Online Appendix for
"Capital Flows and the Global Collateral Cycle"

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This online appendix provides proofs and extensions.

A. AUTARKY EQUILIBRIUM EQUATIONS (SECTION II)

A1. Foreign (Subsection II.A)

The price of the consumption good (or equivalently the riskless bond) is normalized to 1. Each Arrow U pays $(1,0)$ and from the no-profit condition for the intermediaries, trades at a price of $(p^* - d_D)/(d_U - d_D)$.

$$(1 - i_1^*)\frac{(e_c + e_Y p^*)}{(p^* - d_D)/(d_U - d_D)} = (d_U - d_D)e_Y. \tag{A1}$$

$$\frac{\gamma(i_1^*)}{(p^* - d_D)/(d_U - d_D)} = 1. \tag{A2}$$

Equation (A1) is the market clearing condition for Arrow U securities. The market for risk-free commodities (consumption goods and the bond) clears by Walras' Law.The top $1 - i_1^*$ agents spend all of their initial endowment of $e_c + e_Y p^*$ on Arrow U securities at a price of $(p^* - d_D)/(d_U - d_D)$. The aggregate demand has to equal the aggregate supply of Arrow U given by $(d_U - d_D)e_Y$. Equation (A2) states that the marginal buyer is indifferent between holding an Arrow U (left hand side) and a riskless position (right hand side). Notice that these two equations also describe the collateral equilibrium formally presented in Section 2. Let us check that equation A1 implies that the marginal buyer i_1^* is indifferent between holding Y, holding a riskless position, and holding a leveraged position in Y. (Since there are only two states, indifference between any two of these three holdings guarantees indifference to the third.) Equation A1 can be written as $p^* = \gamma(i_1^*)d_U + (1 - \gamma(i_1^*))d_D$,where the left hand side is the ratio $(p^*/1)$ of the prices of Y^* and consumption, and the right hand side is the ratio of marginal their utilities, yielding the first indifference. Hence, the asset is priced according to the marginal buyer's

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beliefs. Equation $p^* = \gamma(i_1^*)d_U + (1-\gamma(i_1^*))d_D$ can also be written as $\frac{\gamma(i_1^*)(d_U - d_D)}{p^* - d_D} = 1$, where the left-hand side is is the return of the leveraged position and the right-hand side is the utility per dollar of the consumption good, yielding the second indifference. Note also that since the number of Arrow U securities is equal to a fixed multiple of the number of Y assets, market clearing of the former is equivalent to market clearing of the latter.

A2. *Home (Subsection II.B)*

As before, we express all the equilibrium equations as if agents were directly trading in Arrow U and D securities that pay $(1,0)$ and $(0,1)$ with prices $(p-\pi_T)/d_U$ and π_T/d_D respectively. Equation (A3) is the market clearing condition for the Arrow U securities:

$$(1-i_1)\frac{(e_c + e_Y p)}{(p-\pi_T)/d_U} = d_U e_Y, \tag{A3}$$

The left hand side is the aggregate demand for Arrow U securities. The top $1-i_1$ agents each have wealth $e_c + e_Y p$, and the price of an Arrow U security is $(p-\pi_T)/d_U$. The right hand side is the aggregate supply of Arrow U securities

Equation (A4) is the market clearing condition for the Arrow D securities:

$$i_2\frac{(e_c + e_Y p)}{\pi_T/d_D} = d_D e_Y. \tag{A4}$$

The left hand side is the aggregate demand for Arrow D securities. Agents $i \leq i_2$ spend all their wealth $e_c + e_Y p$ on Arrow D securities with a price of π_T/d_D. The right hand side is the aggregate supply of Arrow D which equals $d_D e_Y$.

Equation (A5) states that the marginal buyer i_1 is indifferent between the return from an Arrow U (left hand side) and the return from holding the consumption good (right hand side), and equation (A6) states that the marginal buyer i_2 is indifferent between the return from an Arrow D (left hand side) and the return from holding the consumption good (right hand side).

$$\frac{\gamma(i_1)}{(p-\pi_T)/d_U} = 1, \tag{A5}$$

$$\frac{(1-\gamma(i_2))}{\pi_T/d_D} = 1. \tag{A6}$$

B. IE EQUATIONS (SECTION III)

As before we express all the equations in terms of Arrow securities instead of U and D tranches. Equation (B1) states that by no-arbitrage, the return of an Arrow U security obtained from tranching Y (left hand side) is equal to the return of an Arrow U security obtained from leveraging Y^* (right hand side).

$$\frac{1}{(\hat{p}-\hat{\pi}_T)/d_U} = \frac{1}{(\hat{p}^*-d_D)/(d_U-d_D)}. \tag{B1}$$

Equations (B2) and (B3) are the market clearing conditions for Arrow U and D securities respectively:

$$(1-\hat{i}_1)\frac{[(e_c+e_Y\hat{p})+(e_c+e_Y\hat{p}^*)]}{(\hat{p}-\hat{\pi}_T)/d_U} = d_U e_Y + (d_U-d_D)e_Y. \tag{B2}$$

$$\hat{i}_2\frac{[(e_c+e_Y\hat{p})+(e_c+e_Y\hat{p}^*)]}{\hat{\pi}_T/d_D} = d_D e_Y. \tag{B3}$$

Compared to their autarky counterparts (equations (A1), (A3) and (A4), the market clearing conditions in IE include wealth from both countries. The right hand side of both equations also underscores the fact that whereas both countries create Arrow U securities, only Home creates Arrow D securities.

Finally, equations (B4) and (B5) are the optimality conditions for marginal buyers $\hat{i}_1$ and $\hat{i}_2$. Equation (B4) states that the marginal buyer $\hat{i}_1$ is indifferent between an Arrow U security (either via tranching the Home asset or leveraging the Foreign asset) and a safe position (holding consumption goods or riskless bonds)

$$\frac{\gamma(\hat{i}_1)}{(\hat{p}-\hat{\pi}_T)/d_U} = 1. \tag{B4}$$

Equation (B5) states that the marginal buyer $\hat{i}_2$ is indifferent between an Arrow D security and a safe position (holding consumption goods and riskless bonds):

$$\frac{(1-\gamma(\hat{i}_2))}{\hat{\pi}_T/d_D} = 1. \tag{B5}$$

C. Proofs of Propositions

C1. Proof of Proposition 1

From equations (B1), (B4), and (B5), we can write asset prices as

$$\hat{p} = \gamma(\hat{i}_1)d_U + \hat{\pi}_T = \gamma(\hat{i}_1)d_U + \left(1-\gamma(\hat{i}_2)\right)d_D,$$
$$\hat{p}^* = \gamma(\hat{i}_1)(d_U - d_D) + d_D.$$

Hence we have $\hat{\Delta} = d_D(\gamma(\hat{i}_1) - \gamma(\hat{i}_2))$, which is positive since $\hat{i}_1 > \hat{i}_2$ and beliefs are monotonic.

C2. Proof of Corollary 1

We define the basis as $\beta = \hat{\pi}_U + \hat{\pi}_D - 1$. Using equations (B1), (B4), and (B5) we have that
$\beta = (\hat{p} - \hat{\pi}_T)/d_U + \hat{\pi}_T/d_D - 1$=
$(\gamma(\hat{i}_1)d_U + \hat{\pi}_T - \hat{\pi}_T)/d_U + \hat{\pi}_T/d_D - 1 =$
$\gamma(\hat{i}_1)d_U/d_U + (1-\gamma(\hat{i}_2)d_D)/d_D - 1 =$
$\gamma(\hat{i}_1) + \left(1-\gamma(\hat{i}_2)\right) - 1 > 0$since $\hat{i}_1 > \hat{i}_2$ and beliefs are monotonic.
Moreover, from Proposition 1 we have that, $\hat{\Delta} = d_D\beta$ since $\beta = \gamma(\hat{i}_1) + \left(1-\gamma(\hat{i}_2)\right) - 1 = \gamma(\hat{i}_1) - \gamma(\hat{i}_2)$.

C3. Proof of Corollary 2

The difference in wealth between Home and Foreign at $s=0$ is $(e_c + e_Y\hat{p}) + (e_c + e_Y\hat{p}^*) = e_y(\hat{p} - \hat{p}^*) > 0$, which From Proposition 1 is positive.

C4. Proof of Corollary 3

Let $f = \frac{e_c + e_Y\hat{p}}{[(e_c+e_Y\hat{p})+(e_c+e_Y\hat{p}^*)]}$ be the fraction of wealth at $s=0$ collectively held by Home agents. From Proposition 1 and Corollary 2 $f > 1/2$.

Since every Home agent owns the same fraction of wealth between him and his identical twin in Foreign, and since utilities are linear, Home will collectively own its endowment plus the fraction $f > 1/2$ of all asset payoffs in each terminal state, while Foreign will own the same endowment plus the fraction $1-f < 1/2$ of all asset payoffs in each terminal state.

C5. Proof of Proposition 3

(i) Clearly Home must export Arrow D securities, and thus hold fewer than in autarky.

(ii) Note that the ratio of Home wealth to Foreign wealth is less than the ratio of Arrow U Securities created in Home to Arrow U securities created in Foreign. From equations (B1), (B4), and (B5), we have that

$$w = \frac{\hat{p}e_Y + e_c}{\hat{p}^* e_Y + e_c} = \frac{[\gamma(\hat{i}_1)d_U + \left(1-\gamma(\hat{i}_2)\right)d_D]e_Y + e_c}{[\gamma(\hat{i}_1)d_U + \left(1-\gamma(\hat{i}_1)\right)d_D]e_Y + e_c} < \frac{d_U}{d_U - d_D}$$

because dividing both numerators and denominators by d_U and replacing $\frac{d_U}{d_D}$ with R, that is equivalent to

$$\frac{[\gamma(\hat{i}_1) - \gamma(\hat{i}_1)R + (1-R)\left(1-\gamma(\hat{i}_2)\right)R]e_Y + e_c(1-R)}{[\gamma(\hat{i}_1) - \gamma(\hat{i}_1)R + R]e_Y + e_c} < 1$$

which is surely true. It follows from Corollary 2, that Home does not hold as many Arrow U securities as it creates, i.e. Home is a net exporter of Arrow U securities and holds fewer of them than it did in autarky.

C6. Proof of Proposition 2

To prove the proposition we will use the following Lemma.

Lemma 1. *International trade lowers Home's marginal buyer of D tranches and raises Home's marginal buyer of U tranches, while lowering Foreign's marginal buyer of U tranches:i)* $\hat{i}_2 < i_2$*, ii)* $\hat{i}_1 > i_1$*, and iii)* $\hat{i}_1 < i_1^*$.

Proof:

i) $\hat{i}_2 < i_2$

Suppose the opposite. Then at least as many agents would be spending all their money on D tranches, yet buying fewer of them. This could only happen if D tranches are more expensive relative to wealth in global equilibrium than in autarky. Since by hypothesis $1-\gamma(\hat{i}_2) \leq 1-\gamma(i_2)$, the D tranche can only get relatively more expensive if $\hat{i}_1 < i_1$. But that means the number of agents buying safe instruments is less than in autarky, $\hat{i}_1 - \hat{i}_2 < i_1 - i_2$, and each agent is poorer (because the price of Y must have gone down), contradicting our starting observation that Home buys more riskless assets than it did in autarky.

ii) $\hat{i}_1 > i_1$

Suppose the opposite. That means, according to Proposition 4, that at least as many Home agents are spending all their money yet buying fewer U tranches than in autarky. That means U tranches must have gotten more expensive relative to income, which (since $1-\gamma(\hat{i}_2) > 1-\gamma(i_2)$), can only happen if $\hat{i}_1 > i_1$.

iii) $\hat{i}_1 < i_1^*$

Suppose the opposite. That means fewer (or equal numbers) of agents must buy more U tranches than in autarky, when the price of U tranches relative to wealth has become more expensive, which is a contradiction.

The statement of proposition follow directly from the lemma and equations B1, B4, B5, A2 Since $\hat{i}_1 > i_1$and $\hat{i}_2 < i_2$, $\hat{p} > p$. Similarly, since $\hat{i}_1 < i_1^*$, $\hat{p}^* < p^*$.

C7. Proof of Proposition 4

The proposition follows directly from the Home-Bias Neutral equilibrium noting Home holds the fraction $f > .5$ of the securities in *i), ii)* and *iii)* in the statement.

C8. Proof of Proposition 5

In Home-Bias Neutral equilibrium, Home holds the fraction fraction $f > .5$ of world consumption, hence it runs a balance of trade deficit at 0. Furthermore, at 0 Home holds the fraction f of Arrow U securities and the same fraction f of bonds and Arrow D securities. Net cash flows from Foreign to Home at U are then

$$f(d_U + (d_U - d_D)) - d_U + f d_D = d_U(2f - 1) > 0.$$

Home spends this cash flow on consumption, running a trade deficit at U. Similarly, net cash flows from Foreign to Home at D are

$$-(1-f)d_D + f d_D = (2f-1)d_D > 0$$

Again, Home spends this cash flow on consumption, running a trade deficit at D.

Finally, the trade balance for Home at time 0 is

$$TB_0^H = \frac{(e_{c_0} + e_Y \hat{p})}{2e_{c_0} + e_Y(\hat{p} + \hat{p}^*)} - \frac{(e_{c_{0^*}} + e_{Y^*} \hat{p}^*)}{2e_{c_0} + e_Y(\hat{p} + \hat{p}^*)} = \frac{e_Y \hat{\Delta}}{2e_{c_0} + e_Y(\hat{p} + \hat{p}^*)}.$$

D. DYNAMIC MODEL

D1. Budget Set and Collateral Equilibrium in the Dynamic Model

BUDGET SETGiven asset and contract prices in each state s, $(p, (\pi_j)_{j \in J})$, each agent $i \in I$ choses asset holdings y_s of Y, contract trades $\varphi_{j,s}$ and consumption c_s in state s, subject to the budget set defined by

$$
\begin{aligned}
B^i(p,\pi) = \Big\{ & (c_s, y_s, \varphi_s) \in R_+^L \times R^J : \\
& c_0 + p_0 y_0 + \sum_{j\in J} \varphi_{j,0}\pi_{j,0} \le e_c + p_0 e_Y, \\
& \sum_{j\in J} \max(0, -\varphi_{j,0}) \le y_0, \\
& c_U + p_U y_U + \sum_{j\in J} \varphi_{j,U}\pi_{j,U} \le c_0 + p_U y_0 + \sum_{j\in J} \varphi_{j,U} \min(j_U, p_U), \\
& \sum_{j\in J} \max(0, -\varphi_{j,U}) \le y_U, \\
& c_D + p_D y_D + \sum_{j\in J} \varphi_{j,D}\pi_{j,D} \le c_0 + p_D y_0 + \sum_{j\in J} \varphi_{j,U} \min(j_D, p_D), \\
& \sum_{j\in J} \max(0, -\varphi_{j,D}) \le y_D, \\
& c_s = F_s(c_{s^*}, Y_{s^*}) + \sum_{j\in J} \varphi_j \min(j_s, d_s), s \in S_T \Big\}.
\end{aligned}
$$

COLLATERAL EQUILIBRIUMA Collateral Equilibrium in this economy is a price of asset Y, contract prices, asset holdings, contract trades and consumption decisions by all the agents $((p,\pi),(c_0^i, y^i, \varphi^i, c_U^i, c_D^i)_{i\in I}) \in (R_+ \times R_+^J) \times (R_+^L \times R^J)^I$, such that

1) $\int_0^1 c_s^i di = e_{c_s}\ \forall s \in \{0,U,D\}$
2) $\int_0^1 c_s^i di = F(e_{c_{s^*}}, e_{Y_{s^*}})\ \forall s \in S_T$
3) $\int_0^1 y_s^i di = e_{Y_S}\ \forall s \in \{0,U,D\}$
4) $\int_0^1 \varphi_{j,s}^i di = 0, \forall j \in J$ and $\forall s \in \{0,U,D\}$
5) $(c_s^i, y_s^i, \varphi_s^i) \in B^i(p,\pi), \forall i$ and $(c_s, y_s, \varphi_s) \in B^i(p,\pi) \Rightarrow U^i(c_s, y_s) \le U^i(c_s^i, y_s^i), \forall i$

D2. *Equilibrium Equations (Section IV)*

We solve the model numerically for $(e_{c_s}, e_{Y_s}) = (1,1)$, $s = 0,U,D$. $d_{UU} = d_{DU} = 1$. The following equations are the ones that characterize the equilibrium in each case. Unlike the static model in which we stated equations in the space of Arrow securities and the Foreign bond, here we state the equations in the space of the original commodities, assets used as collateral and consumptions. In the dynamic case it is easier to understand the changes of wealth trough time generated by interim scary bad news. Of course, as explained in the paper, the two approaches are equivalent.

AUTARKY FOREIGN. — The first two equations correspond to market clearing for the asset Y^*in states $s=0$ and $s=D$.The next two are the marginal agents' indifferent conditions at states $s=0$ and $s=D$.

$$
\begin{aligned}
&(1-i_0^1)\frac{1+p_0}{p_0-p_D}=1\\
&(i_0^1-i_D^1)\frac{1+p_0}{p_D-d_{DD}}+(1-i_D^1)\frac{1+p_D}{p_D-d_{DD}}=2\\
&\frac{i_0^1(1-p_D)}{p_0-p_D}=i_0^1+(1-i_0^1)\frac{i_0^1(1-d_{DD})}{p_D-d_{DD}}\\
&i_D^1+(1-i_D^1)d_{DD}=p_D
\end{aligned}
$$

AUTARKY HOME. — The first four equations are the ones corresponding to state $s=0:$ the first two are market clearing for the asset Y and the D tranche, and the second two the marginal buyer indifference conditions. The last four equations are the analogous ones for state $s=D$.

$$
\begin{aligned}
&(1-i_0^1)\frac{(1+p_0)}{p_0-\pi_0^T}=1\\
&i_0^2(1+p_0)=\pi_0^T\\
&\frac{i_0^1}{p_0-\pi_0^T}=i_0^1+(1-i_0^1)\frac{i_0^1}{p_D-\pi_D^T}\\
&p_D(1-i_0^2)=\pi_0^T\\
&\frac{(i_0^1-i_D^1)(1+p_0)}{p_D-\pi_D^T}+\frac{(1-i_D^1)(1+p_D)}{p_D-\pi_D^T}=2\\
&\frac{i_D^2}{i_0^2}p_D+i_D^2(1+p_D)=2\pi_D^T\\
&\frac{i_D^1}{p_D-\pi_D^T}=1\\
&d_{DD}(1-i_D^2)=\pi_D^T
\end{aligned}
$$

INTERNATIONAL EQUILIBRIUM. — The first five equations are the ones corresponding to state $s=0:$ the first two are market clearing for the asset Y and the D tranche, and the last three are the marginal buyer indifference conditions and the non arbitrage condition for the Arrow U security (it can be created via tranching Yor via leveraging Y^*. The last five equations are the analogous ones for state $s=D$.

$$
\begin{aligned}
&(1-\hat{i}_0^1)\frac{2+\hat{p}_0+\hat{p}_0^*}{\hat{p}_0-\hat{\pi}_0^T+\hat{p}^*-\hat{p}_D^*}=1\\
&\hat{i}_0^2(2+\hat{p}_0+\hat{p}_0^*)=\hat{\pi}_0^T\\
&\frac{\hat{i}_0^1}{\hat{p}_0-\hat{\pi}_0^T}=\hat{i}_0^1+(1-\hat{i}_0^1)\frac{\hat{i}_0^1}{\hat{p}_D-\hat{\pi}_D^T}\\
&\frac{\hat{i}_0^1}{\hat{p}_0-\hat{\pi}_0^T}=\frac{\hat{i}_0^1(1-\hat{p}_D^*)}{\hat{p}_0^*-\hat{p}_D^*}\\
&\hat{p}_D(1-\hat{i}_0^2)=\hat{\pi}_0^T\\
&(\hat{i}_0^1-\hat{i}_D^1)\frac{2+\hat{p}_0+\hat{p}_0^*}{\hat{p}_D-\hat{\pi}_D^T+\hat{p}_D^*-d_{DD}}+(1-\hat{i}_D^1)\frac{2+\hat{p}_D+\hat{p}_D^*}{\hat{p}_D-\hat{\pi}_D^T+\hat{p}_D^*-d_{DD}}=2\\
&\frac{\hat{i}_D^2}{\hat{i}_0^2}\hat{p}_D+\hat{i}_D^2(2+\hat{p}_D+\hat{p}_D^*)=2\hat{\pi}_D^T\\
&\frac{\hat{i}_D^1}{\hat{p}_D-\hat{\pi}_D^T}=1\\
&\frac{\hat{i}_D^1}{\hat{p}_D-\hat{\pi}_D^T}=\frac{\hat{i}_D^1(1-d_{DD})}{\hat{p}_D^*-d_{DD}}\\
&d_{DD}(1-\hat{i}_D^2)=\hat{\pi}_D^T;
\end{aligned}
$$