Online Appendix for "Matching with Strategic Consistency"*

Marzena Rostek† and Nathan Yoder‡

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S.1 Strategic Consistency and Bargaining Power: Example 3 Revisited

Suppose we want use Theorem 3 to construct the Pareto-optimal profiles for which the outcomes found in Example 3 are stable. To do so, we must first assign cardinal values to the agents' payoffs. Suppose that each agent receives payoff e^2 from their most preferred outcome, e from their second most preferred outcome, and e^{-10} from outcomes that they regard as worse than autarky.[1] We consider two social welfare functions and contrast the profiles they produce. First, consider the Nash product $\phi(x) = \prod_{i \in I} x_i^{\alpha_i}$ with asymmetric weights $(\alpha_a, \alpha_b, \alpha_1, \alpha_2) = (0.3, 0.2, 0.35, 0.15)$. Then in the profile $\{C_i^\phi, \mu_i^\phi\}_{i \in I}$, (4) tells us that the agents choose (and believe others will choose) the contracts in the nonstrategically individually rational outcome that yields the highest value of ϕ among all such outcomes that are subsets of the set of available contracts: i.e., the algorithm (3) can construct $\{C_i^\phi, \mu_i^\phi\}_{i \in I}$

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† University of Wisconsin-Madison, Department of Economics; E-mail: mrostek@ssc.wisc.edu.

‡ University of Georgia, Terry College of Business, John Munro Godfrey, Sr. Department of Economics; E-mail: nathan.yoder@uga.edu.

[1] Formally, we let $u_i(\emptyset) = 1$ for each $i \in I$, and

$$\begin{aligned}
&u_a(\{x_{a1}\}) = e^2; && u_a(\{x_{a2}\}) = e; && u_a(\{x_{a1}, x_{a2}\}) = e^{-10};\\
&u_b(\{x_{b2}\}) = e^2; && u_b(\{x_{b1}\}) = e; && u_b(\{x_{b1}, x_{b2}\}) = e^{-10};\\
&u_1(\{x_{a1}\}) = e^{-10}; && u_1(\{x_{b1}\}) = e; && u_1(\{x_{a1}, x_{b1}\}) = e^2;\\
&u_2(\{x_{a2}\}) = e^2; && u_2(\{x_{b2}\}) = e; && u_2(\{x_{a2}, x_{b2}\}) = e^{-10}.
\end{aligned}$$

using the order $\succ^{\phi}$ given by

$$\{x_{a1}, x_{b1}\} \succ^{\phi} \{x_{a2}, x_{b1}\} \succ^{\phi} \{x_{b2}\} \succ^{\phi} \{x_{a2}\} \succ^{\phi} \{x_{b1}\} \succ^{\phi} \emptyset,$$

and so the stable outcome for $\{C_i^{\phi}, \mu_i^{\phi}\}_{i\in I}$ is given by $\mu^{\phi}(X) = \{x_{a1}, x_{b1}\}$.[2]

Now suppose that we change the weights to be more favorable to firm 2: Let $(\alpha'_a, \alpha'_b, \alpha'_1, \alpha'_2) = (0.3, 0.2, 0.15, 0.35)$, and $\phi'(x) = \prod_{i\in I} x_i^{\alpha'_i}$. Then, the values taken by the social welfare function become higher at outcomes where firm 2 hires Bob relative to those where it hires no one, and at outcomes where firm 2 hires Alice relative to those where it hires Bob: the ordering $\succ^{\phi'}$ is

$$\{x_{a2}, x_{b1}\} \succ^{\phi'} \{x_{a1}, x_{b1}\} \succ^{\phi'} \{x_{a2}\} \succ^{\phi'} \{x_{b2}\} \succ^{\phi'} \{x_{b1}\} \succ^{\phi'} \emptyset,$$

and the stable outcome for $\{C_i^{\phi'}, \mu_i^{\phi'}\}_{i\in I}$ is $\mu^{\phi'}(X) = \{x_{a2}, x_{b1}\}$. ■

S.2 Other Stability Concepts

While the matching literature generally focuses on the solution concept — stability — that we adopt in this paper, many papers also consider other related matching-theoretic solution concepts. Two of the most common are *setwise stability* (Sotomayor, 1999) and *weak setwise stability* (Klaus and Walzl, 2009).

These solution concepts make the same predictions in two-sided one-to-one matching markets (Echenique and Oviedo, 2006; Klaus and Walzl, 2009). But in two-sided many-to-many matching markets (Echenique and Oviedo, 2006; Klaus and Walzl, 2009) or environments with multilateral contracts (Bando and Hirai, 2021), there is, in general, a gap between them. Because it requires that agents' beliefs must be correct, strategic consistency closes this gap between the predictions of stability and weak setwise stability: Whenever a block (in the sense used in stability) is successful, each agent must agree on the set of contracts that will be chosen (Lemma 1). Since the difference between stability and weak setwise stability is that the latter only considers blocks with this property, the two solution concepts must coincide.[3] Hence,

[2] The log values taken by the two social welfare functions ϕ and ϕ' at the nonstrategically individually rational outcomes are given by

$$\begin{array}{lll} \log\phi((u_i(\emptyset))_{i\in I}) = 0; & \log\phi((u_i(\{x_{a2}\}))_{i\in I}) = 0.6; & \log\phi((u_i(\{x_{a2}, x_{b1}\}))_{i\in I}) = 1.15; \\ \log\phi((u_i(\{x_{b1}\}))_{i\in I}) = 0.55; & \log\phi((u_i(\{x_{b2}\}))_{i\in I}) = 0.7; & \log\phi((u_i(\{x_{a1}, x_{b1}\}))_{i\in I}) = 1.5, \\ \log\phi'((u_i(\emptyset))_{i\in I}) = 0; & \log\phi'((u_i(\{x_{a2}\}))_{i\in I}) = 1; & \log\phi'((u_i(\{x_{a2}, x_{b1}\}))_{i\in I}) = 1.35; \\ \log\phi'((u_i(\{x_{b1}\}))_{i\in I}) = 0.35; & \log\phi'((u_i(\{x_{b2}\}))_{i\in I}) = 0.75; & \log\phi'((u_i(\{x_{a1}, x_{b1}\}))_{i\in I}) = 1.1. \end{array}$$

[3] In the literature, weak setwise stability is defined for environments without externalities, but it can be extended to accommodate externalities in the same way as we extended stability. Formally, a set of contracts Y is weakly setwise stable if it is individually rational and there is no $Z \subseteq X$ such that $Z_i = C_i(Z_i \cup Y_i | Z_{-i} \cup Y_{-i})$ for each $i \in N(Z \setminus Y)$.

strategic consistency allows stability to capture outcomes that, with nonstrategic choice, are only captured by weak setwise stability. Example S.1 illustrates.

Example S.1 (Strategic Consistency vs. Nonstrategic Choice). Consider a market with two agents $I = \{1, 2\}$ and two contracts $X = \{x, z\}$ they can sign with each other, and suppose that they have preferences

$$u_1(\{x,z\}) > u_1(\{z\}) > u_1(\{x\}) > u_1(\emptyset); \qquad u_2(\{x\}) > u_2(\{z\}) > u_2(\emptyset) > u_2(\{x,z\}).$$

This is a two-sided market where contracts are substitutable, so with nonstrategic choice, there is a (unique) stable outcome, $\{x\}$. But another outcome is also compelling. Observe that $\{z\}$ is blocked by $\{x\}$ when choice is nonstrategic, since $\hat{C}_1(\{x, z\}) = \{x, z\}$ and $\hat{C}_2(\{x, z\}) = \{x\}$. But this block only occurs because agent 1 incorrectly assumes that agent 2 will choose z as well as x. With strategic consistency, however, the agents must agree: either that the block will result in z being dropped (in profiles where $\{x\}$ is stable) or that it will lead both agents to choose z alone (in profiles where $\{z\}$ is stable). This allows strategic consistency to identify both outcomes as consistent with stability, whereas nonstrategic choice identifies only one. ■

S.3 Indifferences and Multi-Valued Choice

In our main analysis, we assume that agents are never indifferent among sets of contracts that name them: For each $i \in I$ and each distinct $Y, Z \subseteq X_i$ and $X' \subseteq X_{-i}$, $u_i(Y \cup X') \neq u_i(Z \cup X')$. This ensures that nonstrategic choice $\hat{C}_i$ is a function, rather than a correspondence; i.e., for each $Y \subseteq X$, the collection $\hat{C}_i(Y_i|Y_{-i}) := \arg\max_{S \subseteq Y_i} u_i(S \cup Y_{-i})$ has a single element. Hence, with the standard, nonstrategic approach to choice, relaxing this assumption to accommodate indifferences requires extending the definition of stability. We do so in the same way as in Rostek and Yoder (2020).[4]

Definition (Stability with Multi-Valued Choice). Given choice correspondences $\{C_i\}_{i \in I}$, a set of contracts $Y \subseteq X$ is stable if it is

i. *Individually rational*: $Y_i \in C_i(Y_i|Y_{-i})$ for all $i \in I$.

ii. *Unblocked*: There does not exist a nonempty $Z \subseteq (X \setminus Y)$ such that for all $i \in N(Z)$, there exists a $Y' \in C_i((Z \cup Y)_i|(Z \cup Y)_{-i})$ such that $Z_i \subseteq Y'$.

[4] This is not the only way that the definition of stability has been extended to accommodate multi-valued choice. Hatfield et al. (2013) use a universal quantifier rather than n existential one in (ii); i.e., (ii) is replaced with with

ii'. *Unblocked*: There does not exist a nonempty $Z \subseteq (X \setminus Y)$ such that for all $i \in N(Z)$ and for all $Y' \in C_i((Z \cup Y)_i|(Z \cup Y)_{-i})$, we have $Z_i \subseteq Y'$.

The definition of strategic consistency must also be extended slightly to accommodate indifferences. However, strategic consistency still pins down a profile of choice functions, rather than correspondences: Strategic consistency requires each agent to form a belief that specifies a single set of contracts that *will* be chosen by each other agent, rather than multiple sets of contracts they *could* choose. For choices to match those beliefs, then — i.e., for beliefs to be correct — they must be single-valued.

Thus, the only change that we make to the definition of strategic consistency to allow indifferences is to require that at each set of available contracts, choice functions return *an* optimal choice given beliefs, rather than the *unique* optimal choice.

Definition (Strategic Consistency with Indifferences). A profile of choice functions $\{C_i : 2^{X_i} \times 2^{X_{-i}} \to 2^{X_i}\}_{i\in I}$ and beliefs $\{\mu_i : 2^X \to 2^X\}_{i\in I}$ is *strategically consistent* if for each $i \in I$,

i. μ_i is *correct* given $\{C_j\}_{j\neq i}$: For each $Y \subseteq X$, it holds that $\mu_i(Y) = C^{-i}(Y) := \bigcap_{j\neq i}(C_j(Y_j|Y_{-j})\cup Y_{-j})$.

ii. C_i is *optimal* given μ_i: For each $Y \subseteq X$, it holds that $C_i(Y_i|Y_{-i}) \in \arg\max_S u_i(S \cup \mu_i(Y)_{-i})$ s.t. $S \subseteq \mu_i(Y)_i$.

iii. μ_i is *cross-set consistent* given $\{C_i\}_{i\in I}$: For each $Y, Z \subseteq X$, if $Y \supseteq Z \supseteq C_j(Y_j|Y_{-j})$ for all $j \in I$, then $\mu_i(Z) = \mu_i(Y)$.

Consequently, our main results go through with minimal changes to the proofs:

- Lemma 6: replace "=" with "∈" in the statement of the lemma.
- Lemma 2: In the proof of part (b),
  - Beliefs are nonstrategically IR ⇒ Choices are optimal: replace "$\hat{C}_i(\mu(Y)\cap X_i|\mu(Y)\cap X_{-i}) = \mu(Y) \cap X_i = C_i(Y_i|Y_{-i})$" with "$\hat{C}_i(\mu(Y) \cap X_i|\mu(Y) \cap X_{-i}) \ni \mu(Y) \cap X_i = C_i(Y_i|Y_{-i})$".
  - Choices are optimal ⇒ Beliefs are nonstrategically IR: replace "$C_i(Y_i|Y_{-i}) = \hat{C}_i(\mu(Y)\cap X_i|\mu(Y)\cap X_{-i})$" with "$C_i(Y_i|Y_{-i}) \in \hat{C}_i(\mu(Y)\cap X_i|\mu(Y)\cap X_{-i})$" and replace "$\mu(Y)\cap X_i = \hat{C}_i(\mu(Y) \cap X_i|\mu(Y) \cap X_{-i})$" with "$\mu(Y) \cap X_i \in \hat{C}_i(\mu(Y) \cap X_i|\mu(Y) \cap X_{-i})$".
- Lemma 7: In the statement of the lemma, replace "$\mathcal{M} = \{Y \subseteq X|\hat{C}_i(Y_i|Y_{-i}) = Y_i$ for each $i \in I\}$" with "$\mathcal{M} = \{Y \subseteq X|\hat{C}_i(Y_i|Y_{-i}) \ni Y_i$ for each $i \in I\}$"; in the proof, replace "$\hat{C}_i(\emptyset|\emptyset) = \emptyset$" with "$\hat{C}_i(\emptyset|\emptyset) = \{\emptyset\}$".
- Lemma 8: In the proof, replace "$\mathcal{M} = \{Y \subseteq X|\hat{C}_i(Y_i|Y_{-i}) = Y_i$ for each $i \in I\}$" with "$\mathcal{M} = \{Y \subseteq X|\hat{C}_i(Y_i|Y_{-i}) \ni Y_i$ for each $i \in I\}$".

- Theorem 2: In the proof, replace "$\hat{C}_i(\emptyset|\emptyset) = \emptyset$" with "$\hat{C}_i(\emptyset|\emptyset) = \{\emptyset\}$".

- Lemma 9: In the proof, replace "$\mathcal{M} = \{Y \subseteq X | \hat{C}_i(Y_i|Y_{-i}) = Y_i$ for each $i \in I\}$" with "$\mathcal{M} = \{Y \subseteq X | \hat{C}_i(Y_i|Y_{-i}) \ni Y_i$ for each $i \in I\}$".

- Theorem 3: In the statement of the theorem, replace (4) with

$$\mu_i^\phi(Z) \in \arg\max_{S \subseteq Z} \phi((u_i(S))_{i \in I}) \text{ s.t. } \hat{C}_j(S_j|S_{-j}) \ni S_j \ \forall j \in I.$$

The myopic credibility prong of the credibility criterion we use for our forward induction refinements must also be modified to account for multi-valued choice. Formally, given a strategically consistent profile $\{C_i, \mu_i\}_{i \in I}$, we say that Z is a *myopically credible blocking proposal* for Y if, even if agents myopically believe that others will leave their existing contracts intact, each is willing to agree to the contracts in Z, and someone will always reject each of the old contracts in $Y \setminus Z$: $Z_i \in \hat{C}_i((Y \cup Z)_i|(Y \cup Z)_{-i})$ for each $i \in I$, and for each $x \in Y \setminus Z$, there is some $i \in N(x)$ such that $x \notin S_i$ for all $S_i \in \hat{C}_i((Y \cup Z)_i|(Y \cup Z)_{-i})$.

The remainder of our results then go through unchanged, except that Lemma 4 (which shows that with no externalities, Pareto optimality is a stronger refinement than forward induction) requires that there is never a redundant set of contracts; i.e., no two nested outcomes are payoff-equivalent.

- Lemma 4: Replace the statement of the lemma with "Suppose there are no externalities, and for any distinct $S, Z \subseteq X$ with $S \subset Z$, there is an $i \in I$ such that $u_i(Y) \neq u_i(Z)$. If $\{C_i, \mu_i\}_{i \in I}$ is strategically consistent and satisfies Pareto optimality, it also satisfies forward induction." In the proof, replace "By assumption, $u_i(Z_i) \neq u_i(S_i)$, and hence $u_i(Z_i) > u_i(S_i)$, for each $i \in I$ such that $S_i \neq Z_i$." with "If $u_i(Z_i) = u_i(S_i)$ for each $i \in I$, then $S_i \in \hat{C}_i((Y \cup Z)_i|(Y \cup Z)_{-i})$ for each $i \in I$. Then since Z is a myopically credible block of Y, we must have $S \cap Y \setminus Z = \emptyset$, and so $S \subset Z$. Then by hypothesis, $u_i(Z_i) = u_i(Z) \neq u_i(S) = u_i(S_i)$, and hence $u_i(Z_i) > u_i(S_i)$, for some $i \in I$."

- Lemma 10: In the proof, replace "Z is a myopically credible blocking proposal for Y: Since $Z \supset Y$ and Z is nonstrategically individually rational, $Z_i = \hat{C}_i(Z_i|Z_{-i}) = \hat{C}_i((Y \cup Z)_i|(Y \cup Z)_{-i})$ for each $i \in I$." with "Z is a myopically credible blocking proposal for Y: Since $Z \supset Y$ and Z is nonstrategically individually rational, $Z_i \in \hat{C}_i(Z_i|Z_{-i}) = \hat{C}_i((Y \cup Z)_i|(Y \cup Z)_{-i})$ for each $i \in I$. Moreover, $Y \setminus Z = \emptyset$."

- Theorem 5: In the proof of (ii), replace "$\mathcal{M} = \{Z \subseteq X | \hat{C}_i(Z_i|Z_{-i}) = Z_i$ for each $i \in I\}$" with "$\mathcal{M} = \{Z \subseteq X | \hat{C}_i(Z_i|Z_{-i}) \ni Z_i$ for each $i \in I\}$". In the proof of (i), replace "$\hat{C}_i(\emptyset|\emptyset) = \emptyset$" with "$\hat{C}_i(\emptyset|\emptyset) = \{\emptyset\}$".

- Theorem 6: In the proof, replace the first two paragraphs of Step 0 with

  "Let $\mathcal{M} = \{Z \subseteq X | \hat{C}_i(Z_i|Z_{-i}) \ni Z_i \text{ for each } i \in I\}$ denote the set of nonstrategically individually rational outcomes, and define an order $\trianglerighteq$ on $\mathcal{M}$ as follows: $Y \trianglerighteq Z \Leftrightarrow$ either Z is *not* a myopically credible blocking proposal for Y, or $Z = Y$.

  $\trianglerighteq$ is complete: Suppose toward a contradiction that $Y \not\trianglerighteq Z$ and $Z \not\trianglerighteq Y$. Then $Z_i, Y_i \in \hat{C}_i((Y \cup Z)_i|(Y \cup Z)_{-i})$ for all $i \in I$, and for each $x \in (Y \setminus Z) \cup (Z \setminus Y)$, there exists $i \in N(x)$ such that $x \notin S_i$ for each $S_i \in \hat{C}_i((Y \cup Z)_i|(Y \cup Z)_{-i})$. Then for each $x \in Y \setminus Z$, there exists $i \in N(x)$ such that $x \notin Y_i$, which is only possible if $Y \setminus Z = \emptyset$. Likewise, we have $Z \setminus Y = \emptyset$, and so $Y = Z$, a contradiction."

- Theorem 7: Replace the proof with

  "Suppose Y is stable given $\{\hat{C}_i\}_{i \in I}$. Then by definition, it is nonstrategically individually rational: $Y_i \in \hat{C}_i(Y_i|Y_{-i})$ for each $i \in I$. Moreover, there is no $Y' \supset Y$ such that $Y'_i \in \hat{C}_i(Y'_i|Y'_{-i})$ for each $i \in I$: Suppose not, and there exists such a Y'. Then for all $i \in N(Y' \setminus Y)$, we have $Y'_i \setminus Y_i \subseteq Y'_i \in \hat{C}_i(Y'_i|Y'_{-i})$, a contradiction since Y is stable (and therefore unblocked) given $\{\hat{C}_i\}_{i \in I}$. It follows from Theorem 5 that Y is stable for some strategically consistent assessment $\{C_i, \mu_i\}_{i \in I}$ satisfying weak forward induction."

S.4 Multilateral vs. Bilateral Contracts

We emphasize that interactions between pairs of agents specified by a multilateral contract cannot be represented as independent bilateral contracts in a matching on networks model. We illustrate this fact using an example from professional sports.

Example S.2 (Multilateral vs. Bilateral Agreements). On July 31, 2014, Major League Baseball's Detroit Tigers, Tampa Bay Rays, and Seattle Mariners traded a total of five players as part of a single multilateral agreement. Detroit sent two players to Tampa Bay; Tampa Bay sent one player to Detroit; Detroit sent one player to Seattle; and Seattle sent one player to Tampa Bay.[5]

If we tried to model this transaction as three independent bilateral contracts, instead of a single multilateral contract, we would generally fail to predict that it would take place. Indeed, a contract representing the bilateral interaction between the Seattle Mariners and Detroit Tigers would not be individually rational for the Tigers, as they would send Seattle a valuable player without receiving anything in return. Likewise, a contract representing the bilateral interaction between the Mariners and Rays would not be individually rational for the Mariners. These interactions are only possible because they were conducted as part of a single multilateral agreement.

[5]Source: http://mlb.mlb.com/mlb/transactions/index.jsp#month=7&year=2014.

S.5 Strategic Consistency and Nash Equilibrium

Consider the normal form game G defined as follows:

Players The agents $i \in I$.

Actions Sets of contracts $S_i \in 2^{X_i}$.

Payoffs $\pi_i(\{S_i\}_{i\in I}) = u_i\left(\cap_{i\in I}(S_i \cup X_{-i})\right)$.

G extends the link-announcement game discussed in, e.g., Myerson (1991) and Jackson (2010) to our matching with contracts setting. Proposition S.1 shows that an outcome is nonstrategically individually rational if and only if it is the set of contracts signed in a Nash equilibrium of G. The nonstrategically individually rational outcomes are precisely those that can be pinned down by the choice functions and beliefs in a strategically consistent profile (Lemmas 1 and 2), so a profile of choice functions and beliefs is strategically consistent if and only if it is cross-set consistent and maps each set of contracts Y to the contracts signed in a pure strategy Nash equilibrium of G. (Stability then uses this mapping to pin down a stable outcome.)

Proposition S.1 (Nonstrategic Individual Rationality and Nash Equilibrium). *If Y is nonstrategically individually rational, then $\{Y_i\}_{i\in I}$ is a Nash equilibrium of G. Conversely, if $\{S_i(Y)\}_{i\in I}$ is a Nash equilibrium of G, then $S(Y) \equiv \bigcap_{i\in I}(S_i(Y) \cup X_{-i})$ is nonstrategically individually rational.*

Proposition S.2 (Strategic Consistency and Nash Equilibrium). *If $\{C_i, \mu_i\}_{i\in I}$ is a strategically consistent profile of choice functions and beliefs, then for each $Y \subseteq X$, $\{C_i(Y_i|Y_{-i})\}_{i\in I}$ is a Nash equilibrium of G.*

Conversely, if for each $Y \subseteq X$, there is a Nash equilibrium $\{S_i(Y)\}_{i\in I}$ of G such that for each $i \in I$, $\mu_i(Y) = S(Y) \equiv \bigcap_{i\in I}(S_i(Y) \cup X_{-i})$ and $C_i(Y_i|Y_{-i}) = S(Y) \cap X_i$, then the choices $\{C_i\}_{i\in I}$ are optimal given $\{\mu_i\}_{i\in I}$, and the beliefs $\{\mu_i\}_{i\in I}$ are correct given $\{C_i\}_{i\in I}$. If, in addition, we have $S(Y) = S(Z)$ whenever $S(Y) \subseteq Z \subseteq Y$, then $\{C_i, \mu_i\}_{i\in I}$ is a strategically consistent profile.

S.5.1 Proofs for Appendix S.5

Proof of Proposition S.1 (Nonstrategic Individual Rationality and Nash Equilibrium) (Nash Equilibrium $\Rightarrow$ Nonstrategic IR) Suppose that for each $Y \subseteq X$, $\{S_i(Y)\}_{i\in I}$ is a

Nash equilibrium of G. By definition and since $S_i(Y) \subseteq X_i$, we have

$$S_i(Y) \cap \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) = (S_i(Y) \cup X_{-i}) \cap \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \cap X_i = S(Y)_i, \text{ and}$$

$$\left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \cap X_{-i} = (S_i(Y) \cup X_{-i}) \cap \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \cap X_{-i} = S(Y)_{-i}.$$

Then since $\{S_i(Y)\}_{i \in I}$ is a Nash equilibrium of G, for each $Y \subseteq X$ and $i \in I$,

$$S_i(Y) \in \arg \max_{S_i \subseteq X_i} u_i((S_i \cup X_{-i}) \bigcap_{j \neq i} (S_j(Y) \cup X_{-j}));$$

$$\Leftrightarrow S(Y)_i = S_i(Y) \cap \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \in \arg \max_{S_i \subseteq \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right)_i} u_i \left(S_i \cup \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \cap X_{-i} \right);$$

$$\Rightarrow S(Y)_i \in \arg \max_{S_i \subseteq S(Y)_i} u_i \left(S_i \cup \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \cap X_{-i} \right)$$

$$= \arg \max_{S_i \subseteq S(Y)_i} u_i \left(S_i \cup S(Y)_{-i} \right);$$

$$\Leftrightarrow S(Y)_i = \hat{C}_i(S(Y)_i | S(Y)_{-i}),$$

and so $S(Y)$ is nonstrategically individually rational.

(If) Suppose that Y is nonstrategically individually rational. Since $N(x) \geq 2$ for each $x \in X$, $\bigcap_{j \neq i} (Y_j \cup X_{-j}) = Y$. Then for each $i \in I$,

$$Y_i \in \arg \max_{S \subseteq Y_i} u_i(S \cup Y_{-i}) = \arg \max_{S_i \in 2^{Y_i}} u_i \left((S_i \cup X_{-i}) \cap \left(\bigcap_{j \neq i} (Y_j \cup X_{-j}) \right) \right)$$

$$\subseteq \arg \max_{S_i \in 2^{X_i}} u_i \left((S_i \cup X_{-i}) \cap \left(\bigcap_{j \neq i} (Y_j \cup X_{-j}) \right) \right).$$

It follows that $\{Y_i\}_{i \in I}$ is a Nash equilibrium of G. □

Proof of Proposition S.2 (Strategic Consistency and Nash Equilibrium) (Strategic Consistency ⇒ Nash Equilibrium) Suppose that $\{C_i, \mu_i\}_{i \in I}$ is strategically consistent. By Lemmas 1 and 2, at each set $Y \subseteq X$, $\mu_i(Y)$ is nonstrategically individually rational, and $C_i(Y_i | Y_{-i}) = \mu_i(Y) \cap X_i$, for each $i \in I$. Then from Proposition S.1, $\{C_i(Y_i | Y_{-i})\}_{i \in I}$ is a Nash equilibrium of G.

(Nash Equilibrium ⇒ Strategic Consistency) By Proposition S.1, $S(Y)$ is nonstrategically individually rational for each $Y \subseteq X$. Then by Lemma 2, $\{C_i\}_{i \in I}$ are optimal given $\{\mu_i\}_{i \in I}$ and $\{\mu_i\}_{i \in I}$ are correct given $\{C_i\}_{i \in I}$. If, in addition, $S(Y) = S(Z)$ whenever $S(Y) \subseteq Z \subseteq Y$, then for each $i \in I$, the statement $\mu_i(Y) \subseteq Z \subseteq Y$ implies $\mu_i(Y) = \mu_i(Z)$; cross-set (and hence strategic) consistency of $\{C_i, \mu_i\}_{i \in I}$ then follows from Lemma 5. □

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