline{\omega}),1)$, similar to before.

If $r_b>\frac{1+r_z}{x}-1$, then a necessary condition for bank i to be run-proof is

$$V_i(\rho;r_b)=(1+r_b)(c_i+x\widetilde{z}-1)+(r_b-r)(1-\rho)\overset{?}{\geq} 0$$

or equivalently

$$c_i+x\widetilde{z}\overset{?}{\geq} 1-\frac{r_b-r}{1+r_b}(1-\rho)$$

All run-proof banks then have

$$\Delta_i=\rho-(c_i+x\widetilde{z})\leq \rho-\left(1-\frac{r_b-r}{1+r_b}(1-\rho)\right)=-\frac{1+r}{1+r_b}(1-\rho)<0$$

which violates market clearing, as in the proof of Proposition 1, so nothing changes here. ■

Proof of Lemma 10

Suppose depositors at bank i decide whether or not to run on their bank based on whether or not the bank can survive withdrawals by fraction $\overline{\omega}(c_i) \in [\rho, 1]$ of its depositors. Thus, bank i is run-proof if and only if $V_i(\overline{\omega}(c_i); r_b) \geq 0$. Compared to Lemma 9, we are now allowing the degree of pessimism $\overline{\omega}$ to vary across banks. Since banks are distinguished by initial cash holdings, allowing $\overline{\omega}$ to vary across banks reduces to allowing $\overline{\omega}$ to depend on c_i.

If $r_b \leq r_z$, then all banks are run-proof for any $\overline{\omega} \in [\rho, 1]$; see the proof of Lemma 9.

If $r_b \in \left(r_z, \frac{1+r_z}{x} - 1\right)$, then bank i is run-proof if and only if $Y(c_i, r_b) \geq 0$, where

$$Y(c_i, r_b) \equiv c_i - \left(1 - \frac{1+r}{1+r_b}\right)\overline{\omega}(c_i) - \frac{1+r-(1+r_z)\widetilde{z}}{1+r_b}$$

with partial derivatives

$$Y'_{r_b}(c_i, r_b) = \frac{(1+r)(1-\overline{\omega}(c_i)) - (1+r_z)\widetilde{z}}{(1+r_b)^2} < 0$$

and

$$Y'_{c_i}(c_i, r_b) = 1 - \left(1 - \frac{1+r}{1+r_b}\right)\overline{\omega}'(c_i)$$

Assume $\overline{\omega}'(c_i) \leq 0$, in which case $Y'_{c_i}(c_i, r_b) > 0$. Define $\underline{c}(r_b)$ as the unique solution to $Y(\underline{c}(r_b), r_b) = 0$. Then bank i is run-proof if and only if $c_i \geq \underline{c}(r_b)$ and the equilibrium interbank rate r_b' solves $\int_{\{i|c_i \geq \underline{c}(r_b')\}} (\rho - c_i)\, di = 0$. The condition for $r_b' > r_z$ is again $\int_0^1 c_i di < \rho$; this follows from the fact that $Y(c_i, r_z) \geq 0$ for all c_i. The condition for $r_b' < \frac{1+r_z}{x} - 1$ again takes the form of x below some threshold, call it now $\widehat{\widehat{x}}$. This follows from the fact that

$$Y\left(c_i, \frac{1+r_z}{x} - 1\right) = c_i + x\widetilde{z} - 1 + \left(1 - \frac{x(1+r)}{1+r_z}\right)(1 - \overline{\omega}(c_i))$$

which is increasing in c_i and x. Thus, higher x implies lower $\underline{c}\left(\frac{1+r_z}{x} - 1\right)$, which means that $\int_{\{i|c_i \geq \underline{c}\left(\frac{1+r_z}{x}-1\right)\}} (\rho - c_i)\, di$ is negative if and only if x is low. Again, $\underline{c}(r_b') = \overline{c}(r_b^*)$ but the interbank rates will be different.

It is straightforward to show that $r_b > \frac{1+r_z}{x} - 1$ cannot be an equilibrium; see again the proof of Lemma 9. Thus, $\int_0^1 c_i di < \rho$ and $x \geq \widehat{\widehat{x}}$ are associated with $r_b = \frac{1+r_z}{x} - 1$. ■

Proof of Lemma 11

The environment is the same as in Lemma 9 except that r_b is now determined in a market with strategic traders. For brevity, set $x \to 0$ and $\zeta \to \infty$ so that $\ell_i = z_i = 0$ and bank i's only choice is

Δ_i. The Lagrangian for the bank's problem is

$$\begin{aligned}\mathcal{L} &= c_i + \Delta_i - \omega_i \\ &\quad + (1+r_z)\,\widetilde{z} - (1+r_b)\,\Delta_i - (1+r)\,(1-\omega_i) \\ &\quad + \lambda_i\left[c_i + \Delta_i - \omega_i\right]\end{aligned}$$

and the F.O.C. is

$$\frac{\partial \mathcal{L}}{\partial \Delta_i} = \lambda_i - r_b - \alpha_i \Delta_i$$

where $\alpha_i \equiv \frac{\partial r_b}{\partial \Delta_i} > 0$ is the price impact of bank i in the interbank market. If the solution for Δ_i is interior, then

$$\frac{\partial \mathcal{L}}{\partial \Delta_i} = 0 \longrightarrow \Delta_i = \frac{\lambda_i - r_b}{\alpha_i}$$

At pessimism $\overline{\omega}$, any run-proof bank i must satisfy the constraint

$$c_i + \Delta_i \geq \overline{\omega}$$

There are two cases:

1. The constraint is slack, i.e., $\lambda_i = 0$. Then $\Delta_i = -\frac{r_b}{\alpha_i}$ and we need $c_i \geq \overline{\omega} + \frac{r_b}{\alpha_i}$ to confirm that the constraint is indeed slack. The maximized value of the bank is

$$c_i + \widetilde{z} - 1 + r_z \widetilde{z} - r\,(1-\overline{\omega}) + \frac{r_b^2}{\alpha_i} \geq 0$$

   where the inequality follows from the ex ante solvency assumptions, $q_i \equiv c_i + \widetilde{z} - 1 \geq 0$ and $q_i + r_z \widetilde{z} - r \geq 0$. Thus, all banks in this case are run-proof, which means they face $\omega_i = \rho$. The case then corresponds to $c_i \geq \rho + \frac{r_b}{\alpha_i}$.

2. The constraint is binding, i.e., $\lambda_i > 0$. Then $\Delta_i = \overline{\omega} - c_i$ and we can solve for λ_i from the first order condition for Δ_i, which yields that we need $c_i < \overline{\omega} + \frac{r_b}{\alpha_i}$ to confirm that the constraint is indeed binding. The maximized value of the bank is

$$(1+r_z)\,\widetilde{z} - (1+r_b)\,(\overline{\omega} - c_i) - (1+r)\,(1-\overline{\omega})$$

   so, as in Lemma 9, the bank is run-proof if and only if $c_i \geq \overline{c}\,(r_b;\overline{\omega})$. The properties $\frac{\partial \overline{c}(r_b;\overline{\omega})}{\partial \overline{\omega}} > 0$ and $\frac{\partial \overline{c}(r_b;\overline{\omega})}{\partial r_b} > 0$ follow immediately from the proof of Lemma 9. The set of run-proof banks in this case then corresponds to $c_i \in \left[\overline{c}\,(r_b;\overline{\omega})\,,\rho + \frac{r_b}{\alpha_i}\right)$ and all banks in this set have $\Delta_i = \rho - c_i$.

Assume symmetry in price impacts, i.e., $\alpha_i = \alpha$ for all i. With some abuse of notation, we keep the integral notation from the baseline model to denote aggregation, rather than introducing summations over N banks and defining cash sets. This reduces clutter without affecting the intuition.

The market clearing condition is then

$$\int_{\overline{c}(r_b;\overline{\omega})}^{\rho+\frac{r_b}{\alpha}} (\rho - c_i)\, dF(c_i) + \int_{\rho+\frac{r_b}{\alpha}}^{c_{\max}} \left(-\frac{r_b}{\alpha}\right) dF(c_i) = 0 \tag{A.26}$$

which can be rewritten as

$$\underbrace{\int_{\overline{c}(r_b;\overline{\omega})}^{c_{\max}} (\rho - c_i)\, dF(c_i)}_{\equiv W(\overline{c}(r_b;\overline{\omega}))} + \underbrace{\int_{\rho+\frac{r_b}{\alpha}}^{c_{\max}} \left(c_i - \left(\rho + \frac{r_b}{\alpha}\right)\right) dF(c_i)}_{\equiv X\left(\frac{r_b}{\alpha}\right)} = 0$$

to isolate the first term $W(\overline{c}(r_b;\overline{\omega}))$, which is the same as before, and the second term $X\left(\frac{r_b}{\alpha}\right)$, which is the result of strategic trading. The second term can be understood as effective borrowing by lenders who are not lending at their maximum capacity because of the price impact. An increase in price impact would increase the effective borrowing by such lenders both on the extensive margin (by increasing the set of lenders who do not lend at maximum capacity) and on the intensive margin (by increasing the distance between actual lending and maximum capacity at each of these lenders). If the price impact is very small, then the second term disappears, i.e., $\lim_{\alpha\to 0}\left(\rho + \frac{r_b}{\alpha}\right) > c_{\max}$, and we are back to the Walrasian market where each lender lends at maximum capacity.

Differentiate Eq. (A.26) with respect to $\overline{\omega}$:

$$-\left(\rho - \overline{c}(r_b;\overline{\omega})\right) f\left(\overline{c}(r_b;\overline{\omega})\right) \frac{d\overline{c}(r_b;\overline{\omega})}{d\overline{\omega}} - \left[1 - F\left(\rho + \frac{r_b}{\alpha}\right)\right] \frac{d\left(\frac{r_b}{\alpha}\right)}{d\overline{\omega}} = 0$$

$$\iff \quad \underbrace{\frac{\alpha\left(\rho - \overline{c}(r_b;\overline{\omega})\right) f\left(\overline{c}(r_b;\overline{\omega})\right)}{1 - F\left(\rho + \frac{r_b}{\alpha}\right)}}_{>0} \frac{d\overline{c}(r_b;\overline{\omega})}{d\overline{\omega}} + \left(\frac{dr_b}{d\overline{\omega}} - \frac{r_b}{\alpha}\frac{d\alpha}{d\overline{\omega}}\right) = 0 \tag{A.27}$$

where the sign reflects that market clearing requires $\left(\rho - \overline{c}(r_b;\overline{\omega})\right) > 0$; see Eq. (A.26).

Pinning down the price impact α separately from the interbank rate r_b requires an additional equation. We restrict attention to protocols where $\overline{\omega}$ affects α only through $\overline{c}(\cdot)$, i.e., $\frac{d\alpha}{d\overline{\omega}} = \frac{d\alpha}{d\overline{c}(\cdot)}\frac{d\overline{c}(\cdot)}{d\overline{\omega}}$. Any reasonable method for determining the price impact should then have the property $\frac{d\alpha}{d\overline{c}(\cdot)} < 0$. To see this, notice from Eq. (A.26) that a marginal increase in $\overline{c}(\cdot)$ removes only borrowers from the interbank market. The resulting market composition therefore favors borrowers relative to lenders, so an additional unit of borrowing should increase the interbank rate by less than before.

We can now rewrite Eq. (A.27) as

$$\frac{dr_b}{d\overline{\omega}} = -\underbrace{\left(\frac{\alpha\left(\rho - \overline{c}(r_b;\overline{\omega})\right) f\left(\overline{c}(r_b;\overline{\omega})\right)}{1 - F\left(\rho + \frac{r_b}{\alpha}\right)} - \frac{r_b}{\alpha}\frac{d\alpha}{d\overline{c}(r_b;\overline{\omega})}\right)}_{>0} \frac{d\overline{c}(r_b;\overline{\omega})}{d\overline{\omega}} \tag{A.28}$$

where we recall

$$\frac{d\overline{c}\left(r_b;\overline{\omega}\right)}{d\overline{\omega}} \equiv \underbrace{\frac{\partial\overline{c}\left(r_b;\overline{\omega}\right)}{\partial\overline{\omega}}}_{>0} + \underbrace{\frac{\partial\overline{c}\left(r_b;\overline{\omega}\right)}{\partial r_b}}_{>0}\frac{dr_b}{d\overline{\omega}} \tag{A.29}$$

from the proof of Lemma 9. If $\frac{d\overline{c}(r_b;\overline{\omega})}{d\overline{\omega}} \leq 0$, then Eq. (A.28) implies $\frac{dr_b}{d\overline{\omega}} \geq 0$. But $\frac{dr_b}{d\overline{\omega}} \geq 0$ implies $\frac{d\overline{c}(r_b;\overline{\omega})}{d\overline{\omega}} > 0$ from Eq. (A.29), which is a contradiction. Therefore, $\frac{d\overline{c}(r_b;\overline{\omega})}{d\overline{\omega}} > 0$. ■

Online Appendix B – Moral Hazard Extension

The environment is as laid out in Section 6.1. We first study the decentralized equilibrium, including conditions under which an intervention that decreases the interbank rate (in the positive-withdrawal state) will lead to a lower ex ante choice of cash by banks, i.e., a moral hazard channel exists. We then show that loan certificates can still improve welfare under these conditions.

Decentralized Market

In the zero-withdrawal state,

$$V\left(c,\widetilde{\nu}_i;r_b^0,0\right)=f\left(1+q-c-\ell\left(\widetilde{\nu}_i,0\right)\right)+\left(1+r_b^0\right)\left(c+\widetilde{\nu}_i+x\ell\left(\widetilde{\nu}_i,0\right)\right)-\left(1+r\right)$$

Since $c+\widetilde{\nu}_i\geq 0$ for all i, every bank is a net lender on the interbank market, which motivates $r_b^0=0$, in which case

$$V\left(c,\widetilde{\nu}_i;0,0\right)=f\left(1+q-c-\ell\left(\widetilde{\nu}_i,0\right)\right)+c+\widetilde{\nu}_i+x\ell\left(\widetilde{\nu}_i,0\right)-\left(1+r\right)$$

and it is clear from $f'\left(\cdot\right)>1+r$ and $x<1$ that the bank chooses $\ell\left(\widetilde{\nu}_i,0\right)=0$, hence

$$V^*\left(c,\widetilde{\nu}_i;0,0\right)=f\left(1+q-c\right)+c+\widetilde{\nu}_i-\left(1+r\right)$$

In the positive-withdrawal state,

$$V\left(c,\widetilde{\nu}_i;r_b^\rho,\omega\right)=f\left(1+q-c-\ell\left(\widetilde{\nu}_i,\omega\right)\right)-\left(1+r_b^\rho\right)\left(\omega-x\ell\left(\widetilde{\nu}_i,\omega\right)-\widetilde{\nu}_i-c\right)-\left(1+r\right)\left(1-\omega\right)$$

$$\frac{\partial V\left(c,\widetilde{\nu}_i;r_b^\rho,\omega\right)}{\partial\ell}=-f'\left(1+q-c-\ell\left(\widetilde{\nu}_i,\omega\right)\right)+x\left(1+r_b^\rho\right)$$

$$1+r_b^\rho<\frac{f'\left(1+q-c\right)}{x}\longrightarrow\ell\left(\widetilde{\nu}_i,\omega\right)=0 \tag{B.1}$$

Supposing for now (B.1) is true (we return to it below),

$$V^*\left(c,\widetilde{\nu}_i;r_b^\rho,\omega\right)=f\left(1+q-c\right)+\left(1+r_b^\rho\right)\left(c+\widetilde{\nu}_i-\omega\right)-\left(1+r\right)\left(1-\omega\right)$$

$$V^*\left(c,\widetilde{\nu}_i;r_b^\rho,1\right)\geq 0\Longleftrightarrow\widetilde{\nu}_i\geq 1-c-\frac{f\left(1+q-c\right)}{1+r_b^\rho}\equiv\widetilde{\nu}\left(c,r_b^\rho\right)$$

where $\widetilde{\nu}_i\geq\max\left\{-\nu,\widetilde{\nu}\left(c,r_b^\rho\right)\right\}$ is the set of run-proof banks in the positive-withdrawal state.

At $t=0$, the representative bank chooses c to maximize

$$V\left(c;r_b^\rho,0\right)\equiv\pi\int_{\max\left\{-\nu,\widetilde{\nu}\left(c,r_b^\rho\right)\right\}}^{\nu}V^*\left(c,\widetilde{\nu}_i;r_b^\rho,\rho\right)dG\left(\widetilde{\nu}_i\right)+\left(1-\pi\right)\int_{-\nu}^{\nu}V^*\left(c,\widetilde{\nu}_i;0,0\right)dG\left(\widetilde{\nu}_i\right)$$

which expands to

$$
\begin{aligned}
V\left(c; r_b^\rho, 0\right) &= \pi \int_{\max\left\{-\nu, 1-c-\frac{f(1+q-c)}{1+r_b^\rho}\right\}}^{\nu} \left[f\left(1+q-c\right) + \left(1+r_b^\rho\right)\left(c+\widetilde{\nu}_i-\rho\right) - \left(1+r\right)\left(1-\rho\right)\right] dG\left(\widetilde{\nu}_i\right) \\
&\quad + \left(1-\pi\right)\left[f\left(1+q-c\right) + c - \left(1+r\right)\right]
\end{aligned}
$$

Definition B.1. *The decentralized equilibrium is a pair* $\left\{c, r_b^\rho\right\}$ *solving (i) the bank's first order condition*

$$
\frac{\partial V\left(c; r_b^\rho, 0\right)}{\partial c} = 0
$$

and (ii) market clearing in the positive-withdrawal state

$$
\int_{\max\left\{-\nu, 1-c-\frac{f(1+q-c)}{1+r_b^\rho}\right\}}^{\nu} \left(c+\widetilde{\nu}_i-\rho\right) dG\left(\widetilde{\nu}_i\right) = 0
$$

where we check that the solution to these equations satisfies $1+r_b^\rho < \frac{f'(1+q-c)}{x}$ *to confirm* $\ell\left(\widetilde{\nu}_i, \rho\right) = 0$ *for run-proof banks, i.e., banks with* $\widetilde{\nu}_i \in \left[\max\left\{-\nu, 1-c-\frac{f(1+q-c)}{1+r_b^\rho}\right\}, \nu\right]$.

Characterization of Equilibrium There are two cases to consider:

1. The bank chooses c such that $1-c-\frac{f(1+q-c)}{1+r_b^\rho} \leq -\nu$, in which case market clearing will ultimately pin down $c=\rho$. The bank's Lagrangian in this case is

$$
\mathcal{L} = f\left(1+q-c\right) + \left(1+\pi r_b^\rho\right)c - \left(1+r\right) - \pi\rho\left(r_b^\rho - r\right) - \lambda\left(\nu + 1 - c - \frac{f\left(1+q-c\right)}{1+r_b^\rho}\right)
$$

   where $\lambda \geq 0$ is a Lagrange multiplier. The first order condition is

$$
f'\left(1+q-c\right) = 1 + \pi r_b^\rho + \lambda\left(1 - \frac{f'\left(1+q-c\right)}{1+r_b^\rho}\right)
$$

   If $\lambda = 0$, then the equilibrium interbank rate is

$$
r_b^\rho = \frac{f'\left(1+q-\rho\right) - 1}{\pi}
$$

   Confirming $\lambda = 0$ then requires confirming that the constraint on the bank's problem is slack at this solution, i.e.,

$$
\pi \geq \frac{f'\left(1+q-\rho\right) - 1}{\frac{f(1+q-\rho)}{1+\nu-\rho} - 1} \equiv \pi_\lambda
$$

   If instead $\pi < \pi_\lambda$, then $\lambda > 0$, i.e., the constraint binds, which pins down the equilibrium interbank rate as

$$
r_b^\rho = \frac{f\left(1+q-\rho\right)}{1+\nu-\rho} - 1
$$

   as long as we are in this case.

2. The bank chooses c such that $1-c-\frac{f(1+q-c)}{1+r_b^\rho}>-\nu$, in which case

$$\frac{\partial V\left(c;r_b^\rho\right)}{\partial c} = \pi\left[1+r_b^\rho-f'\left(1+q-c\right)\right]\left[\begin{array}{c} 1-G\left(1-c-\frac{f(1+q-c)}{1+r_b^\rho}\right) \\ +\frac{\left(r_b^\rho-r\right)(1-\rho)g\left(1-c-\frac{f(1+q-c)}{1+r_b^\rho}\right)}{1+r_b^\rho} \end{array}\right] - \left(1-\pi\right)\left[f'\left(1+q-c\right)-1\right]$$

With a uniform distribution, the bank's first order condition reduces to

$$0 = \pi\left[1+r_b^\rho-f'\left(1+q-c\right)\right]\left[1-\frac{1+\nu-c-\frac{f(1+q-c)}{1+r_b^\rho}}{2\nu}+\frac{\left(r_b^\rho-r\right)\left(1-\rho\right)}{1+r_b^\rho}\frac{1}{2\nu}\right] - \left(1-\pi\right)\left[f'\left(1+q-c\right)-1\right] \tag{B.2}$$

and market clearing is

$$\left(c-\rho+\frac{\nu+\left(1-c-\frac{f(1+q-c)}{1+r_b^\rho}\right)}{2}\right)\frac{\nu-\left(1-c-\frac{f(1+q-c)}{1+r_b^\rho}\right)}{2\nu}=0$$

Positive interbank trade requires

$$\nu>1-c-\frac{f\left(1+q-c\right)}{1+r_b^\rho}$$

in which case market clearing is simply

$$c-\rho+\frac{\nu+\left(1-c-\frac{f(1+q-c)}{1+r_b^\rho}\right)}{2}=0\longleftrightarrow 1+r_b^\rho=\frac{f\left(1+q-c\right)}{1+\nu+c-2\rho} \tag{B.3}$$

and the condition for positive interbank trade simplifies to $c+\nu>\rho$, which we will assume for now. It is clear that this case will pin down $c<\rho$. Using market clearing to substitute the interbank rate out of the first order condition gives one equation in only c,

$$\underbrace{\frac{f\left(1+q-c\right)}{1+\nu+c-2\rho}}_{LHS}=\underbrace{f'\left(1+q-c\right)+\frac{1-\pi}{\pi}\frac{f'\left(1+q-c\right)-1}{1-\frac{\rho-c}{\nu}+\left(1-\frac{1+r}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)\frac{1-\rho}{2\nu}}}_{RHS}$$

where

$$\frac{\partial LHS}{\partial c}=-\frac{f'\left(1+q-c\right)}{1+\nu+c-2\rho}-\frac{f\left(1+q-c\right)}{\left(1+\nu+c-2\rho\right)^2}<0$$

$$
\begin{aligned}
\frac{\partial RHS}{\partial c} &= -f''(1+q-c)\left(1+\frac{\frac{1-\pi}{\pi}}{1-\frac{\rho-c}{\nu}+\left(1-\frac{1+r}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)\frac{1-\rho}{2\nu}}\right) \\
&\quad -\frac{\frac{1-\pi}{\pi}\frac{1}{\nu}\left[f'(1+q-c)-1\right]}{\left(1-\frac{\rho-c}{\nu}+\left(1-\frac{1+r}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)\frac{1-\rho}{2\nu}\right)^2}\left(1-\frac{1+\frac{f'(1+q-c)}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}}{2}\frac{(1+r)(1-\rho)}{f(1+q-c)}\right)
\end{aligned}
$$

If $f''(\cdot)$ is sufficiently negative, then $\frac{\partial RHS}{\partial c}>0$, in which case we can confirm $c<\rho$ with $LHS<RHS$ at $c=\rho$, or equivalently,

$$
\pi<\frac{\frac{f'(1+q-\rho)-1}{\frac{f(1+q-\rho)}{1+\nu-\rho}-1}}{1+\frac{\frac{f(1+q-\rho)}{1+\nu-\rho}-f'(1+q-\rho)}{\frac{f(1+q-\rho)}{1+\nu-\rho}-1}\left(1-\frac{1+r}{\frac{f(1+q-\rho)}{1+\nu-\rho}}\right)\frac{1-\rho}{2\nu}}\equiv\pi_\rho
$$

We can also confirm $c\geq\nu$ with $LHS\geq RHS$ at $c=\nu$, or equivalently,

$$
\pi\geq\frac{\frac{f'(1+q-\nu)-1}{\frac{f(1+q-\nu)}{1+2(\nu-\rho)}-1}}{1+\frac{\frac{f(1+q-\nu)}{1+2(\nu-\rho)}-f'(1+q-\nu)}{\frac{f(1+q-\nu)}{1+2(\nu-\rho)}-1}\left(\left(1-\frac{1+r}{\frac{f(1+q-\nu)}{1+2(\nu-\rho)}}\right)\frac{1-\rho}{2\nu}-\frac{\rho-\nu}{\nu}\right)}\equiv\pi_\nu \text{ if } 1-\frac{\rho-\nu}{\nu}+\left(1-\frac{1+r}{\frac{f(1+q-\nu)}{1+2(\nu-\rho)}}\right)\frac{1-\rho}{2\nu}>0
$$

and $\pi\geq 0$ if the latter condition is not met. To ensure $\pi_\nu<\pi_\rho$, note

$$
\frac{\partial RHS}{\partial \pi}=-\frac{1}{\pi^2}\frac{f'(1+q-c)-1}{1-\frac{\rho-c}{\nu}+\left(1-\frac{1+r}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)\frac{1-\rho}{2\nu}}
$$

$$
\frac{\partial\left(1-\frac{\rho-c}{\nu}+\left(1-\frac{1+r}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)\frac{1-\rho}{2\nu}\right)}{\partial c}=\frac{1}{\nu}\left(1-\frac{1+\frac{f'(1+q-c)}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}}{2}\frac{(1+r)(1-\rho)}{f(1+q-c)}\right)
$$

$$
\frac{\partial\left(\frac{1+\frac{f'(1+q-c)}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}}{2}\frac{(1+r)(1-\rho)}{f(1+q-c)}\right)}{\partial c}>0
$$

$\implies$

$$
\frac{\partial\left(1-\frac{\rho-c}{\nu}+\left(1-\frac{1+r}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)\frac{1-\rho}{2\nu}\right)}{\partial c}>\frac{1}{\nu}\left(1-\frac{1+\frac{f'(1+q-\rho)}{\frac{f(1+q-\rho)}{1+\nu-\rho}}}{2}\frac{(1+r)(1-\rho)}{f(1+q-\rho)}\right)>0 \text{ for } c\leq\rho
$$

where the second inequality follows from the fact that $\frac{f(1+q-\rho)}{1-\rho}>\frac{f(1-\rho)}{1-\rho}>f'(1-\rho)>$

$(1+r)$. Therefore, $\frac{\partial RHS}{\partial \pi} < 0$ for any $c \in [\nu, \rho]$ if

$$1 - \frac{\rho - \nu}{\nu} + \left(1 - \frac{1+r}{\frac{f(1+q-\nu)}{1+2(\nu-\rho)}}\right)\frac{1-\rho}{2\nu} > 0 \tag{B.4}$$

If (B.4) holds, then $\frac{dc}{d\pi} > 0$, which ensures $\pi_\nu < \pi_\rho$.

Note $\pi_\rho \in (0, \pi_\lambda)$ and $\pi_\lambda \in (0,1)$. Therefore, we conclude that $c \in [\nu, \rho)$ if $\pi \in [\pi_\nu, \pi_\rho)$ and $c = \rho$ if $\pi \in [\pi_\rho, 1]$. An immediate corollary is that all banks are run-proof without aggregate uncertainty.

To summarize, under condition (B.4), we have shown

$$c : \begin{cases} \in [\nu, \rho) \text{ with } \frac{dc}{d\pi} > 0 & if \quad \pi \in [\pi_\nu, \pi_\rho) \\ = \rho & if \quad \pi \in [\pi_\rho, 1] \end{cases}$$

$$r_b^\rho = \begin{cases} \frac{f(1+q-c)}{1+\nu+c-2\rho} - 1 & if \quad \pi \in [\pi_\nu, \pi_\rho) \\ \frac{f(1+q-\rho)}{1+\nu-\rho} - 1 & if \quad \pi \in [\pi_\rho, \pi_\lambda) \\ \frac{f'(1+q-\rho)-1}{\pi} & if \quad \pi \in [\pi_\lambda, 1] \end{cases}$$

Returning to (B.1) to confirm $\ell(\widetilde{\nu}_i, \rho) = 0$, we need

$$x < \frac{f'(1+q-c)}{1+r_b^\rho} = \frac{f'(1+q-c)}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}$$

for $\pi \in [\pi_\nu, \pi_\rho)$. Note that $\frac{f'(1+q-c)}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}$ is increasing in c. Also note that $1 + r_b^\rho$ is weakly decreasing in π for $\pi \in [\pi_\rho, 1]$. Therefore, a sufficient condition for $\ell(\widetilde{\nu}_i, \rho) = 0$ for all $\pi \in [\pi_\nu, 1]$ is

$$x < \frac{f'(1+q-\nu)}{\frac{f(1+q-\nu)}{1+2(\nu-\rho)}} \tag{B.5}$$

This completes the characterization of the equilibrium under conditions (B.4) and (B.5).

Presence of a Moral Hazard Force The first order condition for the representative bank's choice of c having the property $\frac{dc}{dr_b^\rho} > 0$ would represent moral hazard. Differentiating the first order condition,

$$\frac{\partial V\left(c; r_b^\rho, 0\right)}{\partial c} = 0 \longrightarrow \frac{\partial^2 V\left(c; r_b^\rho, 0\right)}{\partial c^2} dc + \frac{\partial^2 V\left(c; r_b^\rho, 0\right)}{\partial r_b^\rho \partial c} dr_b^\rho = 0 \longrightarrow \frac{dc}{dr_b^\rho} = -\frac{\frac{\partial^2 V\left(c; r_b^\rho, 0\right)}{\partial r_b^\rho \partial c}}{\frac{\partial^2 V\left(c; r_b^\rho, 0\right)}{\partial c^2}}$$

Focus on $\pi \in [\pi_\nu, \pi_\rho)$, which is the effective range of parameters for this property. Here,

$$\frac{\partial^2 V\left(c; r_b^\rho, 0\right)}{\partial c^2} = f''(1+q-c)\left[1 - \pi\frac{1 - c - \frac{f(1+q-c)}{1+r_b^\rho} + \nu}{2\nu} + \pi\frac{\left(r_b^\rho - r\right)(1-\rho)}{1+r_b^\rho}\frac{1}{2\nu}\right] + \frac{\pi}{2\nu}\frac{\left[1 + r_b^\rho - f'(1+q-c)\right]^2}{1+r_b^\rho}$$

where we assume $f''(\cdot)$ sufficiently negative for $\frac{\partial^2 V\left(c; r_b^\rho, 0\right)}{\partial c^2} < 0$ to confirm the second order condition. Also,

$$\frac{\partial^2 V\left(c; r_b^\rho, 0\right)}{\partial r_b^\rho \partial c} = \frac{\pi}{2\nu}\left[c + \nu - \rho + \frac{f'(1+q-c)}{1+r_b^\rho}\frac{f(1+q-c) - (1+r)(1-\rho)}{1+r_b^\rho}\right]$$

where a sufficient condition for $\frac{\partial^2 V\left(c; r_b^\rho, 0\right)}{\partial r_b^\rho \partial c} > 0$ is $\rho \leq 2\nu$. Therefore, the solution to the bank's first order condition has the property $\frac{dc}{dr_b^\rho} > 0$ if $\rho \leq 2\nu$ and $f(\cdot)$ is sufficiently concave that $\frac{\partial^2 V\left(c; r_b^\rho, 0\right)}{\partial c^2} < 0$.

Planner's Solution

We now study the planner's choice of cash holdings for values of π such that banks choose $c < \rho$. The planner's choice of c would satisfy

$$\frac{\partial V\left(c; r_b^\rho, 0\right)}{\partial c} + \frac{\partial V\left(c; r_b^\rho, 0\right)}{\partial r_b^\rho}\frac{\partial r_b^\rho}{\partial c} = 0$$

whereas the representative bank's choice only satisfies $\frac{\partial V\left(c; r_b^\rho, 0\right)}{\partial c} = 0$. For $\pi \in (\pi_\nu, \pi_\rho)$,

$$\begin{aligned}\frac{\partial V\left(c; r_b^\rho, 0\right)}{\partial r_b^\rho} &= \underbrace{\pi \int_{1-c-\frac{f(1+q-c)}{1+r_b^\rho}}^{\nu} \left(c + \widetilde{\nu}_i - \rho\right) dG\left(\widetilde{\nu}_i\right)}_{\text{zero by market clearing}} \\ &\quad -\pi \frac{f(1+q-c)}{1+r_b^\rho}\frac{\left(r_b^\rho - r\right)(1-\rho)\, g\left(1 - c - \frac{f(1+q-c)}{1+r_b^\rho}\right)}{1+r_b^\rho} \\ &< 0\end{aligned}$$

With a uniform distribution, market clearing delivers Eq. (B.3), from which we get

$$\frac{\partial r_b^\rho}{\partial c} = -\frac{f'(1+q-c)}{1+\nu+c-2\rho} - \frac{f(1+q-c)}{(1+\nu+c-2\rho)^2} < 0$$

Therefore, the planner's choice of c would satisfy $\frac{\partial V\left(c; r_b^\rho, 0\right)}{\partial c} < 0$, meaning that he would choose a higher c than the representative bank (this follows from the second order condition for the bank's problem, namely the assumption that $f(\cdot)$ is sufficiently concave for $\frac{\partial^2 V\left(c; r_b^\rho, 0\right)}{\partial c^2} < 0$). Thus, there is an inefficiency in the decentralized choice of c.

The simplest solution for the planner to correct this inefficiency is to impose a liquidity requirement at $t = 0$. For the analysis of loan certificates below, we will be interested in cases where the planner also chooses $c < \rho$. In this case, the planner does not find it optimal to make all banks run-proof, leaving scope for liquidity reallocation away from failing banks. If the planner's choice of c satisfies $1-c-\frac{f(1+q-c)}{1+r_b^\rho} > -\nu$ (and hence $c<\rho$ after substituting in r_b^ρ), then

$$\begin{aligned} & \frac{\partial V\left(c;r_b^\rho,0\right)}{\partial c}+\frac{\partial V\left(c;r_b^\rho,0\right)}{\partial r_b^\rho}\frac{\partial r_b^\rho}{\partial c} \\ = \ & \pi\left(\frac{f(1+q-c)}{\nu}-\frac{(1+r)(1-\rho)}{\nu}-1\right)-\left(1-\pi\frac{\rho-c}{\nu}\right)f'(1+q-c)+1 \end{aligned}$$

where we have used the market clearing r_b^ρ from Eq. (B.3). Confirming $c<\rho$ then requires

$$\left.\left(\frac{\partial V\left(c;r_b^\rho,0\right)}{\partial c}+\frac{\partial V\left(c;r_b^\rho,0\right)}{\partial r_b^\rho}\frac{\partial r_b^\rho}{\partial c}\right)\right|_{c=\rho}<0$$

or equivalently

$$\pi<\frac{f'(1+q-\rho)-1}{\frac{f(1+q-\rho)}{\nu}-\frac{(1+r)(1-\rho)}{\nu}-1}\equiv\widetilde{\pi}_\rho$$

Thus, for sufficiently low π, the planner will choose $c<\rho$. We will return to the congruence between $\pi<\widetilde{\pi}_\rho$ and $\pi\in[\pi_\nu,\pi_\rho)$ in numerical examples below. Here, we just note that there is a non-empty set of parameters for which the decentralized equilibrium is $c^e\in[\nu,\rho)$ and the planner's solution is $c^0\in(c^*,\rho)$. The planner can then impose a liquidity requirement $c\geq c^0$ on the decentralized problem at $t=0$ to bring the equilibrium cash holdings up to the constrained efficient level. Next, we explore the role of loan certificates in improving welfare. The main analysis will proceed with loan certificates as the only intervention. At the end of this appendix, we discuss the effect of loan certificates on top of a liquidity regulation.

Loan Certificates

With loan certificates, the constraint on the bank's problem becomes

$$c+\widetilde{\nu}_i+x\ell\left(\widetilde{\nu}_i,\omega\right)+\Delta\left(\widetilde{\nu}_i,\omega\right)+\widetilde{k}\left(\widetilde{\nu}_i,\omega\right)\geq\omega$$

where $\widetilde{k}\left(\widetilde{\nu}_i,\omega\right)$ is the net loan certificate position assigned by the planner. If $\widetilde{k}\left(\widetilde{\nu}_i,\omega\right)<0$, then the bank is a net accepter of certificates. Loan certificates only exist in the positive-withdrawal state. In this state, the above constraint holds with equality, so the bank's value is

$$\begin{aligned} V_k\left(c,\widetilde{\nu}_i;r_b^\rho,r_k,\omega\right) &= f\left(1+q-c-\ell\left(\widetilde{\nu}_i,\omega\right)\right)-\left(1+r_b^\rho\right)\Delta\left(\widetilde{\nu}_i,\omega\right)-\left(1+r_k\right)\widetilde{k}\left(\widetilde{\nu}_i,\omega\right)-(1+r)(1-\omega) \\ &= V\left(c,\widetilde{\nu}_i;r_b^\rho,\omega\right)+\left(r_b^\rho-r_k\right)\widetilde{k}\left(\widetilde{\nu}_i,\omega\right) \end{aligned}$$

The value of the bank in the positive-withdrawal state is therefore $V\left(c,\widetilde{\nu}_i;r_k,\omega\right)$ under the latent interbank rate $r_b^\rho=r_k$ in the model with loan certificates.

Focus on the positive-withdrawal state with $\pi\in(\pi_\nu,\pi_\rho)$. We assume $\ell\left(\widetilde{\nu}_i,\omega\right)=0$; following (B.1), this just requires x sufficiently low. For run-proof banks, the planner implements $\Delta\left(\widetilde{\nu}_i,\rho\right)=0$

and hence

$$\widetilde{k}\left(\widetilde{\nu}_i,\rho\right)=\rho-\widetilde{\nu}_i-c \text{ if } \widetilde{\nu}_i\geq 1-c-\frac{f\left(1+q-c\right)}{1+r_k} \text{ where } \frac{\partial\left(1-c-\frac{f(1+q-c)}{1+r_k}\right)}{\partial r_k}=\frac{f\left(1+q-c\right)}{\left(1+r_k\right)^2}>0$$

The remaining banks will fail, and the planner assigns them $\widetilde{k}\left(\widetilde{\nu}_i,1\right)<0$. Since $\widetilde{k}\left(\cdot\right)$ is defined as net loan certificate position, it should aggregate to zero, i.e.,

$$\int_{-\nu}^{1-c-\frac{f(1+q-c)}{1+r_k}}\widetilde{k}\left(\widetilde{\nu}_i,1\right)dG\left(\widetilde{\nu}_i\right)+\int_{1-c-\frac{f(1+q-c)}{1+r_k}}^{\nu}\left(\rho-\widetilde{\nu}_i-c\right)dG\left(\widetilde{\nu}_i\right)=0 \tag{B.6}$$

The first integral is negative, so the second integral is positive.

We first confirm that loan certificates can implement a lower interbank rate; this is intuitive given the discussion in the main text, so we confirm it for the extended environment only for completeness. Return to Eq. (B.6) and consider $\widetilde{k}\left(\widetilde{\nu}_i,1\right)=-\varepsilon<0$, i.e.,

$$\int_{1-c-\frac{f(1+q-c)}{1+r_k}}^{\nu}\left(\rho-\widetilde{\nu}_i-c\right)d\widetilde{\nu}_i=\left(1+\nu-c-\frac{f\left(1+q-c\right)}{1+r_k}\right)\varepsilon$$

with the uniform distribution. Differentiate to get

$$\begin{aligned}&-\left(\nu-\left(1-c-\frac{f\left(1+q-c\right)}{1+r_k}\right)\right)dc-\left(\rho-1+\frac{f\left(1+q-c\right)}{1+r_k}\right)\left(-dc+\frac{f'\left(1+q-c\right)}{1+r_k}dc+\frac{f\left(1+q-c\right)}{\left(1+r_k\right)^2}dr_k\right)\\ =&\left(-dc+\frac{f'\left(1+q-c\right)}{1+r_k}dc+\frac{f\left(1+q-c\right)}{\left(1+r_k\right)^2}dr_k\right)\varepsilon+\left(1+\nu-c-\frac{f\left(1+q-c\right)}{1+r_k}\right)d\varepsilon\end{aligned}$$

then evaluate at $\varepsilon=0$ (which returns the decentralized equilibrium) to get

$$\left.\frac{dr_k}{d\varepsilon}\right|_{\varepsilon=0}=-\frac{\frac{2(\rho-c)}{c+\nu-\rho}}{\frac{1}{\frac{f(1+q-c)}{(1+\nu+c-2\rho)^2}}+\left(1+\frac{f'(1+q-c)}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)\left.\frac{dc}{dr_k}\right|_{r_k=r_b^{\rho*}}}<0$$

where we recall that we are focusing on parameters where the decentralized equilibrium delivers $c\in[\nu,\rho)$ and we imposed earlier $\rho\leq 2\nu$ to ensure the moral hazard property. Therefore, a perturbation into loan certificates lowers the interest rate.

Welfare Improvement We now turn to welfare. Suppose $1-c-\frac{f(1+q-c)}{1+r_k}>-\nu$ so that some banks are not run-proof. The expected value of a representative bank is

$$\begin{aligned}V\left(c;r_k,0\right)&=\pi\int_{1-c-\frac{f(1+q-c)}{1+r_k}}^{\nu}\left[f\left(1+q-c\right)+\left(1+r_k\right)\left(c+\widetilde{\nu}_i-\rho\right)-\left(1+r\right)\left(1-\rho\right)\right]dG\left(\widetilde{\nu}_i\right)\\&\quad+\left(1-\pi\right)\left[f\left(1+q-c\right)+c-\left(1+r\right)\right]\end{aligned}$$

and the expected value of a representative depositor is

$$E\left(dep\right) = \pi\left[\begin{array}{l}\int_{-\nu}^{1-c-\frac{f(1+q-c)}{1+r_k}}\left(c+\widetilde{\nu}_i+x\left(1+q-c\right)+\widetilde{k}\left(\widetilde{\nu}_i,1\right)\right)dG\left(\widetilde{\nu}_i\right)\\ +\int_{1-c-\frac{f(1+q-c)}{1+r_k}}^{\nu}\left(\rho+\left(1-\rho\right)\left(1+r\right)\right)dG\left(\widetilde{\nu}_i\right)\end{array}\right] + \left(1-\pi\right)\left(1+r\right)$$

so overall welfare is

$$\begin{aligned} V\left(c;r_k,0\right)+E\left(dep\right) &= \pi\int_{-\nu}^{1-c-\frac{f(1+q-c)}{1+r_k}}\left(c+\widetilde{\nu}_i+x\left(1+q-c\right)+\widetilde{k}\left(\widetilde{\nu}_i,1\right)\right)dG\left(\widetilde{\nu}_i\right)\\ &\quad +\pi\int_{1-c-\frac{f(1+q-c)}{1+r_k}}^{\nu}\left[f\left(1+q-c\right)+\left(1+r_k\right)\left(c+\widetilde{\nu}_i-\rho\right)+\rho\right]dG\left(\widetilde{\nu}_i\right)\\ &\quad +\left(1-\pi\right)\left[f\left(1+q-c\right)+c\right] \end{aligned}$$

$\Longleftrightarrow$

$$\begin{aligned} V\left(c;r_k,0\right)+E\left(dep\right) &= c+f\left(1+q-c\right)-\pi\left[f\left(1+q-c\right)-x\left(1+q-c\right)\right]G\left(1-c-\frac{f\left(1+q-c\right)}{1+r_k}\right)\\ &\quad -\pi\left(1+r_k\right)\int_{1-c-\frac{f(1+q-c)}{1+r_k}}^{\nu}\left(\rho-c-\widetilde{\nu}_i\right)dG\left(\widetilde{\nu}_i\right)\\ &\quad +\pi\underbrace{\left[\int_{-\nu}^{1-c-\frac{f(1+q-c)}{1+r_k}}\widetilde{k}\left(\widetilde{\nu}_i,1\right)dG\left(\widetilde{\nu}_i\right)+\int_{1-c-\frac{f(1+q-c)}{1+r_k}}^{\nu}\left(\rho-c-\widetilde{\nu}_i\right)dG\left(\widetilde{\nu}_i\right)\right]}_{\text{zero by Eq. (B.6)}} \end{aligned}$$

$\Longleftrightarrow$

$$\begin{aligned} V\left(c;r_k,0\right)+E\left(dep\right) &= c+f\left(1+q-c\right)-\pi\left[f\left(1+q-c\right)-x\left(1+q-c\right)\right]G\left(1-c-\frac{f\left(1+q-c\right)}{1+r_k}\right)\\ &\quad -\pi\left(1+r_k\right)\int_{1-c-\frac{f(1+q-c)}{1+r_k}}^{\nu}\left(\rho-c-\widetilde{\nu}_i\right)dG\left(\widetilde{\nu}_i\right) \end{aligned}$$

Differentiating with respect to r_k,

$$\begin{aligned} &\frac{d\left(V\left(c;r_k,0\right)+E\left(dep\right)\right)}{dr_k}\\ =\ &\frac{dc}{dr_k}-f'\left(1+q-c\right)\frac{dc}{dr_k}+\pi\left(1+r_k\right)\frac{dc}{dr_k}\\ &-\pi\left[1+r_k-f'\left(1+q-c\right)+x\right]G\left(1-c-\frac{f\left(1+q-c\right)}{1+r_k}\right)\frac{dc}{dr_k}\\ &+\pi\left[\left(1-\rho\right)\left(1+r_k\right)-x\left(1+q-c\right)\right]g\left(1-c-\frac{f\left(1+q-c\right)}{1+r_k}\right)\left(\frac{dc}{dr_k}-\frac{f'\left(1+q-c\right)}{1+r_k}\frac{dc}{dr_k}-\frac{f\left(1+q-c\right)}{\left(1+r_k\right)^2}\right)\\ &-\pi\int_{1-c-\frac{f(1+q-c)}{1+r_k}}^{\nu}\left(\rho-c-\widetilde{\nu}_i\right)dG\left(\widetilde{\nu}_i\right) \end{aligned}$$

Using the uniform distribution and evaluating at r_k equal to the decentralized solution r_b^ρ as given by Eq. (B.3),

$$\begin{aligned}
&\left.\frac{d\left(V\left(c;r_k,0\right)+E\left(dep\right)\right)}{dr_k}\right|_{r_k=r_b^{\rho*}} \\
=\;& -\frac{\pi}{2\nu}\left(1-\rho-\frac{x\left(1+q-c\right)}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)\left(1+\nu+c-2\rho\right) \\
&-\left[\begin{array}{l} f'\left(1+q-c\right)-1-\pi\frac{f(1+q-c)}{1+\nu+c-2\rho}+\frac{\pi x}{\nu}\left(\rho-c+\left(1-\frac{f'(1+q-c)}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)\frac{1+q-c}{2}\right) \\ +\frac{\pi}{\nu}\left[\frac{f(1+q-c)}{1+\nu+c-2\rho}-f'\left(1+q-c\right)\right]\left(\rho-c-\frac{1-\rho}{2}\right)\end{array}\right]\left.\frac{dc}{dr_k}\right|_{r_k=r_b^{\rho*}}
\end{aligned}$$

where moral hazard amounts to $\left.\frac{dc}{dr_k}\right|_{r_k=r_b^{\rho*}}>0$. The first term represents the direct effect of the interest rate on welfare; the second term represents the indirect effect through changes in the bank's choice of cash holdings, i.e., the moral hazard effect. If $x<\frac{1-\rho}{1+\nu+c-2\rho}\frac{f(1+q-c)}{1+q-c}$, then the first term is negative. This means that a perturbation of the interest rate below its decentralized solution increases welfare. The sign of the coefficient on the moral hazard effect is ambiguous, but if $\left.\frac{dc}{dr_k}\right|_{r_k=r_b^{\rho*}}$ is sufficiently small, then the direct effect dominates. Recall

$$\begin{aligned}
\left.\frac{dc}{dr_b^\rho}\right|_{r_b^\rho=r_b^{\rho*}} &= -\frac{\left.\frac{\partial^2V\left(c;r_b^\rho,0\right)}{\partial r_b^\rho\partial c}\right|_{r_b^\rho=r_b^{\rho*}}}{\left.\frac{\partial^2V\left(c;r_b^\rho,0\right)}{\partial c^2}\right|_{r_b^\rho=r_b^{\rho*}}} \\
&= \frac{c+\nu-\rho+\frac{f'(1+q-c)}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\left(1-\frac{(1+r)(1-\rho)}{f(1+q-c)}\right)\left(1+\nu+c-2\rho\right)}{-f''\left(1+q-c\right)\left[\frac{2\nu}{\pi}-2\left(\rho-c\right)+\left(1-\frac{1+r}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)\left(1-\rho\right)\right]-\frac{f(1+q-c)}{1+\nu+c-2\rho}\left(1-\frac{f'(1+q-c)}{\frac{f(1+q-c)}{1+\nu+c-2\rho}}\right)^2}
\end{aligned}$$

so $\left.\frac{dc}{dr_k}\right|_{r_k=r_b^{\rho*}}$ sufficiently small can be delivered by $f\left(\cdot\right)$ sufficiently concave.

Numerical Example A numerical example will illustrate the existence of parameters consistent with these points. Consider $\nu=q$ and the functional form $f\left(y\right)=\alpha y+\beta y^\gamma$. We sketch the computational procedure then present an example. The goal is only to establish that the parameter set is non-empty, not to perform a quantitative analysis with the extended model.

First, solve Eq. (B.2) for $c\left(r_k\right)$. This amounts to solving

$$\begin{aligned}
&\alpha+\beta\gamma\left(1+q-c\right)^{\gamma-1} \\
=\;& 1+\pi r_k-\frac{\pi}{2q}\left[1+r_k-\alpha-\beta\gamma\left(1+q-c\right)^{\gamma-1}\right]\left(\begin{array}{l}\rho+q-c \\ -\frac{\alpha(1+q-c)+\beta(1+q-c)^\gamma}{1+r_k}+\frac{(1-\rho)(1+r)}{1+r_k}\end{array}\right)
\end{aligned}$$

for a vector of possible values of r_k. We focus on the range of r_k that delivers $c\left(r_k\right)\in\left(q,\rho\right)$ and confirm that positive interbank trade is always satisfied, i.e., $c\left(r_k\right)+q>\rho$. We also confirm that the solution $c(r_k)$ over this range always satisfies $\alpha+\beta(1+q-c(r_k))^{\gamma-1}<1+r_k$, which ensures that not all banks are run-proof and hence confirms the use of Eq. (B.2) to solve for c. We also confirm $c'(r_k)>0$ over this range, which confirms the moral hazard property.

Next, solve Eq. (B.3) for $r_b^{\rho*}$. This amounts to solving

$$1+r_b^{\rho*}=\frac{\alpha+\beta\left(1+q-c\left(r_b^{\rho*}\right)\right)^{\gamma-1}}{1-\frac{2\left(\rho-c\left(r_b^{\rho*}\right)\right)}{1+q-c\left(r_b^{\rho*}\right)}}$$

We confirm $\ell(\widetilde{\nu}_i,\rho)=0$ for run-proof banks via condition (B.5), which now simplifies to $x<(1+2(q-\rho))\frac{\alpha+\beta\gamma}{\alpha+\beta}$.

We then evaluate the welfare function

$$\begin{aligned}\mathcal{W}(r_k) &= c(r_k)+f(1+q-c(r_k))-\pi\left[f(1+q-c(r_k))-x(1+q-c(r_k))\right]\frac{1+q-c(r_k)-\frac{f(1+q-c(r_k))}{1+r_k}}{2q}\\ &\quad -\pi(1+r_k)\left(\rho-c(r_k)-\frac{1+q-c(r_k)-\frac{f(1+q-c(r_k))}{1+r_k}}{2}\right)\frac{c(r_k)+q-1+\frac{f(1+q-c(r_k))}{1+r_k}}{2q}\end{aligned}$$

over the range of r_k and confirm $\mathcal{W}'\left(r_b^{\rho*}\right)<0$, which confirms that a perturbation into loan certificates improves welfare. Finally, we confirm that the planner chooses $c<\rho$, which amounts to confirming $\pi<\widetilde{\pi}_\rho$.

The following configuration of parameters generates all of the noted results: $\alpha=1$, $\gamma=\frac{1}{3}$, and $\beta=0.595$ for the output function of the long-term investment project, $r=0.02$ for the deposit rate, $q=0.3$ for non-deposit funding, $\rho=0.5$ for deposit withdrawals in the positive-withdrawal state, $\pi=0.1$ for the probability of the positive-withdrawal state, and any $x\in[0,0.45]$ for the project liquidation value. Given the other parameters, the choice of β delivers an expected decentralized interbank rate $(\pi r_b^{\rho*})$ of 9.4%, which is the interbank rate we calculated for the benchmark model in Section 5.2. While this is not necessary for any of the results, it is convenient for selecting an example from a set of possibilities. Note that the range of x can be expanded to $x\in[0,0.625]$ if we only confirm no liquidations for run-proof banks in the vicinity of $r_b^{\rho*}$. We also note that any $x\geq 0.305$ would deliver a positive measure of banks that could liquidate their way out of a run, eliminating the possibility of an equilibrium where all banks experience runs.

Joint Policy with Liquidity Regulation Suppose the planner imposes a liquidity regulation $c\geq c^0$ at $t=0$. Recall that c^0 strictly exceeds the decentralized choice c^e, so the constraint will bind (with strictly positive Lagrange multiplier) on the bank problem. Accordingly, banks will choose $c=c^0$ and the equilibrium interbank rate in the positive-withdrawal state will be denoted $\widetilde{r}_b^\rho$. The strictly binding constraint means that banks continue to choose $c=c^0$ for interest rates in the vicinity of $\widetilde{r}_b^\rho$. Thus, $\left.\frac{dc}{dr_k}\right|_{r_k=\widetilde{r}_b^\rho}=0$. A perturbation into loan certificates still decreases the interest rate, i.e., the conclusion of $\left.\frac{dr_k}{d\varepsilon}\right|_{\varepsilon=0}<0$ does not change, and the effect of the interest rate on welfare now only occurs through the direct channel so it is easier to conclude $\left.\frac{d(V(c;r_k,0)+E(dep))}{dr_k}\right|_{r_k=\widetilde{r}_b^\rho}<0$. A perturbation into loan certificates therefore delivers a welfare improvement over and above liquidity regulation when the planner's optimal regulation implements $c^0 < \rho$.

In summary, moral hazard can reduce the benefits of loan certificates, but (i) there is still a net benefit for output functions that sufficiently bound the moral hazard and (ii) the moral hazard concern can in principle be corrected with liquidity regulation, specifically a floor on banks' cash holdings. Importantly, liquidity regulation alone cannot achieve the outcome of loan certificates when the planner finds it optimal to mandate $c < \rho$; this follows from the fact that taking into account the effect of c on r_b^ρ in Eq. (B.3) ex ante cannot introduce the first integral in Eq. (B.6), i.e., cannot extract cash out of failing banks for use in market clearing ex post, so a liquidity requirement to increase c at all banks does not make loan certificates redundant.

Online Appendix C – Global Games Extension

This appendix solves for the decentralized equilibrium with an interbank market when each depositor gets a noisy signal about his bank's cash level and depositor strategies are determined via global games.

Simple Case To simplify the initial exposition, assume $x \to 0$ and $\zeta \to \infty$ so that $\ell = z = 0$; this reduces the number of choice variables for the bank. We return to $x > 0$ later in the appendix. A surviving bank's profit (conditional on the choice of Δ and the state variables ω and r_b) is

$$c + \Delta - \omega + (1 + r_z)\,\widetilde{z} - (1 + r_b)\,\Delta - (1 + r)\,(1 - \omega)$$

Also recall that a bank must satisfy

$$c + \Delta - \omega \geq 0$$

to survive until the last period.

We first calculate the payoff differential to a depositor from waiting. If $\omega < c$, then $\Delta = \omega - c < 0$. The payoff differential to a depositor from waiting is r. If instead $\omega > c$, then the bank needs $\Delta = \omega - c > 0$. Rewrite its profit as

$$c + (1 + r_z)\,\widetilde{z} - (1 + r) - r_b\,(\omega - c) + r\omega$$

This expression needs to be non-negative for borrowing to be permitted, i.e.,

$$\omega \leq \frac{(1 + r_b)\,c + (1 + r_z)\,\widetilde{z} - (1 + r)}{r_b - r} \equiv \overline{\omega}\,(c, r_b)$$

where we will focus on solutions $r_b^{**} > r$. If $\omega \leq \overline{\omega}\,(c, r_b)$, then the payoff differential to a depositor from waiting is again r, conditional on $\omega > c$. Note that $\overline{\omega}\,(c, r_b) > c$ reduces to $c + \frac{1+r_z}{1+r}\widetilde{z} > 1$, which is true. Therefore, the payoff differential to a depositor from waiting is r for any $\omega \leq \overline{\omega}\,(c, r_b)$. If instead $\omega > \overline{\omega}\,(c, r_b)$, then the bank cannot borrow. Accordingly, $\Delta = 0$. It follows that the payoff differential to a depositor from waiting is $-\frac{c}{\omega}$ if $\omega > c$, conditional on $\omega > \overline{\omega}\,(c, r_b)$.

We now need to compare the threshold $\overline{\omega}\,(c, r_b)$ to c as well as the limits $\omega \in [\rho, 1]$:

- $\overline{\omega}\,(c, r_b) > \rho$ reduces to

$$c > 1 - \frac{(1 + r_z)\,\widetilde{z}}{1 + r_b} - \frac{(1 - \rho)\,(r_b - r)}{1 + r_b} \equiv c_1\,(r_b)$$

- $\overline{\omega}\,(c, r_b) < c$ reduces to

$$c < 1 - \frac{(1 + r_z)\,\widetilde{z}}{1 + r} \equiv c_2\,(r_b)$$

- $\overline{\omega}\,(c, r_b) < 1$ reduces to

$$c < 1 - \frac{(1 + r_z)\,\widetilde{z}}{1 + r_b} \equiv c_3\,(r_b)$$

It is trivial to see that $c_1(r_b) < c_3(r_b)$. Also note that $c_2(r_b) < c_1(r_b)$ when $r_b > r$ and $(1+r_z)\widetilde{z} > (1+r)(1-\rho)$. The latter rearranges to $\rho + \frac{1+r_z}{1+r}\widetilde{z} > 1$, which is true. Thus, $c_2(r_b) < c_1(r_b) < c_3(r_b)$. It is also straightforward to see that $c < c_1(r_b)$ implies $c < \rho$.

Next, we formally define a depositor's payoff differential $\upsilon(c, \omega; r_b)$ and the integral

$$\Upsilon(c; r_b) \equiv \int_\rho^1 \upsilon(c, \omega; r_b)\, d\omega$$

Fix an r_b. There are three cases:

1. If $c \leq c_1(r_b)$, then $\max\{\overline{\omega}(c, r_b), c\} \leq \rho$ so the case of $\omega \geq \max\{\overline{\omega}(c, r_b), c\}$ always applies. Thus,

$$\upsilon(c, \omega; r_b) = -\frac{c}{\omega} \text{ for all } \omega \in [\rho, 1]$$

and hence

$$\Upsilon(c; r_b) = c\ln(\rho) < 0$$

2. If $c \in (c_1(r_b), c_3(r_b))$, then $\overline{\omega}(c, r_b) \in (\rho, 1)$. Also note that $c > c_1(r_b)$ with $c_1(r_b) > c_2(r_b)$ implies $c > c_2(r_b)$ and therefore $\overline{\omega}(c, r_b) > c$. Thus,

$$\upsilon(c, \omega; r_b) = \begin{cases} r & if \quad \omega \in [\rho, \overline{\omega}(c, r_b)] \\ -\frac{c}{\omega} & if \quad \omega \in (\overline{\omega}(c, r_b), 1] \end{cases}$$

and hence

$$\Upsilon(c; r_b) = r(\overline{\omega}(c, r_b) - \rho) + c\ln(\overline{\omega}(c, r_b)) \tag{C.1}$$

with first derivative

$$\frac{\partial \Upsilon(c; r_b)}{\partial c} = \left(r + \frac{c}{\overline{\omega}(c, r_b)}\right)\frac{1+r_b}{r_b - r} + \ln(\overline{\omega}(c, r_b))$$

At any point $c \in (c_1(r_b), c_3(r_b))$ such that $\Upsilon(c; r_b) = 0$,

$$\left.\frac{\partial \Upsilon(c; r_b)}{\partial c}\right|_{\Upsilon(c; r_b) = 0} = \left(r + \frac{c}{\overline{\omega}(c, r_b)}\right)\frac{1+r_b}{r_b - r} - \frac{r}{c}(\overline{\omega}(c, r_b) - \rho)$$

This expression is positive if and only if

$$\frac{(1+r_b)c^2}{r_b - r} > r\overline{\omega}(c, r_b)\underbrace{\left(\frac{(1+r_z)\widetilde{z} - (1+r)}{r_b - r} - \rho\right)}_{<0}$$

A sufficient condition is $(1+r_z)\widetilde{z} \leq 1 + r$, which we will impose and then return to later. Accordingly, $\frac{\partial \Upsilon(c; r_b)}{\partial c} > 0$ at any point $c \in (c_1(r_b), c_3(r_b))$ where $\Upsilon(c; r_b) = 0$.

3. If $c \geq c_3(r_b)$, then $\overline{\omega}(c, r_b) \geq 1$ so the case of $\omega \leq \overline{\omega}(c, r_b)$ always applies. Thus,

$$\upsilon(c, \omega; r_b) = r \text{ for all } \omega \in [\rho, 1]$$

and hence

$$\Upsilon(c; r_b) = r(1-\rho) > 0$$

We now establish uniqueness of the global games solution, assuming $\Upsilon(c_{\min}; \cdot) < 0$ and $c_{\max} > c_3(r_b)$. These assumptions are akin to the "lower dominance" and "upper dominance" assumptions in Goldstein and Pauzner (2005), respectively. Under these assumptions, there is a unique solution $c^*(r_b)$ to $\Upsilon(c; r_b) = 0$ for any $r_b > r$. The argument proceeds as follows. A solution to $\Upsilon(c; r_b) = 0$ exists by the intermediate value theorem: $\Upsilon(c; r_b)$ is continuous in c with (i) $\Upsilon(c; r_b) < 0$ for any $c \leq c_1(r_b)$ and (ii) $\Upsilon(c; r_b) > 0$ for any $c \geq c_3(r_b)$. Furthermore, any solution to $\Upsilon(c; r_b) = 0$ exists in the interval $c \in (c_1(r_b), c_3(r_b))$. That $\frac{\partial \Upsilon(c; r_b)}{\partial c} > 0$ at any point $c \in (c_1(r_b), c_3(r_b))$ where $\Upsilon(c; r_b) = 0$ then establishes uniqueness.

So far, we have shown that there is a unique $c^*(r_b) \in (c_{\min}, c_{\max})$ for each $r_b > r$. Next, we need to show that market clearing can be satisfied, i.e.,

$$\int_{c^*(r_b)}^{c_{\max}} (\rho - c_i)\, dc_i = 0$$

That is, we need to show that there is an $r_b^{**} > r$ such that

$$c^*(r_b^{**}) = 2\rho - c_{\max}$$

If we can show $\frac{dc^*(r_b)}{dr_b} > 0$, then it will suffice to have parameters such that (i) $c^*(r) < 2\rho - c_{\max}$ and (ii) $c^*(\infty) > 2\rho - c_{\max}$.

To show $\frac{dc^*(r_b)}{dr_b} > 0$, differentiate $\Upsilon(c^*(r_b); r_b) = 0$ to get

$$\frac{\partial \Upsilon(c^*(r_b); r_b)}{\partial c} \frac{dc^*(r_b)}{dr_b} + \frac{\partial \Upsilon(c; r_b)}{\partial r_b} = 0$$

where we have already established $\frac{\partial \Upsilon(c^*(r_b); r_b)}{\partial c} > 0$. Also note that

$$\frac{\partial \Upsilon(c; r_b)}{\partial r_b} = -\left(r + \frac{c}{\frac{(1+r_b)c + (1+r_z)\widetilde{z} - (1+r)}{r_b - r}} \right) \frac{(1+r)c + (1+r_z)\widetilde{z} - (1+r)}{(r_b - r)^2} < 0$$

for any $c \in (c_1(r_b), c_3(r_b))$, which is the interval over which $c^*(r_b)$ exists. Thus, $\frac{dc^*(r_b)}{dr_b} > 0$.

Showing $c^*(r) < 2\rho - c_{\max}$ amounts to showing $\Upsilon(2\rho - c_{\max}; r) > 0$, with $\Upsilon(\cdot)$ as per Eq. (C.1). This is trivially true since $\Upsilon(c; r) = \infty$. Showing $c^*(\infty) > 2\rho - c_{\max}$ amounts to showing $\Upsilon(2\rho - c_{\max}; \infty) < 0$. Note

$$\Upsilon(c; \infty) = r(c - \rho) + c \ln(c)$$

so the desired condition is

$$(2\rho - c_{\max}) \ln(2\rho - c_{\max}) < r(c_{\max} - \rho)$$

which is true for any parameterization with $c_{\max} \in (\rho, 1)$ and $\rho > \widetilde{c}$ as considered in the main analysis of the benchmark model.

The equilibrium r_b^{**} determined by global games solves $\Upsilon\left(2\rho - c_{\max}; r_b^{**}\right) = 0$. This yields $r_b^{**} = 0.104$ for $\{r_z, r, c_{\max}, \widetilde{z}, \rho\}$ as in Section 5.2.1. Note that $(1 + r_z)\,\widetilde{z} \leq 1 + r$ as imposed above in the discussion of $\Upsilon\left(c; r_b\right)$ is satisfied at these parameters. Also note that $\overline{\omega}\left(2\rho - c_{\max}, r_b^{**}\right) = 0.89$, so global games yields the interbank rate that would arise in our benchmark model with $\overline{\omega} = 0.89$.

General Case We now relax the simplifying assumption of $x = 0$. This will generate additional cases, but the analysis follows the same logic as above. Here, we focus on solutions $r_b^{**} \in \left(r, \frac{1+r_z}{x} - 1\right)$.

The payoff differential to a depositor from waiting is still r for any $\omega \leq \overline{\omega}\left(c, r_b\right)$. If instead $\omega > \overline{\omega}\left(c, r_b\right)$, then the bank cannot borrow $\Delta = \omega - c$, which at an interest rate $r_b < \frac{1+r_z}{x} - 1$ means it cannot borrow any lower amount and make up the difference via liquidation since any such combination would lower its profits further. Accordingly, $\Delta = 0$; note here that the bank would never liquidate to lend since $r_b < \frac{1+r_z}{x} - 1$, ruling out $\Delta < 0$. Therefore, the bank needs to liquidate to satisfy $c + x\ell = \omega$ in order to survive, i.e., it will need to liquidate $\ell = \frac{\omega - c}{x}$, which satisfies $\ell \leq \widetilde{z}$ if and only if $\omega \leq c + x\widetilde{z}$. It follows that the payoff differential to a depositor from waiting is $\frac{1+r_z}{x}\frac{c+x\widetilde{z}-\omega}{1-\omega} - 1$ if $\omega \leq c + x\widetilde{z}$ and $-\frac{c+x\widetilde{z}}{\omega}$ if $\omega > c + x\widetilde{z}$, conditional on $\omega > \overline{\omega}\left(c, r_b\right)$. The rest of the analysis depends on the ordering of $c_1\left(r_b\right)$ and $c_2\left(r_b\right)$, where $c_2\left(r_b\right)$ is now given by

$$c_2\left(r_b\right) \equiv 1 - \frac{(1 + r_z)\,\widetilde{z}}{1 + r} + \frac{(r_b - r)\,x\widetilde{z}}{1 + r}$$

and $c < c_2\left(r_b\right)$ implies $\overline{\omega}\left(c, r_b\right) < c + x\widetilde{z}$.

The condition for $c_1\left(r_b\right) \geq c_2\left(r_b\right)$ is

$$r_b \leq \frac{1 + r_z}{x} - 1 - \frac{(1 - \rho)(1 + r)}{x\widetilde{z}} \tag{C.2}$$

In this case,

$$\Upsilon\left(c; r_b\right) = \begin{cases} (c + x\widetilde{z})\ln(\rho) & if \quad c \leq c_1\left(r_b\right) \\ r\left(\overline{\omega}\left(c, r_b\right) - \rho\right) + (c + x\widetilde{z})\ln\left(\overline{\omega}\left(c, r_b\right)\right) & if \quad c \in \left(c_1\left(r_b\right), c_3\left(r_b\right)\right) \\ r\,(1 - \rho) & if \quad c \geq c_3\left(r_b\right) \end{cases}$$

and the steps to prove (i) existence and uniqueness of a solution $c^*\left(r_b\right)$ to $\Upsilon\left(c; r_b\right) = 0$ and (ii) $\frac{dc^*(r_b)}{dr_b} > 0$ are the same as for $x = 0$.[8]

Suppose instead $c_1\left(r_b\right) < c_2\left(r_b\right)$. Then there are four cases:

1. If $c \leq c_1\left(r_b\right)$, then there are two subcases:

[8]Note that the sufficient condition $(1 + r_z)\,\widetilde{z} \leq 1 + r$ for $\frac{\partial \Upsilon(c; r_b)}{\partial c} > 0$ at any point $c \in \left(c_1\left(r_b\right), c_3\left(r_b\right)\right)$ where $\Upsilon\left(c; r_b\right) = 0$ is also sufficient when $x > 0$.

(a) If $c_{\min} + x\widetilde{z} \geq \rho$, then

$$\upsilon(c,\omega;r_b) = \begin{cases} \frac{1+r_z}{x}\frac{c+x\widetilde{z}-\omega}{1-\omega} - 1 & if \quad \omega \in [\rho, c+x\widetilde{z}] \\ -\frac{c+x\widetilde{z}}{\omega} & if \quad \omega \in (c+x\widetilde{z}, 1] \end{cases}$$

and hence

$$\begin{aligned} \Upsilon(c;r_b) &= -\frac{(1+r_z)(1-c-x\widetilde{z})}{x}\ln\left(\frac{1-\rho}{1-c-x\widetilde{z}}\right) \\ &\quad + \left(\frac{1+r_z}{x} - 1\right)(c+x\widetilde{z}-\rho) + (c+x\widetilde{z})\ln(c+x\widetilde{z}) \end{aligned}$$

with first and second derivatives

$$\frac{\partial\Upsilon(c;r_b)}{\partial c} = \frac{1+r_z}{x}\ln\left(\frac{1-\rho}{1-c-x\widetilde{z}}\right) + \ln(c+x\widetilde{z})$$

$$\frac{\partial^2\Upsilon(c;r_b)}{\partial c^2} = \frac{1+r_z}{x}\frac{1}{1-c-x\widetilde{z}} + \frac{1}{c+x\widetilde{z}} > 0$$

(b) If $c_{\min} + x\widetilde{z} < \rho$, then define c_u such that $c_u + x\widetilde{z} = \rho$. The expression for $\Upsilon(c;r_b)$ in the previous subcase applies for any $c \in (c_u, c_1(r_b))$, whereas $\upsilon(c,\omega;r_b) = -\frac{c+x\widetilde{z}}{\omega}$ for all $\omega \in [\rho, 1]$ and hence $\Upsilon(c;r_b) = (c+x\widetilde{z})\ln(\rho) < 0$ for any $c \leq c_u$.

2. If $c \in (c_1(r_b), c_2(r_b)]$, then

$$\upsilon(c,\omega;r_b) = \begin{cases} r & if \quad \omega \in [\rho, \overline{\omega}(c,r_b)] \\ \frac{1+r_z}{x}\frac{c+x\widetilde{z}-\omega}{1-\omega} - 1 & if \quad \omega \in (\overline{\omega}(c,r_b), c+x\widetilde{z}] \\ -\frac{c+x\widetilde{z}}{\omega} & if \quad \omega \in (c+x\widetilde{z}, 1] \end{cases}$$

and hence

$$\begin{aligned} \Upsilon(c;r_b) &= r(\overline{\omega}(c,r_b)-\rho) + \frac{(1+r_z)(c+x\widetilde{z}-1)}{x}\ln\left(\frac{1-\overline{\omega}(c,r_b)}{1-c-x\widetilde{z}}\right) \\ &\quad + \left(\frac{1+r_z}{x} - 1\right)(c+x\widetilde{z}-\overline{\omega}(c,r_b)) + (c+x\widetilde{z})\ln(c+x\widetilde{z}) \end{aligned}$$

with first and second derivatives

$$\begin{aligned} \frac{\partial\Upsilon(c;r_b)}{\partial c} &= \left(1 + r_b + \frac{1+r_z}{x}\frac{c + \frac{(1+r_z)\widetilde{z}}{1+r} - \frac{(r_b-r)x\widetilde{z}}{1+r} - 1}{1 - c - \frac{(1+r_z)\widetilde{z}}{1+r_b}}\right)\frac{1+r}{r_b-r} \\ &\quad + \frac{1+r_z}{x}\ln\left(\frac{\frac{1+r_b-(1+r_b)c-(1+r_z)\widetilde{z}}{r_b-r}}{1-c-x\widetilde{z}}\right) + \ln(c+x\widetilde{z}) \end{aligned}$$

$$\frac{\partial^2\Upsilon(c;r_b)}{\partial c^2} = \frac{\left(\frac{\frac{1+r_z}{x}}{1+r_b} - 1\right)^2(1+r_z)x\widetilde{z}^2}{(1-c-x\widetilde{z})\left(1-c-\frac{(1+r_z)\widetilde{z}}{1+r_b}\right)^2} + \frac{1}{c+x\widetilde{z}} > 0$$

3. If $c \in (c_2(r_b), c_3(r_b))$, then

$$\Upsilon(c; r_b) = r(\overline{\omega}(c, r_b) - \rho) + (c + x\widetilde{z}) \ln(\overline{\omega}(c, r_b))$$

where we already know that $\frac{\partial \Upsilon(c; r_b)}{\partial c} > 0$ at any point $c \in (c_2(r_b), c_3(r_b))$ where $\Upsilon(c; r_b) = 0$.

4. If $c \geq c_3(r_b)$, then $\Upsilon(c; r_b) = r(1 - \rho) > 0$.

The proof of a unique solution $c^*(r_b)$ to $\Upsilon(c; r_b) = 0$ when $c_1(r_b) < c_2(r_b)$ proceeds as follows. For a given r_b, define the smallest solution to the global game as

$$c_0(r_b) \equiv \arg\min_c \{c | c \in (c_{\min}, c_{\max}), \Upsilon(c; r_b) = 0\}$$

Assuming $\Upsilon(c_{\min}; \cdot) < 0$ and $c_{\max} > c_3(r_b)$ again implies $c_0(r_b)$ exists, i.e., there is at least one solution, and $\frac{\partial \Upsilon(c_0(r_b); r_b)}{\partial c} > 0$. The rest of the proof establishes that $c_0(r_b)$ is the only solution. Suppose $c_0(r_b) \in (c_L, c_2(r_b)]$, where $c_L = c_u$ if $c + x\widetilde{z} < \rho$ and $c_L = c_{\min}$ if $c + x\widetilde{z} \geq \rho$.[9] Then by the convexity of $\Upsilon(c; r_b)$ over $c \in (c_L, c_2(r_b)]$, there cannot be any $\widetilde{c}_0(r_b) \in (c_0(r_b), c_2(r_b)]$ such that $\Upsilon(\widetilde{c}_0(r_b); r_b) = 0$ and $\frac{\partial \Upsilon(\widetilde{c}_0(r_b); r_b)}{\partial c} < 0$, which means there are no additional solutions to $\Upsilon(c; r_b) = 0$ on the interval $c \in (c_L, c_2(r_b)]$. We already know there cannot be any $\widetilde{c}_0(r_b) \in (c_2(r_b), c_3(r_b))$ such that $\Upsilon(\widetilde{c}_0(r_b); r_b) = 0$ and $\frac{\partial \Upsilon(\widetilde{c}_0(r_b); r_b)}{\partial c} < 0$, and clearly there is no $\widetilde{c}_0(r_b) \geq c_3(r_b)$ such that $\Upsilon(\widetilde{c}_0(r_b); r_b) = 0$. Therefore, there are no additional solutions to $\Upsilon(c; r_b) = 0$ on the interval $c > c_2(r_b)$. This establishes uniqueness if $c_0(r_b) \in (c_L, c_2(r_b)]$. If instead $c_0(r_b) \in (c_2(r_b), c_3(r_b))$, then uniqueness follows from the same arguments about the properties of $\Upsilon(c; r_b)$ on the interval $c > c_2(r_b)$.

Take again the parameters in Section 5.2.1, but now also using $x = 0.75$ rather than $x = 0$ as above. It is straightforward to confirm that the right-hand side of (C.2) is negative and hence we must use the $\Upsilon(c; r_b)$ defined with $c_1(r_b) < c_2(r_b)$. Solving $\Upsilon(2\rho - c_{\max}; r_b^{**}) = 0$ yields $r_b^{**} = 0.095$, which is associated with $\overline{\omega} = 0.98$. That is, global games now yields the interbank rate that would arise in our benchmark model with $\overline{\omega} = 0.98$.

[9] That there is never a solution $c \leq c_u$ if $c + x\widetilde{z} < \rho$ follows from $\Upsilon(c; r_b) < 0$ for any $c \leq c_u$.

Online Appendix D – Additional Results with Information Pooling

This appendix provides formal derivations for the discussion in Section 6.3.3.

Decentralized Equilibrium with General $\overline{\omega}$ The analysis that follows starts from an equilibrium where each bank only experiences withdrawals ρ under information suppression then checks if patient depositors at any one bank have an incentive to deviate when assessing expected solvency under the assumption that their bank experiences withdrawals $\overline{\omega} \in [\rho, 1]$. We maintain the assumption of a uniform cash distribution as in Lemma 4.

Consider first $r_b \in \left(r_z, \frac{1+r_z}{x} - 1\right)$. Banks can observe everyone's cash positions so a bank that experiences withdrawals $\overline{\omega}$ can participate in the interbank market if and only if $c_i \geq \overline{c}\left(r_b; \overline{\omega}\right)$ as defined in the proof of Lemma 9. With market clearing from Eq. (A.25) and the uniform distribution,

$$\overline{c}\left(r_b^*; \overline{\omega}\right) = 2\rho - c_{\max}$$

and hence

$$1 + r_b^* = \frac{\left(1 + r_z\right)\widetilde{z} - \left(1 + r\right)\left(1 - \overline{\omega}\right)}{c_{\max} + \overline{\omega} - 2\rho}$$

from the definition of $\overline{c}\left(r_b; \overline{\omega}\right)$. A sufficient condition for $r_b^* > r_z$ is $c_{\max} < \widetilde{z} + 2\rho - 1$. With $c_{\min} + \widetilde{z} = 1$, this sufficient condition simplifies to $\widetilde{c} < \rho$, which is true for the context we consider. To confirm $r_b^* < \frac{1+r_z}{x} - 1$, we need

$$\overline{\omega} + \frac{1 + r}{\frac{1+r_z}{x}}\left(1 - \overline{\omega}\right) - x\widetilde{z} > 2\rho - c_{\max}$$

Now consider $r_b = \frac{1+r_z}{x} - 1$. The definition of $\overline{c}\left(r_b; \overline{\omega}\right)$ pins down

$$\overline{c}\left(\frac{1 + r_z}{x} - 1; \overline{\omega}\right) = \overline{\omega} + \frac{1 + r}{\frac{1+r_z}{x}}\left(1 - \overline{\omega}\right) - x\widetilde{z}$$

and Eq. (A.25) must instead determine aggregate liquidations for market clearing at this interbank rate. Therefore,

$$\overline{c} = \begin{cases} 2\rho - c_{\max} & if \quad \overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}\left(1 - \overline{\omega}\right) - x\widetilde{z} > 2\rho - c_{\max} \\ \overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}\left(1 - \overline{\omega}\right) - x\widetilde{z} & if \quad \overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}\left(1 - \overline{\omega}\right) - x\widetilde{z} \leq 2\rho - c_{\max} \end{cases}$$

or equivalently

$$\overline{c} = \min\left\{2\rho - c_{\max}, \overline{\omega} + \frac{1 + r}{\frac{1+r_z}{x}}\left(1 - \overline{\omega}\right) - x\widetilde{z}\right\} \tag{D.1}$$

where we write $\overline{c}$ without arguments for simplicity.

A bank with $c_i \geq \overline{c}$ can borrow/liquidate its way out of $\overline{\omega}$ withdrawals so that a patient depositor gets 1 from running and $1 + r$ from waiting. A bank with $c_i < \overline{c}$ would instead have to liquidate $\ell_i = \frac{\overline{\omega} - c_i}{x}$, where $\ell_i \leq \widetilde{z}$ if and only if $c_i \geq \overline{\omega} - x\widetilde{z}$. Thus, a patient depositor gets $\frac{c_i + x\widetilde{z}}{\overline{\omega}}$ from running and 0 from waiting if $c_i < \overline{\omega} - x\widetilde{z}$, and 1 from running and $\min\left\{\frac{(1+r_z)\left(\widetilde{z}-\frac{\overline{\omega}-c_i}{x}\right)}{1-\overline{\omega}}, 1+r\right\}$ from waiting if $c_i \geq \overline{\omega} - x\widetilde{z}$. Note that $\min\left\{\frac{(1+r_z)\left(\widetilde{z}-\frac{\overline{\omega}-c_i}{x}\right)}{1-\overline{\omega}}, 1+r\right\} = 1+r$ corresponds to $c_i \geq \overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}(1-\overline{\omega}) - x\widetilde{z}$. However, we know from Eq. (D.1) that $\overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}(1-\overline{\omega}) - x\widetilde{z} \geq \overline{c}$, so it is impossible to have $\min\left\{\frac{(1+r_z)\left(\widetilde{z}-\frac{\overline{\omega}-c_i}{x}\right)}{1-\overline{\omega}}, 1+r\right\} = 1+r$ if $c_i < \overline{c}$. Thus, a patient depositor gets 1 from running and $\frac{(1+r_z)\left(\widetilde{z}-\frac{\overline{\omega}-c_i}{x}\right)}{1-\overline{\omega}}$ from waiting if $c_i \geq \overline{\omega} - x\widetilde{z}$ when $c_i < \overline{c}$.

We have three cases around two thresholds, $\overline{c}$ and $\overline{\omega} - x\widetilde{z}$:

1. If $\overline{\omega} - x\widetilde{z} \leq c_{\min}$, then the expected values of a patient depositor at bank i are

$$E_{run} = 1$$

$$E_{wait} = \int_{c_{\min}}^{\overline{c}} \frac{(1+r_z)\left(\widetilde{z}-\frac{\overline{\omega}-c_i}{x}\right)}{1-\overline{\omega}} dF(c_i) + \int_{\overline{c}}^{c_{\max}} (1+r)\, dF(c_i)$$

which yields a no-run condition of

$$\frac{1+r_z}{1-\overline{\omega}}\left(\widetilde{z} - \frac{\overline{\omega}}{x} + \frac{\overline{c}+c_{\min}}{2x}\right)(\overline{c}-c_{\min}) + (1+r)(c_{\max}-\overline{c}) \geq c_{\max} - c_{\min}$$

2. If $\overline{\omega} - x\widetilde{z} \in (c_{\min}, \overline{c})$, then the expected values of a patient depositor at bank i are

$$E_{run} = \int_{c_{\min}}^{\overline{\omega}-x\widetilde{z}} \frac{c_i + x\widetilde{z}}{\overline{\omega}} dF(c_i) + \int_{\overline{\omega}-x\widetilde{z}}^{c_{\max}} dF(c_i)$$

$$E_{wait} = \int_{\overline{\omega}-x\widetilde{z}}^{\overline{c}} \frac{(1+r_z)\left(\widetilde{z}-\frac{\overline{\omega}-c_i}{x}\right)}{1-\overline{\omega}} dF(c_i) + \int_{\overline{c}}^{c_{\max}} (1+r)\, dF(c_i)$$

which yields a no-run condition of

$$\frac{1+r_z}{1-\overline{\omega}}\frac{(\overline{c}+x\widetilde{z}-\overline{\omega})^2}{2x} + (1+r)(c_{\max}-\overline{c}) \geq c_{\max} + x\widetilde{z} - \frac{\overline{\omega}}{2} - \frac{(c_{\min}+x\widetilde{z})^2}{2\overline{\omega}}$$

3. If $\overline{\omega} - x\widetilde{z} \geq \overline{c}$, then the expected values of a patient depositor at bank i are

$$E_{run} = \int_{c_{\min}}^{\overline{c}} \frac{c_i + x\widetilde{z}}{\overline{\omega}} dF(c_i) + \int_{\overline{c}}^{c_{\max}} dF(c_i)$$

$$E_{wait} = \int_{\overline{c}}^{c_{\max}} (1+r)\, dF(c_i)$$

which yields a no-run condition of

$$(1+r)(c_{\max}-\overline{c}) \geq \frac{\overline{c}-c_{\min}}{\overline{\omega}}\left(\frac{\overline{c}+c_{\min}}{2} + x\widetilde{z}\right) + c_{\max} - \overline{c}$$

We now go through the three cases for each of the two possible values of $\overline{c}$ in Eq. (D.1). If $c_{\max} \geq 2\rho + x\widetilde{z} - \left(\overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}\left(1-\overline{\omega}\right)\right)$, then $\overline{c} = 2\rho - c_{\max}$ and hence:

- $\overline{\omega} \leq c_{\min} + x\widetilde{z}$ has a no-run condition of

$$\begin{aligned} & c_{\max}^2 + 2\left(\overline{\omega} + \frac{(1+2r)\left(1-\overline{\omega}\right)}{\frac{1+r_z}{x}} - x\widetilde{z} - 2\rho\right) c_{\max} \\ & +2\left(2\rho - c_{\min}\right)\left(\rho + x\widetilde{z} + \frac{c_{\min}}{2} - \overline{\omega}\right) - \frac{2\left(1-\overline{\omega}\right)\left(2\left(1+r\right)\rho - c_{\min}\right)}{\frac{1+r_z}{x}} \\ \geq\ & 0 \end{aligned}$$

  The roots of the associated quadratic are

$$c_{\max} = \underbrace{2\rho + x\widetilde{z} - \left(\overline{\omega} + \frac{1+2r}{\frac{1+r_z}{x}}\left(1-\overline{\omega}\right)\right)}_{<2\rho + x\widetilde{z} - \left(\overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}\left(1-\overline{\omega}\right)\right)} \pm \sqrt{\begin{array}{l}\left(2\rho + x\widetilde{z} - \overline{\omega} - \frac{(1+2r)(1-\overline{\omega})}{\frac{1+r_z}{x}}\right)^2 \\ -2\left(2\rho - c_{\min}\right)\left(\rho + x\widetilde{z} + \frac{c_{\min}}{2} - \overline{\omega}\right) \\ + \frac{2(1-\overline{\omega})(2(1+r)\rho - c_{\min})}{\frac{1+r_z}{x}}\end{array}}$$

  Note that the smaller root violates the condition for $\overline{c} = 2\rho - c_{\max}$, thus the no-run condition here is $c_{\max}$ greater than or equal to the larger root, i.e., a lower bound on $c_{\max}$.

- $\overline{\omega} \geq 2\rho + x\widetilde{z} - c_{\max}$ (or equivalently $c_{\max} \geq 2\rho + x\widetilde{z} - \overline{\omega}$) has a no-run condition of

$$c_{\max}^2 - 2\left(2\rho + 2r\overline{\omega} + x\widetilde{z}\right)c_{\max} + 4\left(\rho + r\overline{\omega} + x\widetilde{z}\right)\rho - \left(c_{\min} + 2x\widetilde{z}\right)c_{\min} \leq 0$$

  The roots of the associated quadratic are

$$c_{\max} = 2\rho + 2r\overline{\omega} + x\widetilde{z} \pm \sqrt{4\left(\rho + r\overline{\omega} + x\widetilde{z}\right)r\overline{\omega} + \left(c_{\min} + x\widetilde{z}\right)^2}$$

  The relevant values of $c_{\max}$ are smaller than the larger root, so the no-run condition is

$$c_{\max} \geq 2\rho + 2r\overline{\omega} + x\widetilde{z} - \sqrt{4\left(\rho + r\overline{\omega} + x\widetilde{z}\right)r\overline{\omega} + \left(c_{\min} + x\widetilde{z}\right)^2}$$

  However, we now have two lower bounds on $c_{\max}$ that depend on $\overline{\omega}$, specifically

$$c_{\max} \geq \max\left\{2\rho + x\widetilde{z} - \overline{\omega}, 2\rho + 2r\overline{\omega} + x\widetilde{z} - \sqrt{4\left(\rho + r\overline{\omega} + x\widetilde{z}\right)r\overline{\omega} + \left(c_{\min} + x\widetilde{z}\right)^2}\right\}$$

  Note that the condition

$$2\rho + 2r\overline{\omega} + x\widetilde{z} - \sqrt{4\left(\rho + r\overline{\omega} + x\widetilde{z}\right)r\overline{\omega} + \left(c_{\min} + x\widetilde{z}\right)^2} \overset{?}{\geq} 2\rho + x\widetilde{z} - \overline{\omega}$$

simplifies to

$$\overline{\omega}^2 - \frac{4r\left(\rho + x\widetilde{z}\right)}{1+4r}\overline{\omega} - \frac{\left(c_{\text{min}} + x\widetilde{z}\right)^2}{1+4r} \overset{?}{\geq} 0$$

and the associated quadratic has roots

$$\overline{\omega} = \frac{2r\left(\rho + x\widetilde{z}\right)}{1+4r} \pm \sqrt{\left(\frac{2r\left(\rho + x\widetilde{z}\right)}{1+4r}\right)^2 + \frac{\left(c_{\text{min}} + x\widetilde{z}\right)^2}{1+4r}}$$

Therefore, the no-run condition is

$$c_{\text{max}} \geq 2\rho + 2r\overline{\omega} + x\widetilde{z} - \sqrt{4\left(\rho + r\overline{\omega} + x\widetilde{z}\right)r\overline{\omega} + \left(c_{\text{min}} + x\widetilde{z}\right)^2} \equiv c_L\left(\overline{\omega}\right) \tag{D.2}$$

for any

$$\overline{\omega} \geq \frac{2r\left(\rho + x\widetilde{z}\right)}{1+4r} + \sqrt{\left(\frac{2r\left(\rho + x\widetilde{z}\right)}{1+4r}\right)^2 + \frac{\left(c_{\text{min}} + x\widetilde{z}\right)^2}{1+4r}} \tag{D.3}$$

- $\overline{\omega} - x\widetilde{z} \in \left(c_{\text{min}}, \overline{c}\right)$ has a no-run condition of

$$\begin{aligned} & c_{\text{max}}^2 + 2\left(\overline{\omega} + \frac{\left(1+2r\right)\left(1-\overline{\omega}\right)}{\frac{1+r_z}{x}} - x\widetilde{z} - 2\rho\right)c_{\text{max}} \\ & + \frac{\left(c_{\text{min}} + x\widetilde{z}\right)^2}{\frac{1+r_z}{x}}\frac{1-\overline{\omega}}{\overline{\omega}} + \left(2\rho + x\widetilde{z} - \overline{\omega}\right)^2 + \frac{\overline{\omega}\left(1-\overline{\omega}\right)}{\frac{1+r_z}{x}} - \frac{2\left(1-\overline{\omega}\right)\left(2\left(1+r\right)\rho + x\widetilde{z}\right)}{\frac{1+r_z}{x}} \\ \geq\ & 0 \end{aligned}$$

The roots of the associated quadratic are

$$c_{\text{max}} = 2\rho + x\widetilde{z} - \left(\overline{\omega} + \frac{1+2r}{\frac{1+r_z}{x}}\left(1-\overline{\omega}\right)\right) \pm \sqrt{\begin{array}{c} \left(2\rho + x\widetilde{z} - \overline{\omega} - \frac{(1+2r)(1-\overline{\omega})}{\frac{1+r_z}{x}}\right)^2 - \frac{(c_{\text{min}}+x\widetilde{z})^2}{\frac{1+r_z}{x}}\frac{1-\overline{\omega}}{\overline{\omega}} \\ -\left(2\rho + x\widetilde{z} - \overline{\omega}\right)^2 - \frac{\overline{\omega}(1-\overline{\omega})}{\frac{1+r_z}{x}} + \frac{2(1-\overline{\omega})(2(1+r)\rho + x\widetilde{z})}{\frac{1+r_z}{x}} \end{array}}$$

Note that the no-run condition here is c_{max} greater than or equal to the larger root for the same reasons as in the first bullet. Also note that this case corresponds to

$$c_{\text{min}} + x\widetilde{z} < \overline{\omega} < \frac{2r\left(\rho + x\widetilde{z}\right)}{1+4r} + \sqrt{\left(\frac{2r\left(\rho + x\widetilde{z}\right)}{1+4r}\right)^2 + \frac{\left(c_{\text{min}} + x\widetilde{z}\right)^2}{1+4r}}$$

If $c_{\text{max}} < 2\rho + x\widetilde{z} - \left(\overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}\left(1-\overline{\omega}\right)\right)$, then $\overline{c} = \overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}\left(1-\overline{\omega}\right) - x\widetilde{z}$. Note that this value of $\overline{c}$ is incongruent with $\overline{\omega} > \overline{c} + x\widetilde{z}$, so there are only two cases to consider here:

- $\overline{\omega} \leq c_{\text{min}} + x\widetilde{z}$ has a no-run condition of

$$c_{\text{max}} \geq \overline{c} + \left(1 - \frac{1+r_z}{1-\overline{\omega}}\left(\widetilde{z} - \frac{\overline{\omega}}{x} + \frac{\overline{c} + c_{\text{min}}}{2x}\right)\right)\frac{\overline{c} - c_{\text{min}}}{r}$$

This lower bound on $c_{\max}$ does not breach the upper bound that delivers the considered value of $\overline{c}$ if and only if

$$2\rho + x\widetilde{z} - \left(\overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}(1-\overline{\omega})\right) \geq \overline{c} + \left(1 - \frac{1+r_z}{1-\overline{\omega}}\left(\widetilde{z} - \frac{\overline{\omega}}{x} + \frac{\overline{c}+c_{\min}}{2x}\right)\right)\frac{\overline{c}-c_{\min}}{r}$$

or equivalently

$$\begin{aligned}
& \left(\frac{1+r_z}{2rx} + \frac{1+3r}{2r}\frac{1+r}{\frac{1+r_z}{x}} - \frac{1+2r}{r}\right)(1-\overline{\omega})^2 \\
& -2\left(\frac{1-c_{\min}-x\widetilde{z}}{2r}\left(\frac{1+r_z}{x}-1\right) - (1-\rho-x\widetilde{z})\right)(1-\overline{\omega}) + \frac{(1+r_z)(1-c_{\min}-x\widetilde{z})^2}{2rx} \\
\leq \; & 0
\end{aligned}$$

The roots of the associated quadratic are

$$1-\overline{\omega} = \frac{\frac{1-c_{\min}-x\widetilde{z}}{2r}\left(\frac{1+r_z}{x}-1\right) - (1-\rho-x\widetilde{z}) \pm \sqrt{\begin{array}{l}\left(\frac{1-c_{\min}-x\widetilde{z}}{2r}\left(\frac{1+r_z}{x}-1\right) - (1-\rho-x\widetilde{z})\right)^2 \\ -\left(\frac{1+r_z}{2rx} + \frac{1+3r}{2r}\frac{1+r}{\frac{1+r_z}{x}} - \frac{1+2r}{r}\right)\frac{(1+r_z)(1-c_{\min}-x\widetilde{z})^2}{2rx}\end{array}}}{\frac{1+r_z}{2rx} + \frac{1+3r}{2r}\frac{1+r}{\frac{1+r_z}{x}} - \frac{1+2r}{r}}$$

If the larger root exceeds 1, then the condition for this case to be valid reduces to an upper bound on $\overline{\omega}$, specifically

$$\overline{\omega} \leq 1 - \frac{\frac{1-c_{\min}-x\widetilde{z}}{2r}\left(\frac{1+r_z}{x}-1\right) - (1-\rho-x\widetilde{z}) - \sqrt{\begin{array}{l}\left(\frac{1-c_{\min}-x\widetilde{z}}{2r}\left(\frac{1+r_z}{x}-1\right) - (1-\rho-x\widetilde{z})\right)^2 \\ -\left(\frac{1+r_z}{2rx} + \frac{1+3r}{2r}\frac{1+r}{\frac{1+r_z}{x}} - \frac{1+2r}{r}\right)\frac{(1+r_z)(1-c_{\min}-x\widetilde{z})^2}{2rx}\end{array}}}{\frac{1+r_z}{2rx} + \frac{1+3r}{2r}\frac{1+r}{\frac{1+r_z}{x}} - \frac{1+2r}{r}}$$

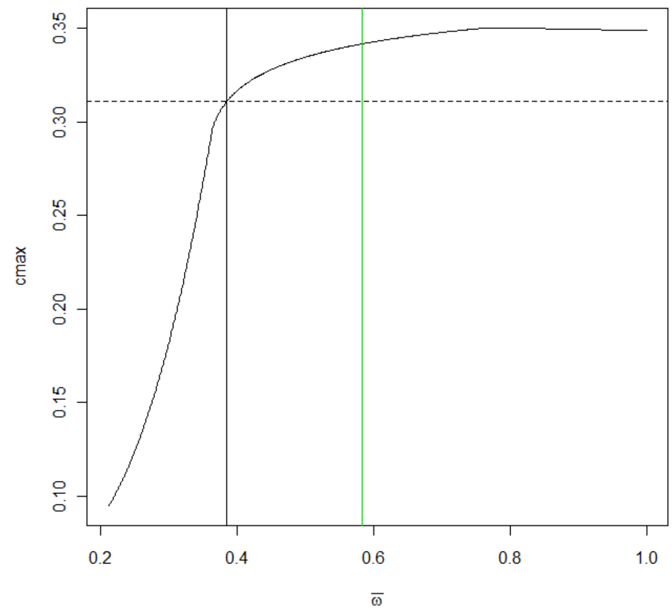
- $\overline{\omega} > c_{\min} + x\widetilde{z}$ has a no-run condition of

$$c_{\max} \geq \overline{c} + \frac{1}{r}\left(\overline{c} + x\widetilde{z} - \frac{\overline{\omega}}{2} - \frac{(c_{\min}+x\widetilde{z})^2}{2\overline{\omega}} - \frac{1+r_z}{1-\overline{\omega}}\frac{(\overline{c}+x\widetilde{z}-\overline{\omega})^2}{2x}\right)$$

Threshold $c_{\max}$ for No-Run Equilibrium In Figure D.1, we use the above derivations to plot the lower bound on $c_{\max}$ as a function of $\overline{\omega} \in [\rho, 1]$ keeping all other parameters as in Section 5.2.1:

Figure D.1:

Lower Bound on $c_{\max}$ for No-Run Equilibrium



The solid black curve plots the lowest value of $c_{\max}$ for which there exists a no-run equilibrium under information suppression as a function of $\overline{\omega}$. The dashed horizontal line is the 1873 value of $c_{\max}$ (0.311). The vertical black line indicates the $\overline{\omega}$ at which 0.311 is the lowest value of $c_{\max}$ for which there exists a no-run equilibrium under information suppression. The vertical green line indicates the $\overline{\omega}$ at which 0.311 is the lowest value of $c_{\max}$ for which there exists a no-run solution with loan certificates under information suppression.

The parameterization involves a high value of x (0.75). The non-monotonicity illustrated in Figure D.1 can be established formally for high x; we do this before demonstrating how the figure changes for low x. Return to the definition of $c_L(\overline{\omega})$ in Eq. (D.2). The derivative with respect to $\overline{\omega}$ is

$$c_L'(\overline{\omega}) = -2r\left(\frac{\rho + 2r\overline{\omega} + x\widetilde{z}}{\sqrt{4(\rho + r\overline{\omega} + x\widetilde{z})\, r\overline{\omega} + (c_{\min} + x\widetilde{z})^2}} - 1\right)$$

The condition for $c_L'(\overline{\omega}) < 0$ reduces to $c_{\min} < \rho$, which is true. The next step is to confirm that $c_L(\overline{\omega})$ is indeed the relevant lower bound on $c_{\max}$ at high $\overline{\omega}$. This will establish a downward-sloping part of the curve at high $\overline{\omega}$. To confirm the relevance of $c_L(\overline{\omega})$, we must confirm (D.3). With $c_{\min} < \rho$, a sufficient condition for (D.3) at $\overline{\omega} = 1$ is $\rho + x\widetilde{z} \leq 1$. Therefore, the lower bound on $c_{\max}$ is locally decreasing in $\overline{\omega}$ for $\overline{\omega}$ close to 1 if $\rho + x\widetilde{z} \leq 1$. The last step is to find a range of $\overline{\omega}$ where the lower bound on $c_{\max}$ is increasing in $\overline{\omega}$. Recall that the case of $\overline{\omega} \leq c_{\min} + x\widetilde{z}$ has $E_{run} = 1$ and

$$E_{wait} = \frac{\int_{c_{\min}}^{\overline{c}} \frac{(1+r_z)\left(\widetilde{z} - \frac{\overline{\omega} - c_i}{x}\right)}{1-\overline{\omega}} dc_i + \int_{\overline{c}}^{c_{\max}} (1+r)\, dc_i}{c_{\max} - c_{\min}}$$

Differentiate E_{wait} to get

$$\frac{dE_{wait}}{d\overline{\omega}} = \frac{\frac{\frac{1+r_z}{x}}{(1-\overline{\omega})^2}\int_{c_{\min}}^{\overline{c}}\left(c_i + x\widetilde{z} - 1\right)dc_i + \left(\frac{\frac{1+r_z}{x}\left(\overline{c}+x\widetilde{z}-\overline{\omega}\right)}{1-\overline{\omega}} - (1+r)\right)\frac{d\overline{c}}{d\overline{\omega}}}{c_{\max} - c_{\min}}$$

$$\frac{dE_{wait}}{dc_{\max}} = \frac{1 + r - E_{wait} + \left(\frac{\frac{1+r_z}{x}\left(\overline{c}+x\widetilde{z}-\overline{\omega}\right)}{1-\overline{\omega}} - (1+r)\right)\frac{d\overline{c}}{dc_{\max}}}{c_{\max} - c_{\min}}$$

where $\overline{c}$ is given by Eq. (D.1). Thus, for parameters such that $\overline{c} = \overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}\left(1-\overline{\omega}\right) - x\widetilde{z}$, we have

$$\frac{dE_{wait}}{d\overline{\omega}} = \frac{\frac{\frac{1+r_z}{x}}{(1-\overline{\omega})^2}\int_{c_{\min}}^{\overline{c}}\left(c_i + x\widetilde{z} - 1\right)dc_i}{c_{\max} - c_{\min}} < 0$$

$$\frac{dE_{wait}}{dc_{\max}} = \frac{1 + r - E_{wait}}{c_{\max} - c_{\min}} > 0$$

and, for parameters such that $\overline{c} = 2\rho - c_{\max}$, we have

$$\frac{dE_{wait}}{d\overline{\omega}} = \frac{\frac{\frac{1+r_z}{x}}{(1-\overline{\omega})^2}\int_{c_{\min}}^{\overline{c}}\left(c_i + x\widetilde{z} - 1\right)dc_i}{c_{\max} - c_{\min}} < 0$$

$$\frac{dE_{wait}}{dc_{\max}} = \frac{1 + r - E_{wait} + \frac{\frac{1+r_z}{x}}{1-\overline{\omega}}\left[c_{\max} - \left(2\rho + x\widetilde{z} - \left(\overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}\left(1-\overline{\omega}\right)\right)\right)\right]}{c_{\max} - c_{\min}} > 0$$

where we have used the sufficient condition $\rho + x\widetilde{z} \leq 1$ to sign $\frac{dE_{wait}}{d\overline{\omega}}$ when $\overline{c} = 2\rho - c_{\max}$. For both values of $\overline{c}$ then, $\frac{dE_{wait}}{d\overline{\omega}} < 0$ and $\frac{dE_{wait}}{dc_{\max}} > 0$. Recall that the lower bound on $c_{\max}$ is such that $E_{wait} = 1$, so it follows that an increase in $\overline{\omega}$ requires an increase in $c_{\max}$ to preserve this equality. This establishes an upward-sloping part of the curve at low $\overline{\omega}$, conditional on the case of $\overline{\omega} \leq c_{\min} + x\widetilde{z}$ being valid. Assuming $\rho < c_{\min} + x\widetilde{z}$ so that the case of $\overline{\omega} \leq c_{\min} + x\widetilde{z}$ applies for at least some values of $\overline{\omega}$ completes the proof.

What happens if x is low? We now establish that the solid black curve in Figure D.1 would be strictly decreasing. To see this, consider $x \to 0$. The three cases become $\overline{\omega} \leq c_{\min}$, $\overline{\omega} \in (c_{\min}, \overline{c})$, and $\overline{\omega} \geq \overline{c}$. Only the case of $\overline{\omega} \geq \overline{c}$ is valid. This follows from $\overline{\omega} \geq \rho$ and the fact that market clearing implies $\overline{c} < \rho$ (otherwise every bank in the interbank market would be a net lender, which is impossible). However, when $\overline{\omega} \geq \overline{c}$, the lower bound $c_L\left(\overline{\omega}\right)$ applies, and we have already shown that $c_L'\left(\overline{\omega}\right) < 0$. Thus, the entire curve is downward-sloping.

Derivations for Loan Certificates We now turn to the derivations behind the green vertical line in Figure D.1.

Set $\zeta \to \infty$ so that $z_i = 0$. The planner chooses a cutoff $\overline{\overline{c}}_l$ along with the loan certificate allocation such that (i) $c_i < \overline{\overline{c}}_l$ have to rely solely on liquidations and (ii) $c_i \geq \overline{\overline{c}}_l$ can borrow (or borrow/liquidate in the limiting case of $r_k = \frac{1+r_z}{x} - 1$) their way out of $\overline{\omega}$ withdrawals at the latent interest rate of $r_b = r_k$. We consider the loan certificate allocation in Proposition 5, specifically

$$k_i = \begin{cases} \max\left\{\overline{k} - c_i, 0\right\} & if \quad c_i < \overline{\overline{c}}_l \\ \nu_i & if \quad c_i \geq \overline{\overline{c}}_l \end{cases}$$

with $\overline{k} = \widetilde{c} + \overline{\nu} - \rho$, which simplifies to

$$\overline{k} = c_{\max} - \rho$$

with $\widetilde{c} + \overline{\nu} = c_{\max}$ as in Section 5.2.1.

The cutoff $\overline{\overline{c}}_l$ is pinned down by $\overline{k} = \int_0^1 k_i di$, which defines the same cutoff as loan certificates without information suppression, i.e., $\overline{\overline{c}}_l$ solves Eq. (12). We obtain a closed-form solution for $\overline{\overline{c}}_l$ using a uniform cash distribution. If $\overline{\overline{c}}_l \leq \overline{k}$, then Eq. (12) delivers $\overline{\overline{c}}_l = c_{\max} - \frac{c_{\max}^2 - c_{\min}^2}{2\rho}$ and confirming $\overline{\overline{c}}_l \leq \overline{k}$ requires $c_{\max} \geq \sqrt{2\rho^2 + c_{\min}^2}$. If instead $\overline{\overline{c}}_l > \overline{k}$, then Eq. (12) with $\overline{k} = c_{\max} - \rho$ delivers $\overline{\overline{c}}_l = c_{\max} - \sqrt{c_{\max}^2 - c_{\min}^2 - \rho^2}$ and confirming $\overline{\overline{c}}_l > \overline{k}$ requires $c_{\max} < \sqrt{2\rho^2 + c_{\min}^2}$. Thus,

$$\overline{\overline{c}}_l = \begin{cases} c_{\max} - \sqrt{c_{\max}^2 - c_{\min}^2 - \rho^2} & if \quad c_{\max} < \sqrt{2\rho^2 + c_{\min}^2} \\ c_{\max} - \frac{c_{\max}^2 - c_{\min}^2}{2\rho} & if \quad c_{\max} \geq \sqrt{2\rho^2 + c_{\min}^2} \end{cases}$$

The 1873 value of $c_{\max}$ satisfies $c_{\max} \geq \sqrt{2\rho^2 + c_{\min}^2}$, hence $\overline{\overline{c}}_l = c_{\max} - \frac{c_{\max}^2 - c_{\min}^2}{2\rho}$.

If $c_i \geq \overline{\overline{c}}_l$, then a patient depositor gets 1 from running and $1 + r$ from waiting. A bank with $c_i < \overline{\overline{c}}_l$ would have to liquidate $\ell_i = \frac{\overline{\omega} - c_i - k_i + \overline{k}}{x}$, where $\ell_i \leq \widetilde{z}$ if and only if $c_i \geq \overline{\omega} - x\widetilde{z} - k_i + \overline{k}$. Using the posited loan certificate allocation to substitute out k_i, this is equivalent to liquidating $\ell_i = \min\left\{\frac{\overline{\omega} - \max\left\{c_i - \overline{k}, 0\right\}}{x}, \widetilde{z}\right\}$. There are two cases when $c_i < \overline{\overline{c}}_l$. If $\max\left\{c_i - \overline{k}, 0\right\} < \overline{\omega} - x\widetilde{z}$, then a patient depositor gets $\frac{c_i + x\widetilde{z} - \min\left\{c_i, \overline{k}\right\}}{\overline{\omega}}$ from running and 0 from waiting. If $\max\left\{c_i - \overline{k}, 0\right\} \geq \overline{\omega} - x\widetilde{z}$, then a patient depositor gets 1 from running and

$$\min\left\{\frac{(1 + r_z)\left(\widetilde{z} - \frac{\overline{\omega} - c_i}{x}\right) - \left(\frac{1 + r_z}{x} - (1 + r_k)\right)\min\left\{c_i, \overline{k}\right\}}{1 - \overline{\omega}}, 1 + r\right\}$$

from waiting. It is straightforward to show that $\overline{k} + \overline{\omega} - x\widetilde{z} \geq \overline{\overline{c}}_l$ when $\overline{\overline{c}}_l = c_{\max} - \frac{c_{\max}^2 - c_{\min}^2}{2\rho}$. Therefore, the case of $\max\left\{c_i - \overline{k}, 0\right\} \geq \overline{\omega} - x\widetilde{z}$ applies for all $c_i < \overline{\overline{c}}_l$ when $\overline{\omega} \leq x\widetilde{z}$ while the case of $\max\left\{c_i - \overline{k}, 0\right\} < \overline{\omega} - x\widetilde{z}$ applies for all $c_i < \overline{\overline{c}}_l$ when $\overline{\omega} > x\widetilde{z}$.

We thus have the following cases:

1. If $\overline{\omega} \leq x\widetilde{z}$, then $E_{run} = 1$ and

$$\begin{aligned} E_{wait} &= \int_{c_{\min}}^{\overline{\overline{c}}_l} \min\left\{\frac{(1 + r_z)\left(\widetilde{z} - \frac{\overline{\omega} - c_i}{x}\right) - \left(\frac{1 + r_z}{x} - (1 + r_k)\right)\min\left\{c_i, \overline{k}\right\}}{1 - \overline{\omega}}, 1 + r\right\} dF(c_i) \\ &+ \int_{\overline{\overline{c}}_l}^{c_{\max}} (1 + r)\, dF(c_i) \end{aligned}$$

If $1+r_k < \frac{1+r_z}{x}$, then the function in the first integral of E_{wait} must be less than $1+r$, otherwise the bank could borrow its way out of $\overline{\omega}$ withdrawals at the latent interest rate of $r_b = r_k$. If $1+r_k = \frac{1+r_z}{x}$, then this function is less than $1+r$ if and only if $c_i < \overline{\omega} + \frac{1+r}{\frac{1+r_z}{x}}(1-\overline{\omega}) - x\widetilde{z} = \overline{c}\left(\frac{1+r_z}{x} - 1; \overline{\omega}\right)$, where we recall the definition of $\overline{c}(r_b; \overline{\omega})$ in the proof of Lemma 9.

2. If $\overline{\omega} > x\widetilde{z}$, then

$$E_{run} = \int_{c_{\min}}^{\overline{\overline{c}}_l} \frac{c_i + x\widetilde{z} - \min\left\{c_i, \overline{k}\right\}}{\overline{\omega}} dF(c_i) + \int_{\overline{\overline{c}}_l}^{c_{\max}} dF(c_i)$$

$$E_{wait} = \int_{\overline{\overline{c}}_l}^{c_{\max}} (1+r)\, dF(c_i)$$

Consider $r_k < \frac{1+r_z}{x} - 1$. Recall that the resource constraint at $t = 1$ of a bank that will survive until $t = 2$ is $c_i + k_i - \overline{k} + x\ell_i + \Delta_i \geq \omega_i$, where $k_i - \overline{k}$ is reallocation via loan certificates. If $c_i \geq \overline{\overline{c}}_l$, then $\ell_i = 0$ and the resource constraint simplifies to $\Delta_i = \omega_i - \rho$. If the bank were to face withdrawals of $\overline{\omega}$, it would need to be able to profitably borrow $\Delta_i = \overline{\omega} - \rho$ at $r_b = r_k$, which it can only do if $c_i \geq \overline{c}(r_k; \overline{\omega})$. Thus, r_k^* solves

$$\overline{c}(r_k^*; \overline{\omega}) = \overline{\overline{c}}_l$$

which yields

$$1 + r_k^* = \frac{(1+r_z)\widetilde{z} - (1+r)(1-\overline{\omega})}{\overline{\omega} - \overline{\overline{c}}_l}$$

after using the definition of $\overline{c}(r_b; \overline{\omega})$. Confirming $r_k^* < \frac{1+r_z}{x} - 1$ then requires

$$\overline{\omega} > \frac{\overline{\overline{c}}_l + x\widetilde{z} - \frac{1+r}{\frac{1+r_z}{x}}}{1 - \frac{1+r}{\frac{1+r_z}{x}}} \equiv \gamma$$

Note that $\overline{c}\left(\frac{1+r_z}{x} - 1; \overline{\omega}\right) \leq \overline{\overline{c}}_l$ is equivalent to $\overline{\omega} \leq \gamma$.

If $\overline{\overline{c}}_l < (1 - x\widetilde{z}) \frac{1+r}{\frac{1+r_z}{x}}$, which is true at the 1873 value of $c_{\max}$, then $\gamma < x\widetilde{z}$ and there are three cases around two thresholds:

- If $\overline{\omega} \leq \gamma$, then the no-run condition is

$$\int_{c_{\min}}^{\overline{c}\left(\frac{1+r_z}{x}-1;\overline{\omega}\right)} \frac{(1+r_z)\left(\widetilde{z} - \frac{\overline{\omega}-c_i}{x}\right)}{1-\overline{\omega}} dF(c_i) + \int_{\overline{c}\left(\frac{1+r_z}{x}-1;\overline{\omega}\right)}^{c_{\max}} (1+r)\, dF(c_i) \geq 1$$

which works out to

$$
\begin{aligned}
&\left(1-\frac{1+r}{\frac{1+r_z}{x}}\right)^2\overline{\omega}^2-2\left(\left(1-\frac{1+r}{\frac{1+r_z}{x}}\right)\left(c_{\min}+x\widetilde{z}-\frac{1+r}{\frac{1+r_z}{x}}\right)-\frac{r\left(c_{\max}-c_{\min}\right)}{\frac{1+r_z}{x}}\right)\overline{\omega} \\
&+\left(c_{\min}+x\widetilde{z}-\frac{1+r}{\frac{1+r_z}{x}}\right)^2-\frac{2r\left(c_{\max}-c_{\min}\right)}{\frac{1+r_z}{x}} \\
\leq\;& 0
\end{aligned}
$$

so this case is only valid if $\overline{\omega}$ is also between the roots of the associated quadratic. For the parameters in Figure D.1, γ is strictly less than the larger root. Thus, the no-run condition is valid for all $\overline{\omega}\leq\gamma$, which means the vertical green line cannot be in this case.

- If $\overline{\omega}\in(\gamma,x\widetilde{z}]$, then the no-run condition is

$$
\int_{c_{\min}}^{\overline{\overline{c}}_l}\frac{(1+r_z)\left(\widetilde{z}-\frac{\overline{\omega}-c_i}{x}\right)-\left(\frac{1+r_z}{x}-(1+r_k)\right)\min\left\{c_i,\overline{k}\right\}}{1-\overline{\omega}}dF\left(c_i\right)+\int_{\overline{\overline{c}}_l}^{c_{\max}}(1+r)\,dF\left(c_i\right)\geq 1
$$

which works out to

$$
\begin{aligned}
&\frac{\frac{1+r_z}{x}}{1-\overline{\omega}}\left(\frac{\overline{\overline{c}}_l+c_{\min}}{2}+x\widetilde{z}-\overline{\omega}\right)\left(\overline{\overline{c}}_l-c_{\min}\right) \\
&-\frac{\frac{1+r_z}{x}-(1+r_k)}{1-\overline{\omega}}\int_{c_{\min}}^{\overline{\overline{c}}_l}\min\left\{c_i,\overline{k}\right\}dc_i+(1+r)\left(c_{\max}-\overline{\overline{c}}_l\right) \\
\geq\;& c_{\max}-c_{\min}
\end{aligned}
$$

where we know

$$
\int_{c_{\min}}^{\overline{\overline{c}}_l}\min\left\{c_i,\overline{k}\right\}dc_i=\left(\rho-\frac{c_{\max}+\overline{\overline{c}}_l}{2}\right)\left(c_{\max}-\overline{\overline{c}}_l\right)
$$

from Eq. (12). After some algebra, the no-run condition becomes

$$
\begin{aligned}
&\left(\overline{\overline{c}}_l-c_{\min}+\frac{c_{\min}+rc_{\max}-(1+r)\overline{\overline{c}}_l}{\frac{1+r_z}{x}}\right)\overline{\omega}^2 \\
&-\left(\begin{array}{l}\frac{\left(c_{\min}+rc_{\max}-(1+r)\overline{\overline{c}}_l\right)\left(1+\overline{\overline{c}}_l\right)}{\frac{1+r_z}{x}}+\left(\frac{\overline{\overline{c}}_l+c_{\min}}{2}+\overline{\overline{c}}_l+x\widetilde{z}\right)\left(\overline{\overline{c}}_l-c_{\min}\right)\\ -\left(1-\frac{1+r}{\frac{1+r_z}{x}}\right)\left(\rho-\frac{c_{\max}+\overline{\overline{c}}_l}{2}\right)\left(c_{\max}-\overline{\overline{c}}_l\right)\end{array}\right)\overline{\omega} \\
&-\left(\begin{array}{l}\left(\overline{\overline{c}}_l+x\widetilde{z}-\frac{1+r}{\frac{1+r_z}{x}}\right)\left(\rho-\frac{c_{\max}+\overline{\overline{c}}_l}{2}\right)\left(c_{\max}-\overline{\overline{c}}_l\right)\\ +\frac{\left(c_{\min}+rc_{\max}-(1+r)\overline{\overline{c}}_l\right)\overline{\overline{c}}_l}{\frac{1+r_z}{x}}+\left(\frac{\overline{\overline{c}}_l+c_{\min}}{2}+x\widetilde{z}\right)\left(\overline{\overline{c}}_l-c_{\min}\right)\overline{\overline{c}}_l\end{array}\right) \\
\leq\;& 0
\end{aligned}
$$

so this case is only valid if $\overline{\omega}$ is also between the roots of the associated quadratic. For the parameters in Figure D.1, the larger root is strictly less than $x\widetilde{z}$. Thus, the no-run condition is valid for all $\overline{\omega}$ less than or equal to this root, which is where the vertical green line is drawn.

- If $\overline{\omega} > x\widetilde{z}$, then the no-run condition is

$$\int_{\overline{\overline{c}}_l}^{c_{\max}} (1+r)\, dF(c_i) \geq \int_{c_{\min}}^{\overline{\overline{c}}_l} \frac{c_i + x\widetilde{z} - \min\left\{c_i, \overline{k}\right\}}{\overline{\omega}} dF(c_i) + \int_{\overline{\overline{c}}_l}^{c_{\max}} dF(c_i)$$

which works out to

$$\left(r\overline{\omega} + \rho - \frac{c_{\max} + \overline{\overline{c}}_l}{2}\right)\left(c_{\max} - \overline{\overline{c}}_l\right) \geq \left(\frac{\overline{\overline{c}}_l + c_{\min}}{2} + x\widetilde{z}\right)\left(\overline{\overline{c}}_l - c_{\min}\right)$$

so this case is only valid if $\overline{\omega}$ also satisfies

$$\overline{\omega} \geq \frac{1}{r}\left(\frac{c_{\max} + \overline{\overline{c}}_l}{2} + \left(\frac{\overline{\overline{c}}_l + c_{\min}}{2} + x\widetilde{z}\right)\frac{\overline{\overline{c}}_l - c_{\min}}{c_{\max} - \overline{\overline{c}}_l} - \rho\right)$$

which is incongruent with $\overline{\omega} \leq 1$ at the parameters being considered. Thus, the no-run condition is violated for all $\overline{\omega} > x\widetilde{z}$ at the 1873 value of $c_{\max}$.

More General Specification for $\overline{\omega} = 1$ Consider a loan certificate allocation by the planner of the form

$$k_i = \begin{cases} \kappa(c_i) & if \quad c_i < \overline{\overline{c}}_l \\ \nu_i & if \quad c_i \geq \overline{\overline{c}}_l \end{cases} \tag{D.4}$$

where $\kappa(c_i) \geq 0$. If $r_k^* < \frac{1+r_z}{x} - 1$, then $\ell_i = 0$ for a bank with $c_i \geq \overline{\overline{c}}_l$ so $\overline{k} = \widetilde{c} + \overline{\nu} - \rho$ is necessary and sufficient to satisfy the resource constraint of any such bank at $t = 1$ in a no-run equilibrium. However, $\overline{\overline{c}}_l$ need not be the same as before. The cutoff $\overline{\overline{c}}_l$ is pinned down by $\overline{k} = \int_0^1 k_i di$, which is now

$$\overline{k} = \int_{c_{\min}}^{\overline{\overline{c}}_l} \kappa(c_i)\, dF(c_i) + \int_{\overline{\overline{c}}_l}^{c_{\max}} \nu_i dF(c_i)$$

or equivalently

$$\int_{c_{\min}}^{\overline{\overline{c}}_l} \kappa(c_i)\, dF(c_i) = \overline{k}F\left(\overline{\overline{c}}_l\right) - \int_{\overline{\overline{c}}_l}^{c_{\max}} (\rho - c_i)\, dF(c_i)$$

after recalling $\nu_i = \widetilde{c} + \overline{\nu} - c_i$ and hence $\nu_i = \overline{k} + \rho - c_i$. With a uniform distribution,

$$\int_{c_{\min}}^{\overline{\overline{c}}_l} \kappa(c_i)\, dc_i = \overline{k}\left(\overline{\overline{c}}_l - c_{\min}\right) - \left(\rho - \frac{c_{\max} + \overline{\overline{c}}_l}{2}\right)\left(c_{\max} - \overline{\overline{c}}_l\right) \tag{D.5}$$

Note that Eq. (D.5) can be rewritten as

$$\int_{c_{\min}}^{\overline{\overline{c}}_l} \left(\overline{k} - \kappa(c_i)\right) dc_i = \left(2\rho - c_{\max} - \overline{\overline{c}}_l\right)\frac{c_{\max} - \overline{\overline{c}}_l}{2}$$

where the left-hand side is the total cash pulled from banks at the bottom of the distribution. We focus on systems of loan certificates where the left-hand side is non-negative. Accordingly, Eq. (D.5) delivers $\overline{\overline{c}}_l \leq 2\rho - c_{\max}$. The condition to confirm $r_k^* < \frac{1+r_z}{x} - 1$ is still $\overline{\omega} > \gamma$, which simplifies to $\overline{\overline{c}}_l + x\widetilde{z} < 1$ at $\overline{\omega} = 1$. With $\overline{\overline{c}}_l \leq 2\rho - c_{\max}$, a sufficient condition for $\overline{\overline{c}}_l + x\widetilde{z} < 1$ is

$2\rho - c_{\max} < 1 - x\widetilde{z}$, which is satisfied at our parameters.

Setting $\overline{\omega} = 1$, a bank with $c_i < \overline{\overline{c}}_l$ would have to liquidate $\ell_i = \frac{1 - c_i - \kappa(c_i) + \overline{k}}{x}$, where $\ell_i \leq \widetilde{z}$ if and only if $c_i \geq 1 - x\widetilde{z} - \kappa(c_i) + \overline{k}$. Therefore, a patient depositor gets $c_i + x\widetilde{z} + \kappa(c_i) - \overline{k}$ from running and 0 from waiting if $c_i < \min\left\{\overline{\overline{c}}_l, 1 - x\widetilde{z} - \kappa(c_i) + \overline{k}\right\}$.

Suppose $\overline{\overline{c}}_l \leq 1 - x\widetilde{z} - \kappa(c_i) + \overline{k}$ for all $c_i \leq \overline{\overline{c}}_l$. That is, suppose $\kappa(\cdot)$ always satisfies $\kappa(\cdot) \leq 1 - x\widetilde{z} + \overline{k} - \overline{\overline{c}}_l$. Then,

$$E_{run} = \int_{c_{\min}}^{\overline{\overline{c}}_l} \left[c_i + x\widetilde{z} - \kappa(c_i)\right] dF(c_i) + \int_{\overline{\overline{c}}_l}^{c_{\max}} dF(c_i)$$

$$E_{wait} = \int_{\overline{\overline{c}}_l}^{c_{\max}} (1 + r)\, dF(c_i)$$

The no-run condition $E_{wait} \geq E_{run}$ is then

$$r\left(c_{\max} - \overline{\overline{c}}_l\right) \geq \left(\overline{\overline{c}}_l - c_{\min}\right)\left(\frac{\overline{\overline{c}}_l + c_{\min}}{2} + x\widetilde{z}\right) - \int_{c_{\min}}^{\overline{\overline{c}}_l} \kappa(c_i)\, dc_i$$

where we can use Eq. (D.5) to substitute out the integral and rewrite the no-run condition as

$$\overline{\overline{c}}_l^2 - \frac{c_{\min}^2 + c_{\max}^2}{2} + \left(x\widetilde{z} - \overline{k}\right)\left(\overline{\overline{c}}_l - c_{\min}\right) + (\rho - r)\left(c_{\max} - \overline{\overline{c}}_l\right) \leq 0$$

With $\widetilde{c} + \overline{\nu} = c_{\max}$, we have $\overline{k} = c_{\max} - \rho$ and hence the no-run condition becomes

$$\overline{\overline{c}}_l^2 + (r + x\widetilde{z} - c_{\max})\,\overline{\overline{c}}_l + (\rho - r)\, c_{\max} - (\rho + x\widetilde{z})\, c_{\min} - \frac{(c_{\max} - c_{\min})^2}{2} \leq 0$$

The associated quadratic has roots

$$\overline{\overline{c}}_l = -\frac{r + x\widetilde{z} - c_{\max}}{2} \pm \sqrt{\left(\frac{r + x\widetilde{z} - c_{\max}}{2}\right)^2 + (\rho + x\widetilde{z})\, c_{\min} - (\rho - r)\, c_{\max} + \frac{(c_{\max} - c_{\min})^2}{2}}$$

The no-run condition therefore requires $\overline{\overline{c}}_l$ between these roots, which will be impossible if the larger root is less than $c_{\min}$, i.e., if

$$-\frac{r + x\widetilde{z} - c_{\max}}{2} + \sqrt{\left(\frac{r + x\widetilde{z} - c_{\max}}{2}\right)^2 + (\rho + x\widetilde{z})\, c_{\min} - (\rho - r)\, c_{\max} + \frac{(c_{\max} - c_{\min})^2}{2}} < c_{\min}$$

or equivalently

$$-(\rho - r)(c_{\max} - c_{\min}) + \frac{(c_{\max} - c_{\min})^2}{2} + c_{\min}(c_{\max} - c_{\min}) < 0$$

which reduces to

$$\widetilde{c} < \rho - r$$

Thus, there does not exist a system of loan certificates of the form in (D.4) with $\kappa(\cdot) \leq 1 - x\widetilde{z} + \overline{k} - \overline{\overline{c}}_l$

that makes information suppression work at $\overline{\omega} = 1$ if $\widetilde{c} < \rho - r$, which is satisfied at our parameters.

Now suppose there exists at least some $c_i < \overline{\overline{c}}_l$ for which $\kappa(c_i) > 1 - x\widetilde{z} + \overline{k} - \overline{\overline{c}}_l$. Without borrowing or liquidation, the available resources of a bank at $t = 1$ are $c_i + \kappa(c_i) - \overline{k}$. Therefore, a bank with $\kappa(c_i) > 1 - x\widetilde{z} + \overline{k} - \overline{\overline{c}}_l$ has resources of at least $c_i + 1 - x\widetilde{z} - \overline{\overline{c}}_l$ at $t = 1$. A bank with $c_i \geq \overline{\overline{c}}_l$ instead has resources $c_i + \nu_i - \overline{k}$, which simplifies to ρ. A bank with $c_i < \overline{\overline{c}}_l$ cannot have more resources than a bank with $c_i \geq \overline{\overline{c}}_l$. Accordingly, if $c_{\min} + 1 - x\widetilde{z} - \overline{\overline{c}}_l \geq \rho$, then there cannot exist any $c_i < \overline{\overline{c}}_l$ for which $\kappa(c_i) > 1 - x\widetilde{z} + \overline{k} - \overline{\overline{c}}_l$. With $\overline{\overline{c}}_l \leq 2\rho - c_{\max}$, a sufficient condition is $2\rho - c_{\max} \leq c_{\min} + 1 - \rho - x\widetilde{z}$. This reduces to $c_{\max} \geq 3\rho + x\widetilde{z} - 1 - c_{\min}$, which is satisfied at our parameters.