Supplemental Appendix

PROOF OF PROPOSITION 1:

Consider an impulse deviation in which the planner chooses action x for an infinitesimal time ε, raising the stock by εx, after which σ^c is resumed. The planner's deviation payoff is

$$\hat{\mathcal{W}}^c(x,\varepsilon,X) = \left(\int_0^\varepsilon Z(q^c(X),\tau)e^{-r\tau}\,d\tau\right)u(x) + \int_\varepsilon^{+\infty} e^{-r\tau}Z(q^c(X),\tau)u(\sigma^c(X^c(\tau-\varepsilon,X+\varepsilon x)))\,d\tau. \tag{A.1}$$

Observe that $q^c(X) = \frac{qe^{-\Delta T^c(X)}}{1-q+qe^{-\Delta T^c(X)}}$ evolves according to

$$\sigma^c(X)\dot{q}^c(X) = -\Delta q^c(X)(1-q^c(X)),\ \text{with } q^c(0)=q. \tag{A.2}$$

Observe now that the survival probability satisfies the simple rule

$$Z(q^c(X),\tau+\varepsilon) = Z(q^c(X),\tau) - q^c(X)\left(1-e^{-\Delta\varepsilon}\right)e^{-\Delta\tau}.$$

Equipped with this condition and changing variables, we rewrite the benefit of a deviation as

$$\hat{\mathcal{W}}^c(x,\varepsilon,X) = \int_0^\varepsilon Z(q^c(X),\tau)e^{-r\tau}u(x)d\tau$$

$$+e^{-r\varepsilon}\Big(\mathcal{W}^c(X+x\varepsilon) + \int_0^{+\infty} e^{-r\tau}\left(Z(q^c(X),\tau) - Z(q^c(X+x\varepsilon),\tau)\right)u(\sigma^c(X^c(\tau,X+x\varepsilon)))d\tau$$

$$-q^c(X)\left(1-e^{-\Delta\varepsilon}\right)\varphi^c(X+x\varepsilon)\Big).$$

Taking a first-order Taylor approximation in ε yields

$$\hat{\mathcal{W}}^c(x,\varepsilon,X) = \mathcal{W}^c(X) + \varepsilon\frac{\partial\hat{\mathcal{W}}^c}{\partial\varepsilon}(x,0,X) + o(\varepsilon).$$

At an optimum $(\mathcal{W}^c(X),\sigma^c(X))$ of the planner's problem, any impulse deviation must be weakly dominated. Hence, we must have

$$0 = \max_{x\in\mathbb{R}_+}\frac{\partial\hat{\mathcal{W}}^c}{\partial\varepsilon}(x,0,X)$$

or

$$0 = \max_{x\in\mathbb{R}_+}\left(-r\mathcal{W}^c(X) - \Delta q^c(X)\varphi^c(X) + u(x) + x\left(\dot{\mathcal{W}}^c(X) + \dot{q}^c(X)\frac{\mathcal{W}^c(X)-\varphi^c(X)}{1-q^c(X)}\right)\right).$$

The necessary conditions for optimality are then

$$r\mathcal{W}^c(X) + \Delta q^c(X)\varphi^c(X) = \max_{x\in\mathbb{R}_+} u(x) + x\left(\dot{\mathcal{W}}^c(X) + \frac{\dot{q}^c(X)}{1-q^c(X)}\left(\mathcal{W}^c(X)-\varphi^c(X)\right)\right), \tag{A.3}$$

$$\sigma^c(X) \in \arg\max_{x\in\mathbb{R}_+} u(x) + x\left(\dot{\mathcal{W}}^c(X) + \frac{\dot{q}^c(X)}{1-q^c(X)}\left(\mathcal{W}^c(X)-\varphi^c(X)\right)\right). \tag{A.4}$$

Given strict concavity of the maximand above, an interior solution is given by (1). Inserting (1) into (A.3) and using (A.2) also yields

$$r\mathcal{W}^c(X) + q^c(X)\Delta\varphi^c(X) = u(\sigma^c(X)) + \sigma^c(X)\dot{\mathcal{W}}^c(X) - q^c(X)\Delta(\mathcal{W}^c(X)-\varphi^c(X)) \tag{A.5}$$

with the boundary condition $\mathcal{W}^c(\overline{X}) = \mathcal{V}_\infty$.

PROOF OF PROPOSITION 2:

We define the deviation payoff $\hat{\mathcal{W}}^*(x, \varepsilon, X)$ as

$$\hat{\mathcal{W}}^*(x,\varepsilon,X) = \left(\int_0^\varepsilon e^{-r\tau} Z(p,\tau)d\tau\right) u(x) + \int_\varepsilon^{+\infty} e^{-r\tau} Z(p,\tau) u(\sigma^*(X^*(\tau-\varepsilon, X + x\varepsilon)))d\tau.$$

Changing variables, we rewrite this expression as

$$\hat{\mathcal{W}}^*(x,\varepsilon,X) = \left(\int_0^\varepsilon e^{-r\tau} Z(p,\tau)d\tau\right) u(x) + e^{-r\varepsilon}\int_0^{+\infty} e^{-r\tau} Z(p,\tau+\varepsilon) u(\sigma^*(X^*(\tau, X + x\varepsilon)))d\tau.$$

Using the identity $Z(p, \varepsilon + s) = Z(p, \varepsilon)Z(p(\varepsilon), s)$ we have

$$\hat{\mathcal{W}}^*(x,\varepsilon,X) = \left(\int_0^\varepsilon e^{-r\tau} Z(p,\tau)d\tau\right) u(x) + e^{-r\varepsilon} Z(p,\varepsilon)\Big(\int_0^{+\infty} e^{-r\tau} Z(p(\varepsilon),\tau) u(\sigma^*(X^*(\tau, X + x\varepsilon)))d\tau \tag{A.6}$$

where $p(\varepsilon) = \frac{pe^{-\Delta\varepsilon}}{1-p+pe^{-\Delta\varepsilon}}$ and, for ε small enough, $p(\varepsilon) = p - \Delta p(1-p)\varepsilon + o(\varepsilon)$. The deviation payoff can actually be expressed as

$$\hat{\mathcal{W}}^*(x,\varepsilon,X) = \left(\int_0^\varepsilon e^{-r\tau} Z(p,\tau)d\tau\right) u(x)$$
$$+e^{-r\varepsilon} Z(p,\varepsilon)\left(\mathcal{W}^*(X+x\varepsilon) - \frac{p(\varepsilon)-p}{1-p}\left(\mathcal{W}^*(X+x\varepsilon) - \varphi^*(X+x\varepsilon)\right)\right).$$

Assuming $\mathcal{W}^*$ differentiable, a first-order Taylor approximation in ε yields

$$\hat{\mathcal{W}}^*(x,\varepsilon,X) = \mathcal{W}^*(X) + \varepsilon\left(-(r+p\Delta)\mathcal{W}^*(X) + u(x) + x\dot{\mathcal{W}}^*(X) - \frac{\dot{p}(0)}{1-p}\left(\mathcal{W}^*(X) - \varphi^*(X)\right)\right) + o(\varepsilon)$$

or, using $\dot{p}(0) = -\Delta p(1-p)$,

$$\hat{\mathcal{W}}^*(x,\varepsilon,X) = \mathcal{W}^*(X) + \varepsilon\left(-r\mathcal{W}^*(X) - p\Delta\varphi^*(X) + u(x) + x\dot{\mathcal{W}}^*(X)\right) + o(\varepsilon)$$

We rewrite this Taylor expansion as

$$\hat{\mathcal{W}}^*(x,\varepsilon,X) = \mathcal{W}^*(X) + \varepsilon\frac{\partial\hat{\mathcal{W}}^*}{\partial\varepsilon}(x,0,X) + o(\varepsilon).$$

At a *SBE* $(\mathcal{W}^*(X), \sigma^*(X))$, any impulse deviation must be weakly dominated for the current decision-maker. We should thus have

$$0 = \max_{x\in\mathbb{R}_+}\frac{\partial\hat{\mathcal{W}}^*}{\partial\varepsilon}(x,0,X) = \max_{x\in\mathbb{R}_+}\left(-r\mathcal{W}^*(X) - \Delta p\varphi^*(X) + u(x) + x\dot{\mathcal{W}}^*(X)\right). \tag{A.7}$$

Given strict concavity of the maximand above, an interior solution is given by (2). Inserting into the maximand of (A.7) yields

$$r\mathcal{W}^*(X) + p\Delta\varphi^*(X) = u(\sigma^*(X)) + \sigma^*(X)\dot{\mathcal{W}}^*(X). \tag{A.8}$$

PROOF OF PROPOSITION 3:

Inserting (2) into (A.8), we obtain

$$r\mathcal{W}^*(X) + p\Delta\varphi^*(X) = \frac{(\zeta + \dot{\mathcal{W}}^*(X))^2}{2}.$$

Taking the highest root yields

$$\text{(A.9)} \qquad \begin{cases} \dot{\mathcal{W}}^*(X) = -\zeta + \sqrt{2\left(r\mathcal{W}^*(X) + p\Delta\varphi^*(X)\right)} & \text{if } X \in [0, \overline{X}), \\ \mathcal{W}^*(X) = \mathcal{V}_\infty & \text{if } X \geq \overline{X}. \end{cases}$$

Finally, we obtain

$$\text{(A.10)} \qquad \sigma^*(X) = \begin{cases} \sqrt{2\left(r\mathcal{W}^*(X) + p\Delta\varphi^*(X)\right)} & \text{if } X \in [0, \overline{X}), \\ \zeta & \text{if } X \geq \overline{X}. \end{cases}$$

Backwards and Forwards System of Ordinary Differential Equations. We now form a system of ODEs for $\varphi^*(X)$ and $\mathcal{W}^*(X)$. Studying its properties provides existence and uniqueness of a solution. Observe that, for $X \in [0, \overline{X})$, φ^* is differentiable when σ^* is, with

$$\dot{\varphi}^*(X) = \int_0^{+\infty} e^{-\lambda\tau} u'(\sigma^*(X^*(\tau, X))\dot{\sigma}^*(X^*(\tau, X))\frac{\partial X^*}{\partial X}(\tau, X)d\tau$$

where $\frac{\partial X^*}{\partial X}(\tau, X) = \frac{\sigma^*(X^*(\tau,X))}{\sigma^*(X)}$. Manipulating and integrating by parts yields the ODE for φ^*:

$$\text{(A.11)} \qquad \sigma^*(X)\dot{\varphi}^*(X) = \lambda\varphi^*(X) - u(\sigma^*(X)).$$

Using (A.10), we rewrite this condition as

$$\text{(A.12)} \qquad \dot{\varphi}^*(X) = \frac{\lambda\varphi^*(X) - u(\sigma^*(X))}{\sqrt{2\left(r\mathcal{W}^*(X) + p\Delta\varphi^*(X)\right)}}.$$

Together with the definition of σ^* in (A.10), (A.9) and (A.12) define a system of ODEs whose trajectories may be represented in the $(\varphi, \mathcal{W})$ plan. The terminal conditions for this system are

$$\text{(A.13)} \qquad \mathcal{W}^*(\overline{X}) = \varphi^*(\overline{X}) = \mathcal{V}_\infty$$

where equalities follow from the fact that uncertainty is resolved for all decision-makers once the stock reaches $\overline{X}$.

Existence and Unicity. The locus L_1 of points such that $\dot{\varphi}^*(X) = 0$ is defined as $\lambda\varphi = u(\sigma^*)$ or

$$\sigma^* = \zeta \pm \sqrt{2\lambda(\mathcal{V}_\infty - \varphi)}.$$

Using the expression of σ^* from (A.10), we find that L_1 is made of two branches:

$$\text{(A.14)} \qquad 2(r\mathcal{W} + p\Delta\varphi) = \left(\zeta \pm \sqrt{2\lambda(\mathcal{V}_\infty - \varphi)}\right)^2.$$

It turns out that $\dot{\varphi}^*(X) > 0$ (resp. < 0) when the trajectory lies below (resp. above) this locus. Observe also that $\mathcal{W}^*(\overline{X}) = \varphi^*(\overline{X}) = \mathcal{V}_\infty$ lies below L_1 since $2(r + p\Delta)\mathcal{V}_\infty < \zeta^2 = 2\lambda\mathcal{V}_\infty$.

Similarly, the locus L_2 of points such that $\dot{\mathcal{W}}^*(X) = 0$ is, from (A.9),

$$\text{(A.15)} \qquad 2(r\mathcal{W} + p\Delta\varphi) = \zeta^2.$$

We have $\dot{\mathcal{W}}^*(X) < 0$ (resp. > 0) when the trajectory lies below (resp. above) this locus.

Observe that L_1 and L_2 intersect at

$$\mathcal{W}_\infty = \left(1 + \frac{1-p}{r}\Delta\right)\mathcal{V}_\infty \text{ and } \varphi_\infty = \mathcal{V}_\infty. \tag{A.16}$$

Consider now the backward trajectories of the system (A.9)-(A.12). Accordingly, let $\mathcal{W}_*(Y) = \mathcal{W}^*(\overline{X} - Y)$ and $\varphi_*(Y) = \varphi^*(\overline{X} - Y)$ be the solution to the backward system; thus considering values of the stock $X = \overline{X} - Y \leq \overline{X}$ for $Y \geq 0$ and viewing $Y = 0$ as the initial point of those backward trajectories. For this backward system, the initial conditions are the same as the terminal conditions (A.13) of the forward system, i.e.,

$$\mathcal{W}_*(0) = \varphi_*(0) = \mathcal{V}_\infty. \tag{A.17}$$

Using (A.9)-(A.12), we can write this backward system as

$$\dot{\mathcal{W}}_*(Y) = \zeta - \sqrt{2\left(r\mathcal{W}^*(Y) + p\Delta\varphi_*(Y)\right)} \tag{A.18}$$

$$\dot{\varphi}_*(Y) = \frac{u(\sigma_*(Y)) - \lambda\varphi_*(Y)}{\sqrt{2\left(r\mathcal{W}_*(Y) + p\Delta\varphi_*(Y)\right)}} \tag{A.19}$$

where, from (A.10), we define

$$\sigma_*(Y) = \sigma^*(\overline{X} - Y) = \sqrt{2\left(r\mathcal{W}_*(Y) + p\Delta\varphi_*(Y)\right)}. \tag{A.20}$$

Trajectories of the backward system, if they converge when Y goes to infinity (or equivalently, considering $\overline{X}$ going to infinity in the forward system), do so towards the stationary point $(\varphi_\infty, \mathcal{W}_\infty)$ where L_1 and L_2 intersect.

Observe that the action $\sigma^*(\overline{X}^-)$ is given by (A.10), i.e.,

$$\sigma^*(\overline{X}^-) = \zeta\sqrt{1 - \frac{1-p}{\lambda}\Delta} < \zeta. \tag{A.21}$$

The backward system (A.18)-(A.19) satisfies a Lipschitz condition at $Y = 0$ with

$$\dot{\mathcal{W}}_*(0) = \zeta\left(1 - \sqrt{1 - \frac{1-p}{\lambda}\Delta}\right) > 0 \text{ and } \dot{\varphi}_*(0) = \frac{u(\sigma^*(\overline{X}^-)) - \lambda\mathcal{V}_\infty}{\sigma^*(\overline{X}^-)} < 0 \tag{A.22}$$

where the last inequality follows from the fact that $\lambda\mathcal{V}_\infty = \frac{\zeta^2}{2}$ is maximizing flow payoff. From Cauchy-Lipschitz Theorem, locally in a right-neighborhood of $Y = 0$, there exists a unique solution to the backward system (A.18)-(A.19) together with the initial condition (A.17). This solution can be extended over the whole interval $\left[0, \overline{X}\right]$. Observe that the initial conditions (A.17) of the backward system are the same for all possible values of $\overline{X}$ and that the derivatives $\dot{\mathcal{W}}_*(0)$ and $\dot{\mathcal{W}}_*(0)$ in (A.22) are also independent of $\overline{X}$ since, from (A.21), $\sigma^*(\overline{X}^-) = \sigma_*(0^+)$ is itself so. For all values of $\overline{X}$, solutions of the forward system thus lie on the same one-dimensional locus $\mathcal{L}$ in the $(\varphi, \mathcal{W})$ space. Hence, fixing a particular value $\overline{X}$ amounts in fact to choosing a point along the locus $\mathcal{L}$ that corresponds to the initial values $(\mathcal{W}^*(0), \varphi^*(0))$ for the forward system.

Long-run behavior of the backward system. Because $(\varphi_*(0), \mathcal{W}_*(0))$ lies on the diagonal, below both L_1 and L_2, the trajectory of the backward system starting from $\varphi_*(0) = \mathcal{W}_*(0) = \mathcal{V}_\infty$ is such that $\mathcal{W}_*$ is increasing while φ_* is first decreasing (and always so for all values of $Y \in \left[0, \overline{X}\right]$ when $\overline{X}$ is small) before it eventually reaches L_1 and is increasing afterwards (case where $\overline{X}$ is large). Observe that φ_*, once it has already crossed L_1 at some Y_1 and lies above L_1 cannot cross it one more time at some finite $Y_2 > Y_1$ since when it crosses L_1 , it must cross it necessarily with $\dot{\varphi}_*(Y_2) = 0$ but this cannot be for a trajectory coming from above L_1 unless it is at point where L_1 admits a vertical tangent; and the only such point corresponds to $\varphi = \mathcal{V}_\infty$, which is only reached for $Y_2 = +\infty$.

For the forward system, we have the reverse pattern. First, $\mathcal{W}^*$ is always decreasing starting from $\mathcal{W}^*(0)$ and going to $\mathcal{W}^*(\overline{X}) = \mathcal{W}_*(0) = \mathcal{V}_\infty$. Hence Item 1. is proved.

Second, when $\overline{X}$ is large enough, $(\varphi^*(0), \mathcal{W}^*(0))$ lies above L_1 and thus φ^* is first decreasing starting from $\varphi^*(0)$ before it necessarily reaches L_1 and becomes increasing afterwards towards $\varphi^*(\overline{X}) = \mathcal{V}_\infty$. When $\overline{X}$ is small enough, $(\varphi^*(0), \mathcal{W}^*(0))$ lies below L_1 and thus φ^* is always increasing starting from $\varphi^*(0)$. Hence, Item 2. is proved.

That $\mathcal{W}^*$ is always decreasing in the forward system, also implies that $\sigma^*(X) < \zeta$, for $X \in [0, \overline{X})$. When $\overline{X}$ increases towards $+\infty$, $(\varphi^*(0), \mathcal{W}^*(0))$ converges towards $(\varphi_\infty, \mathcal{W}_\infty)$ which is the stationary point of the backward system. Moreover, by definition, we have $\mathcal{W}^*(X) \le \lambda \mathcal{V}_\infty \int_0^{+\infty} e^{-r\tau} \left(1 - p + pe^{-\Delta\tau}\right) d\tau = \mathcal{W}_\infty$. Hence, a trajectory of the backward system crossing L_2 stays below the horizontal line $\mathcal{W} = \mathcal{W}_\infty$ when converging towards $(\varphi_\infty, \mathcal{W}_\infty)$. Moreover, in the neighborhood of $(\varphi_\infty, \mathcal{W}_\infty)$, the backward system can be linearized as

$$\dot{\mathcal{W}}_*(Y) = -\frac{r}{\zeta}(\mathcal{W}_*(Y) - \mathcal{W}_\infty) - \frac{p\Delta}{\zeta}(\varphi_*(Y) - \varphi_\infty) \tag{A.23}$$

$$\dot{\varphi}_*(Y) = -\frac{\lambda}{\zeta}(\varphi_*(Y) - \varphi_\infty). \tag{A.24}$$

This linear system has two negative eigenvalues $\chi_1 = -\frac{\lambda}{\zeta}$ and $\chi_2 = -\frac{r}{\zeta}$ and $(\varphi_\infty, \mathcal{W}_\infty)$ is thus a stable node. The solutions to the linearized system are of the form

$$\mathcal{W}_*(Y) - \mathcal{W}_\infty = p\phi_0 e^{-\frac{\lambda}{\zeta}Y} + w_0 e^{-\frac{r}{\zeta}Y} \tag{A.25}$$

$$\varphi_*(Y) - \varphi_\infty = \phi_0 e^{-\frac{\lambda}{\zeta}Y} \tag{A.26}$$

for (ϕ_0, w_0) arbitrary constants. We can eliminate Y from those two equations to get

$$\mathcal{W}_*(Y) - \mathcal{W}_\infty = p(\varphi_*(Y) - \varphi_\infty) + w_0 \left(\frac{\varphi_*(Y) - \varphi_\infty}{\phi_0}\right)^{\frac{r}{\lambda}} \tag{A.27}$$

When $Y \to +\infty$, $\varphi_*(Y) \to \varphi_\infty$ and $\mathcal{W}_*(Y) \to \mathcal{W}_\infty$ and the behavior of these solutions is thus given by the dominant terms corresponding to the highest eigenvalue χ_2, i.e., $\mathcal{W}_*(Y) - \mathcal{W}_\infty \sim w_0 \left(\frac{\varphi_*(Y) - \varphi_\infty}{\phi_0}\right)^{\frac{r}{\lambda}}$. In the $(\varphi, \mathcal{W})$ spaces these solutions are thus tangent to the vertical line $\varphi = \varphi_\infty = \mathcal{V}_\infty$ but remain above loci L_1 since we know that $\varphi_*(Y)$ do not cross L_1 twice.

For $\overline{X}$ large enough, $(\varphi^*(0), \mathcal{W}^*(0))$ thus lies close to the vertical line $\varphi = \mathcal{V}_\infty$ with $\mathcal{W}^*(0) < \mathcal{W}_\infty$ and $\varphi^*(0) < \mathcal{V}_\infty$ but $\varphi^*(0) - \mathcal{V}_\infty$ is of a much lower order of magnitude than $\mathcal{W}^*(0) - \mathcal{W}_\infty$. Hence, we must have $\mathcal{W}^*(0) < \left(1 + \frac{1-p}{r}\Delta\right) \varphi^*(0)$ for $\overline{X}$ large enough.

For $\overline{X}$ small enough, $(\varphi^*(0), \mathcal{W}^*(0))$ lies close to $(\mathcal{V}_\infty, \mathcal{V}_\infty)$ and again $\mathcal{W}^*(0) < \left(1 + \frac{1-p}{r}\Delta\right) \varphi^*(0)$. Using (A.8) and (A.12) at $X = 0$, we now compute

$$\sigma^*(0)\left(\dot{\mathcal{W}}^*(0) - \dot{\varphi}^*(0)\right) = r\left(\mathcal{W}^*(0) - \left(1 + \frac{1-p}{r}\Delta\right)\varphi^*(0)\right). \tag{A.28}$$

Hence, whether $\overline{X}$ is large or small enough, we have $\dot{\mathcal{W}}^*(0) - \dot{\varphi}^*(0) < 0$ which proves Item 3.

Phase Diagram Illustration. This section illustrates the phase–plane analysis of the backward system defined in equations (A.25)-(A.26).We fix parameters

$$\zeta = 1, \quad \lambda = 1.01, \quad \Delta = 1, \quad r = 0.01, \quad p = 0.5, \quad \mathcal{V}_\infty = \frac{1}{2.02}.$$

Figure 1 displays the loci

$$L_1: \quad 2(r\mathcal{W} + p\Delta\varphi) = \left(\zeta - \sqrt{2\lambda(\mathcal{V}_\infty - \varphi)}\right)^2, \qquad L_2: \quad 2(r\mathcal{W} + p\Delta\varphi) = \zeta^2,$$

together with backward trajectories $(\varphi_*(Y), \mathcal{W}_*(Y))$ for $\overline{X} \in \{1, 10, 100\}$. All trajectories start from $(\mathcal{V}_\infty, \mathcal{V}_\infty)$ and initially move to the right. When $\overline{X}$ is sufficiently large, the trajectory crosses the lower branch of L_1 once and thereafter remains strictly above it.

The horizontal axis on Figure 1 is zoomed around $\varphi = \mathcal{V}_\infty$, while the vertical axis is extended to display convergence toward the stationary point $(\varphi_\infty, \mathcal{W}_\infty)$. All trajectories start on the right of the lower branch of L_1, cross it and converge monotonically without further crossings.

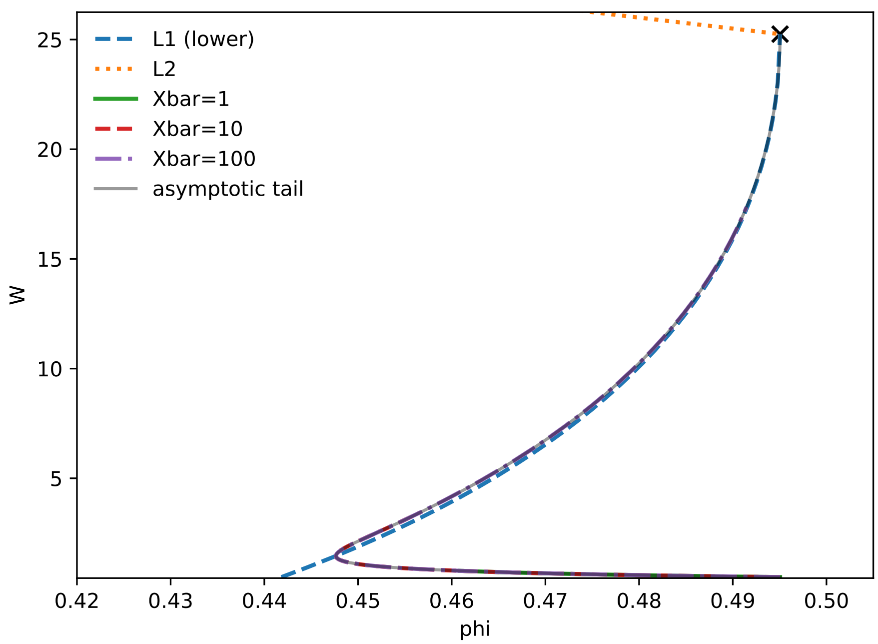

Figure 1. Backward trajectories $(\varphi_*(Y), \mathcal{W}_*(Y))$ for $\overline{X} \in \{1, 10, 100\}$ together with the lower branch of L_1 and locus L_2.

PROOF OF PROPOSITION 4:

Using the identity $Z(q, \varepsilon + s) = Z(q, \varepsilon)Z(q(\varepsilon), s)$, the planner's expected payoff in (4) writes also as

$$\mathcal{W}^r(x, \varepsilon) = \left(\int_0^\varepsilon e^{-r\tau} Z(q, \tau) d\tau\right) u(x) + e^{-r\varepsilon} Z(q, \varepsilon) \int_0^{+\infty} e^{-r\tau} Z(q(\varepsilon), \tau) u(\sigma^*(X^*(\tau, x\varepsilon))) d\tau \tag{A.29}$$

First, we rewrite (A.29) and obtain

$$\mathcal{W}^r(x,\varepsilon) = \left(\int_0^{\varepsilon} e^{-r\tau} Z(q,\tau) d\tau\right) u(x) + e^{-r\varepsilon} Z(q,\varepsilon) \left(\mathcal{W}^*(x\varepsilon) + \frac{p-q(\varepsilon)}{1-p}\left(\mathcal{W}^*(x\varepsilon) - \varphi^*(x\varepsilon)\right)\right). \tag{A.30}$$

We observe that $q(\varepsilon)$ admits the following Taylor approximation when ε is close to zero $q(\varepsilon) = q - \Delta q(1-q)\varepsilon + o(\varepsilon)$. We obtain a first-order Taylor approximation of $\mathcal{W}^r(x,\varepsilon)$ for ε close to zero:

$$\begin{aligned}\mathcal{W}^r(x,\varepsilon) = \mathcal{W}^r(0,0) &+ \varepsilon\left(u(x) + x\left(\dot{\mathcal{W}}^*(0) + \frac{p-q}{1-p}\left(\dot{\mathcal{W}}^*(0) - \dot{\varphi}^*(0)\right)\right)\right)\\ &-\varepsilon\left((r+q\Delta)\mathcal{W}^r(0,0) - \frac{q(1-q)\Delta}{1-p}\left(\mathcal{W}^*(0) - \varphi^*(0)\right)\right) + o(\varepsilon).\end{aligned} \tag{A.31}$$

Using $\mathcal{W}^r(0,0) = \mathcal{W}^*(0) - \frac{q-p}{1-p}\left(\mathcal{W}^*(0) - \varphi^*(0)\right)$ and (A.8) for $X = 0$, namely $r\mathcal{W}^*(0) + p\Delta\varphi^*(0) = u(\sigma^*(0)) + \sigma^*(0)\dot{\mathcal{W}}^*(0)$, and (A.28), we obtain

$$\mathcal{W}^r(x,\varepsilon) = \mathcal{W}^r(0,0) + \varepsilon\left(u(x) - u(\sigma^*(0)) + (x - \sigma^*(0))\left(\dot{\mathcal{W}}^*(0) + \frac{p-q}{1-p}\left(\dot{\mathcal{W}}^*(0) - \dot{\varphi}^*(0)\right)\right)\right) + o(\varepsilon). \tag{A.32}$$

It follows from (A.32) that the optimal impulse regulation entails

$$x^r \in \arg\max_{x\in\mathbb{R}_+} u(x) + x\left(\dot{\mathcal{W}}^*(0) + \frac{p-q}{1-p}\left(\dot{\mathcal{W}}^*(0) - \dot{\varphi}^*(0)\right)\right).$$

An interior solution thus satisfies (5). For $p > q$, we have

$$x^r < \sigma^*(0) \Leftrightarrow \dot{\mathcal{W}}^*(0) - \dot{\varphi}^*(0) < 0$$

Using (3), we conclude that, for $\overline{X}$ large (resp. small) enough and $p > q$, a cap on actions is optimal and (5) immediately follows.