Supplemental Appendix: A Generalized Control Function Approach to Production Function Estimation

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January 7, 2026

Proof of Theorem 1. Let $f^0(k_{it}, v_{it})$ denote the true production function. Because $q_{it} - f^0(k_{it}, v_{it}) = \omega_{it} + \varepsilon_{it}$, Assumption 1 implies that $f^0(k_{it}, v_{it})$ satisfies moment condition (3) for the control function $h^0\left(z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right) = \mathrm{E}\left[q_{it} - f^0(k_{it}, v_{it})|\, z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right]$. Let $\tilde{f}(k_{it}, v_{it})$ be some production function that also satisfies moment condition (3) for some control function $\tilde{h}\left(z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right)$. To show that $\frac{\partial f(k_{it}, v_{it})}{\partial v_{it}}$ is nonparametrically point-identified, we show that, for any k_{it}, the difference $f^0(k_{it}, v_{it}) - \tilde{f}(k_{it}, v_{it})$ does not change with v_{it} almost surely.

Because both (f^0, h^0) and $(\tilde{f}, \tilde{h})$ satisfy moment condition (3), taking the difference yields

$$
\begin{aligned}
& \mathrm{E}\left[f^0(k_{it}, v_{it}) - \tilde{f}(k_{it}, v_{it}) \middle| z_{it}, \mathrm{E}[q_{it-1}|x_{it-1}] \right] \\
= \;& \mathrm{E}\left[\tilde{h}\left(z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right) - h^0\left(z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right) \middle| z_{it}, \mathrm{E}[q_{it-1}|x_{it-1}] \right] \\
= \;& \tilde{h}\left(z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right) - h^0\left(z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right).
\end{aligned}
$$

*We are grateful to Aviv Nevo for useful comments and suggestions.

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This implies the exclusion restriction

$$
\begin{aligned}
& \mathrm{E}\left[f^0(k_{it}, v_{it}) - \tilde{f}(k_{it}, v_{it}) \middle| z_{it}, \mathrm{E}[q_{it-1}|x_{it-1}] \right] \\
= \; & \mathrm{E}\left[f^0(k_{it}, v_{it}) - \tilde{f}(k_{it}, v_{it}) \middle| z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}] \right].
\end{aligned}
$$

Assumption 2 ensures that, for any function κ, the exclusion restriction

$$
\begin{aligned}
& \mathrm{E}\left[\kappa\left(v_{it}, z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right) | \, z_{it}, \mathrm{E}[q_{it-1}|x_{it-1}] \right] \\
= \; & \mathrm{E}\left[\kappa\left(v_{it}, z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right) | \, z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}] \right]
\end{aligned}
$$

holds only if κ does not change with v_{it} almost surely. Recalling that $k_{it} \in z^c_{it}$ and setting $\kappa\left(v_{it}, z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right) = f^0(k_{it}, v_{it}) - \tilde{f}(k_{it}, v_{it})$ therefore establishes the result.

Proof of Theorem 2. We divide the proof into three parts.

We first prove that moment condition (3) implies moment condition (5). Recalling the definitions of $\tilde{\varphi}_{it}$ and $\tilde{q}_{it} - \tilde{f}_{it}$ in Theorem 2, moment condition (3) implies

$$
\mathrm{E}\left[q_{it} - f(k_{it}, v_{it}) - \left(\tilde{q}_{it} - \tilde{f}_{it}\right) \middle| z_{it}, \mathrm{E}[q_{it-1}|x_{it-1}] \right] = 0. \tag{6}
$$

This is because moment condition (3) implies $h(z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]) = \mathrm{E}[q_{it} - f(k_{it}, v_{it}) | z_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]]$ so that the control function has to satisfy $h(z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]) = \mathrm{E}[q_{it} - f(k_{it}, v_{it}) | z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]] = \tilde{q}_{it} - \tilde{f}_{it}$. Because $\varphi(z_{it}) - \tilde{\varphi}_{it}$ is pinned down by $(z_{it}, \mathrm{E}[q_{it-1}|x_{it-1}])$, we have

$$
\begin{aligned}
& \mathrm{E}\left[(\varphi(z_{it}) - \tilde{\varphi}_{it}) \left(q_{it} - f(k_{it}, v_{it}) - \left(\tilde{q}_{it} - \tilde{f}_{it}\right) \right) \right] \\
= \; & \mathrm{E}\left[(\varphi(z_{it}) - \tilde{\varphi}_{it}) \, \mathrm{E}\left[q_{it} - f(k_{it}, v_{it}) - \left(\tilde{q}_{it} - \tilde{f}_{it}\right) \middle| z_{it}, \mathrm{E}[q_{it-1}|x_{it-1}] \right] \right] \\
= \; & 0,
\end{aligned}
$$

where the first equality is due to the law of iterated expectations and the second equality to equation (6).

Next, we prove that moment condition (5) implies moment condition (4) with $h\left(z^c_{it}, \mathrm{E}[q_{it-1}|x_{it-1}]\right) =$

$\tilde{q}_{it} - \tilde{f}_{it}$. Note that

$$
\begin{aligned}
& \mathrm{E}\left[\tilde{\varphi}_{it}\left(q_{it} - f(k_{it}, v_{it}) - \left(\tilde{q}_{it} - \tilde{f}_{it}\right)\right)\right] \qquad (7) \\
= \; & \mathrm{E}\left[\tilde{\varphi}_{it}\mathrm{E}\left[q_{it} - f(k_{it}, v_{it}) - \left(\tilde{q}_{it} - \tilde{f}_{it}\right)\middle| z_{it}^c, \mathrm{E}[q_{it-1}|x_{it-1}]\right]\right] \\
= \; & 0,
\end{aligned}
$$

where the first equality is due to the law of iterated expectations and the second equality to $\tilde{q}_{it} - \tilde{f}_{it} = \mathrm{E}[q_{it} - f(k_{it}, v_{it})|z_{it}^c, \mathrm{E}[q_{it-1}|x_{it-1}]] = 0$ by the definition of $\tilde{q}_{it} - \tilde{f}_{it}$. Adding moment condition (5) and moment condition (7) implies moment condition (4) with $h\left(z_{it}^c, \mathrm{E}[q_{it-1}|x_{it-1}]\right) = \tilde{q}_{it} - \tilde{f}_{it}$.

Finally, we prove that moment condition (5) is Neyman orthogonal. Define the shorthand $\tilde{z}_{it}^c = (z_{it}^c, \mathrm{E}[q_{it-1}|x_{it-1}])$. Define $\mathcal{F}$ to be the set of function tuples (η, ζ) such that $\mathrm{E}[\eta^2(\tilde{z}_{it}^c)] < \infty$ and $\mathrm{E}[\zeta^2(\tilde{z}_{it}^c)] < \infty$. Because $\varphi(z_{it})$ and $q_{it} - f(k_{it}, v_{it})$ have finite L^2 norms, for any $(\eta, \zeta) \in \mathcal{F}$ and for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, we have

Is the notation okay here?

$$
\mathrm{E}\left|(\varphi(z_{it}) - \tilde{\varphi}_{it} - \lambda_1\eta(\tilde{z}_{it}^c))\left(q_{it} - f(k_{it}, v_{it}) - \left(\tilde{q}_{it} - \tilde{f}_{it} + \lambda_2\zeta(\tilde{z}_{it}^c)\right)\right)\right| < \infty.
$$

Moment condition (5) is therefore well-defined for any L^2-integrable perturbation of $\left(\tilde{\varphi}_{it}, \tilde{q}_{it} - \tilde{f}_{it}\right)$. Next, note that

$$
\begin{aligned}
& \left.\frac{\partial \mathrm{E}(\varphi(z_{it}) - \tilde{\varphi}_{it} - \lambda_1\eta(\tilde{z}_{it}^c))(q_{it} - f(k_{it}, v_{it}) - (\tilde{q}_{it} - \tilde{f}_{it} + \lambda_2\zeta(\tilde{z}_{it}^c)))}{\partial\lambda}\right|_{\lambda=0} \\
= \; & \begin{pmatrix} \mathrm{E}\left[\eta(\tilde{z}_{it}^c)\left(q_{it} - f(k_{it}, v_{it}) - \left(\tilde{q}_{it} - \tilde{f}_{it}\right)\right)\right] \\ -\mathrm{E}\left[(\varphi(z_{it}) - \tilde{\varphi}_{it})\zeta(\tilde{z}_{it}^c)\right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}
$$

where the second equality is due to the law of iterated expectations and the definitions of $\tilde{\varphi}_{it}$ and $\tilde{q}_{it} - \tilde{f}_{it}$.

GMM estimator. Corresponding to moment condition (5), define the moment function

$$
m_{it}(\theta) = q_{it} - f(k_{it}, v_{it}; \theta) - \left(\tilde{q}_{it} - \tilde{f}_{it}(\theta)\right), \qquad (8)
$$

where we make the parameterization of the production function explicit.

We solve the GMM problem

$$\min_{\theta} \left(\frac{1}{NT} \sum_{i,t} \left(\varphi(z_{it}) - \tilde{\varphi}_{it} \right) m_{it}(\theta) \right)^{\top} W \left(\frac{1}{NT} \sum_{i,t} \left(\varphi(z_{it}) - \tilde{\varphi}_{it} \right) m_{it}(\theta) \right),$$

where the superscript $\top$ denotes the transpose. We use the weighting matrix

$$W = \left(\frac{1}{NT-1} \sum_{i,t} \left(\left(\varphi(z_{it}) - \tilde{\varphi}_{it} \right) m_{it}\left(\theta^0\right) - \hat{\mu} \right)^{\top} \left(\left(\varphi(z_{it}) - \tilde{\varphi}_{it} \right) m_{it}\left(\theta^0\right) - \hat{\mu} \right) \right)^{-1},$$

where

$$\hat{\mu} = \frac{1}{NT} \sum_{i,t} \left(\varphi(z_{it}) - \tilde{\varphi}_{it} \right) m_{it}\left(\theta^0\right)$$

and θ^0 denotes the true parameters.

Recall that $\tilde{\varphi}_{it} = \mathrm{E}\left[\varphi(z_{it}) | z_{it}^c\right]$. To the extent that terms in $\varphi(z_{it})$ can be perfectly predicted by the complete set of Hermite polynomials of total degree d in the variables in z_{it}^c, the matrix Φ with rows $\varphi(z_{it}) - \tilde{\varphi}_{it}$ is rank deficient. In solving the GMM problem, we therefore use a selection of columns from the matrix Φ that has full rank. To construct this selection, we start with an empty matrix and keep adding columns from the matrix Φ as long as this increases the rank.