Supplemental Appendix

Competing Diffusions in a Social Network

Arthur Campbell, C. Matthew Leister, Philip Ushchev, and Yves Zenou

A Model with a general threshold adoption rule

A.1 Model

In this section, we relax the assumption that $d(k) = 1$ for all k. Instead, we assume that the threshold is given by a discrete function $d(k)$ in the number of contacts k of an individual for which we assume $d(k)$ is non-decreasing, $d(k) \leq k/2$ for each $k > 1$ and $d(1) = 1$.[1] Importantly, the assumption $d(k) \leq k/2$ for each $k > 1$ embodies the notion of horizontal differentiation in our setting (i.e., agents are more inclined to adopt goods of their own type because they require less than 50% of their contacts have been using the product in the previous period in order to choose it in the current period).

Proceeding as in Section 3.1, the quantities $x_{M,t}, x_{N,t}, m_t, n_t$ are defined as before and can be written as follows:

$$x_{M,t} = \alpha_M m_t + (1-\alpha_M)(\rho m_t + (1-\rho)(1-n_t)) \tag{A.1}$$

$$x_{N,t} = \alpha_N n_t + (1-\alpha_N)\left((1-\rho)n_t + \rho(1-m_t)\right), \tag{A.2}$$

where the probabilities m_t and n_t evolve according to[2]

$$m_t = 1 - \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1 - x_{M,t-1}\right)^{k-j} \left(x_{M,t-1}\right)^j \tag{A.3}$$

$$n_t = 1 - \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1 - x_{N,t-1}\right)^{k-j} \left(x_{N,t-1}\right)^j \tag{A.4}$$

[1] Since $k = 1, 2, \cdots$ takes discrete values, $d(k)$ is a discrete function.

[2] Both of these equations can be equivalently represented as sums from $d(k)$ to k, such that

$$m_t = \sum_k p_k \sum_{j=d(k)}^{k} \binom{k}{j} \left(1 - x_{M,t-1}\right)^{k-j} \left(x_{M,t-1}\right)^j$$

and

$$n_t = \sum_k p_k \sum_{j=d(k)}^{k} \binom{k}{j} \left(1 - x_{N,t-1}\right)^{k-j} \left(x_{N,t-1}\right)^j.$$

To understand these equations, consider first $x_{M,t}$ (expression (A.1)). In period $t+1$, a type-M individual observes a contact of the same type with probability α_M who has adopted a type-M good in the period t with probability m_t. Otherwise, she observes a contact uniformly at random from the population (with probability $1-\alpha_M$), and then with probability ρ, the contact is of type M, while with probability $1-\rho$, she is of type N. Each of these types has adopted a type-M good in period t with probability m_t and n_t, respectively. One may similarly understand the relationship for a type N in (A.2).

Now, consider the evolution of the likelihood that a type-M consumer adopts good M in period t in equation (A.3). In period t, a type-M consumer with k contacts adopts the type-N content only if she observes that *at most* $d(k)-1$ of her contacts had adopted the type-N good in the previous period. With probability p_k, a consumer finds k contacts, and the probabilities that each contact adopted type-M and type$-N$ good in the previous period are $x_{M,t-1}$ and $1-x_{M,t-1}$ respectively. The expectation of this quantity is given by:

$$\sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1-x_{M,t-1}\right)^{k-j} \left(x_{M,t-1}\right)^j .$$

Thus, the expression for adopting a type-M good by a type-M individual is 1 minus this expectation. A similar explanation is true for n_t (expression (A.4)).

A.2 General characterization result

A steady-state equilibrium (n^*, m^*) satisfies $n_{t-1} = n_t = n^*$ and $m_{t-1} = m_t = m^*$. We say that there is a *mass-market equilibrium* when $x_M^* = m^* = 1$ and $x_N^* = n^* = 0$ and a *niche-market equilibrium* when $x_M^* = m^* = 0$ and $x_N^* = n^* = 1$. We say that there is *mixed-market equilibrium* when $\rho < x_M^* < m^* < 1$ and $0 < x_N^* < n^* < 1$.

Define the following positive values:

$$B_N = \frac{1-\left[\alpha_N + (1-\alpha_N)(1-\rho)\right] p_1}{1-(1-\alpha_M)(1-\rho)-\left[\alpha_M(1-\rho)+\alpha_N\rho\right] p_1} \tag{A.5}$$

and

$$B_M = \frac{1-\left[\alpha_M + (1-\alpha_M)\rho\right] p_1}{1-(1-\alpha_N)\rho-\left[\alpha_M(1-\rho)+\alpha_N\rho\right] p_1}. \tag{A.6}$$

Denote by $\hat{k} \geq 1$ the largest k, such that $d(k)=1$, and by $\sum_{k=1}^{\hat{k}} kp_k$ the truncated mean for all individuals for which $d(k)=1$.[3] We have the following result:[4]

[3]When $\hat{k}=\infty$, $d(k)=1$, for all k, and $\sum_{k=1}^{\infty} kp_k = \mathbb{E}[k]$, the expected or mean number of contacts for each individual.

[4]The proof of Theorem A1 can be found in Section A.4 below.

Theorem A1.

- *If* $\sum_{k=1}^{\widehat{k}} kp_k < B_N$*, then the niche-market steady-state equilibrium is asymptotically stable. Otherwise, when* $\sum_{k=1}^{\widehat{k}} kp_k > B_N$*, it is unstable.*
- *If* $\sum_{k=1}^{\widehat{k}} kp_k < B_M$*, then the mass-market steady-state equilibrium is asymptotically stable. Otherwise, when* $\sum_{k=1}^{\widehat{k}} kp_k > B_M$*, it is unstable.*
- *If* $\sum_{k=1}^{\widehat{k}} kp_k > \max\{B_M, B_N\}$*, then there exists at least one mixed-market steady-state equilibrium that is asymptotically stable.*

The "extreme"good (either niche or mass) markets are always steady-state equilibria, since if every agent chooses an extreme action, say niche, then everybody will choose the same action. The key question we ask in Theorem A1 is if these steady-state equilibria are (asymptotically) stable; that is, if a small fraction of agents adopts the other good, do the dynamics of the system take it to the extreme good steady state? Consider the extreme niche-market equilibrium and let us show under which condition it is stable. In a niche-market equilibrium, independent of their type, all agents only observe others adopting the niche good (i.e., $x_M^* = 0$ and $x_N^* = 1$) and so adopt a niche good themselves (i.e., $m^* = 0$ and $n^* = 1$). Suppose a small fraction of individuals in the neighborhood of this steady state adopts the type-M good. These individuals are then potentially observed by others in the following period. The conditions for the stability of this steady state can be understood by whether, in expectation, the fraction of people adopting the alternative good is increasing or decreasing. In the neighborhood of the niche-market steady state, this is determined by the fraction of type-M individuals with a threshold of $d(k) = 1$ (and any niche individuals with a degree of 1), since all other individuals must observe two or more people adopting for them to adopt themselves.

In other words, we look at all agents with threshold $d(k) = 1$ and thus consider the mean number of contacts among them, such that $\sum_{k=1}^{\widehat{k}} kp_k$. If this truncated mean is small enough, the niche-market equilibrium is stable. The same reasoning applies for the mass-market equilibrium. When $\sum_{k=1}^{\widehat{k}} kp_k$ is large enough, only a mixed-market equilibrium in which both goods survive in equilibrium is possible. Thus, the presence (lack thereof) of individuals with threshold $d(k) = 1$ is important for the stability of the extreme steady states. In particular, smaller $\widehat{k}$ (more people who require at least two observations of their own type) makes it easier to satisfy the condition for the extreme equilibria to be stable.

More generally, Theorem A1 shows that if the truncated mean $\sum_{k=1}^{\widehat{k}} kp_k$ is smaller than B_M and B_N,[5] then each individual extreme steady-state equilibrium, namely the

[5] We cannot rank B_M and B_N, since they depend on α_N and α_M, and we make no assumption on whether α_N is greater or smaller than α_M. However, when $\alpha_N = \alpha_M = \alpha$, then $B_N > B_M$, since $\rho > 1/2$.

mass- or niche-market, is asymptotically stable. This provides a threshold on connectedness beneath which niche- and mass-market goods cannot survive separately in equilibrium. Interestingly, when $\sum_{k=1}^{\widehat{k}} kp_k > \max\{B_M, B_N\}$, then neither the niche-market nor the mass-market extreme steady-state equilibrium is stable. In this case, only the mixed-market equilibrium is stable. In other words, higher connectedness makes possible the existence of both communities of niche- and mass-market goods because consumers of both markets find a sufficient number of like-minded agents to whom they pass on their type's information and preserve their market share.

Let us provide more intuition on the results in Theorem A1. Assume that $p_1 = 0$, such that all agents have at least two contacts, and no homophily ($\alpha_M = \alpha_N = 0$). Then, since $\rho > 1/2$, $B_N = 1/\rho < 2$ and $B_M = 1/(1-\rho) > 2$, where ρ is the fraction of mainstream individuals. Thus, when the network is such that $B_N < \sum_{k=1}^{\widehat{k}} kp_k < B_M$, only the mass-market goods will survive in steady state. Consequently, this mechanism will tend to *amplify* the majority action. When there is homophily, this is not necessary true, since $B_N = 1/[(1-\alpha_M)(1-\rho)]$ and $B_M = 1/[(1-\alpha_N)\rho]$, where the denominators are respectively the probability that a contact of a type-M and type-N individual is of the same type. Indeed, we see that α_M and α_N are negatively related to B_N and B_M, respectively, and that B_N increases with ρ while B_M decreases with ρ. When individuals are not too homophilous, or when the fraction of mainstream individuals is not too large (it has to be greater than 1/2), then we are more likely to have a niche-market or a mass-market equilibrium. In this case, the threshold condition may be written as

$$\sum_{k=1}^{\widehat{k}} kp_k(Pr[\text{contact same type}]) \lesseqgtr 1.$$

If $\epsilon \approx 0$ individuals of a given type are choosing their own type in period t, then, in period $t+1$, approximately $\epsilon \sum_{k=1}^{\widehat{k}} kp_k(Pr[\text{contact same type}])$ choose it in the following period. When this is bigger than ϵ, then, the dynamics move away from the extreme steady state, which cannot be stable.

A.3 Simulations for a regular network

In Theorem A1, we provide sufficient conditions for the existence and asymptotic stability of the extreme and mixed steady states. There are cases in which multiple steady states are asymptotically stable, and so comparative static exercises are not always possible in the general framework that we consider.[6] Indeed, as we show below, both extremes may co-exist and so even comparative static exercises on the "highest" and "lowest" steady states become problematic. Rather, in this section, we present what we feel are

[6]In Section 3, we consider the case $d(k) = 1$ and show that there is essentially a unique steady state; we are also able to obtain comparative static results.

informative simulations of the behavior of our system for a regular network ($p_k = 1$ for some k). In particular, we are interested in understanding how features such as a uniform threshold rule $d(k) = d$ for all k and degree k of the regular network affect the types/number/characteristics of asymptotically stable steady states that emerge and, in the cases in which there are more than one, the relative sizes of the basins of attraction for each.

Our first set of simulations considers a regular network with zero homophily $\alpha_M = \alpha_N = 0$. When there is zero homophily, our system of two equations may reduce to a single equation in which the relevant state is the fraction h_t of the population adopting the mass-market good, and a steady state is denoted by h^*. In each case, we consider a grid of starting points for h_0 and find the steady state to which each starting point converges. The results are displayed in Figure A1. Each panel shows the set of steady states reached and, for each set of starting values h_0 that approach each state, the basin of attraction. Each panel also shows these outcomes for different degrees k holding fixed a given threshold rule $d(k) = d$ for all k. We consider different values for the threshold from $d = 1$ (panel (a)), $d = 2$ (panel (b)), $d = 3$ (panel (c)), to $d = 1$ (panel (d)). The case $d = 1$ has a unique stable steady state indicated by the solid line. In the remaining cases $d = 2, 3, 4$, both extreme steady states exist for all values of k, and the basins of attraction for each are separated by the dashed line shown in the figures. We also see that, beyond a threshold value of k, a mixed steady state also exists (as well as the two extremes), in which case the basin of attraction for the mixed steady state separates the basins of attraction for the two extremes.

It is possible for both extreme equilibria and a single mixed equilibrium to arise in this setting. When $d = 1$, a unique stable steady state arises for all values $h_0 > 0$ (panel (a)). In this case, the uniquely stable steady state is the mass-market steady state in low-degree networks; however, above a threshold degree, the unique stable steady state becomes mixed and decreases as the degree increases. The uniqueness and tractability of this case lends itself to further analysis, which we pursue in Section 3. The three other cases $d = 2, 3, 4$ are all qualitatively similar to one another. Both extreme steady states are always present, and then beyond a threshold degree a third mixed steady state emerges.

We emphasize three properties of our system from these simulations. First, in all cases, when the mixed steady state emerges, it has a large basin of attraction; that is, a non-trivial set of starting points $h_0 \in (0, 1)$ results in the mixed steady state. Moreover, in the cases where $d > 1$, the mixed steady state that emerges is strictly away from the extremes (that is, unlike the case where $d = 1$, where the mixed steady state emerges close to the mass-market extreme and gradually moves away as k increases). This suggests that for $d > 1$, the system exhibits a phase transition whereby around a critical density k^{crit},

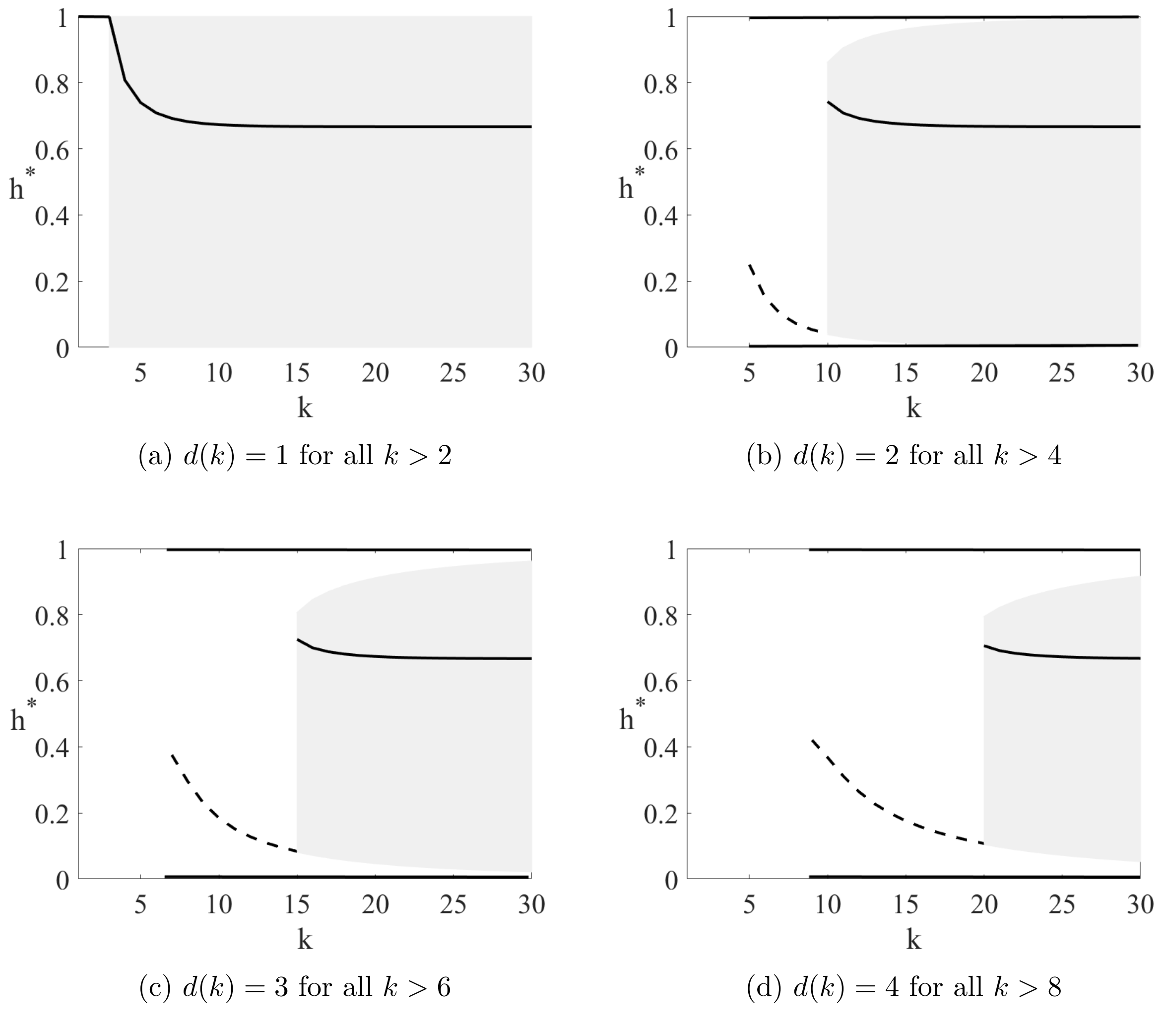

Figure A1: Steady states h^* (solid black lines) as a function of k ($p_k = 1$) for cases $d(k) = 1, 2, 3, 4$; dashed line separates basins of attraction to extreme steady states, and the gray area gives the basin of attraction to interior steady states the system distinctly changes behavior from always exhibiting extreme steady states to exhibiting a mixed steady state with a non-negligible basin of attraction.

Second, the process of diffusion confers an advantage to the mass-market good. This is evident in two features of the simulations. Indeed, across all simulations, the basin of attraction for the mass-market steady state is larger than the niche-market steady state, and, when the mixed steady state exists, it amplifies the mass-market good relative to the share of type-M people in the population, such that $h^* > \rho$. This characteristic is particularly pronounced in the case that $d = 1$, since all steady states exhibit the property that $h^* > \rho$, which we investigate further in Section 3.

Third, higher thresholds for adopting one's own type of good (larger d) confers advantages to the extreme steady states. Specifically, it makes the threshold density for the mixed steady state to exist greater and, when it does exist, the basin of attraction for the mixed steady state is smaller. One way to interpret a higher threshold is that it corresponds to a reduction in the amount of horizontal differentiation between the two goods.[7] This suggests that as the goods become closer substitutes, it becomes less likely that both can concurrently exist. The reason for the lack of co-existence is that this similarity tends to re-enforce the dynamics that lead to one or the other of the extremes.

A.4 Proof of Theorem A1

The dynamics of the system are described by:

$$\begin{aligned} x_t &= \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} (1 - w_{n,t-1})^{k-j} (w_{n,t-1})^j \\ y_t &= \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} (1 - w_{m,t-1})^{k-j} (w_{m,t-1})^j \end{aligned}$$

where

$$\begin{aligned} w_{n,t-1} &= \alpha_n (1 - x_{t-1}) + (1 - \alpha_n) [(1 - \rho)(1 - x_{t-1}) + \rho y_{t-1}] \\ w_{m,t-1} &= \alpha_m (1 - y_{t-1}) + (1 - \alpha_m) [(1 - \rho) x_{t-1} + \rho (1 - y_{t-1})] \end{aligned}$$

[7]For example, the threshold for a type-N individual to adopt a type-M good is $k - d$, and so the difference in the thresholds for each type to adopt the type-M good is given by $k-2d$, which is decreasing in d.

substituting in $w_{n,t-1}, w_{m,t-1}$ into the right-hand side and defining

$$\begin{aligned} f_x\left(x_{t-1}, y_{t-1}\right) &= \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1-\left(\alpha_n\left(1-x_{t-1}\right)+\left(1-\alpha_n\right)\left[\left(1-\rho\right)\left(1-x_{t-1}\right)+\rho y_{t-1}\right]\right)\right)^{k-j} \\ &\quad \times \left(\alpha_n\left(1-x_{t-1}\right)+\left(1-\alpha_n\right)\left[\left(1-\rho\right)\left(1-x_{t-1}\right)+\rho y_{t-1}\right]\right)^j \\ f_y\left(x_{t-1}, y_{t-1}\right) &= \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1-\left(\alpha_m\left(1-y_{t-1}\right)+\left(1-\alpha_m\right)\left[\left(1-\rho\right)x_{t-1}+\rho\left(1-y_{t-1}\right)\right]\right)\right)^{k-j} \\ &\quad \times \left(\alpha_m\left(1-y_{t-1}\right)+\left(1-\alpha_m\right)\left[\left(1-\rho\right)x_{t-1}+\rho\left(1-y_{t-1}\right)\right]\right)^j \end{aligned}$$

we can write this system as

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} f_x\left(x_{t-1}, y_{t-1}\right) \\ f_y\left(x_{t-1}, y_{t-1}\right) \end{bmatrix} = f\left(x_{t-1}, y_{t-1}\right)$$

where we note that $f_j\left(x_{t-1}, y_{t-1}\right)$ for $j = m, n$ defines a $C1$ map $R^2 \to R^2$ and define the Jacobian matrix of the system by $A:$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_x}{\partial x_{t-1}} & \frac{\partial f_x}{\partial y_{t-1}} \\ \frac{\partial f_y}{\partial x_{t-1}} & \frac{\partial f_y}{\partial y_{t-1}} \end{bmatrix}$$

Lemma A7. *The sign of the partial derivatives of the mapping f are*

$$\begin{aligned} \frac{\partial f_x}{\partial x_{t-1}} &\geq 0 \\ \frac{\partial f_x}{\partial y_{t-1}} &\leq 0 \\ \frac{\partial f_y}{\partial x_{t-1}} &\leq 0 \\ \frac{\partial f_y}{\partial y_{t-1}} &\geq 0 \end{aligned}$$

where a sufficient condition for the inequality to be strict is $w_{n,t-1}, w_{m,t-1} \neq 0, 1$.

Proof. First, we present a useful result for the first derivative of a function of the form:

$$\begin{aligned} f\left(d\left(k\right), w\right) &= \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1-w\right)^{k-j} \left(w\right)^j \\ &= \sum_k p_k \sum_{j=0}^{d(k)-1} \frac{k!}{(k-j)!j!} \left(1-w\right)^{k-j} \left(w\right)^j \end{aligned}$$

Taking the first derivative we find:

$$\frac{\partial f}{\partial w}=\sum_k p_k\sum_{j=0}^{d(k)-1}\frac{k!}{(k-j)!\,(j-1)!}\,(1-w)^{k-j}\,(w)^{j-1}-\frac{k!}{(k-j-1)!j!}\,(1-w)^{k-j-1}\,(w)^{j}$$

Consider two consecutive terms in the sum $j=z,z+1$. For $j=z$

$$\frac{k!}{(k--1)!\,(z-1)!}\,(1-w)^{k-z}\,(w)^{z-1}-\frac{k!}{(k-z-1)!z!}\,(1-w)^{k-z-1}\,(w)^{z}$$

and for $j=z+1$

$$\frac{k!}{(k-z-1)!\,(z)!}\,(1-w)^{k-z-1}\,(w)^{z}-\frac{k!}{(k-z)!\,(z+1)!}\,(1-w)^{k-z}\,(w)^{z+1}$$

where the second term for $j=z$ cancels out the first term for $j=z+1$. Hence, the evaluation of the summation

$$\sum_{j=0}^{d(k)-1}\frac{k!}{(k-j)!\,(j-1)!}\,(1-w)^{k-j}\,(w)^{j-1}-\frac{k!}{(k-j-1)!j!}\,(1-w)^{k-j-1}\,(w)^{j}$$

results in only second term for the upper limit $j=d(k)-1$, note that the first term for the lower limit is 0. The derivative is given by:

$$\frac{\partial f}{\partial w}\,(d(k)\,,w)=\sum_k p_k\left(-\frac{k!}{(k-d(k))!\,(d(k)-1)!}\,(1-w)^{k-d(k)}\,(w)^{d(k)-1}\right)$$

Using the chain rule we evaluate $\frac{\partial f_x}{\partial x_{t-1}}=\frac{\partial f_x}{\partial w_{n,t-1}}\frac{\partial w_{n,t-1}}{\partial x_{t-1}}$ and $\frac{\partial f_x}{\partial y_{t-1}}=\frac{\partial f_x}{\partial w_{n,t-1}}\frac{\partial w_{n,t-1}}{\partial y_{t-1}}$ where the above result can be used to evaluate

$$\frac{\partial f_x}{\partial w_{n,t-1}}=\sum_k p_k\left(-\frac{k!}{(k-d(k))!\,(d(k)-1)!}\,(1-w_{n,t-1})^{k-d(k)}\,(w_{n,t-1})^{d(k)-1}\right)$$

and we can readily observe that:

$$\begin{aligned}\frac{\partial w_{n,t-1}}{\partial x_{t-1}}&=-\alpha_n-(1-\alpha_n)\,(1-\rho)<0\\ \frac{\partial w_{n,t-1}}{\partial y_{t-1}}&=(1-\alpha_n)\,\rho>0\end{aligned}$$

Thus, for $0 < w_{n,t-1} < 1$ $\frac{\partial f_x}{\partial w_{n,t-1}} < 0$ and hence

$$\begin{aligned}\frac{\partial f_x}{\partial x_{t-1}} &> 0\\ \frac{\partial f_x}{\partial y_{t-1}} &< 0\end{aligned}$$

Similarly,

$$\frac{\partial f_y}{\partial w_{m,t-1}} = \sum_k p_k \left(-\frac{k!}{(k-d(k))!\,(d(k)-1)!} (1-w_{m,t-1})^{k-d(k)} (w_{m,t-1})^{d(k)-1} \right)$$

and

$$\begin{aligned}\frac{\partial w_{m,t-1}}{\partial x_{t-1}} &> (1-\alpha_m)(1-\rho)\\ \frac{\partial w_{m,t-1}}{\partial y_{t-1}} &< -\alpha_m - (1-\alpha_m)\rho\end{aligned}$$

so, for $0 < w_{n,t-1} < 1$ $\frac{\partial f_y}{\partial w_{m,t-1}} < 0$ and hence

$$\begin{aligned}\frac{\partial f_y}{\partial x_{t-1}} &< 0\\ \frac{\partial f_y}{\partial y_{t-1}} &> 0\end{aligned}$$

This completes the proof. □

Each element (see derivation in Lemma A7) is given by

$$\begin{aligned}\frac{\partial f_x}{\partial x_{t-1}} &= (\alpha_n + (1-\alpha_n)(1-\rho)) \sum_k p_k \left(-\frac{k!}{(k-d(k)-1)!d(k)!} (1-w_n)^{k-d(k)} (w_n)^{d(k)-1} \right) \geq 0\\ \frac{\partial f_x}{\partial y_{t-1}} &= -((1-\alpha_n)\rho) \sum_k p_k \left(-\frac{k!}{(k-d(k)-1)!d!} (1-w_n)^{k-d(k)} (w_n)^{d(k)-1} \right) \leq 0\\ \frac{\partial f_y}{\partial x_{t-1}} &= -(1-\alpha_m)(1-\rho) \sum_k p_k \left(-\frac{k!}{(k-d(k)-1)!d(k)!} (1-w_m)^{k-d(k)} (w_m)^{d(k)-1} \right) \leq 0\\ \frac{\partial f_y}{\partial x_{t-1}} &= (\alpha_m + (1-\alpha_m)\rho) \sum_k p_k \left(-\frac{k!}{(k-d(k)-1)!d(k)!} (1-w_m)^{k-d(k)} (w_m)^{d(k)-1} \right) \geq 0\end{aligned}$$

where the inequalities are strict for $0 < w_j < 1$ $j = m, n$.

A useful result is Theorem 4.11 on page 221 in Elaydi (2007),[8] which is reproduced here:

[8]Elaydi, Saber N. (2007). *Discrete Chaos: With Applications in Science and Engineering.* Second Edition, London: Chapman & Hall / CRC.

Theorem 4.11 (Elaydi (2007)) *Let $f : G \subset R^2 \to R^2$ be a $C1$ map, where G is an open subset of R^2, X^* is a fixed point of f, and $A = Df(X^*)$. Then the following statements hold true:*

1. *If $\rho(A) < 1$, then X^* is asymptotically stable.*
2. *If $\rho(A) > 1$, then X^* is unstable.*
3. *If $\rho(A) = 1$, then X^* may or may not be stable*

Where $\rho(A) = \max\{|\lambda_1|, |\lambda_2|\}$, and f and A are defined as they are above. Hence, if $\rho(A) < 1$ (> 1) at a steady state $X^* = (x^*, y^*)$ of our system then the steady state is asymptotically stable (unstable).

Lemma A8. *If* $\max\{a_{11}, a_{22}\} > 1$ *then* $\rho(A) > 1$. *Otherwise, suppose* $\max\{a_{11}, a_{22}\} \leq 1$ *and* $(1 - a_{22})(1 - a_{11}) > a_{12}a_{21}$ *(*$< a_{12}a_{21}$*) then* $\rho(A) < 1$ (> 1).

Proof. The characteristic equation for the 2×2 matrix A is

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

with roots λ_1, λ_2

$$\begin{aligned}
\lambda_1 &= \frac{(a_{11} + a_{22}) + \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2} \\
\lambda_2 &= \frac{(a_{11} + a_{22}) - \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}
\end{aligned}$$

We know from lemma A7 that $a_{11}, a_{22} > 0$ and $a_{12}, a_{21} < 0$, hence, both roots are real and $\rho(A) = \lambda_1$. Now, note that if $a_{11}, a_{22} > 1$ then this immediately implies $\lambda_1 > 1$. Now, suppose $\max\{a_{11}, a_{22}\} \leq 1$ and that

$$\begin{aligned}
(1 - a_{22})(1 - a_{11}) &> a_{12}a_{21} \\
1 - (a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) &> 0 \\
(a_{11} + a_{22})^2 + 4(1 - (a_{11} + a_{22})) &> (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) \\
(2 - (a_{11} + a_{22}))^2 &> (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})
\end{aligned}$$

which implies that

$$\begin{aligned}
2 - (a_{11} + a_{22}) &> \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})} \\
1 &> \frac{(a_{11} + a_{22}) + \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2} \\
1 &> \lambda_1
\end{aligned}$$

the same steps apply for the case $(1-a_{22})(1-a_{11}) < a_{12}a_{21}$ with the opposite inequality.

$\square$

Returning to the main proof. In the niche market steady state $(x^*, y^*) = (0,1)$, $w_n^* = 1$, $w_m^* = 0$ and the Jacobian is given by:

$$A(0,1) = \begin{bmatrix} (\alpha_n + (1-\alpha_n)(1-\rho)) p_1 & -(1-\alpha_n)\rho p_1 \\ -(1-\alpha_m)(1-\rho)\sum_{k=1}^{\widehat{k}} kp_k & (\alpha_m + (1-\alpha_m)\rho)\sum_{k=1}^{\widehat{k}} kp_k \end{bmatrix}$$

If $\sum_{k=1}^{\widehat{k}} kp_k < B_n$:

$$\begin{aligned} \sum_{k=1}^{\widehat{k}} kp_k &< \frac{1-(\alpha_n+(1-\alpha_n)(1-\rho))p_1}{1-(1-\alpha_m)(1-\rho)-p_1[\alpha_m(1-\rho)+\alpha_n\rho]} \\ \sum_{k=1}^{\widehat{k}} kp_k &< \frac{1-(\alpha_n+(1-\alpha_n)(1-\rho))p_1}{\rho+\alpha_m-\rho\alpha_m-\alpha_m p_1+\rho\alpha_m p_1-\rho\alpha_n p_1} \\ \sum_{k=1}^{\widehat{k}} kp_k\left[(1-\alpha_m)(1-\rho)(1-\alpha_n)\rho p_1\right] &< \begin{array}{c}[1-(\alpha_n+(1-\alpha_n)(1-\rho))p_1] \\ \times\left[1-(\alpha_m+(1-\alpha_m)\rho)\sum_{k=1}^{\widehat{k}} kp_k\right]\end{array} \\ a_{12}a_{21} &< (1-a_{11})(1-a_{22}) \end{aligned}$$

and we conclude by lemma A8 above that $\rho(A) < 1$ and so (using Theorem 4.11 above) that the steady state is aymptotically stable. Similarly the same steps apply for the case where $\sum_{k=1}^{\widehat{k}} kp_k > B_n$ to find that $\rho(A) > 1$ and the steady state is unstable.

In the mass market steady state $(x^*, y^*) = (1,0)$, $w_n^* = 0$, $w_m^* = 1$ and the Jacobian is given by:

$$A(1,0) = \begin{bmatrix} (\alpha_n + (1-\alpha_n)(1-\rho))\sum_{k=1}^{\widehat{k}} kp_k & -(1-\alpha_n)\rho\sum_{k=1}^{\widehat{k}} kp_k \\ -(\alpha_m + (1-\alpha_m)\rho) p_1 & (1-\alpha_m)(1-\rho) p_1 \end{bmatrix}$$

We note that the niche market condition is the same as the mass market condition with the subscripts m, n switched and parameter ρ replaced by $1-\rho$. Then, applying identical steps to the above we establish the result.

Define the steady state relations $x^*(y)$ and $y^*(x)$ by:

$$\begin{aligned} x^* &= f_x(x^*, y) \\ y^* &= f_y(x, y^*) \end{aligned}$$

In the above $f_x(x^*, y)$ is a continuous function that is increasing in x^* and decreasing in y and $f_x(0,1) = 0$; $f_x(1,0) = 1$. Hence, for each $y \in [0,1]$ there exists a non-empty set $x^*(y) = \{x : 0 \leq x \leq 1; x = f_x(x,y)\}$.

Define $\underline{x}^*(0) = \min\{x : 0 \le x \le 1; x = f_x(x,0)\}$ and $\overline{x}^*(1) = \max\{x : 0 \le x \le 1; x = f_x(x,1)\}$. By virtue of $\underline{x}^*(0)$ being the minimum value of x that satisfies $x = f_x(x,0)$ and $f_x(0,0) > 0$ we conclude that $f_x(x,0) > x$ for all $x < \underline{x}^*(0)$, and by virtue of $\overline{x}^*(1)$ being the maximum value of x that satisfies $x = f_x(x,1)$ and $f_x(0,1) < 0$ we conclude that $f_x(x,0) < x$ for all $x > \overline{x}^*(1)$. Now, $\frac{\partial f_x(x^*,y)}{\partial y} < 0$ for $0 < y < 1$ so $f_x(\underline{x}^*(0), y) > \underline{x}^*(0)$ and $f_x(\overline{x}^*(1), y) < \overline{x}^*(1)$ and by the continuity of $f_x(x,y)$ in x and y and $\frac{\partial f_x(x^*,y)}{\partial y} < 0$ then for every $x \in [\overline{x}^*(1), \underline{x}^*(0)]$ there is a unique $0 \le y \le 1$ such that $x = f_x(x,y)$.

Define a function $g(x) : [\overline{x}^*(1), \underline{x}^*(0)] \to [0,1]$ as the solution y to $x = f_x(x,y)$ for each $x \in [\overline{x}^*(1), \underline{x}^*(0)]$. The function $g(x)$ is continuous in x on $[\overline{x}^*(1), \underline{x}^*(0)]$ and by definition $g(\overline{x}^*(1)) = 1, g(\underline{x}^*(0)) = 0$. We can (similarly to $g(x)$) define a function $h(y)$ using $f_y(x, y^*)$ that is continuous on $\left[\overline{y}^*(1), \underline{y}^*(0)\right]$ and $h(\overline{y}^*(1)) = 1, h\left(\underline{y}^*(0)\right) = 0$ where the quantities are defined as they are above with x, y interchanged.

By definition any interior point (x^*, y^*) such that $(x^*, g(x^*)) = (h(y^*), y^*)$ is a steady state of our system. We know that the points $(\overline{x}^*(1), 1)$ and $(\underline{x}^*(0), 0)$ lie on $g(x)$ and are connected by a continuous function in x and similarly the points $\left(0, \underline{y}^*(0)\right)$ and $(1, \overline{y}^*(1))$ lie on $h(y)$ and are connected by a continuous function in y. Hence, the graphs $(x^*, g(x^*))$ and $(h(y^*), y^*)$ are guaranteed to intersect on the interior of $[0,1] \times [0,1]$ provided that they do not intersect on the boundary at $(0,1)$ or $(1,0)$. In the event that one or both do intersect on the boundary then a sufficient condition to guarantee that they will intersect on the interior is that the orientation of the slope at the exterior points is such that

$$\frac{dg}{dx}\frac{dh}{dy} > 1$$

where $\frac{dg}{dy} = \frac{a_{12}}{1-a_{11}}, \frac{dh}{dx} = \frac{a_{21}}{1-a_{22}}$. Hence, a sufficient condition for the existence of an interior steady state is that each extreme steady state is unstable $\sum_{k=1}^{\widehat{k}} kp_k > B_m, B_n$, such that in the neighborhood of $(0,1)$ $g(x) > h^{-1}(x)$ and in the neighborhood of $(1,0)$ $g(x) < h^{-1}(x)$.

Finally, we establish our result by showing that, under this condition, there is at least one interior steady state where $\max\{a_{11}, a_{22}\} < 1$ and

$$\frac{dg}{dx}\frac{dh}{dy} < 1$$

implying that

$$a_{12}a_{21} < (1 - a_{11})(1 - a_{22})$$

In particular, the graphs $(x, g(x))$ and $(h(y), y)$ cross at least once in $[\overline{x}^*(1), \underline{x}^*(0)] \times \left[\overline{y}^*(1), \underline{y}^*(0)\right]$. By virtue of $g(x), h(y)$ being functions (i.e. single valued) then if there is a single point of crossing $\frac{dg}{dx}\frac{dh}{dy} < 1$ and $\frac{dg}{dx}, \frac{dh}{dy} < 0$ or in the case where there are multiple points of crossing then at least one has the property $\frac{dg}{dx}\frac{dh}{dy} < 1$. Hence, by lemma A8 that steady state is asymptotically stable.

B Mean-preserving spread (MPS) results

Lemma B9. *The threshold B weakly decreases in response to a MPS.*

Proof. First, B only depends on the degree distribution through p_1. Second, p_1 weakly increases in response to a MPS, while B decreases w.r.t. p_1, which can be verified by differentiating B w.r.t. p_1:

$$\frac{\partial B}{\partial p_1} = -B^2 \frac{(1-\alpha)^2(1-\rho)\rho}{[1-p_1(\alpha+(1-\alpha)\rho)]^2} < 0.$$

Hence, B weakly decreases in response to a MPS. This completes the proof. □

C Influencers (Section 5)

C.1 Derivation of the first-order conditions for "influencers"

The steady state conditions are equal to:

$$m = 1 - (1-e_M)f(z_M), \tag{C.1}$$

$$n = 1 - (1-e_N)f(z_N), \tag{C.2}$$

where z_M and z_N are given by

$$z_M := \alpha(1-m) + (1-\alpha)(1-h), \tag{C.3}$$

$$z_N := \alpha(1-n) + (1-\alpha)h, \tag{C.4}$$

while h is given by

$$h := \rho m + (1-\rho)(1-n). \tag{C.5}$$

The FOC of the mass-market influencer is given by:

$$\frac{\partial h^*(e_M, e_N)}{\partial e_M} = \rho C'(\rho e_M). \tag{C.6}$$

while, the FOC of the niche-market influencer is equal to:

$$\frac{\partial\left[1-h^*(e_M,e_N)\right]}{\partial e_M}=(1-\rho)C'\left((1-\rho)e_N\right). \tag{C.7}$$

To state the FOCs in an operational form, we need to find the expressions for $\frac{\partial h^*(e_M,e_N)}{\partial e_M}$ and $\frac{\partial[1-h^*(e_M,e_N)]}{\partial e_M}$.

We start with deriving $\frac{\partial h^*(e_M,e_N)}{\partial e_M}$. By differentiating both sides of the steady-state condition (C.1) w.r.t. e_M, we get:

$$\frac{\partial m}{\partial e_M}=f(z_M)-(1-e_M)f'(z_M)\frac{\partial z_M}{\partial e_M}. \tag{C.8}$$

To find $\frac{\partial z_M}{\partial e_M}$, let us differentiate both sides of (C.3) w.r.t. e_M. This yields:

$$\frac{\partial z_M}{\partial e_M}=-\alpha\frac{\partial m}{\partial e_M}-(1-\alpha)\frac{\partial h}{\partial e_M}. \tag{C.9}$$

From plugging (C.9) into (C.8),

$$\frac{\partial m}{\partial e_M}=f(z_M)+(1-e_M)f'(z_M)\left(\alpha\frac{\partial m}{\partial e_M}+(1-\alpha)\frac{\partial h}{\partial e_M}\right),$$

hence,

$$\frac{\partial m}{\partial e_M}=\frac{f(z_M)}{1-\alpha(1-e_M)f'(z_M)}+\frac{(1-\alpha)(1-e_M)f'(z_M)}{1-\alpha(1-e_M)f'(z_M)}\frac{\partial h}{\partial e_M}. \tag{C.10}$$

Similarly, from differentiating both sides of the steady-state condition (C.2) w.r.t. e_M,

$$\frac{\partial n}{e_M}=-(1-e_N)f'(z_N)\frac{\partial z_N}{\partial e_M}. \tag{C.11}$$

To find $\frac{\partial z_N}{\partial e_M}$, let us differentiate both sides of (C.4) w.r.t. e_M. This yields:

$$\frac{\partial z_N}{\partial e_M}=-\alpha\frac{\partial n}{e_M}+(1-\alpha)\frac{\partial h}{\partial e_M}. \tag{C.12}$$

From plugging (C.12) into (C.11),

$$\frac{\partial n}{e_M}=(1-e_N)f'(z_N)\left(\alpha\frac{\partial n}{e_M}-(1-\alpha)\frac{\partial h}{\partial e_M}\right),$$

hence,

$$\frac{\partial n}{e_M}=-\frac{(1-\alpha)(1-e_N)f'(z_N)}{1-\alpha(1-e_N)f'(z_N)}\frac{\partial h}{\partial e_M}. \tag{C.13}$$

Let us now differentiate both sides of (C.5) w.r.t. e_M. This yields:

$$\frac{\partial h}{\partial e_M}=\rho\frac{\partial m}{\partial e_M}-(1-\rho)\frac{\partial n}{\partial e_M}. \tag{C.14}$$

Multiplying (C.10) by ρ and subtracting from it (C.10) multiplied by $1-\rho$, we get:

$$\frac{\partial h}{\partial e_M} = \rho\frac{f(z_M)}{1-\alpha(1-e_M)f'(z_M)} + (1-\alpha)\left[\rho\frac{(1-e_M)f'(z_M)}{1-\alpha(1-e_M)f'(z_M)} + (1-\rho)\frac{(1-e_N)f'(z_N)}{1-\alpha(1-e_N)f'(z_N)}\right]\frac{\partial h}{\partial e_M},$$

hence,

$$\frac{\partial h}{\partial e_M} = \frac{\rho\frac{f(z_M)}{1-\alpha(1-e_M)f'(z_M)}}{1-(1-\alpha)\left[\rho\frac{(1-e_M)f'(z_M)}{1-\alpha(1-e_M)f'(z_M)} + (1-\rho)\frac{(1-e_N)f'(z_N)}{1-\alpha(1-e_N)f'(z_N)}\right]}.$$

From this, multiplying both sides by $(1-\alpha(1-e_M)f'(z_M))\,(1-\alpha(1-e_N)f'(z_N))$, we get:

$$\frac{\partial h^*(e_M, e_N)}{\partial e_M} = \rho\frac{f(z_M)}{\Delta}\left[1-\alpha(1-e_N)f'(z_N)\right],$$

where

$$\begin{aligned}\Delta &:= (1-\alpha(1-e_M)f'(z_M))\,(1-\alpha(1-e_N)f'(z_N)) \\ &- (1-\alpha)\left[\rho(1-e_M)f'(z_M)\,(1-\alpha(1-e_M)f'(z_M)) + (1-\rho)(1-e_N)f'(z_N)\,(1-\alpha(1-e_N)f'(z_N\end{aligned}$$

or, after simplifications,

$$\begin{aligned}\Delta &= \alpha\,(1-(1-e_M)f'(z_M))\,(1-(1-e_N)f'(z_N)) \\ &+ (1-\alpha)\left[1-\rho(1-e_M)f'(z_M)-(1-\rho)(1-e_N)f'(z_N)\right].\end{aligned} \tag{C.15}$$

Hence, the mass-market influencer's FOC (C.6) becomes

$$\frac{f(z_M)}{\Delta}\left[1-\alpha(1-e_N)f'(z_N)\right] = C'(\rho e_M),$$

where Δ is given by (C.15).

Let us now derive $\frac{\partial[1-h^*(e_M,e_N)]}{\partial e_N}$. By differentiating both sides of the steady state condition (C.1), we get:

$$\frac{\partial m}{\partial e_N} = -(1-e_M)f'(z_M)\frac{\partial z_M}{\partial e_N}. \tag{C.16}$$

To find $\frac{\partial z_M}{\partial e_N}$, let us differentiate both sides of (C.3) w.r.t. e_M. This yields:

$$\frac{\partial z_M}{\partial e_N} = -\alpha\frac{\partial m}{\partial e_N} - (1-\alpha)\frac{\partial h}{\partial e_N}. \tag{C.17}$$

From plugging (C.17) into (C.16),

$$\frac{\partial m}{\partial e_N} = (1-e_M)f'(z_M)\left[\alpha\frac{\partial m}{\partial e_N} + (1-\alpha)\frac{\partial h}{\partial e_N}\right],$$

hence,

$$\frac{\partial m}{\partial e_N} = \frac{(1-\alpha)(1-e_M)f'(z_M)}{1-\alpha(1-e_M)f'(z_M)}\frac{\partial h}{\partial e_N}. \tag{C.18}$$

Similarly, from differentiating both sides of the steady-state condition (C.2) w.r.t. e_N,

$$\frac{\partial n}{\partial e_N} = f(z_N) - (1-e_N)f'(z_N)\frac{\partial z_N}{\partial e_N}. \tag{C.19}$$

To find $\frac{\partial z_N}{\partial e_N}$, let us differentiate both sides of (C.4) w.r.t. e_N. This yields:

$$\frac{\partial z_N}{\partial e_N} = -\alpha\frac{\partial n}{e_N} + (1-\alpha)\frac{\partial h}{\partial e_N}. \tag{C.20}$$

From plugging (C.20) into (C.19),

$$\frac{\partial n}{\partial e_N} = f(z_N) + (1-e_N)f'(z_N)\left[\alpha\frac{\partial n}{e_N} - (1-\alpha)\frac{\partial h}{\partial e_N}\right],$$

hence,

$$\frac{\partial n}{\partial e_N} = \frac{f(z_N)}{1-\alpha(1-e_N)f'(z_N)} - \frac{(1-\alpha)(1-e_N)f'(z_N)}{1-\alpha(1-e_N)f'(z_N)}\frac{\partial h}{\partial e_N}. \tag{C.21}$$

Let us now differentiate both sides of (C.5) w.r.t. e_N. This yields:

$$\frac{\partial h}{\partial e_N} = \rho\frac{\partial m}{\partial e_N} - (1-\rho)\frac{\partial n}{\partial e_N}. \tag{C.22}$$

Multiplying (C.10) by ρ and subtracting from it (C.10) multiplied by $1-\rho$, and using (C.19), we get:

$$\frac{\partial h}{\partial e_N} = \frac{-(1-\rho)f(z_N)}{1-\alpha(1-e_N)f'(z_N)} + \left[\rho\frac{(1-\alpha)(1-e_M)f'(z_M)}{1-\alpha(1-e_M)f'(z_M)} + (1-\rho)\frac{(1-\alpha)(1-e_N)f'(z_N)}{1-\alpha(1-e_N)f'(z_N)}\right]\frac{\partial h}{\partial e_N},$$

hence,

$$\frac{\partial h}{\partial e_N} = \frac{\frac{-(1-\rho)f(z_N)}{1-\alpha(1-e_N)f'(z_N)}}{1-(1-\alpha)\left[\rho\frac{(1-e_M)f'(z_M)}{1-\alpha(1-e_M)f'(z_M)} + (1-\rho)\frac{(1-e_N)f'(z_N)}{1-\alpha(1-e_N)f'(z_N)}\right]},$$

multiplying both sides of which by $(1-\alpha(1-e_M)f'(z_M))\,(1-\alpha(1-e_N)f'(z_N))$, we get:

$$\frac{\partial h^*(e_M, e_N)}{\partial e_N} = -(1-\rho)\frac{f(z_N)}{\Delta}\left[1-\alpha(1-e_M)f'(z_M)\right],$$

where Δ is given by (C.15). Hence, the niche-market influencer's FOC takes the following form:

$$\frac{f(z_N)}{\Delta}\left[1-\alpha(1-e_M)f'(z_M)\right] = C'\left((1-\rho)e_N\right).$$

To sum up: the FOCs (24) and (5) in the main text are equal to

$$C'\left(\rho e_M\right) = \frac{f(z_M)}{\Delta}\left[1-\alpha(1-e_N)f'(z_N)\right],$$

$$C'\left((1-\rho)e_N\right) = \frac{f(z_N)}{\Delta}\left[1-\alpha(1-e_M)f'(z_M)\right],$$

where

$$\begin{aligned}\Delta &= \alpha\left[1-(1-e_M^*)f'(z_M^*)\right]\left[1-(1-e_N^*)f'(z_N^*)\right]\\ &+ (1-\alpha)\left[1-\rho(1-e_M^*)f'(z_M^*)-(1-\rho)(1-e_N^*)f'(z_N^*)\right].\end{aligned}$$

C.2 Each influencer has an impact on both markets

Let us demonstrate that each influencer has an impact on both markets. The steady-state equilibrium $\left(m^*\left(e_M,e_N\right),n^*\left(e_M,e_N\right)\right)$ is defined by the steady-state conditions:

$$m = 1-(1-e_M)f(z_M),$$

$$n = 1-(1-e_N)f(z_N),$$

where z_M and z_N are given by

$$z_M := \left[\alpha+(1-\alpha)\rho\right](1-m)+(1-\alpha)(1-\rho)n,$$

$$z_N := (1-\alpha)\rho m+\left[\alpha+(1-\alpha)(1-\rho)\right](1-n).$$

To show that each influencer affects both markets, let show that the cross derivatives,

$$\frac{\partial m^*\left(e_M,e_N\right)}{\partial e_N} \quad \text{and} \quad \frac{\partial n^*\left(e_M,e_N\right)}{\partial e_M},$$

are non-zero. Differentiating both sides of the steady-state conditions yields the Jacobian of the equilibrium mapping $(e_M,e_N)\to\left(m^*\left(e_M,e_N\right),n^*\left(e_M,e_N\right)\right)$:

$$\begin{pmatrix}\frac{\partial m}{\partial e_M} & \frac{\partial m}{\partial e_N}\\ \frac{\partial n}{\partial e_M} & \frac{\partial n}{\partial e_N}\end{pmatrix} = \begin{pmatrix} f(z_M) & 0\\ 0 & f(z_N)\end{pmatrix} -$$

$$-\begin{pmatrix}(1-e_M)f'(z_M) & 0\\ 0 & (1-e_N)f'(z_N)\end{pmatrix}\begin{pmatrix}\frac{\partial z_M}{\partial m} & \frac{\partial z_M}{\partial n}\\ \frac{\partial z_N}{\partial m} & \frac{\partial z_N}{\partial n}\end{pmatrix}\begin{pmatrix}\frac{\partial m}{\partial e_M} & \frac{\partial m}{\partial e_N}\\ \frac{\partial n}{\partial e_M} & \frac{\partial n}{\partial e_N}\end{pmatrix}.$$

From the expressions for z_M and z_N,

$$\begin{pmatrix} \frac{\partial z_M}{\partial m} & \frac{\partial z_M}{\partial n} \\ \frac{\partial z_N}{\partial m} & \frac{\partial z_N}{\partial n} \end{pmatrix} = -\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - (1-\alpha) \begin{pmatrix} \rho & -(1-\rho) \\ -\rho & 1-\rho \end{pmatrix},$$

which is a non-diagonal matrix. Hence, the Jacobian of the equilibrium mapping is non-diagonal either. Indeed, solving for the Jacobian, we get

$$\begin{pmatrix} \frac{\partial m}{\partial e_M} & \frac{\partial m}{\partial e_N} \\ \frac{\partial n}{\partial e_M} & \frac{\partial n}{\partial e_N} \end{pmatrix} =$$

$$= \frac{1}{\Delta} \begin{pmatrix} 1 - [\alpha + (1-\alpha)(1-\rho)]\,(1-e_N) f'(z_N) f(z_M) & -(1-\alpha)(1-\rho)(1-e_M) f'(z_M) f(z_N) \\ -(1-\alpha)\rho(1-e_N) f'(z_N) f(z_M) & 1 - [\alpha + (1-\alpha)\rho]\,(1-e_M) f'(z_M) f(z_N) \end{pmatrix},$$

where

$$\Delta := \det \begin{pmatrix} 1 - [\alpha + (1-\alpha)\rho]\,(1-e_M) f'(z_M) f(z_N) & (1-\alpha)(1-\rho)(1-e_M) f'(z_M) f(z_N) \\ (1-\alpha)\rho(1-e_N) f'(z_N) f(z_M) & 1 - [\alpha + (1-\alpha)(1-\rho)]\,(1-e_N) f'(z_N) f(z_M) \end{pmatrix}.$$

Hence, the cross-derivatives are given by:

$$\frac{\partial m^*(e_M, e_N)}{\partial e_N} = -(1-\alpha)(1-\rho)(1-e_M) f'(z_M) \frac{f(z_N)}{\Delta} \neq 0;$$

$$\frac{\partial n^*(e_M, e_N)}{\partial e_M} = -(1-\alpha)\rho(1-e_N) f'(z_N) \frac{f(z_M)}{\Delta} \neq 0.$$

This demonstrates that each influencer has an impact on both markets.

D Homophily: Choosing the number of connections

Consider a setting in which each agent $j \in \{M, N\}$ chooses their number of connections k_j. As above, since we have a directed network, each individual j decides, at the beginning of the period—prior to the model's analysis—their level of connection k_j, anticipating the steady-state equilibrium values of m^* and n^*.

As in Section 6, we consider a *regular* network so that the steady state conditions are given by the system (28) but with $\alpha_M = \alpha_N = \alpha$ and two distinct values of k_M and k_N, that is,

$$m^* = 1 - [\alpha(1-m^*) + (1-\alpha)(1-h)]^{k_M}, \tag{D.1}$$

$$n^* = 1 - [\alpha(1-n^*) + (1-\alpha)h]^{k_N}. \tag{D.2}$$

This uniquely determines a steady-state equilibrium $(m^*(k_M, k_N), n^*(k_M, k_N))$. As in Section 6, each mass-type individual chooses k_M that maximizes $k_M\,[\alpha m^* + (1-\alpha)h^* - D(k_M)]$

while each niche-type individual chooses k_N that maximizes $k_N\left[\alpha n^* + (1-\alpha)(1-h^*) - D(k_N)\right]$, where $\alpha_M = \alpha_N = \alpha$. Denote

$$z_M(m^*, n^*) = \left[\alpha + (1-\alpha)\rho\right](1-m^*) + (1-\alpha)(1-\rho)n^*,$$

$$z_N(m^*, n^*) = (1-\alpha)\rho m^* + \left[\alpha + (1-\alpha)(1-\rho)\right](1-n^*).$$

Assume that $D(k_j) = \frac{\beta}{2}k_j^2$, where $\beta > 0$ is the marginal cost. Then, for a type $j \in \{M, N\}$ agent, this maximization program is equivalent to

$$\max_{k_j}\left[k_j(1 - z_j(m^*, n^*)) - \frac{\beta}{2}k_j^2\right].$$

Since each agent's individual choice is negligible to the market, it does not affect the equilibrium value of $z_j(m^*, n^*)$. Hence, the j-type agent's first-order condition is given by:

$$1 - z_j(m^*, n^*) = \beta k_j. \tag{D.3}$$

Clearly, the choice of k_j will be the same across all agents within the same type (but possibly different across types).

Lemma D10. *The equilibrium values k_M^* and k_N^* of connection number choices satisfy the following equilibrium conditions:*

$$\beta k_M^* = 1 - \beta k_N^* + \frac{\alpha}{(1-\rho)(1-\alpha)}\left[1 - \beta k_N^* - (1-\beta k_N^*)^{k_N^*}\right], \tag{D.4}$$

$$\beta k_N^* = 1 - \beta k_M^* + \frac{\alpha}{\rho(1-\alpha)}\left[1 - \beta k_M^* - (1-\beta k_M^*)^{k_M^*}\right]. \tag{D.5}$$

Proof. Since k_M is the same for all mass-market agents, and k_N is the same for all niche market agents, the equilibrium conditions take the form:

$$m = 1 - z_M^{k_M}, \tag{D.6}$$

$$n = 1 - z_N^{k_N}, \tag{D.7}$$

where z_M and z_N are given by

$$z_M = \left[\alpha + (1-\alpha)\rho\right](1-m) + (1-\alpha)(1-\rho)n;$$

$$z_N = (1-\alpha)\rho m + \left[\alpha + (1-\alpha)(1-\rho)\right](1-n).$$

Or, equivalently,

$$1 - z_M = [\alpha + (1-\alpha)\rho]\, m + (1-\alpha)(1-\rho)(1-n)$$

$$1 - z_N = (1-\alpha)\rho(1-m) + [\alpha + (1-\alpha)(1-\rho)]\, n.$$

From combining the equilibrium conditions (D.6)-(D.7) with the FOCs (D.3), we obtain the equilibrium conditions in terms of of k_M and k_N:

$$\beta k_M = [\alpha + (1-\alpha)\rho] \left[1 - (1-\beta k_M)^{k_M}\right] + (1-\alpha)(1-\rho)\,(1-\beta k_N)^{k_N}\,, \tag{D.8}$$

$$\beta k_N = (1-\alpha)\rho\,(1-\beta k_M)^{k_M} + [\alpha + (1-\alpha)(1-\rho)] \left[1 - (1-\beta k_N)^{k_N}\right]. \tag{D.9}$$

Adding up these conditions, we get

$$\beta k_M + \beta k_N = \alpha \left[1 - (1-\beta k_M)^{k_M} + 1 - (1-\beta k_N)^{k_N}\right] + (1-\alpha).$$

Hence,

$$\alpha = \frac{\beta k_M + \beta k_N - 1}{1 - (1-\beta k_M)^{k_M} - (1-\beta k_N)^{k_N}}. \tag{D.10}$$

Plugging the expression (D.10) for α back into (D.8)-(D.9), we get after simplifications:

$$\rho\left[1 - \beta k_M - (1-\beta k_M)^{k_M}\right] = (1-\rho)\left[1 - \beta k_N - (1-\beta k_N)^{k_N}\right]. \tag{D.11}$$

Let us now restate (D.10) as follows:

$$\beta\,(k_M + k_N) = 1 + \frac{\alpha}{1-\alpha}\left[1 - \beta k_M - (1-\beta k_M)^{k_M} + 1 - \beta k_N - (1-\beta k_N)^{k_N}\right].$$

This, combined with (D.11), implies

$$\beta k_M = 1 - \beta k_N + \frac{\alpha}{(1-\rho)(1-\alpha)}\left[1 - \beta k_N - (1-\beta k_N)^{k_N}\right]$$

$$\beta k_N = 1 - \beta k_M + \frac{\alpha}{\rho(1-\alpha)}\left[1 - \beta k_M - (1-\beta k_M)^{k_M}\right].$$

This completes the proof. □

Figure A2 illustrates how the equilibrium values $k_M^*(\rho)$ and $k_N^*(\rho)$ evolve as the mass-market share ρ rises from 1/2 to 1. Parameter values are $\alpha = 0.1$ and $\beta = 0.2$. The figure shows that $k_M^*(\rho)$ increases in ρ, while $k_N^*(\rho)$ decreases in ρ. Equivalently, each type's equilibrium number of connections rises in the relative size of its preferred market: the larger the preferred market share, the higher the number of links drawn in equilibrium.

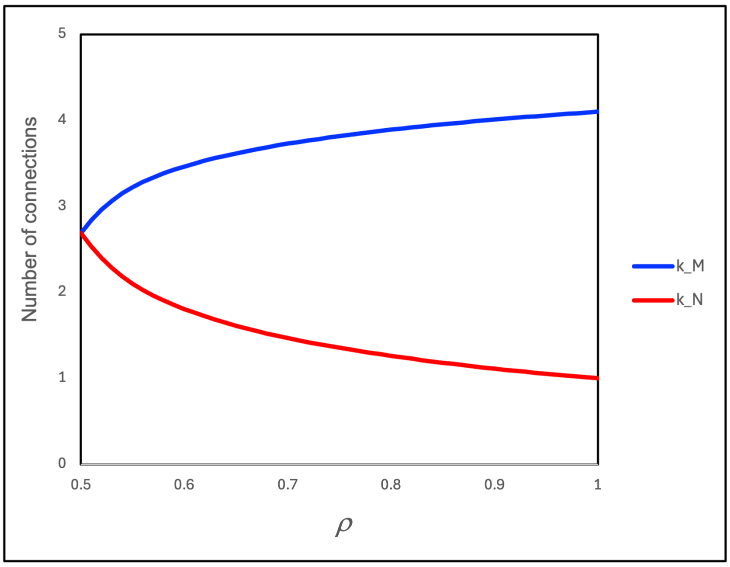

Figure A2: How equilibrium numbers of connections vary with the mass market relative size

To obtain sharper results, assume $\rho = 1/2$ and focus on a symmetric equilibrium for which $k_M^* = k_N^* = k^*$. Then, after some calculations, equations (D.4) and (D.5) reduce to

$$\beta k^* = \frac{1+\alpha}{2} - \alpha \left(1 - \beta k^*\right)^{k^*}. \tag{D.12}$$

The expression $(1 - \beta k^*)^{k^*}$ requires $1 - \beta k^* > 0$. Thus, the natural domain is $0 < k < 1/\beta$. Therefore, if β is sufficiently small, the feasible range $[0, 1/\beta]$ becomes large and equilibria with $k > 1$ become possible.

Proposition D1. *Assume $\rho = \frac{1}{2}$. Then, a stable symmetric equilibrium k^* always exists. Furthermore, there exists a threshold value $\widehat{\alpha}(\beta) \in (0,1)$, which is V-shaped in β, has a minimum value of $\widehat{\alpha}(1/2) \approx 0.59$, and satisfies $\lim_{\beta\to 0} \widehat{\alpha}(\beta) = \lim_{\beta \to 1} \widehat{\alpha}(\beta) = 1$, such that:*

(i) If $\alpha < \widehat{\alpha}(\beta)$, there exists a unique stable symmetric equilibrium k^* given by (D.12), *with*

$$\frac{\partial k^*}{\partial \alpha} \gtreqless 0 \iff \beta \lesseqgtr \frac{1}{2} \qquad \text{and} \qquad \frac{\partial k^*}{\partial \beta} < 0.$$

(ii) If $\alpha > \widehat{\alpha}(\beta)$, there exist two stable symmetric equilibria, $\underline{k}^*$ and $\overline{k}^* > \underline{k}^*$, with $\underline{k}^*$ decreasing in α and $\overline{k}^*$ increasing in α.

Proof. Denote $y = \beta k$ and let us restate the symmetric equilibrium equation (D.12) as follows:

$$y = \frac{1}{2} + \alpha \left[\frac{1}{2} - (1-y)^{\frac{y}{\beta}} \right]$$

Let us denote the RHS by $g(y, \alpha, \beta)$:

$$g(y, \alpha, \beta) := \frac{1+\alpha}{2} - \alpha \left(1-y\right)^{\frac{y}{\beta}}$$

In this proof, subscripts indicate partial derivatives. In particular, $g_y := \partial g / \partial y$. We call $y^* \in (0, 1)$ a *stable symmetric equilibrium* if y^* satisfies

$$y^* = g\left(y^*, \alpha, \beta\right) \qquad \text{and} \qquad g_y\left(y^*, \alpha, \beta\right) < 1.$$

We start by observing that

$$g(0, \alpha, \beta) = \frac{1-\alpha}{2} > 0;$$

$$g(1, \alpha, \beta) = \frac{1+\alpha}{2} < 1;$$

$$g_y(y, \alpha, \beta) = \alpha \frac{1}{\beta} \left(1-y\right)^{\frac{y}{\beta}} \left[\frac{y}{1-y} - \ln(1-y) \right] > 0, \quad \forall y \in (0, 1).$$

Thus, $g(\cdot, \alpha, \beta)$ continuously maps $[0, 1]$ into $\left[\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right] \subset [0, 1]$, hence, there exists at least one stable equilibrium $y^* \in \left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right)$.

By differentiating $g(y, \alpha, \beta)$ twice with respect to y, we obtain:

$$g_{yy}(y, \alpha, \beta) = \frac{\alpha}{\beta} \left(1-y\right)^{\frac{y}{\beta}-2} \left[2 - y - \frac{1}{\beta} \left(y - (1-y)\ln(1-y)\right)^2 \right].$$

It is readily verified that, as y increases from 0 to 1, $\left[2 - y - \frac{1}{\beta}\left(y - (1-y)\ln(1-y)\right)^2\right]$ changes its sign (from "+" to "−") only once. Hence, there exists a threshold $y_0(\alpha, \beta) \in (0, 1)$, such that $g(y, \alpha, \beta)$ is strictly convex w.r.t. y over $(0, y_0(\alpha, \beta))$, and strictly concave w.r.t. y over $(y_0(\alpha, \beta), 1)$. To sum up, $g(y, \alpha, \beta)$ is an increasing S-shaped function. Hence, two cases may arise: either there is a unique stable equilibrium y^*, or there are two stable equilibria, $\underline{y}^*$ and $\overline{y}^* > \underline{y}^*$, with an unstable equilibrium (that is, such that $g_y > 1$) in between.[9] To find the conditions for each equilibrium pattern to arise, consider the extreme cases: $\alpha = 0$ and $\alpha = 1$. When $\alpha = 0$, the RHS of the equilibrium condition becomes $g(y, 0, \beta) \equiv \frac{1}{2}$, hence $y^* = \frac{1}{2}$ is a unique equilibrium. Furthermore, it is clear that when α is positive but sufficiently close to zero, the equilibrium is unique, as

[9]There is also a zero measure case of two equilibria one of which is a tangency point of the g-function with the 45°-line.

$g_y(y, \alpha, \beta) < 1$ for all $y \in (0, 1)$. On the contrary, when $\alpha = 1$, the RHS of the equilibrium condition takes the form

$$g(y, 1, \beta) = 1 - (1 - y)^{\frac{y}{\beta}},$$

and one can readily show that, for each $\beta \in (0, 1)$, we get:

$$g(0, 1, \beta) = 0, \qquad g(1, 1, \beta) = 1,$$

$$g_y(0, 1, \beta) = 0, \qquad g_y(1, 1, \beta) = 0.$$

Hence, $\overline{y}^* = 1$ and $\underline{y}^* = 0$ are two stable equilibria (and there is an unstable equilibrium in between). By continuity, when α is sufficiently close to one, there must be two stable equilibria, $\overline{y}^*$ and $\underline{y}^*$, satisfying $\frac{1+\alpha}{2} < \overline{y}^* < \underline{y}^* < \frac{1+\alpha}{2}$, and again, with an unstable equilibrium in between. Thus, there must exist a threshold value $\widehat{\alpha}(\beta) \in (0, 1)$, such that: $\alpha < \widehat{\alpha}(\beta) \Longrightarrow$ unique stable equilibrium y^* and $\alpha > \widehat{\alpha}(\beta) \Longrightarrow$ two stable equilibria, $\underline{y}^*$ and $\overline{y}^* > \underline{y}^*$ (with an unstable equilibrium in between). We relegate the characterization of the threshold $\widehat{\alpha}(\beta)$ to the end of the proof.

Consider now the comparative statics of the stable equilibria. First, assume that $\alpha < \widehat{\alpha}(\beta)$, so that the stable equilibrium y^* is unique. Then,

$$\frac{\partial y^*}{\partial \alpha} = \frac{g_\alpha(y^*, \alpha, \beta)}{1 - g_y(y^*, \alpha, \beta)} \Longrightarrow \text{sign}\left\{\frac{\partial y^*}{\partial \alpha}\right\} = \text{sign}\left\{g_\alpha(y^*, \alpha, \beta)\right\}.$$

Furthermore, one can show that

$$g_\alpha(y^*, \alpha, \beta) \gtreqless 0 \Longleftrightarrow y^* \gtreqless \widehat{y}(\beta),$$

where $\widehat{y}(\beta)$ is the unique solution to the equation

$$(1 - y)^{\frac{y}{\beta}} = \frac{1}{2}.$$

One can show that

$$y^* \gtreqless \widehat{y}(\beta) \Longleftrightarrow \widehat{y}(\beta) \lesseqgtr \frac{1}{2} \Longleftrightarrow \beta \lesseqgtr \frac{1}{2}.$$

This proves that

$$\frac{\partial y^*}{\partial \alpha} \gtreqless 0 \Longleftrightarrow \beta \lesseqgtr \frac{1}{2}.$$

As for the case of multiple stable equilibria, which takes place when $\alpha > \widehat{\alpha}(\beta)$, one can show that

$$\underline{y}^* < \widehat{y}(\beta) < \overline{y}^*.$$

Hence,

$$\frac{\partial \underline{y}^*}{\partial \alpha} = \frac{g_\alpha\left(\underline{y}^*, \alpha, \beta\right)}{1 - g_y\left(\underline{y}^*, \alpha, \beta\right)} < 0 < \frac{g_\alpha\left(\overline{y}^*, \alpha, \beta\right)}{1 - g_y\left(\overline{y}^*, \alpha, \beta\right)} = \frac{\partial \overline{y}^*}{\partial \alpha}.$$

To establish comparative statics with respect to β, observe that, for any stable equilibrium y^* (no matter unique or not) we have

$$g_\beta\left(y^*, \alpha, \beta\right) = \frac{\alpha}{\beta^2}\left(1 - y^*\right)^{\frac{y^*}{\beta}} y^* \ln(1 - y^*) < 0.$$

$$\frac{\partial y^*}{\partial \beta} = \frac{g_\beta\left(y^*, \alpha, \beta\right)}{1 - g_y\left(y^*, \alpha, \beta\right)} \Longrightarrow \operatorname{sign}\left\{\frac{\partial y^*}{\partial \beta}\right\} = \operatorname{sign}\left\{g_\beta\left(y^*, \alpha, \beta\right)\right\} < 0.$$

Recall that $y^* = \beta k^*$. Since a reduction in β (lower marginal costs) leads to an increase in y^*, it leads to an increase in k^*.

It remains to characterize the threshold $\widehat{\alpha}(\beta)$. For that, let us restate the equilibrium condition as follows:

$$\alpha = \frac{y - \frac{1}{2}}{\frac{1}{2} - (1 - y)^{\frac{y}{\beta}}}. \tag{D.13}$$

The RHS of (D.13) is equal to 1 at $y = 0$ and at $y = 1$, and has a vertical asymptote at $y = \widehat{y}(\beta)$. Furthermore, the RHS of (D.13) has a unique interior local minimizer, $\widetilde{y}(\beta)$, which satisfies $\widetilde{y}(\beta) \gtreqless \widehat{y}(\beta) \iff \beta \gtreqless \frac{1}{2}$. The minimizer $\widetilde{y}(\beta)$ can be found as the unique root of the following equation:

$$(1 - y)^{y/\beta}\left[1 + \frac{1}{\beta}\left(y - \frac{1}{2}\right)\left(\frac{y}{1 - y} - \ln(1 - y)\right)\right] = \frac{1}{2}. \tag{D.14}$$

The threshold $\widehat{\alpha}(\beta)$ is given by the value of the RHS of (D.13) evaluated at the minimizer $\widetilde{y}(\beta)$:

$$\widehat{\alpha}(\beta) := \frac{\widetilde{y}(\beta) - \frac{1}{2}}{\frac{1}{2} - (1 - \widetilde{y}(\beta))^{\frac{\widetilde{y}(\beta)}{\beta}}}. \tag{D.15}$$

For each $\beta \in (0, 1)$, the value of $\widehat{\alpha}(\beta)$ can be found by numerically solving (D.14) and plugging the result into (D.15). Figure A3 shows how $\widehat{\alpha}(\beta)$ is obtained from varying β from 0 to 1. One can see that $\widehat{\alpha}(\beta)$ is V-shaped with respect to β and is minimized at $\beta = \frac{1}{2}$, with

$$\widehat{\alpha}(\beta) = \frac{1}{1 + \ln 2} \approx 0.59.$$

Finally, it is readily verified that $\lim_{\beta \to 0} \widetilde{y}(\beta) = 1$ and $\lim_{\beta \to 1} \widetilde{y}(\beta) = 0$. Plugging each of these values into (D.15) implies $\lim_{\beta \to 0} \widehat{\alpha}(\beta) = \lim_{\beta \to 1} \widehat{\alpha}(\beta) = 1$. This completes the proof.

□

Proposition D1 shows that when the homophily rate is not too high and is at most

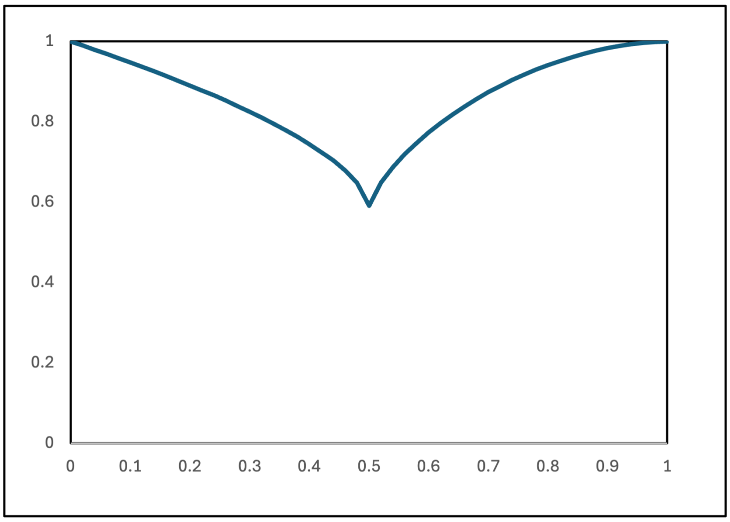

Figure A3: The V-shaped threshold $\widehat{\alpha}(\beta)$.

less than 0.59, a unique symmetric equilibrium k^* exists. In each period, every agent draws k^* contacts from the population. To obtain an interpretable value for k^*, we can use a rounding rule that discretizes equilibrium solutions to integers. The rule is as follows: for any real number $x \in \mathbb{R}$, x is rounded up when its fractional part satisfies $x - \lfloor x \rfloor \geq 0.5$, and rounded down otherwise. Then, when homophily α increases, the equilibrium number of contacts k^* increases (decreases) only if β is lower (greater) than 1/2. When $\beta < 1/2$, marginal costs are low enough that the increased expected payoff from same-type interactions dominates, so the optimal k^* rises. When $\beta > 1/2$, marginal costs increase, outweighing the homophily-induced gain, pushing k^* down.

Consider the numerical calibration $(\alpha, \beta) = (0.5, 0.2)$. Note that $0.5 = \alpha < \widehat{\alpha}_{\min}(\beta) = \widehat{\alpha}(1/2) \approx 0.59$, which lies below the minimum feasible homophily threshold. This inequality guarantees the existence of a unique symmetric interior equilibrium. Substituting (α, β) into the equilibrium mapping gives $k^* \approx 3.74$. Using the rounding rule described above, $k^* \approx 3.74$ is rounded up to $k^* = 4$, meaning that each agent draws four contacts at random from the population in every period.