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# What Matters for Money Competition: Payments versus Reserves

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## A. Reserve Money: Characterizing the Non-Payment Equilibrium

For completeness, we characterize the non-payment equilibrium and its equilibrium conditions. The non-payment equilibrium serves as a benchmark, in which neither money is used for payments and both are hoarded as reserve money.

**Definition 3.** *In a non-payment equilibrium, both monies are used as reserve monies in that agents never make a payment upon a payment shock.*

**Proposition 3.** *A non-payment equilibrium exists if $c_{-i} \geq r z_{-i}$ for both $i \in \{1,2\}$.*

**Proof.** Consider an agent's continuation value, defined at the beginning of any $dt$-interval. Let $W_{mn}$ denote the continuation value when her money portfolio contains $m \in \{0,1\}$ units of Money 1 and $n \in \{0,1\}$ units of Money 2. Under a non-payment equilibrium, the system of Bellman equations is given by:

$$rW_{11} = 0, \tag{A1}$$

$$rW_{01} = -c_1, \tag{A2}$$

$$rW_{10} = -c_2, \tag{A3}$$

$$rW_{00} = -c_1 - c_2. \tag{A4}$$

*Payment incentives.* We verify that the proposed payment behavior is optimal. Specifically, because making a payment is never optimal, it implies that

$$z_1 \leq W_{11} - W_{01}\,, \tag{A5}$$

$$z_1 \leq W_{10} - W_{00}\,, \tag{A6}$$

$$z_2 \leq W_{11} - W_{10}\,, \tag{A7}$$

$$z_2 \leq W_{01} - W_{00}\,. \tag{A8}$$

Plugging (A1)–(A4) into (A5)–(A8) yields the stated condition. □

## B. Proofs of Results in the Main Text

### *B.1. Proof of Proposition 1*

**Proof.** Without loss of generality, we consider $i = 1$, that is, a type-1 money-division equilibrium. Consider an agent's continuation value, defined at the beginning of any $dt$-interval. Let $W_{mn}$ denote the continuation value when her money portfolio contains $m \in \{0,1\}$ units of Money 1 and $n \in \{0,1\}$ units of Money 2. Recall that payment opportunities arrive at rate $\lambda$. If the agent has Money 1, she pays with it; the payment settles with probability $\mu_1$, yields transaction gain $z_1$, and if it settles she gives up Money 1. Hence, under a type-1 money-division equilibrium, the system of Bellman equations is given by:

(B1) $$rW_{11} = \lambda\mu_1 (z_1 + W_{01} - W_{11}),$$

(B2) $$rW_{01} = -c_1 + \lambda\mu_1 (W_{11} - W_{01}),$$

(B3) $$rW_{10} = -c_2 + \lambda\mu_1 (z_1 + W_{00} - W_{10}),$$

(B4) $$rW_{00} = -c_1 - c_2 + \lambda\mu_1 (W_{10} - W_{00}).$$

We analyze this system. First, subtract (B2) from (B1):

(B5) $$W_{11} - W_{01} = \frac{c_1 + \lambda\mu_1 z_1}{r + 2\lambda\mu_1}.$$

Similarly, subtract (B4) from (B3):

(B6) $$W_{10} - W_{00} = \frac{c_1 + \lambda\mu_1 z_1}{r + 2\lambda\mu_1}.$$

Hence,

(B7) $$W_{11} - W_{01} = W_{10} - W_{00}.$$

Next, compare states that differ only in Money 2. Subtract (B3) from (B1) and use that the payment terms are identical up to relabeling the post-payment state:

(B8) $$W_{11} - W_{10} = \frac{c_2}{r}.$$

Likewise, $W_{01} - W_{00} = c_2/r$.

*Payment incentives.* We now verify that the proposed payment behavior is optimal by deriving the necessary and sufficient conditions.

(i) An agent with Money 1 always pays with Money 1. Consider state $(1,1)$ (a similar argument applies to $(1,0)$). Upon a payment opportunity, paying with Money 1 yields expected value $\mu_1(z_1 + W_{01}) + (1-\mu_1)W_{11}$, while not paying yields $W_{11}$. Thus paying with Money 1 is optimal iff

(B9) $$\mu_1(z_1 + W_{01}) + (1-\mu_1)W_{11} \geq W_{11}.$$

Using (B5), condition (B9) becomes

(B10) $$c_1 \leq (r + \lambda\mu_1)z_1.$$

(ii) An agent with Money 2 does not pay with Money 2. Consider state $(0,1)$. If the agent were to pay with Money 2, the payment would settle with probability $\mu_2$ and yield $z_2$; if it settles she moves to $(0,0)$, otherwise she stays in $(0,1)$. Thus paying with Money 2 is weakly optimal iff

$$\mu_2(z_2 + W_{00}) + (1-\mu_2)W_{01} \geq W_{01},$$

that is

$$W_{01} - W_{00} \leq z_2.$$

Therefore, to ensure no payment with Money 2 we require the reverse inequality:

(B11) $$W_{01} - W_{00} \geq z_2.$$

Using $W_{01} - W_{00} = c_2/r$ from (B8), this becomes

$$c_2 \geq r z_2. \tag{B12}$$

Note that an agent with both monies, that is, an agent at state (1,1), would never prefer to pay with Money 2 because doing so sacrifices the continuation value $c_2/r$ while Money 1 is available; more formally, under (B10) and (B12) the potential "pay-with-Money-2" deviation is dominated by "pay-with-Money-1."

*Acceptance incentives.* Under the proposed equilibrium, Money 1 is always accepted by construction. Off the equilibrium path, if Money 2 were tendered, the recipient compares holding Money 2 versus not holding it. By (B8),

$$W_{11} - W_{10} = \frac{c_2}{r} \geq 0 \qquad \text{and} \qquad W_{01} - W_{00} = \frac{c_2}{r} \geq 0,$$

so accepting Money 2 is (weakly) optimal. Thus acceptance constraints do not bind.

Taken together, combining (B10) and (B12) yields the stated conditions. □

### *B.2. Proof of Generalized Proposition 2*

We state and prove Proposition 4, a generalized version of Proposition 2 concerning the co-payment equilibrium, without imposing the assumption of $r \to 0$.

**Proposition 4.** *For $i \in \{1,2\}$, a type-$i$ co-payment equilibrium exists if*

(B13)

$$\begin{cases} \frac{c_i}{\lambda} + \frac{\mu_{-i}}{\lambda\left(\mu_{-i}+\frac{r}{\lambda}\right)} c_{-i} - \left(\mu_i + \frac{r}{\lambda}\right) z_i + \frac{\mu_i \mu_{-i}^2}{\left(\mu_{-i}+\frac{r}{\lambda}\right)\left(2\mu_i+\mu_{-i}+\frac{r}{\lambda}\right)} z_{-i} \leq 0, & \textit{(IC1)} \\ \frac{\mu_i}{\lambda} c_i - \frac{\mu_{-i}\left(\mu_i+\frac{r}{\lambda}\right)}{\lambda\left(\mu_{-i}+\frac{r}{\lambda}\right)} c_{-i} - \mu_i\left(\mu_i+\frac{r}{\lambda}\right) z_i + \mu_{-i}\left(2\mu_i + \frac{r}{\lambda} - \frac{\mu_i\left(3\mu_i\mu_{-i}+2\mu_i\frac{r}{\lambda}+2\mu_{-i}\frac{r}{\lambda}+\left(\frac{r}{\lambda}\right)^2\right)}{\left(\mu_{-i}+\frac{r}{\lambda}\right)\left(2\mu_i+\mu_{-i}+\frac{r}{\lambda}\right)}\right) z_{-i} \leq 0, & \textit{(IC2)} \\ \frac{c_{-i}}{\lambda} - \frac{\left(\mu_i+\frac{r}{\lambda}\right)\left(\mu_{-i}+\frac{r}{\lambda}\right)+\mu_i\frac{r}{\lambda}}{2\mu_i+\mu_{-i}+\frac{r}{\lambda}} z_{-i} \leq 0, & \textit{(IC3)} \\ -\frac{c_i}{\lambda} - \frac{\mu_{-i}}{\lambda\left(\mu_{-i}+\frac{r}{\lambda}\right)} c_{-i} - \mu_i z_i + \mu_{-i}\frac{\mu_i\mu_{-i}+2\mu_i\frac{r}{\lambda}+\mu_{-i}\frac{r}{\lambda}+\left(\frac{r}{\lambda}\right)^2}{\left(\mu_{-i}+\frac{r}{\lambda}\right)\left(2\mu_i+\mu_{-i}+\frac{r}{\lambda}\right)} z_{-i} \leq 0, & \textit{(IC4)} \end{cases}$$

It is easy to verify that Proposition 2 in the main text holds by imposing $r \to 0$ and recognize that (IC4) is automatically satisfied whenever (IC1) holds when $r \to 0$.

**Proof.** Without loss of generality, we consider $i = 1$, that is, a type-1 co-payment equilibrium. Consider an agent's continuation value at the beginning of any $dt$-interval. Similarly, let $W_{mn}$ denote the continuation value when her portfolio contains $m \in \{0,1\}$ units of Money 1 and $n \in \{0,1\}$ units of Money 2. Following similar logic underlying the proof of Proposition 1, the system of Bellman equations of a type-1 co-payment equilibrium is given by:

$$rW_{11} = \lambda\mu_1\left(z_1 + W_{01} - W_{11}\right), \tag{B14}$$

$$rW_{01} = -c_1 + \lambda\mu_2\left(z_2 + W_{00} - W_{01}\right) + \lambda\mu_1\left(W_{11} - W_{01}\right), \tag{B15}$$

$$rW_{10} = -c_2 + \lambda\mu_1\left(z_1 + W_{00} - W_{10}\right) + \lambda\mu_2\left(W_{11} - W_{10}\right), \tag{B16}$$

$$rW_{00} = -c_1 - c_2 + \lambda\mu_1\left(W_{10} - W_{00}\right). \tag{B17}$$

The system (B14)–(B17) is linear in $(W_{11}, W_{01}, W_{10}, W_{00})$. Solving it yields:
(B18)

$$\begin{pmatrix} W_{11} \\ W_{01} \\ W_{10} \\ W_{00} \end{pmatrix} = \begin{pmatrix} -\frac{\mu_1}{r(2\mu_1+s)}c_1 - \frac{\mu_1\mu_2}{r(2\mu_1+s)(\mu_2+s)}c_2 + \frac{\mu_1(\mu_1+s)}{s(2\mu_1+s)}z_1 + \frac{\mu_1\mu_2\left(\mu_1\mu_2+2\mu_1 s+\mu_2 s+s^2\right)}{s(2\mu_1+s)(\mu_2+s)(2\mu_1+\mu_2+s)}z_2 \\ -\frac{\mu_1+s}{r(2\mu_1+s)}c_1 - \frac{\mu_2(\mu_1+s)}{r(2\mu_1+s)(\mu_2+s)}c_2 + \frac{\mu_1^2}{s(2\mu_1+s)}z_1 + \frac{\mu_2(\mu_1+s)\left(\mu_1\mu_2+2\mu_1 s+\mu_2 s+s^2\right)}{s(2\mu_1+s)(\mu_2+s)(2\mu_1+\mu_2+s)}z_2 \\ -\frac{\mu_1}{r(2\mu_1+s)}c_1 - \frac{\mu_1\mu_2+2\mu_1 s+s^2}{r(2\mu_1+s)(\mu_2+s)}c_2 + \frac{\mu_1(\mu_1+s)}{s(2\mu_1+s)}z_1 + \frac{\mu_1\mu_2^2(\mu_1+s)}{s(2\mu_1+s)(\mu_2+s)(2\mu_1+\mu_2+s)}z_2 \\ -\frac{\mu_1+s}{r(2\mu_1+s)}c_1 - \frac{\mu_1\mu_2+2\mu_1 s+\mu_2 s+s^2}{r(2\mu_1+s)(\mu_2+s)}c_2 + \frac{\mu_1^2}{s(2\mu_1+s)}z_1 + \frac{\mu_1^2\mu_2^2}{s(2\mu_1+s)(\mu_2+s)(2\mu_1+\mu_2+s)}z_2 \end{pmatrix}.$$

*Payment incentives.* Similarly, we verify that the proposed payment behavior is optimal by deriving the necessary and sufficient conditions.

(i) An agent with both Money 1 and Money 2 pays with Money 1 rather than not paying. If the agent follows the equilibrium rule and tenders Money 1, her expected value is $\mu_1(z_1 + W_{01}) + (1-\mu_1)W_{11}$; if she does not pay, it is $W_{11}$. Thus paying with Money 1 is optimal iff

$$W_{11} - W_{01} \leq z_1. \tag{B19}$$

(ii) An agent with only Money 1 pays with Money 1 rather than not paying. Similarly, paying with Money 1 in state $(1,0)$ is optimal iff

$$W_{10} - W_{00} \leq z_1. \tag{B20}$$

(iii) An agent with both Money 1 and Money 2 pays with Money 1 rather than using Money 2. Paying with Money 2 yields $\mu_2(z_2 + W_{10}) + (1-\mu_2)W_{11}$. Hence paying with Money 1 is optimal relative to paying with Money 2 iff

$$\mu_1\left(W_{11} - W_{01} - z_1\right) \leq \mu_2\left(W_{11} - W_{10} - z_2\right). \tag{B21}$$

(iv) An agent with only Money 2 pays with Money 2 rather than not paying. If the agent tenders Money 2, her expected value is $\mu_2(z_2 + W_{00}) + (1-\mu_2)W_{01}$; if she does not pay, it is $W_{01}$. Thus paying with Money 2 is optimal iff

$$W_{01} - W_{00} \leq z_2. \tag{B22}$$

Before proceeding, it is useful to establish an identity showing that under a co-payment equilibrium, the two payment monies are substitutes. Let

$$D \equiv W_{11} + W_{00} - W_{01} - W_{10}.$$

We have

$$\begin{aligned} rD &= \lambda\mu_1\left(z_1 + W_{01} - W_{11} + W_{10} - W_{00}\right) - \lambda\mu_1\left(z_1 + W_{00} - W_{10} + W_{11} - W_{01}\right) - \lambda\mu_2\left(z_2 + W_{00} - W_{01} + W_{11} - W_{10}\right) \\ &= -2\lambda\mu_1 D - \lambda\mu_2(z_2 + D), \end{aligned}$$

so that

$$\left(r + 2\lambda\mu_1 + \lambda\mu_2\right)D = -\lambda\mu_2 z_2.$$

Since $s = r/\lambda$. Then

$$D = -\frac{\mu_2 z_2}{2\mu_1 + \mu_2 + s} < 0.$$

Equivalently,

$$(W_{10} - W_{00}) - (W_{11} - W_{01}) = -\left(W_{11} + W_{00} - W_{01} - W_{10}\right) = \frac{\mu_2 z_2}{2\mu_1 + \mu_2 + s} > 0. \tag{B23}$$

Now, note the redundancy of (B19): By (B23), we have $W_{10} - W_{00} > W_{11} - W_{01}$. Therefore (B20) implies (B19). Hence among the four payment incentive constraints, it is without loss to keep (B20), (B21) and (B22), noting that (B19) is then automatically satisfied.

*Acceptance incentives.* We also verify the willing-to-accept incentive constraints. In this equilibrium, a payee who currently holds portfolio $(i, j)$ must be willing to accept a payment that (upon successful settlement) increases her holdings by one unit of the tendered money. The potentially relevant acceptance conditions are: for type-$(0,0)$,

$$W_{00} \leq W_{10}\ ; \tag{B24}$$

for type-$(0,1)$,

$$W_{01} \leq W_{11}\ ; \tag{B25}$$

and for type-$(1,0)$,

$$W_{10} \leq W_{11}\ . \tag{B26}$$

We evaluate (B24)–(B26) using the closed-form continuation values in (B18).

(a) Condition (B24) is equivalent to $W_{10} - W_{00} \geq 0$. By (B18), this difference simplifies to

$$W_{10} - W_{00} = \frac{1}{\lambda(2\mu_1 + s)} c_1 + \frac{\mu_2}{\lambda(2\mu_1 + s)(\mu_2 + s)} c_2 + \frac{\mu_1}{2\mu_1 + s} z_1 + \frac{\mu_1 \mu_2^2}{(2\mu_1 + s)(\mu_2 + s)(2\mu_1 + \mu_2 + s)} z_2,$$

which is weakly positive for $(c_1, c_2, z_1, z_2) \geq 0$. Hence (B24) always holds.

(b) Condition (B25) is equivalent to $W_{11} - W_{01} \geq 0$. By (B18), this difference is

(B27)

$$W_{11} - W_{01} = \frac{1}{\lambda(2\mu_1 + s)} c_1 + \frac{\mu_2}{\lambda(2\mu_1 + s)(\mu_2 + s)} c_2 + \frac{\mu_1}{2\mu_1 + s} z_1 - \frac{\mu_2 \left(\mu_1 \mu_2 + 2\mu_1 s + \mu_2 s + s^2\right)}{(2\mu_1 + s)(\mu_2 + s)(2\mu_1 + \mu_2 + s)} z_2,$$

so (B25) is equivalent to (B27).

(c) Condition (B26) is equivalent to $W_{11} - W_{10} \geq 0$. By (B18), this difference simplifies to

$$W_{11} - W_{10} = \frac{1}{\lambda(\mu_2 + s)} c_2 + \frac{\mu_1 \mu_2}{(\mu_2 + s)(2\mu_1 + \mu_2 + s)} z_2,$$

which is weakly positive for $(c_2, z_2) \geq 0$. Hence (B26) always holds.

Taken together, substituting the closed-form solutions from (B18) into the three nonredundant payment incentive constraints (B20), (B21), (B22), and also into the only potentially binding acceptance constraint (B27) yields exactly the four inequalities labeled (IC1)–(IC4) in (2). □