Supplemental Appendix: Reputation and competitive selection

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This appendix contains the proofs of Lemmata 5 and 6 (used for Proposition 4) and the proofs of the results in Section IV.

SA 1. Proof of Lemma 5

The proof of Lemma 5 involves arguments similar to those in the proof of Lemma 2, and we will use the analysis already presented in the proof of Lemma 2 in the main appendix.

Consider reputational equilibria where $c < \hat{c}^{RE}$. We first establish that competent firms must exert effort in period 1 when there are many initial entrants, allowing us for $T \geq 2$ to use the bound on ϕ^T established in the Proof of Lemma 2. We then use this bound to show that competent firms must typically exert effort in periods $1,...,T-1$, so that $\mu^T > 1-\varepsilon'$.

Given that there are at least two active firms in period 1, a consumer in period 2 must believe that a firm with output g has a higher reputation than a firm with output b. This implies such active firms must certainly exert effort with strictly positive probability in period 1. Given this, a firm's reputation in period 2 after a g output in period 1 must be strictly greater than ψ. So, such a firm will certainly be active in period 2, but will not be active if it produces output b.

Consider an on-path history $\underline{h}^t$ in which all active firms entered at time 1 and have never previously produced output b. Let $\eta(\underline{h}^t)$ be the probability that such active firms exerts effort; since all active firms entered at the same time they must exert effort with the same probability. Let $\hat{\eta}(\underline{h}^t)$ be the probability that consumers *believe* such a firm will exert effort. Fix an arbitrary competent active firm $i \in J(\underline{h}^t)$ at such a history. Let this firm's expected continuation value at complete history $\underline{h}_i^t$ be $V_i^{e,t}(\underline{h}^t)$ if it exerts effort and $V_i^{n,t}(\underline{h}^t)$ if it does not, and let $V_i^t(\underline{h}^t) := \max\{V_i^{e,t}(\underline{h}^t), V_i^{n,t}(\underline{h}^t)\}$. We will denote $p(\underline{h}^{t+1}|\underline{h}^t, g_i)$ as the probability of complete history $\underline{h}^{t+1}$ given $\underline{h}^t$ and that firm i produces output g in period t. On the equilibrium path in period 1, the value functions must satisfy:

$$V_i^{e,1}(\underline{h}^1) = \frac{U(\hat{\eta}(\underline{h}^1))(1-c)}{|J(\underline{h}^1)|} + \alpha\delta \sum_{h^2 \in H^2_{g_i}} p(\underline{h}^2|h^1, g_i)V^2(\underline{h}^2)$$

$$V_i^{n,1}(\underline{h}^1) = \frac{U(\hat{\eta}(\underline{h}^1))}{|J(\underline{h}^1)|} + \beta\delta \sum_{h^2 \in H^2_{g,i}} p(\underline{h}^2|h^1, g_i)V^2(\underline{h}^2)$$

where we use the fact that a firm will cease to be active if it produces output b at the end of the first period. This clearly implies:

$$V_i^{e,1}(\underline{h}^1) - V_i^{n,1}(\underline{h}^1) = -\frac{U(\hat{\eta}(\underline{h}^1))}{|J(\underline{h}^1)|}c + (\alpha-\beta)\delta \sum_{h^2 \in H^2_{g_i}} p(\underline{h}^2|h^1, g_i)V_i^2(\underline{h}^2) \geq -\frac{\alpha c}{|J(\underline{h}^1)|} + (\alpha-\beta)\delta\frac{\beta}{|J(\underline{h}^1)|},$$

using the fact that the firm's continuation payoff at any history $\underline{h}^2 \in \underline{H}^2_{g_i}$ is at least $\beta/|J(\underline{h}^1)|$. The RHS is positive for any $c < \hat{c}^{RE} = \delta(\alpha-\beta)\beta/\alpha$, and so any competent firm must exert effort in period 1. More generally, for any on-path history $\underline{h}^t$, where all active firms entered at the same time

1, we have:

$$V_i^{e,t}(\underline{h}^t) = \frac{U(\hat{\eta}(\underline{h}^t))(1-c)}{|J(\underline{h}^t)|} + \alpha\delta \sum_{\underline{h}^{t+1}\in \underline{H}_{g_i}^{t+1}\setminus \underline{H}_x^{t+1}} p(\underline{h}^{t+1}|\underline{h}^t, g_i)V_i^{t+1}(\underline{h}^{t+1})$$
$$+\delta(1-\alpha)(1-U(\eta(\underline{h}^t)))^{|J(\underline{h}^t)|-1}\overline{V}_i^{t+1}(\underline{h}^t, b)$$
$$V_i^{n,t}(\underline{h}^t) = \frac{U(\hat{\eta}(\underline{h}^t))}{|J(\underline{h}^t)|} + \beta\delta \sum_{\underline{h}^{t+1}\in \underline{H}_{g_i}^{t+1}\setminus \underline{H}_x^{t+1}} p(\underline{h}^{t+1}|\underline{h}^t, g_i)V_i^{t+1}(\underline{h}^{t+1})$$
$$+\delta(1-\beta)(1-U(\eta(\underline{h}^t)))^{|J(\underline{h}^t)|-1}\overline{V}_i^{t+1}(\underline{h}^t, b)$$

where $\overline{V}_i^{t+1}(\underline{h}^t, b) \leq 1/(1-\delta)$ is a firm's expected time $t+1$ continuation payoff conditional on history $\underline{h}^t$ and all active firms in period t producing output b.

We can establish a bound on conditional probabilities in the summand of these expressions:

$$p(\underline{H}_{g_i}^{t+1}\setminus \underline{H}_x^{t+1}|\underline{h}^t, g_i) = 1 - p(\underline{H}_{g_i}^{t+1}\cap \underline{H}_x^{t+1}|\underline{h}^t, g_i) = 1 - \frac{p(\underline{H}_{g_i}^{t+1}\cap \underline{H}_x^{t+1}|\underline{h}^t)}{U(\eta(\underline{h}^t))} \geq 1 - \frac{p(\underline{H}_x^{t+1}|\underline{h}^t)}{\beta}.$$

Now let $\underline{H}_{n,M}^t$ be the set of period t complete histories at which there are at least M active firms at period t, each of which entered in period 1 and have previously always produced g but exert effort with probability strictly less than 1 when competent in period t. If $\underline{h}^t \in \underline{H}_{n,M}^t$, the value functions of an arbitrary competent firm $i \in J(\underline{h}^t)$ must satisfy:

$$\text{(SA1)}\quad 0 \geq V_i^{e,t}(\underline{h}^t) - V^{n,t}(\underline{h}^t) \geq -\frac{c\alpha}{|J(\underline{h}^t)|} + \frac{\delta(\alpha-\beta)\beta(1-\frac{p(\underline{H}_x^{t+1}|\underline{h}^t)}{\beta})}{|J(\underline{h}^t)|} - \frac{\delta(\alpha-\beta)(1-\beta)^{|J(\underline{h}^t)|-1}}{1-\delta}.$$

We will assume $|J(\underline{h}^t)| \geq M > -1/ln(1-\beta)$, which implies $-|J(\underline{h}^t)|(1-\beta)^{|J(\underline{h}^t)|-1} \geq -M(1-\beta)^{M-1}$. We can apply this bound to the inequality in (II.1) before multiplying it by $p(\underline{h}^t)|J(\underline{h}^t)|$ and summing over $\underline{h}^t \in \underline{H}_{n,M}^t$ to establish the second inequality of:

$$\text{(SA2)}\quad p(\underline{H}_x^{t+1}) \geq p(\underline{H}_x^{t+1}\cap \underline{H}_{n,M}^t) \geq p(\underline{H}_{n,M}^t)\frac{\delta(\alpha-\beta)\beta - c\alpha - \frac{M\delta(\alpha-\beta)(1-\beta)^{M-1}}{1-\delta}}{\delta(\alpha-\beta)} \geq Kp(\underline{H}_{n,M}^t)$$

where

$$K = \frac{\delta(\alpha-\beta)\beta - c\alpha}{2\delta(\alpha-\beta)} > 0.$$

The first inequality in (II.2) holds by definition and the final inequality holds for all sufficiently large M because $\lim_{M\to\infty} M(1-\beta)^{M-1} = 0$. However, from equation (A.2) it is immediate that:

$$p(\underline{H}_x^t) = p(\underline{H}_x^t \cap \underline{H}_g^t) + p(\underline{H}_x^t \setminus \underline{H}_g^t) \leq \phi^t \frac{\psi}{\hat{\mu}_g(\psi)-\psi} + \phi^t = \phi^t \frac{\hat{\mu}_g(\psi)}{\hat{\mu}_g(\psi)-\psi}$$

And so, for any $\varepsilon'' > 0$, and T we can ensure that $p(\underline{H}_x^{t+1}) \leq \varepsilon''$ for $t \leq T$, by choosing a sufficiently large number of initial entrants, $\hat{M}$, implying that $p(\underline{H}_{n,M}^t) \leq \varepsilon''/K$.

Let $\underline{H}^t_{\leq M}$ be the set of period $t \leq T$ histories for which fewer than M incumbents produced output g in period $t-1$. For each $\underline{h}^t \in \underline{H}^t_{\leq M}$, there must exist some previous history h^s with $s \in \{1,...,t-1\}$ such that there was entry by $M(h^s) \geq \hat{M}$ firms in period s, but only some positive number less than M have produced output g in period $s,...,t-1$ periods. The probability that a single firm will produce output b at least once in $t-s \leq T$ periods is at most $(1-\beta^T)$. The probability that at least $M(h^s)-M$ out of $M(h^s)$ firms produce at least one output of b in less than $t-s \leq T$ periods is therefore at most:

$$\sum_{i=0}^{M} \binom{M(h^s)}{i} (1-\beta^T)^{M(h^s)-i} \leq M \binom{M(h^s)}{M} (1-\beta^T)^{M(h^s)-M} \leq M\hat{M}^M (1-\beta^T)^{\hat{M}-M},$$

where the first inequality follows for $M(h^s) \geq M/2$. Using l'Hopital's rule we get $\lim_{\hat{M}\to\infty} M\hat{M}^M(1-\beta^T)^{\hat{M}-M} = \lim_{\hat{M}\to\infty} M.M!(1-\beta^T)^{\hat{M}-M}/(-ln(1-\beta^T))^M = 0$. This establishes $p(\underline{H}^t_{\leq M}) \leq \varepsilon''$ for all $t \leq T$, and for all sufficiently large $\hat{M}$.

Next, let $\underline{H}^{t+1}_{e,g}$ be the set of histories for which, each incumbent firm in period $t+1$ exerted effort with probability 1, if competent, in every period $1,...,t-1$ and produced output g in every period $1,...,t-1$. For $\underline{h}^{t+1} \notin \underline{H}^{t+1}_{e,g}$ it must be there was some preceding history h^s with $s \leq t \leq T$ where either: fewer than M incumbents produced output g (so $\underline{h}^s \in H^s_{\leq M}$); or there were at least M active firms but competent firms did not exert effort with probability 1 (so $\underline{h}^s \in \underline{H}^s_{n,M}$); or there were at least M incumbents that produced g, but there was successful entry (so $\underline{h}^s \in \underline{H}^s_x$). We must therefore have:

$$p(\underline{H}^{T+1}_{e,g}) \geq 1 - \sum_{s=1}^{T} p(\underline{H}^s_{\leq M}) - \sum_{s=2}^{T-1} p(\underline{H}^s_{n,M}) - \sum_{s=2}^{T} p(\underline{H}^s_x) \geq 1 - T\varepsilon'' - T\varepsilon''/K - T\varepsilon''$$

where the second inequality must hold for any $\varepsilon'' > 0$, for all $\hat{M}$ large enough. For any $\varepsilon' > 0$, choosing $T = T_{\varepsilon'}$ such that $\hat{\mu}^{T-1}_g(\psi) > 1-\varepsilon'/2$ and $\varepsilon'' > 0$ sufficiently small, we must then have $\mu^T \geq p(\underline{H}^{T+1}_{e,g})\hat{\mu}^{T-1}_g(\psi) \geq 1-\varepsilon'$, given sufficiently many initial entrants. And hence, $\mu^t > 1-\varepsilon'$ for $t \geq T_{\varepsilon'}$ by Lemma 1. □

SA 2. Proof of Lemma 6

Assume without loss, that $\varepsilon > 0$ is sufficiently small that $\hat{\mu}_b(1-\varepsilon) > \psi$. As in the proof of Lemma 3, if $p^t_1\mu^t_1 \geq 1-\varepsilon$ for some t then a monopolist never faces successful entry at t or subsequently, $p^s_1\mu^s_1 \geq 1-\varepsilon$ for all $s \geq t$.

Given any $\varepsilon' > 0$, by Lemma 2, there is some $\bar{E}_{\varepsilon'} \in [3,\infty)$, $\bar{\xi}_{\varepsilon'} \in (0,\beta/3)$ and $T_{\varepsilon'}$ such that $\mu^t > 1-\varepsilon'$ for any $t \geq T_{\varepsilon'}$, $\xi < \bar{\xi}_{\varepsilon'}$ and $E > \bar{E}_{\varepsilon'}$. Given any $\varepsilon'' > 0$, Lemma 4 ensures an upper bound $\overline{M}$ on the number of firms and the existence of some $T_{\varepsilon''}$ such that with probability greater than $1-\varepsilon''$ any active non-monopolist at period t is either not active at time $t+T_{\varepsilon''}$ or at some time between t and $t+T_{\varepsilon''}-1$ has been the only firm to produce output g.

Suppose, therefore, towards a contradiction, that $p^t_1\mu^t_1 < 1-\varepsilon$ for all t. Let μ^t_{-1} be the average equilibrium reputation of active firms which are not a monopolist at time t. Assume that $t \geq T_{\varepsilon'}$. For such t, we need $p^t_1\mu^t_1 + (1-p^t_1)\mu^t_{-1} \geq 1-\varepsilon'$. The left hand side is certainly smaller than $1-\varepsilon+(1-p^t_1)$ and so we get $p^t_1 \leq 1-\varepsilon+\varepsilon'$. Next consider some arbitrary $\varepsilon''' > 0$ and suppose that $\mu^t_{-1} < 1-\varepsilon'''$. This implies that $p^t_1 + (1-p^t_1)(1-\varepsilon''') \geq p^t_1\mu^t_1 + (1-p^t_1)\mu^t_{-1} \geq 1-\varepsilon'$ and so $p^t_1 \geq 1-\varepsilon'/\varepsilon'''$. By choosing ε' sufficiently small we get a contradiction, which implies that we must indeed have $\mu^t_{-1} \geq 1-\varepsilon'''$ when ε' is small.

Suppose the equilibrium has generic beliefs for $t \geq T$ and consider arbitrary $t \geq T + T_{\varepsilon'} + T_{\varepsilon''}$. We will show that at such times, non-monopolists that were not active at $T_{\varepsilon'} + T$ must have a high reputation. Let $\underline{H}^t_{A^s}$ be the set of period t complete histories at which all active firms at period t were also active at time $t - s$. Also let $\underline{H}^t_1$ be the set of period t histories at which there is a monopolist. Lemma 4 and $t - T_{\varepsilon''} \geq T$ implies that $p(\underline{H}^t_{A^{T_{\varepsilon''}}} \setminus \underline{H}^t_1) \leq \varepsilon''$. To see this notice that the probability that in period $s \in \{t - T_{\varepsilon''}, ..., t\}$ at most one firm produced g and any others b is at least $1 - \varepsilon''$ and after such an event the g firm either be becomes a monopolist and remains one until period t, or is replaced in period $s' \in \{s, ..., t\}$ (given generic beliefs it cannot become an active non-monopolist). Given $p^t_1 \leq 1 - \varepsilon + \varepsilon'$ we must have $p(\underline{H}^t_1 \cup \underline{H}^t_{A^{T_{\varepsilon''}}}) = p^t_1 + p(\underline{H}^t_{A^{T_{\varepsilon''}}} \setminus \underline{H}^t_1) \leq 1 - \varepsilon + \varepsilon' + \varepsilon''$. Using the above facts and $\mu^t_{-1} = \sum_{\underline{h}^t \notin \underline{H}^t_1} \mu(\underline{h}^t) p(\underline{h}^t)/(1 - p^t_1) \geq 1 - \varepsilon'''$, where $\mu(\underline{h}^t)$ is the average reputation of an active firm at complete history $\underline{h}^t$, we get

$$\frac{\sum_{\underline{h}^t \notin \underline{H}^t_1 \cup \underline{H}^t_{A^{T_{\varepsilon''}}}} \mu(\underline{h}^t) p(\underline{h}^t)}{1 - p(\underline{H}^t_1 \cup \underline{H}^t_{A^{T_{\varepsilon''}}})} \geq 1 - \varepsilon''' - \frac{\sum_{\underline{h}^t \in \underline{H}^t_{A^{T_{\varepsilon''}}} \setminus \underline{H}^t_1} p(\underline{h}^t)(\mu(\underline{h}^t) - (1 - \varepsilon'''))}{1 - p(\underline{H}^t_1 \cup \underline{H}^t_{A^{T_{\varepsilon''}}})} \geq 1 - \varepsilon''' - \frac{\varepsilon''' \varepsilon''}{\varepsilon - \varepsilon' - \varepsilon''}.$$

Clearly, the right hand side can be made greater than $1 - 2\varepsilon'''$ for ε' and ε'' sufficiently small, as we shall henceforth assume. That implies $q^t_B + (1 - q^t_B)(1 - \sqrt{2\varepsilon'''}) \geq 1 - 2\varepsilon'''$ where $q^t_B = p\big(\mu(\underline{h}^t) \geq 1 - \sqrt{2\varepsilon'''} | \underline{h}^t \notin \underline{H}^t_1 \cup \underline{H}^t_{A^{T_{\varepsilon''}}}\big)$, which rearranges to $q^t_B \geq 1 - \sqrt{2\varepsilon'''}$.

All active firms at history $\underline{h}^t \notin \underline{H}^t_1 \cup \underline{H}^t_{A^{T_{\varepsilon''}}}$ were new entrants in some period $s' \geq t - T_{\varepsilon''}$, produced the same outputs in periods $s \geq s'$, exerted effort with the same probability if competent, and thus had the same reputation. If such firms were active in period s and $s+1$, we must have:

$$\frac{\mu(\underline{h}^{s+1})}{1 - \mu(\underline{h}^{s+1})} \leq \frac{\mu(\underline{h}^s)}{1 - \mu(\underline{h}^s)} \left(\frac{e(\underline{h}^s)(\alpha - \beta)}{\beta} + 1 \right).$$

where $e(\underline{h}^s)$ is the firm's probability of effort at history $\underline{h}^s$ if competent. And so, for a firm that was not active at time $t - T_{\varepsilon''}$ to obtain a reputation of $1 - \sqrt{2\varepsilon'''} > \psi$ by time t we must have:

$$\frac{1 - \sqrt{2\varepsilon'''}}{\sqrt{2\varepsilon'''}} \leq \frac{\mu(h^t)}{1 - \mu(h^t)} \leq \frac{\psi}{1 - \psi} \prod_{s = t - T_{\varepsilon''}}^{t-1} \left(\frac{e(\underline{h}^s)(\alpha - \beta)}{\beta} + 1 \right).$$

This implies, for each such firm we need:

$$\max_{s \in \{t - T_{\varepsilon''}, ..., t-1\}} e(\underline{h}^s) \geq \underline{e} := \left(\left(\frac{1 - \sqrt{2\varepsilon'''}}{\sqrt{2\varepsilon'''}} \frac{1 - \psi}{\psi} \right)^{1/T_{\varepsilon''}} - 1 \right) \frac{\beta}{\alpha - \beta} > 0.$$

Let $\underline{H}^t_{M, N^s, e}$ be the set of period t complete histories at which there are exactly M active firms that were not active at time $t - s$, which exert effort with probability greater than e if competent. Our analysis above, in particular the fact that $1 - p(\underline{H}^t_1 \cup \underline{H}^t_{A^{T_{\varepsilon''}}}) \geq \varepsilon - \varepsilon' - \varepsilon''$ for all $t \geq T + T_{\varepsilon'} + T_{\varepsilon''}$, ensures that for each $k \in \mathbb{N}_+$ there is some period t between $T + T_{\varepsilon'} + (k+1)(T_{\varepsilon''} + 1)$ and $T_{\varepsilon'} + (k + 2)(T_{\varepsilon''} + 1) - 1\}$ and some number of active firms $M^t \in \{2, ..., E\}$ for which

$$p(\underline{H}^t_{M^t, N^{T_{\varepsilon''}}, \underline{e}}) \geq \underline{p} = \frac{(\varepsilon - \varepsilon' - \varepsilon'')(1 - \sqrt{2\varepsilon'''})}{E T_{\varepsilon''}}.$$

We now consider the implications of consumers' belief updating in period $t+1$. Suppose a firm with reputation μ and likelihood ratio $x=(1-\mu)/\mu$, exerts effort with probability e if competent. Let $\check{\mu}_q(x,e)$ denote the firm's updated reputation if it subsequently produces output q. The difference in the firm's reputation after producing g as opposed to b is then:

$$D(x,e) := \check{\mu}_g(x,e) - \check{\mu}_b(x,e) = \frac{1}{1+xG(e)} - \frac{1}{1+xB(e)} \geq 0$$

where

$$G(e) = \frac{\beta}{\beta + e(\alpha-\beta)} \leq B(e) = \frac{(1-\beta)}{1-\beta-e(\alpha-\beta)}$$

Notice that

$$\frac{\partial D(x,e)}{\partial x} = \frac{(B(e)-G(e))(1-B(e)G(e)x^2)}{(1+xB(e))^2(1+xG(e))^2}$$

is positive for all $x^2 \leq 1/(B(e)G(e))$ and negative otherwise, implying that $D(x,e)$ is minimized either by having x as large as possible, or as small as possible. Also notice that the likelihood ratio of a firm at some history $\underline{h}^t \in \underline{H}^t_{\hat{m},N^{T_{\varepsilon''}},\underline{e}}$ is bounded above and below respectively by

$$\bar{x} = \frac{1-\psi}{\psi}\left(\frac{1-\beta}{1-\alpha}\right)^{T_{\varepsilon''}} \qquad \text{and} \qquad \underline{x} = \frac{1-\psi}{\psi}\left(\frac{\beta}{\alpha}\right)^{T_{\varepsilon''}}.$$

Finally, notice that $\partial D(x,e)/\partial e > 0$.

Slightly abusing notation, let the event $h^{t+1}_{c,M} = (1, M-1, \cdot)$ be the $t+1$ period consumer histories at which one firm has produced output g and $M-1$ firms have produced output b; there may be more than one consumer history consistent with this event, associated with different numbers of new entrants in period $t+1$. We adopt the same definitions developed in Lemma 1 and Lemma 3, where in particular $\underline{H}^t_M$ is the set of complete period t histories that occur with positive probability in our equilibrium in which there are exactly M active firms, and $\underline{H}^t(h^t_c)$ is the set of period t complete histories consistent with consumer history h^t_c. We let $\mu_{c,q}(h^t_c)$ be consumers' (average) belief that a firm is competent given their history h^t_c and let $\mu_q(\underline{h}^t)$ be the average probability of competence of firms that produced output q in period t, conditional on the complete history $\underline{h}^t$. Given $p(\underline{H}^t_{M^t,N^{T_{\varepsilon''}},\underline{e}}) \geq \underline{p}$ for some $M^t \in \{2,...,E\}$ and some t between $T+T_{\varepsilon'}+(k+1)(T_{\varepsilon''}+1)$ and $T+T_{\varepsilon'}+(k+2)(T_{\varepsilon''}+1)-1$, we have:

$$\begin{aligned}
\mu^t \leq & \mu^{t+1} + p(h^{t+1}_{c,M^t})\left(\frac{\mu^{t+1}_{c,g}(\check{h}^{t+1}_{c,M^t}) + \mu^{t+1}_{c,b}(\check{h}^{t+1}_c)(M^t-1)}{M^t} - \max\{\mu^{t+1}_{c,g}(\check{h}^{t+1}_c), \check{\mu}^{t+1}_{c,b}, \psi\}\right) \\
\leq & \mu^{t+1} + p(h^{t+1}_{c,M^t})\frac{M^t-1}{M^t}(\mu^{t+1}_{c,b}(\check{h}^{t+1}_c) - \mu^{t+1}_{c,g}(\check{h}^{t+1}_c)) \\
= & \mu^{t+1} + \frac{M^t-1}{M^t}\sum_{h^t \in \underline{H}^t_{M^t}} p(h^t) \sum_{h^{t+1} \in H^{t+1}(h^{t+1}_{c,M^t})} p(h^{t+1}|h^t)(\mu_b(\underline{h}^{t+1}) - \mu_g(\underline{h}^{t+1})) \\
\leq & \mu^{t+1} - \frac{1}{2}\left(\underline{p}\beta(1-\alpha)^{E-1}\min\{D(\underline{x},\underline{e}), D(\bar{x},\underline{e})\}\right) + \varepsilon''.
\end{aligned}$$

The first inequality follows from (A1) and the second from $\mu^{t+1}_{c,g}(\check{h}^{t+1}_c) \leq \max\{\mu^{t+1}_{c,g}(\check{h}^{t+1}_c), \check{\mu}^{t+1}_{c,b}, \psi\}$.

The equality follows by definition. The final inequality follows from $p(\underline{H}^t_{M^t,N^{T_{\varepsilon''}},\underline{e}}) \geq \underline{p}$, from $p(H^t_{M^t} \setminus H^t_{M^t,N^{T_{\varepsilon''}},0}) \leq \varepsilon''$, from $\mu_g(\underline{h}^{t+1}) - \mu_b(\underline{h}^{t+1}) \geq \min\{D(\underline{x},\underline{e}), D(\overline{x},\underline{e})\}$ for $\underline{h}^t \in \underline{H}^t_{M^t,N^{T_{\varepsilon''}},\underline{e}}$, from $\mu_g(\underline{h}^{t+1}) - \mu_b(\underline{h}^{t+1}) \geq 0$ for $\underline{h}^t \in \underline{H}^t_{M^t,N^{T_{\varepsilon''}},0} \setminus H^t_{M^t,N^{T_{\varepsilon''}},\underline{e}}$ (as conditional on $\underline{h}^t \in \underline{H}^t_{M^t,N^{T_{\varepsilon''}},0}$ all firms have the same reputation and exert effort with the same probability), from $p(h^{t+1}|h^t) \geq (1-\alpha)^{E-1}\beta$ for $h^{t+1} \in H^{t+1}(h^{t+1}_{c,M^t})$, and from $(M^t-1)/M^t \geq 1/2$. This, however, implies that $\mu^{t+1} - \mu^t \geq \hat{\varepsilon}$ for some $\hat{\varepsilon} > 0$ when ε'' is sufficiently small and so (given Lemma 1) we have $\mu^{T+T_{\varepsilon'}+(k+1)(T_{\varepsilon''}+1)} \geq k\hat{\varepsilon}$, which is clearly impossible for large enough k. This establishes the lemma's claims given eventually generic beliefs.

Suppose instead, that the equilibrium rewards success for $t \geq T$. Consider $t \geq T + T_{\varepsilon'}$. We showed above that our assumption $p^t_1\mu^t_1 < 1-\varepsilon$ implies $p^t_1 \leq 1-\varepsilon+\varepsilon' < 1$ for $\varepsilon' < \varepsilon$. As argued in Lemma 4 there is an upper bound on the number of firms $\overline{M} < \infty$ and in each period there is a probability greater than $\beta(1-\alpha)^{\overline{M}-1}$ that exactly one firm gets output g and all others get b. If the probability of new entry when *at most one firm* produces output g (and any others producing b) is less than r, then:

$$p^{t+1}_1 \geq (p^t_1 + (1-p^t_1)\beta(1-\alpha)^{\overline{M}-1})(1-r), \tag{SA3}$$

We can then choose $r \in (0,1)$ such that $\beta(1-\alpha)^{\overline{M}-1}(1-r)/(1-(1-r)(1-\beta(1-\alpha)^{\overline{M}-1})) > 1-\varepsilon+\varepsilon'$. For such r there exists $T_r < \infty$ such that for any $s \geq T_{\varepsilon'}$, if (SA3) held for $t \in \{s,...,s+T_r-1\}$ then $p^{s+T_r}_1 \geq 1-\varepsilon+\varepsilon'$, a contradiction. Hence, to avoid this conclusion in some period $t \in \{s,...,s+T_r-1\}$ there must be entry with probability greater than r when at most one firm produces output g. The total probability of entry in period t when at most one firm produces g, is then at least $\beta(1-\alpha)^{\overline{M}-1}r$. For arbitrary $M > 0$, given $\xi < \beta/(M+1)$ and $E \geq M$ there will be at least M new entrants after such an event (as they will get a share of total sales of at least β). A slight adaption of the argument at the start of Lemma 5 then implies that for a sufficiently large number of active firms in period t, any competent firm will exert effort with probability 1. Formally, given some private history h^t_i of some competent firm i let its continuation value be $V^{t,e}_i(h^t_i)$ if it exerts effort and $V^{t,n}_i(h^t_i)$ if it does not. Let $V^t_i(h^t_i) = \max\{V^{t,n}_i(h^t_i), V^{t,e}_i(h^t_i)\}$ and let $\overline{V}^{t+1}_i(h^t_i,b)$ be the firm's continuation value if all firms subsequently produce output b. Also let: $\hat{\eta}(h^t_i)$ be the probability consumers expect a firm to exert effort at that history, let $\eta_j(h^t_i)$ be the probability firm i expects firm j to exert effort, and $J(h^t_i)$ be the set of active firms. We have:

$$V^{e,t}_i(h^t_i) = \frac{U(\hat{\eta}(h^t_i))(1-c)}{|J(h^t_i)|} + \alpha\delta\sum_{h^{t+1}_i} p(h^{t+1}_i|h^t_i,g_i)V^{t+1}_i(h^{t+1}_i) + \delta(1-\alpha)\prod_{j\neq i}(1-U(\eta_j(h^t_i))\overline{V}^{t+1}_i(h^t_i,b)$$

$$V^{n,t}_i(h^t_i) = \frac{U(\hat{\eta}(h^t_i))}{|J(h^t_i)|} + \beta\delta\sum_{h^{t+1}_i} p(h^{t+1}|h^t_i,g_i)V^{t+1}_i(h^{t+1}_i) + \delta(1-\beta)\prod_{j\neq i}(1-U(\eta(h^t_i))\overline{V}^{t+1}_i(h^t_i,b).$$

And so:

$$V^{e,t}_i(h^t_i) - V^{n,t}_i(h^t_i) \geq -\frac{cU(\hat{\eta}(h^t_i))}{|J(h^t_i)|} + \delta(\alpha-\beta)\Big(\frac{\beta}{|J(h^t_i)|} - \frac{(1-\beta)^{|J(h^t_i)|-1}}{1-\delta}\Big) > 0$$

where the first inequality follows because $V^{t+1}_i(h^{t+1}_i) \geq \beta/|J(h^t_i)|$ since the equilibrium rewards success at such t and $\prod_{j\neq i}(1-U(\eta(h_i^t)))\leq(1-\beta)^{|J(h_i^t)|-1}$ and the second inequality follows for all sufficiently large $|J(h_i^t)|$ since $\lim_{M\to\infty}M(1-\beta)^{M-1}=0$, $U(\hat{\eta}(h_i^t))\leq\alpha$ and $c<\hat{c}^{RE}$.

Given full-effort when there are a large number of firms M^t and a probability greater than $\beta(1-\alpha)^{\overline{M}-1}r/\overline{M}$ of new entry by a large number M^t of firms in period t, we can simply repeat the argument in Lemma 3 that for some $\hat{\varepsilon}>0$, we have $\mu^{t+1}\geq\mu^t+\hat{\varepsilon}$ (at every consumer history inconsistent with exactly one of M^t firms producing g, consumer selection still weakly increases reputations). We can then repeat this argument k times to see that $\mu^{T+T_{\varepsilon'}+kT_r}\geq k\hat{\varepsilon}$ which clearly cannot hold for sufficiently large k. □

SA 3. Proof of Proposition 6

Consider consumer behavior consistent with a strong-incentive equilibrium and a competent firm i with private history h_i^t that expects to be active in the current period with sales of R, when it decides whether to exert effort. Let its continuation value be $V_i^{e,t}(h_i^t)$ if it exerts efforts and $V_i^{n,t}(h_i^t)$ and let $V_i^t(h_i^t)=\max\{V_i^{e,t}(h_i^t),V_i^{n,t}(h_i^t)\}$. Also let g_i be the event where i produces output g in period t. Continuation values must satisfy:

$$V_i^{e,t}(h_i^t)=R-cR+\delta\alpha\sum_{h_i^{t+1}}p(h_i^{t+1}|h_i^t,g_i)V_i^{t+1}(h_i^{t+1})$$
$$V_i^{n,t}(h_i^t)=R+\delta\beta\sum_{h_i^{t+1}}p(h_i^{t+1}|h_i^t,g_i)V_i^{t+1}(h_i^{t+1})$$

When a firm expects $J(h_i^t)$ active firms we must have $V_i^{t+1}(h_i^{t+1})\geq\beta(1-c)\sum_{k=0}^{\infty}\delta^k\alpha^k/|J(h_i^t)|$ and so

$$V_i^{e,t}(h_i^t)-V_i^{n,t}(h_i^t)\geq-cR+\frac{\delta(\alpha-\beta)\beta(1-c)}{|J(h_i^t)|(1-\delta\alpha)}.$$

If firm i is active and receives revenue $R\in[0,1/|J(h_i^t)|]$, then given $cR\leq c/|J(h_i^t)|$ and $c<\bar{c}^{SI}$, we then immediately get $V_i^{e,t}(h_i^t)-V_i^{n,t}(h_i^t)>0$. Hence, a competent firm always exerts effort if it is active (even if only one consumer purchased from it), and so consumers select those firms with the highest reputations.

Without loss of generality, any such equilibrium is a full-effort equilibrium (requiring inactive, low reputation but competent firms to exert effort won't change outcomes or incentives), which can't exist for all large E and small ξ by Proposition 2. In fact, the proof of Proposition 2 relied only on consumers selecting firms with the highest reputation on the equilibrium path, and those firms always exerting effort, which we have established must be true for the strong-incentive equilibria above. lemmatas 1, 2 and 3 must then apply to the strong incentive equilibria above, but Lemma 3 (the emergence of a competent monopolist) contradicts the definition of a strong-incentive equilibrium. □

SA 4. The firm age model

In this section, we study the *firm age model* introduced in Section IV.B, in which consumers *do not* observe calendar time, but *do* observe the age of any incumbent firms.

The age of a firm is the number of periods since it entered, where that age is 1 when it enters and 2 in the next period (this simplifies notation compared to assuming age 0 on entry). Potential new entrants also don't observe calendar time, but do observe the age of any incumbent. The latter assumption ensures that entrants cannot signal the calendar time and thus information about the incumbents through their entry decision.

We focus on full-effort equilibria with generic beliefs, so an incumbent firm never has a reputation of exactly ψ. Hence, all active firms will have the same age. Given this, we can summarize on path consumer histories by $h_c = (m_g, m_b, m_x, a)$, which specifies the number of incumbent firms with past outputs g and b, the number of new entrants, and the age of incumbents a (where $a = 1$ when there are no incumbents, and $a \geq 2$ otherwise). We similarly define $h_e = (m_g, m_b, a)$ as a pre-entry history that is observed by the first potential entrant in some period. Let a period t complete history be denoted $\underline{h}^t$, as before.

Consumers must form beliefs about calendar time from the history that they observe. To ensure such beliefs are well-defined, we follow Pei (2025) and assume that the game ends with probability $1-\rho$ in each period, where $\rho \in [\delta, 1)$. Future payoffs are still discounted by δ where $\delta/\rho \in (0,1]$ is discounting due to impatience. Given this, consumers who do not observe calendar time should assign prior probability $(1-\rho)\rho^{t-1}$ that they are in calendar time period t before observing the age and outputs of any incumbent firms. Given a complete history $\underline{h}_t$ we let $p(\underline{h}_t)$ denote the equilibrium probability of $\underline{h}_t$ conditional on the game not ending before time t, so that $\rho^{t-1}p(\underline{h}_t)$ is the unconditional probability of that history. All other elements of the model are the same as our baseline model.

In order to establish Proposition 7's claim thatequilibria with full-effort and generic beliefs cannot exist when there is almost-free entry, we first show that we can focus on a class of equilibria, where entry decisions are stationary.

Definition 1 (**Equilibrium with stationary entry**)**.** An equilibrium with stationary entry is one in which entry decisions are the same at every pre-entry history for which any incumbent's reputation is strictly below ψ.

This means that in a full-effort equilibrium with generic beliefs and stationary entry, the distribution of the number of firms that enter when the reputation of all incumbent firms is below ψ, does not depend on the number of incumbents with b or g outputs or the age of incumbents in that period.

Definition 2 (**Outcome equivalence**)**.** We say two equilibria σ and σ^* are outcome-equivalent if they generate the same distribution over pre-entry histories h_e.

Lemma SA 1. *Given any full-effort equilibrium with generic beliefs where there is at least one new entrant whenever all incumbents have reputations below ψ, there exists an outcome-equivalent full-effort equilibrium with stationary entry and generic beliefs.*

Proof of Lemma SA 1: Let $\underline{H}^t(h_e)$ be the set of period t complete histories consistent with potential entrants observing pre-entry history h_e in period t. Denote $p(m|h_e)$ the probability of m entrants in period t conditional on potential entrants observing the pre-entry history. Then define $q(m)$ as the probability of $m > 0$ new entrants conditional on pre-entry histories where there will be entry by at least one firm. This satisfies:

$$q(m) = \frac{\sum_t \sum_{h_e} \sum_{\underline{h}^t \in \underline{H}^t(h_e)} p(m|h_e)p(\underline{h}^t)\rho^{t-1}}{\sum_s \sum_{h'_e} \sum_{\underline{h}^s \in H^s(h'_e)} \sum_{m'>0} p(m'|h'_e)p(\underline{h}^s)\rho^{s-1}}.$$

Clearly $\sum_{m>0} q(m) = 1$. Next for any $h_e = (m_g, m_b, a) \neq (0,0,1)$, define $q(h_e|m)$ as the probability of that pre-entrant history in period $t+a-1$ and any incumbents having been active in periods $t, ..., t+a-2$ given m firms successfully entered the market in period t; this does not depend on the period t or the pre-entry history at which the incumbent firms entered given full-effort. It can be defined as:

$$q(h_e = (m_g, m_b, a)|m) = \rho^{a-1} \sum_{\underline{h}^a \in H^a(h_e)} p(\underline{h}^a | h_c = (0,0,m,1)).$$

Given this, the probability of reaching pre-entrant history $h_e \neq (0,0,1)$ is just $\sum_m q(m)q(h_e|m)$. We can then identify a new full-effort equilibrium where for any pre-entry history where $p(m|h_e) > 0$ for some m we let $p(m'|h_e) = q(m')$ in the new equilibrium. Clearly this has the same probability of any pre-entry history.

It remains to show that such entry probabilities can be obtained as part of optimal potential entrant behavior. Let Π_m be the expected post entry profits of each of m new entrants that become active firms (in any period $t \geq 1$). Clearly if $q(m) > 0$, then we must have $\Pi_m \geq \xi$. Also if $\Pi_m > \xi$ for some $m \leq E$ then we must have $q(m') = 0$ for all $m' < m$ (or some potential entrant would deviate to entering, knowing that if $m-1$ firms have entered the E^{th} potential entrant will enter etc). Hence if $q(m) > 0$, we must have $\Pi_{m'} \leq \xi$ for $m' > m$. In the new equilibrium, let firm m enter with probability $\sum_{m' \geq m} q(m')/(\sum_{m' \geq m-1} q(m'))$ and so the probability the first m firms have entered is $\sum_{m' \geq m} q(m')$ and the probability exactly m have entered is $q(m)$. Letting $m^1 = \min\{m \geq 1 : q(m) > 0\} > 0$ then the first m^1 firms enter for sure, and since $\Pi_m \geq \xi$ for all $m \geq m^1$ with $q(m) > 0$, such firms won't deviate on the equilibrium path. Similarly for $m > m^1$, because $\Pi_m \leq \xi$ with $\Pi_m = \xi$ if $q(m) > 0$ then on the equilibrium path no firm can benefit by deviating (entering when the above strategies calls for it not to, or not entering when called to). Off-the equilibrium path, some optimal strategies can be identified by backward induction. □

Given an equilibrium with stationary entry, for $h_c = (m_g, m_b, m_x, a)$ we define $p(h_c) = \sum_{\underline{h}^a \in \underline{H}^a(h^c)} p(\underline{h}^a)$. Let $\underline{H}^a_z$ be the set of complete period a histories for which all *active* firms have age a. We then have $p(\underline{H}^a_z) = \sum_{h_c = (m_g, m_b, 0, a)} p(\underline{h}_c)$. As previously: let $J(\underline{h}^t)$ denote the set of active firms at history $\underline{h}^t$ as previously and let $\mu_i(\underline{h}^t)$ be the probability that firm i is competent given $\underline{h}^t$. Also let $\mu^t_{c,q}(h_c)$ be the consumer's belief that a firm with output q is competent given h_c.

Lemma SA 2 (Increasing reputations). *In any full-effort equilibrium with generic beliefs where there is at least one new entrant whenever all incumbents have reputations below ψ, consumers' average belief that an arbitrary active firm is competent given age a, denoted μ^a, is weakly increasing in a.*

Proof of Lemma SA 2: Recall that a firm in period t who entered in period 1 has age $a = t$. Given Lemma SA 1 we can restrict attention to equilibria with stationary entry, in which consumers beliefs about any firm of age a are identical to their beliefs given the additional information that such a firm entered in period 1. The result is then a straightforward adaption of Lemma SA 2, using the facts that the belief about any particular firm must be a martingale, and consumers positively select based on reputation.

The average reputation of active firms of age a satisfies:

$$\psi = \mu^1 \leq \mu^a = \frac{1}{p(\underline{H}_z^a)} \sum_{h_c=(m_g,m_b,0,a)} \sum_{\underline{h}^a \in \underline{H}^a(h_c)} p(\underline{h}^a) \sum_{i \in J(\underline{h}^a)} \frac{\mu_i(\underline{h}^a)}{|J(\underline{h}^a)|}$$

$$= \frac{1}{p(\underline{H}_z^a)} \sum_{h_c=(m_g,m_b,0,a)} \sum_{\underline{h}^a \in \underline{H}^a(h_c)} p(\underline{h}^a) \sum_{\underline{h}^{a+1} \in \underline{H}^{a+1}} p(\underline{h}^{a+1}|\underline{h}^a) \sum_{i \in J(\underline{h}^a)} \frac{\mu_i(\underline{h}^{a+1})}{|J(\underline{h}^a)|}$$

$$= \frac{1}{p(\underline{H}_z^a)} \sum_{h_c=(m_g,m_b,m_x,a+1)} p(h_c) \frac{\mu_{c,g}(h_c)m_g + \mu_{c,b}(h_c)m_b}{m_g+m_b}$$

$$\text{(SA4)} \qquad \leq \frac{1}{p(\underline{H}_z^{a+1})} \sum_{h_c=(m_g,m_b,0,a+1)} p(h_c) \max\{\mu_{c,g}(h_c)\mathbb{1}_{m_g>0}, \mu_{c,b}(h_c)\mathbb{1}_{m_b>0}\} = \mu^{a+1}$$

where on the first line, the inequality holds by induction, and the equalities by definition. The second line follows from the first by the martingale property of beliefs $E[\mu_i(\underline{h}^{t+1})|\underline{h}^t] = \mu_i(\underline{h}^t)$. The third line follows by definition (consumers beliefs must on average be correct). The inequality on the fourth line is then because an average of two values is less than their maximum and because $m_x > 0$ when incumbents reputations are below ψ but not above ψ: more precisely consumer histories with age $a+1$ incumbents and $m_x = 0$ occur only if $\max\{\mu_{c,g}(h_c)\mathbb{1}_{m_g>0}, \mu_{c,b}(h_c)\mathbb{1}_{m_b>0}\} \geq \psi$ and consumer histories with age $a+1$ incumbents and $m_x > 0$ only occur if $\max\{\mu_{c,g}(h_c)\mathbb{1}_{m_g>0}, \mu_{c,b}(h_c)\mathbb{1}_{m_b>0}\} \leq \psi$; the latter histories are inconsistent with $\underline{H}_z^{a+1}$. The final equality then follows by definition given consumers selection rule. □

Lemma SA 3 (Arbitrarily high average reputations for old active firms). *Fix ψ, α, β and $\varepsilon > 0$. There exists $A < \infty$, $\bar{E} < \infty$, $\bar{\xi} > 0$ such that if $E \geq \bar{E}$ and $\xi \leq \bar{\xi}$, in any full-effort equilibrium, we have $\mu^a \geq 1-\varepsilon$ for $a \geq A$ and $p(\underline{H}_z^A) \geq 1-\varepsilon$ where μ^a is the average reputation of an age a firm.*

Proof of Lemma SA 3: Recall the notation from the proofs of lemmatas 2 and 5: $\underline{H}_g^t :=$ The set of t-period complete histories in which each incumbent firm in period t produced output g in every period $1,\ldots,t-2$. $\underline{H}_x^t :=$ The set of t-period complete histories in which there is successful entry in period t *despite* at least one firm producing output g in period $t-1$. $\underline{H}_b^t :=$ The set of t-period complete histories in which each incumbent firm in period t has produced output g in every period $1,\ldots,t-2$ but produced b in period $t-1$. Let $S^{t+1}(A) \subset \underline{H}^{t+1}$ be the set of $t+1$-period complete histories that succeed some t-period complete history $\underline{h}^t \in A \subset \underline{H}^t$. Also recall $\underline{H}^t(h_c)$ is the set of time t complete histories consistent with h_c.

For arbitrary $\hat{M} \geq 2$, it is clear that for any $E \geq \hat{M}$ and $\xi \in (0, \beta/\hat{M})$ we can ensure there must be at least $\hat{M}$ active firms in any period in which entry would be successful, and in particular there will be at least $\hat{M}$ initial entrants. Henceforth, we assume this, and assume without loss that there is stationary entry.

Recall that $\hat{\mu}_g^{t-1}(\psi)$ is the reputation of a firm in period t that is known to have produced output g in every period $1,\ldots,t-1$ and exerted effort if competent in those periods (as is true in full-effort equilibrium). Clearly, $\hat{\mu}_g^{t-1}(\psi) \geq \hat{\mu}_g(\psi) > \psi$ for $t \geq 2$, and $\hat{\mu}_g^{t-1}(\psi) \to 1$ as $t \to \infty$.

Also notice that in a full-effort equilibrium a firm's reputation conditional on a complete history $\underline{h}^t$ cannot go down at succeeding history $\underline{h}^{t+1}$ in which it produced output g in period t. Hence an incumbent firm i's reputation conditional on any history $\underline{h}^t \in \underline{H}_g^t \cap \underline{H}_{g_i}^t$ must be at least $\hat{\mu}_g(\psi)$.

Consider a consumer history $h_c = (m_g, m_b, m_x, a)$ in which there is successful entry $m_x > 0$ despite at least one incumbent firm having produced output g in the previous period, $m_g > 0$. This requires that consumers must believe that the incumbent with output g is competent with probability less than ψ, and so we must have:

$$\psi \geq \frac{p(\underline{H}_g^a \cap \underline{H}^a(h_c) \cap \underline{H}_x^a)\hat{\mu}_g(\psi)}{p(\underline{H}_g^a \cap \underline{H}^a(h_c) \cap \underline{H}_x^a) + p(\underline{H}^a(h_c) \cap (\underline{H}_x^a \setminus \underline{H}_g^a))}$$

or equivalently $p(\underline{H}^a(h_c) \cap (\underline{H}_x^a \setminus \underline{H}_g^a)) \geq p(\underline{H}_g^a \cap \underline{H}^a(h_c) \cap \underline{H}_x^a)(\hat{\mu}_g(\psi) - \psi)/\psi$. And so:

$$\begin{aligned} p(\underline{H}^a \setminus \underline{H}_g^a) \geq & \sum_{h_c=(m_g,m_b,m_x,a)} p(\underline{H}^a(h_c) \cap (\underline{H}_x^a \setminus \underline{H}_g^a)) \\ & \geq \sum_{h_c=(m_g,m_b,m_x,a)} p(\underline{H}_g^a \cap \underline{H}^a(h_c) \cap \underline{H}_x^a) \frac{\hat{\mu}_g(\psi) - \psi}{\psi} = p(\underline{H}_g^a \cap \underline{H}_x^a) \frac{\hat{\mu}_g(\psi) - \psi}{\psi}. \end{aligned}$$

Next notice that $p(\underline{H}_b^a)$ is smaller than the probability that each initial entrant (at time 1) would have produced b in some period $1,...,a-1$, if new entry after period 1 was impossible. If new entry was impossible after period 1, the probability that a given initial entrant produced output g in periods $1,...,a-1$ would be at least β^{a-1}. Given at least $\hat{M}$ initial entrants, the probability that this would not be true for any of them would then be less than $(1-\beta^{a-1})^{\hat{M}}$. And so, $p(\underline{H}_b^a) \leq (1-\beta^{a-1})^{\hat{M}}$. As noted in the proof of lemmatas 2 and 5, we have $\underline{H}_g^{a+1} \supset S^{a+1}(\underline{H}_g^a \setminus (\underline{H}_b^a \cup \underline{H}_x^a))$ and so:

$$\begin{aligned} p(\underline{H}_g^{a+1}) \geq & p(\underline{H}_g^a) - p(\underline{H}_g^a \cap H_x^a) - p(\underline{H}_b^a) \\ \geq & p(\underline{H}_g^a) - (1 - p(\underline{H}_g^a)) \frac{\psi}{\hat{\mu}_g(\psi) - \psi} - (1-\beta^{a-1})^{\hat{M}}. \end{aligned}$$

For any a and $\varepsilon_{a+1} > 0$ and we can then choose $\varepsilon_a > 0$ such that if $p(\underline{H}_g^a) \geq 1 - \varepsilon_a$ for all large enough $\hat{M}$ we must have $p(\underline{H}_g^{a+1}) \geq 1 - \varepsilon_{a+1}$. For any A and $\varepsilon_{A+1} > 0$ we can then choose $\varepsilon_a > 0$ for $a \leq A$ and $\hat{M}$ large enough that $p(\underline{H}_g^{A+1}) \geq 1 - \varepsilon_{A+1}$ if $p(\underline{H}_g^1) \geq 1 - \varepsilon_1$ but since $p(\underline{H}_g^1) = 1$ this is certainly satisfied.

And so, for any $\varepsilon' > 0$ such that $(1-\varepsilon')^2 \geq 1 - \varepsilon$ we can choose A such that $\hat{\mu}_g^{A-1}(\psi) > 1 - \varepsilon'$. Given such an A, if there are sufficiently many entrants, $p(\underline{H}_z^A) \geq p(\underline{H}_g^{A+1}) \geq 1 - \varepsilon'$. And so, $\mu^A \geq \hat{\mu}_g^{A-1}(\psi) p(\underline{H}_g^{A+1}) \geq (1-\varepsilon')^2 \geq 1 - \varepsilon$. Given Lemma SA 2, we must also have $\mu^a \geq \mu^A$ for $a \geq A$. □

Lemma SA 4 (Emergence of a "competent monopolist" in full-effort equilibria). *Given any ψ, α, β and $\varepsilon > 0$, there exists $\bar{E} < \infty$, $\bar{\xi} > 0$ such that for any $\delta \in (0,1)$, $\xi < \bar{\xi}$, $E > \bar{E}$, there is some $A < \infty$ such that in any full-effort equilibrium with generic beliefs we have $p_1^a \mu_1^a > 1 - \varepsilon$ for all $a \geq A$, where μ_1^a is the average reputation of a monopolist of age a, and p_1^a is the equilibrium probability that in period a there is a single active firm who entered in period 1 conditional on the game not ending before period a.*

Proof of Lemma SA 4: We know that for any α, β, ψ and small $\varepsilon > 0$, we can find A and $\bar{E}$ and $\bar{\xi}$ such that if $E \geq \bar{E}$ and $\xi \leq \bar{\xi}$ and $a \geq A$, then reputations of active firms of an age a firm are greater than $1-\varepsilon$ and $p(\underline{H}_z^A) \geq 1 - \varepsilon$. Furthermore, we can assume the equilibrium has stationary entry. Given this, we first want to show that the probability that all firms of age A can be replaced when they have age $a \geq A$ is small. Let $\overline{J}(\underline{h}^t)$ be the set of *incumbent* firms at complete history t, while

$J(\underline{h}^t)$ remains the set of active firms. Define $\overline{\mu}^A = \mu^A$ and for $a \geq A+1$, define:

$$\overline{\mu}^a = \frac{1}{p(\underline{H}_z^A)}\Big(\sum_{h_c=(m_g,m_b,0,a)}\sum_{\underline{h}^a\in\underline{H}^a(h_c)} p(\underline{h}^a)\sum_{i\in J(\underline{h}^a)}\frac{\mu_i(\underline{h}^a)}{|J(\underline{h}^a)|}$$
$$+\sum_{a'\in\{A+1,\dots,a\}}\sum_{h'_c=(m'_g,m'_b,m'_x,a'):m'_x>0}\sum_{\underline{h}^{a'}\in\underline{H}^{a'}(h'_c)} p(\underline{h}^{a'})\sum_{i\in\overline{J}(\underline{h}^{a'})}\frac{\mu_i(\underline{h}^{a'})}{|\overline{J}(\underline{h}^{a'})|}\Big)$$

This is effectively an imagined average reputation for firms of age a who were active in period A, and either remain active or are replaced in some period $a' \in \{A+1,\dots,a\}$. The first line is the probability weighted sum of the reputations of firms who are still active in period a, and the second line is the probability weighted sum of the reputations of those who were instead replaced in period $a' \in \{A+1,\dots,a\}$. This imagined reputation is increasing in a, indeed it can strictly increase because there is still consumer selection between firms when some incumbents produce b and some g of whom some remain active, $m_x = 0$. Formally:

$$\begin{aligned}
\overline{\mu}^a &= \frac{1}{p(\underline{H}_z^A)}\Big(\sum_{h_c=(m_g,m_b,0,a+1)} p(h_c)\frac{\mu_{c,g}(h_c)m_g+\mu_{c,b}(h_c)m_b}{m_g+m_b} \\
&\quad+\sum_{a'\in\{A+1,\dots,a+1\}}\sum_{h'_c=(m'_g,m'_b,m'_x,a'):m'_x>0} p(h'_c)\frac{\mu_{c,g}(h'_c)m'_g+\mu_{c,b}(h'_c)m'_b}{m'_g+m'_b}\Big) \\
&\leq \frac{1}{p(\underline{H}_z^A)}\Big(\sum_{h_c=(m_g,m_b,0,a+1)} p(h_c)\max\{\mathbb{1}_{m_g>0}\mu_{c,g}(h_c)+\mathbb{1}_{m_b>0}\mu_{c,b}(h_c)\} \\
&\quad+\sum_{a'\in\{A+1,\dots,a+1\}}\sum_{h'_c=(m'_g,m'_b,m'_x,a'):m'_x>0} p(h'_c)\frac{\mu_{c,g}(h'_c)m'_g+\mu_{c,b}(h'_c)m'_b}{m'_g+m'_b}\Big) = \overline{\mu}^{a+1},
\end{aligned}$$

where the first equality follows from the Martingale property on beliefs and the inequality from the fact that the maximum of two values is larger than its average, and the second equality is by definition.

For any $a \geq A$ we must have

$$\begin{aligned}
\overline{\mu}^a p(\underline{H}_z^A) =& \sum_{h_c=(m_g,m_b,0,a)} p(h_c)\max\{\mathbb{1}_{m_g>0}\mu_{c,g}(h_c)+\mathbb{1}_{m_b>0}\mu_{c,b}(h_c)\} \\
&+\sum_{a'\in\{A+1,\dots,a\}}\sum_{h'_c=(m'_g,m'_b,m'_x,a'):m'_x>0} p(h'_c)\frac{\mu_{c,g}(h'_c)m'_g+\mu_{c,b}(h'_c)m'_b}{m'_g+m'_b} \\
\text{(SA5)}\qquad &\leq p(\underline{H}_z^A)-(1-\psi)p^{A,a}
\end{aligned}$$

where

$$p^{A,a} = \sum_{a'\in\{A+1,\dots,a\}}\sum_{h'_c=(m'_g,m'_b,m'_x,a'):m'_x>0} p(h'_c),$$

is the probability that active firms in period A entered in period 1 and are replaced between period $A+1$ and a, conditional on the game lasting a periods. The equality in (SA5) is by definition, and the inequality follows from $\max\{\mathbb{1}_{m_g>0}\mu_{c,g}(h_c), \mathbb{1}_{m_b>0}\mu_{c,b}(h_c)\} \leq 1$ and $\sum_{h_c=(m_g,m_b,0,a)} p(h_c) \leq$

$p(\underline{H}_z^A) - p^{A,a}$ and $(\mu_{c,g}(h'_c)m'_g + \mu_{c,b}(h'_c)m'_b)/(m'_g+m'_b) \le \psi$ for all $h'_c = (m'_g, m'_b, m'_x, a')$ with $m'_x > 0$ and $p(h'_c) > 0$. Given the above inequality, we have $p^{A,a}/p(\underline{H}_z^A) \le (1-\overline{\mu}^a)/(1-\psi)$. Given that we $\overline{\mu}^a \ge \mu^A \ge 1-\varepsilon$, we have that the probability of all active firms of age A are replaced by $a \ge A$ conditional on the game lasting that long is arbitrarily small for small ε. The probability some of the period 1 firms remain active is $p(\underline{H}_z^a) = p(\underline{H}_z^A) - p^{A,a} \ge p(\underline{H}_z^A)(1-\varepsilon/(1-\psi))$.

Given that there are at most E firms in any period on the equilibrium path, the probability that some active firm gets output g while all others get b is at least $\beta(1-\alpha)^{E-1}$. Therefore, the probability that this does *not* happen in any period $s \in \{A,\dots,A'-1\}$ conditional on all active firms in period A having age A and the game not ending before A', is most $(1-\beta(1-\alpha)^{E-1})^{A'-A}$, which can be made strictly less than $\varepsilon > 0$ by choosing A' large enough. That is: $(p(\underline{H}_z^{A'}) - p_1^{A'})/p(\underline{H}_z^A) \le \varepsilon$ or equivalently $1-\varepsilon p(\underline{H}_z^A)/p(\underline{H}_z^{A'}) \le p_1^{A'}/p(\underline{H}_z^{A'})$.

Combining this with $p(\underline{H}_z^{A'})/p(\underline{H}_z^A) \ge (1-\psi-\varepsilon)/(1-\psi)$ we have $p_1^{A'}/p(\underline{H}_z^{A'}) \ge 1-\varepsilon(1-\psi)/(1-\psi-\varepsilon)$. Furthermore, since $\mu^{A'} \le 1-(1-\mu_1^{A'})p_1^{A'}/p(H_z^{A'})$ because non-monopolists can't have a reputation above 1, we must have $\mu_1^{A'}p_1^{A'}/p(H_z^{A'}) \ge \mu^{A'} - 1 + p_1^{A'}/p(H_z^{A'}) \ge 1-\varepsilon-\varepsilon(1-\psi)/(1-\psi-\varepsilon)$. In fact, since $p(\underline{H}_z^A) \ge 1-\varepsilon$ we have $p(\underline{H}_z^{A'}) \ge (1-\varepsilon)(1-\psi-\varepsilon)/(1-\psi)$ and so $\mu_1^{A'}p_1^{A'} \ge (1-\varepsilon)^2(1-\psi-\varepsilon)/(1-\psi) - \varepsilon(1-\varepsilon)$.

Given this, for any α, β, ψ and $\varepsilon > 0$ such that $\mu_b(1-\varepsilon) > \psi$ we can choose $\bar{E} < \infty$ and $\bar{\xi} > 0$ such that for any given $E \ge \bar{E}$ and $\xi \le \bar{\xi}$, there exists A such that $\mu_1^A p_1^A/p(H_z^A) \ge \mu_1^A p_1^A \ge 1-\varepsilon$. Given $\mu_1^a p_1^a/p(H_z^a) \ge \mu_1^a p_1^a \ge 1-\varepsilon$ for some a, we have $\hat{\mu}_b(\mu_1^a) > \psi$ and so $\mu_1^{a+1}p_1^{a+1}/p(\underline{H}_z^{a+1}) \ge \mu_1^{a+1}p_1^{a+1} \ge 1-\varepsilon$, hence the premise holds for all $a \ge A$. □

Proof of Proposition 7: Formally, consider any ψ, α, β, $\varepsilon' \in (0, 1-\psi)$ and $\varepsilon > 0$ such that $\hat{\mu}_b(1-\varepsilon) > 1-\varepsilon' > \psi$. By Lemma SA 4, for all sufficiently large E and sufficiently small ξ, there exists A, such that $\mu_1^a \ge \mu_1^a p_1^a > 1-\varepsilon$ for all $a \ge A$. Therefore, any monopolist of age $a \ge A+1$ will have a reputation strictly above $1-\varepsilon' > \psi$ even after output b and so will not be replaced. We are interested in the incentives of such a monopolist to exert effort. Let $V^{e,a}(R)$ be a competent monopolist's expected continuation payoff when it has sales of R if it exerts effort and let $V_q^{n,a}(R)$ be its payoff if it doesn't exert effort. We must have:

$$V^{e,a}(R) = R(1-c) + \delta\left(\alpha V^{e,a+1}(U(\hat{\mu}_g(\mu_1^a))) + (1-\alpha)V^{e,a+1}(U(\hat{\mu}_b(\mu_1^a)))\right)$$
$$V^{n,a}(R) = R + \delta\left(\beta V^{e,a+1}(U(\hat{\mu}_g(\mu_1^a))) + (1-\beta)V^{e,a+1}(U(\hat{\mu}_b(\mu_1^a)))\right)$$

In a full-effort equilibrium, we have

$$\begin{aligned} &V^{e,a+1}(U(\hat{\mu}_g(\mu_1^a))) - V^{e,a+1}(U(\hat{\mu}_b(\mu_1^a))) \\ &\quad = (1-c)(U(\hat{\mu}_g(\mu_1^a)) - U(\hat{\mu}_b(\mu_1^a))) = (1-c)(\alpha-\beta)(\hat{\mu}_g(\mu_1^a)-\hat{\mu}_b(\mu_1^a)) \end{aligned}$$

In order for the firm to exert effort given age $a \ge A+1$ and sales of R we need:

$$\text{(SA6)} \quad \begin{aligned} 0 \le V^{e,a}(R) - V^{n,a}(R) &= -cR + \delta(\alpha-\beta)(V^{e,a+1}(U(\hat{\mu}_b(\mu_1^a))) - V^{e,a+1}(U(\hat{\mu}_b(\mu_1^a)))) \\ &= -cR + \delta(\alpha-\beta)(1-c)(U(\hat{\mu}_g(\mu_1^a)) - U(\hat{\mu}_b(\mu_1^a))) \\ &\le -cR + (\alpha-\beta)^2\varepsilon' \end{aligned}$$

In particular, consider this inequality when $R \ge U(0) = \beta$ so $cR \ge \beta > 0$. Clearly, there is some $\varepsilon' > 0$ small enough such that this inequality cannot hold, which means that the firm will have an incentive to deviate from full-effort. Therefore, no full-effort equilibrium with generic beliefs can exist. □

Proof of Proposition 8: Suppose $\bar{x}_i \in (0,\beta)$ so at least one firm enters if incumbents' reputations fall below ξ. In a full-effort equilibrium, an age $a < \bar{a}$ firm must then have a reputation $\hat{\mu}_g^{a-1}(\psi) > \psi$ after output g and $\hat{\mu}_b(\hat{\mu}_g^{a-2}(\psi)) < \psi$ after output b. However, its reputation when it has age $a \geq \bar{a}$ must be either $\hat{\mu}_g^{\bar{a}-1}(\psi)$ or $\hat{\mu}_b(\hat{\mu}_g^{\bar{a}-2}(\psi))$ both of which strictly exceed ψ; there is no further selection.

In line with Proposition 1, an age $a \geq \bar{a}-1$ monopolist will then exert effort if and only if $c \leq \bar{c}^{LC,FA} = \hat{c}(\mu_g^{\bar{a}-2}(\psi))$. As shown in the proof of Proposition 3, a competent firm's continuation value after a g output is at least $(1-c)U(\hat{\mu}_g(\psi))/(1-\delta\alpha)$, and so if it expects to be replaced in the next period if it produces output b giving a payoff 0, it exerts effort if $c \leq \bar{c}^{LC} = \delta(\alpha - \beta)/(\delta(\alpha-\beta)+1-\delta\alpha)$. That proof also shows the cutoff $\bar{c}^{LC} < \hat{c}(\mu)$ for all μ, so $\bar{c}^{LC,FA} < \bar{c}^{LC}$. The proof of Proposition 3 also shows that off the equilibrium path when there is more than one incumbent firm if consumers believe g output firms have reputation $\hat{\mu}^g(\psi) > \psi$ and b output firms have reputation $\hat{\mu}^b(\psi) < \psi$, then a competent firm which expects itself and competitor to be active will optimally exert effort when $c \leq \bar{c}^{LC}$. If it expects only itself to be active at such histories, it will exert effort if $c \leq \bar{c}^{LC,FA}$. Hence, full-effort equilibria exist in the firm-age model if and only if $c \leq \bar{c}^{LC,FA}$ where $\bar{c}^{LC,FA} < \bar{c}^{LC}$. It is straightforward to verify that $\hat{c}(\psi) = \bar{c}^{EM}$ can be larger or smaller than $\hat{c}(\hat{\mu}_g^{\bar{a}-2}(\psi))$ depending on ψ, α and β; in particular, since $\hat{c}(\mu)$ is not monotonic in μ and approaches 0 for $\mu \approx 0$ and $\mu \approx 1$.

Expected quality is identical to the full-effort equilibrium in the baseline model in periods $a < \bar{a}$, but in periods $t \geq \bar{a}$, firms with age $a \geq \bar{a}$ and output b are not replaced by new entrants with lower reputations of ψ; such firms would have been replaced in the baseline model. This guarantees that the average reputation of active firms in period $t \geq \bar{a}$ is higher than in the baseline model, and so expected quality is also higher. Since average reputations of active firms are strictly higher than in the exogenous monopoly model from period 2 onward, so is expected quality. □

References

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