Supplemental appendix to 'When should pre-trends be parallel?' by Ghanem, Sant'Anna, and Wüthrich

AUXILIARY LEMMA

Here we provide a general necessary and sufficient condition that is helpful for proving Propositions 1–2. It can be interpreted as a generalization of Theorem 3.1 in Ghanem, Sant'Anna and Wüthrich (2025) to the case where Assumptions PPT and PT hold jointly (specialized to model (1)). To prove the result, we impose the following additional weak regularity condition.

ASSUMPTION REG: $P(E[\varepsilon_{i1}-\varepsilon_{i0}|\omega_i]>0)<1$ *and* $P(E[\varepsilon_{i0}-\varepsilon_{i(-1)}|\omega_i]>0)<1$.

Assumption REG (together with Assumption SEL) ensures that the specific selection mechanisms we use to prove the "only if" direction in the proof of the lemma are nondegenerate.

LEMMA 1: *Suppose that Assumptions SEL and REG hold and the model is* (1)*. Then, Assumptions PPT and PT hold jointly for all nondegenerate* $g\in\mathscr{G}_\omega$ *if and only if* $E[\varepsilon_{i1}-\varepsilon_{i0}|\omega_i]=E[\varepsilon_{i0}-\varepsilon_{i(-1)}|\omega_i]=0$.

PROOF:

We adopt the proof strategy in Ghanem, Sant'Anna and Wüthrich (2025, Theorem 3.1). The "if" direction follows by the LIE and Assumption SEL. We proceed to show the "only if" direction. Assumption PPT and PT holding for all $g\in\mathscr{G}_\omega$ implies that they hold for the following two mechanisms,

$$\begin{aligned} G_i &= g_1(\omega_i,\nu_i)=1\{\nu_i>\nu\}1\{E[\varepsilon_{i0}-\varepsilon_{i(-1)}|\omega_i]\leq 0\},\\ G_i &= g_2(\omega_i,\nu_i)=1\{\nu_i>\nu\}1\{E[\varepsilon_{i1}-\varepsilon_{i0}|\omega_i]\leq 0\}. \end{aligned}$$

Both of these selection mechanisms are nondegenerate under the maintained assumptions.

We first consider the implications of Assumptions PPT and PT holding jointly for g_1. Under Assumption SEL, by Lemmas G.1 and G.2 in Ghanem, Sant'Anna and Wüthrich (2025), Assumption PPT holding for g_1 implies that

$$E[\varepsilon_{i0}-\varepsilon_{i(-1)}|\omega_i]=0. \tag{A1}$$

Similarly, Assumption PT holding for g_1 implies that

$$E[1\{E[\varepsilon_{i0}-\varepsilon_{i(-1)}|\omega_i]\leq 0\}E[\varepsilon_{i1}-\varepsilon_{i0}|\omega_i]]=0, \tag{A2}$$

where we have used Assumption SEL. Since $1\{E[\varepsilon_{i0}-\varepsilon_{i(-1)}|\omega_i]\leq 0\}=1$ by (A1), it follows that the last equality implies $E[\varepsilon_{i1}-\varepsilon_{i0}]=0$, which holds by assumption.

Next, we consider the implications of Assumptions PPT and PT holding jointly for g_2. Under Assumption SEL, by Lemmas G.1 and G.2 in Ghanem, Sant'Anna and Wüthrich (2025), Assumption PT holding for g_2 implies

$$E[\varepsilon_{i1}-\varepsilon_{i0}|\omega_i]=0. \tag{A3}$$

Similarly, Assumption PPT holding for g_2 implies that

$$E[1\{E[\varepsilon_{i1}-\varepsilon_{i0}|\omega_i]\leq 0\}E[\varepsilon_{i0}-\varepsilon_{i(-1)}|\omega_i]]=0, \tag{A4}$$

where we have used Assumption SEL. Since $1\{E[\varepsilon_{i1}-\varepsilon_{i0}|\omega_i]\leq 0\}=1$ by (A3), (A4) implies $E[\varepsilon_{i0}-\varepsilon_{i(-1)}]=0$, which holds by assumption.

It follows that Assumption PPT and PT holding jointly for all $g\in\mathscr{G}_\omega$ imply $E[\varepsilon_{i0}-\varepsilon_{i(-1)}|\omega_i]=0$ and $E[\varepsilon_{i1}-\varepsilon_{i0}|\omega_i]=0$. Q.E.D.

Proofs of propositions

PROOF OF PROPOSITION 1:

The result follows from an application of Lemma 1 with $\omega_i = (\alpha_i, \mu_i)$. Q.E.D.

PROOF OF PROPOSITION 2:

The result follows from an application of Lemma 1 with $\omega_i = (\alpha_i, \varepsilon_i^0, \mu_i, \eta_i^0)$. Q.E.D.

Derivation of necessary and sufficient conditions with covariates

Here we provide a detailed derivation of the necessary and sufficient conditions with covariates. By the arguments in Appendix D of Ghanem, Sant'Anna and Wüthrich (2025) with X_i replaced by X_i^0 combined with the arguments in the proof of Lemma 1, the necessary and sufficient conditions for Assumptions PPTX and PTX to hold jointly for all $g \in \mathscr{G}_{\omega,X^0}$ are

$$\text{(C1)} \qquad E[Y_{i0}(0) - Y_{i(-1)}(0)|X_i^0, \omega_i] = E[Y_{i0}(0) - Y_{i(-1)}(0)|X_i^0],$$

$$\text{(C2)} \qquad E[Y_{i1}(0) - Y_{i0}(0)|X_i^0, \omega_i] = E[Y_{i1}(0) - Y_{i0}(0)|X_i^0].$$

Plugging model (3) into (C1) yields

$$\begin{aligned} &E[\gamma_0(X_{i0}) + \alpha_i + \lambda_0 + \varepsilon_{i0} - (\gamma_{-1}(X_{i(-1)}) + \alpha_i + \lambda_{-1} + \varepsilon_{i(-1)})|X_i^0, \omega_i] \\ &= E[\gamma_0(X_{i0}) + \alpha_i + \lambda_0 + \varepsilon_{i0} - (\gamma_{-1}(X_{i(-1)}) + \alpha_i + \lambda_{-1} + \varepsilon_{i(-1)})|X_i^0]. \end{aligned}$$

Simplifying this expression and noting that X_i^0 contains X_{i0} and $X_{i(-1)}$ yields

$$E[\varepsilon_{i0} - \varepsilon_{i(-1)}|X_i^0, \omega_i] = E[\varepsilon_{i0} - \varepsilon_{i(-1)}|X_i^0].$$

Noting that $E[\varepsilon_{i0} - \varepsilon_{i(-1)}|X_i^0] = E[E[\varepsilon_{i0} - \varepsilon_{i(-1)}|X_i]|X_i^0] = 0$ because $E[\varepsilon_{it}|X_i] = 0$ by assumption yields condition (4).

Plugging model (3) into (C2) yields

$$\begin{aligned} &E[\gamma_1(X_{i1}) + \alpha_i + \lambda_1 + \varepsilon_{i1} - (\gamma_0(X_{i0}) + \alpha_i + \lambda_0 + \varepsilon_{i0})|X_i^0, \omega_i] \\ &= E[\gamma_1(X_{i1}) + \alpha_i + \lambda_1 + \varepsilon_{i1} - (\gamma_0(X_{i0}) + \alpha_i + \lambda_0 + \varepsilon_{i0})|X_i^0]. \end{aligned}$$

Simplifying this expression and noting that X_i^0 contains X_{i0} but not X_{i1} yields

$$E[\gamma_1(X_{i1}) + \varepsilon_{i1} - \varepsilon_{i0}|X_i^0, \omega_i] = E[\gamma_1(X_{i1}) + \varepsilon_{i1} - \varepsilon_{i0}|X_i^0].$$

Rearranging this expression and noting that $E[\varepsilon_{i1} - \varepsilon_{i0}|X_i^0] = 0$ by $E[\varepsilon_{it}|X_i] = 0$ and the LIE as before yields condition (5).