

Supplemental Appendix

Climate change and long-run factor shares

Tianzi Liu Ivan Rudik Zebang Xu

A Derivations

A.1 Proof for Lemma 1

Proof. By the chain rule,

$$\frac{\partial}{\partial K_t} F(A_t K_t, B_t L_t) = F_1(A_t K_t, B_t L_t) \cdot A_t, \quad \frac{\partial}{\partial L_t} F(A_t K_t, B_t L_t) = F_2(A_t K_t, B_t L_t) \cdot B_t,$$

which gives equation (3). Multiplying by K_t/Y_t and L_t/Y_t yields equations (4) and (5). Finally, since F is homogeneous of degree one, Euler's theorem implies $F(AK, BL) = AKF_1(AK, BL) + BLF_2(AK, BL)$ for all (AK, BL) . Then dividing both sides by $Y = F(\cdot)$ yields $s_{K,t} + s_{L,t} = 1$. ■

A.2 Proof for Household Euler Equation

The representative dynasty maximizes

$$\sum_{t=0}^{\infty} \beta^t L_t \frac{c_t^{1-\theta}}{1-\theta}$$

subject to the aggregate budget constraint

$$c_t L_t = Y_t - K_{t+1} + (1 - \delta)K_t.$$

Form the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[L_t \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t (Y_t - K_{t+1} + (1 - \delta)K_t - c_t L_t) \right].$$

The first-order condition with respect to c_t is

$$\beta^t \left(L_t c_t^{-\theta} - \lambda_t L_t \right) = 0 \quad \Rightarrow \quad \lambda_t = c_t^{-\theta}.$$

The first-order condition with respect to K_{t+1} is

$$-\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \left(\frac{\partial Y_{t+1}}{\partial K_{t+1}} + 1 - \delta \right) = 0.$$

Substituting $\lambda_t = c_t^{-\theta}$ and defining $r_{t+1} \equiv \partial Y_{t+1} / \partial K_{t+1}$ yields

$$c_t^{-\theta} = \beta c_{t+1}^{-\theta} (1 + r_{t+1} - \delta),$$

which can be rearranged to obtain the Euler equation:

$$\left(\frac{c_{t+1}}{c_t} \right)^\theta = \beta (1 + r_{t+1} - \delta).$$

A.3 Proof for Lemma 2

Proof. From Lemma 1, $r_t = A_t F_1(A_t K_t, B_t L_t)$. Using equation (2) and the detrended variables,

$$A_t K_t = A_T(T_t) A_t^e K_t = A_T(T_t) (B_t^e L_t) k_t, \quad B_t L_t = B_T(T_t) (B_t^e L_t).$$

Since $F_1(\cdot)$ is homogeneous of degree 0, $F_1(\lambda AK, \lambda BL) = F_1(AK, BL)$ for any $\lambda > 0$, so

$$F_1(A_t K_t, B_t L_t) = F_1(A_T(T_t) k_t, B_T(T_t)),$$

which implies equation (11). ■

A.4 Proof for Proposition 1

Proof. Fix T and suppress time subscripts for steady-state values. Define the effective capital-labor ratio

$$\tilde{k} \equiv \frac{AK}{BL}.$$

CRS implies there exists a strictly concave function f such that $Y = F(AK, BL) = BLf(\tilde{k})$, where $f(\tilde{k}) \equiv F(\tilde{k}, 1)$. The marginal product of capital can be written as $r \equiv \frac{\partial Y}{\partial K} = AF_1(AK, BL) = Af'(\tilde{k})$, so the BGP condition $r = r^*$ implies

$$f'(\tilde{k}) = \frac{r^*}{A} = \frac{r^*}{A^e A_T(T)}. \quad (20)$$

Capital's share can be written as a function of \tilde{k}

$$s_K = \frac{rK}{Y} = \frac{Af'(\tilde{k})K}{BLf(\tilde{k})} = \frac{\tilde{k}f'(\tilde{k})}{f(\tilde{k})}. \quad (21)$$

Log-differentiate equation (21) yields

$$d \log s_K = d \log \tilde{k} + d \log f'(\tilde{k}) - d \log f(\tilde{k}).$$

Since $d \log f(\tilde{k}) = \frac{\tilde{k}f'(\tilde{k})}{f(\tilde{k})} d \log \tilde{k} = s_K d \log \tilde{k}$, we have

$$d \log s_K = (1 - s_K) d \log \tilde{k} + d \log f'(\tilde{k}).$$

To find $\frac{d \log s_K}{dT}$, we now solve for $\frac{d \log \tilde{k}}{dT}$ and $\frac{d \log f'(\tilde{k})}{dT}$. Differentiating equation (20) with respect to T yields

$$d \log f'(\tilde{k}) = -d \log A_T(T).$$

Also, since $d \log f'(\tilde{k}) = \frac{f''(\tilde{k})\tilde{k}}{f'(\tilde{k})} d \log \tilde{k}$, we have

$$d \log \tilde{k} = -\frac{f'(\tilde{k})}{f''(\tilde{k})\tilde{k}} d \log A_T(T). \quad (22)$$

Note that under CRS, we have $F_1(\tilde{k}, 1) = f'(\tilde{k})$, $F_2(\tilde{k}, 1) = f(\tilde{k}) - \tilde{k}f'(\tilde{k})$, $F_{12}(\tilde{k}, 1) = -\tilde{k}f''(\tilde{k})$. Plugging these back into the elasticity of substitution definition yields

$$\sigma_{KL} = \frac{f'(\tilde{k})(f(\tilde{k}) - \tilde{k}f'(\tilde{k}))}{-\tilde{k}f(\tilde{k})f''(\tilde{k})}. \quad (23)$$

Using equation (21), $f(\tilde{k}) - \tilde{k}f'(\tilde{k}) = f(\tilde{k})(1 - s_K)$. Equation (23) then rearranges to

$$-\frac{f'(\tilde{k})}{\tilde{k}f''(\tilde{k})} = \frac{\sigma_{KL}}{1 - s_K}. \quad (24)$$

Combine equations (22) and (24) to obtain

$$d \log \tilde{k} = \frac{\sigma_{KL}}{1 - s_K} d \log A_T(T). \quad (25)$$

Substituting in $d \log \tilde{k}$ from equation (25) and $d \log f'(\tilde{k}) = -d \log A_T(T)$ yields

$$d \log s_K = (1 - s_K) \left(\frac{\sigma_{KL}}{1 - s_K} d \log A_T(T) \right) - d \log A_T(T) = (\sigma_{KL} - 1) d \log A_T(T).$$

Dividing by dT yields the first expression in equation (12). The second follows from $s_L = 1 - s_K$, i.e. $d \log s_L/dT = -(s_K/s_L) d \log s_K/dT$. ■

A.5 Proof of Lemma 3

Proof. The period utility of the Boppart–Krusell balanced-growth class can be expressed as:

$$u(c, h) = \frac{(c v(x))^{1-\theta} - 1}{1 - \theta}, \quad x \equiv h c^{\nu/(1-\nu)}, \quad \nu < 1, \quad (26)$$

with the standard logarithmic variant for $\theta = 1$. We first derive the intratemporal marginal rate of substitution (MRS) $-u_h/u_c$ that determines wage.

$$w = -\frac{u_h(c, h)}{u_c(c, h)} = c^{1/(1-\nu)} \Phi(x), \quad \Phi(x) \equiv -\frac{v'(x)}{v(x) + \frac{\nu}{1-\nu} x v'(x)}. \quad (27)$$

This expression corresponds to the MRS structure emphasized in Boppart and Krusell (2020): the dependence on c is through $c^{1/(1-\nu)}$ and all remaining dependence is through

the composite $x = hc^{\nu/(1-\nu)}$.

Now use the key balanced-growth scaling property. Take any $\lambda > 0$ and define the scaled bundle

$$c' \equiv c \lambda^{1-\nu}, \quad h' \equiv h \lambda^{-\nu}.$$

Then the composite x is unchanged:

$$x' = h'(c')^{\nu/(1-\nu)} = h\lambda^{-\nu} (c\lambda^{1-\nu})^{\nu/(1-\nu)} = hc^{\nu/(1-\nu)} = x.$$

Using equation (27), we obtain

$$-\frac{u_h(c', h')}{u_c(c', h')} = (c')^{1/(1-\nu)} \Phi(x') = (c\lambda^{1-\nu})^{1/(1-\nu)} \Phi(x) = \lambda c^{1/(1-\nu)} \Phi(x) = \lambda \left(-\frac{u_h(c, h)}{u_c(c, h)} \right).$$

Thus, scaling wages by λ scales the MRS by λ when (c, h) are scaled by $(\lambda^{1-\nu}, \lambda^{-\nu})$.

Finally, apply this one-period with $\lambda = g_w$, where $g_w \equiv w_{t+1}/w_t = g_B$ is the gross wage growth rate on the balanced growth path. Starting from any (c_t, h_t) that satisfies the intratemporal condition at wage w_t , the scaled pair

$$c_{t+1} = c_t g_w^{1-\nu}, \quad h_{t+1} = h_t g_w^{-\nu}$$

satisfies the intratemporal condition at wage $w_{t+1} = g_w w_t$. Therefore, along a BGP,

$$\frac{c_{t+1}}{c_t} = g_w^{1-\nu} = g_B^{1-\nu}, \quad \frac{h_{t+1}}{h_t} = g_w^{-\nu} = g_B^{-\nu}.$$

The special case $\nu = 0$ implies $h_{t+1} = h_t$ for all t (the standard KPR case). ■

We also show that wage growth on a balanced growth path is pinned down by the growth rate of labor-augmenting productivity in the following lemma.

Lemma 4 (Wage growth on a BGP) *Suppose the economy is on a balanced growth path with constant long-run temperature T (so $A_T(T)$ and $B_T(T)$ are constant). Then the competitive wage per unit of labor input (i.e., per hour in the elastic-labor extension) grows at the*

gross rate

$$g_w \equiv \frac{w_{t+1}}{w_t} = g_B.$$

Proof. From the firm's first-order condition in Lemma 1, the competitive wage satisfies

$$w_t = \frac{\partial Y_t}{\partial L_t} = B_t F_2(A_t K_t, B_t L_t).$$

Because F has constant returns to scale, $F_2(\cdot)$ is homogeneous of degree 0, so it can be written as a function of the effective capital–labor ratio, $F_2(A_t K_t, B_t L_t) = F_2\left(\frac{A_t K_t}{B_t L_t}, 1\right) \equiv \phi\left(\frac{A_t K_t}{B_t L_t}\right)$.

On a BGP (and with constant long-run temperature), the effective ratio $\tilde{k}_t \equiv \frac{A_t K_t}{B_t L_t}$ is constant over time: indeed, $A_t = A_T(T)A_t^e$ and $B_t = B_T(T)B_t^e$ and $A_T(T), B_T(T)$ are constant, while the BGP requires the detrended ratio $k_t \equiv A_t^e K_t / (B_t^e L_t)$ to be stationary. Hence $\tilde{k}_t = (A_T(T)/B_T(T))k_t$ is constant on the BGP. Therefore $F_2(A_t K_t, B_t L_t)$ is constant on the BGP, implying

$$\frac{w_{t+1}}{w_t} = \frac{B_{t+1}}{B_t}.$$

Finally, since $B_t = B_T(T)B_t^e$ and $B_T(T)$ is constant on the BGP, we have $\frac{B_{t+1}}{B_t} = \frac{B_{t+1}^e}{B_t^e} = g_B$ by the exogenous law of motion $B_{t+1}^e = g_B B_t^e$. This proves $g_w = g_B$. ■

A.6 Proof of Proposition 2

Proof. Fix T and suppress time subscripts for BGP values. We define the effective capital–labor ratio as $\tilde{k} \equiv \frac{AK}{BhN}$. And the rest of the proof follows the proof of Proposition 1 in Section Appendix A.4. ■