ONLINE APPENDIX

Market Power and Capital Constraints

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Appendix A generalizes our benchmark model in various ways. Appendix B provides additional information about our empirics, including statistical tests, and robustness analysis. Proofs are in Appendix C. Appendix tables and figures are at the end.

A Model extensions

We provide a series of model extensions. In Appendix A.1, we introduce inventory positions. In Appendix A.2, we explain how to adjust our framework to analyze trade settings. In Appendix A.3, we illustrate how to model and analyze different types of constraints. In Appendix A.4, we show that the main prediction of our model, which is about how price and price impact depend on capital constraints, can generalize to a discriminatory price auction.

A.1 Inventory positions

Here we explain how to extend the model to account for privately observed inventory positions. To reduce the number of parameters, we shut off the trader's signal about the return of the asset, ϵ_i, by setting $\underline{\epsilon} = \overline{\epsilon} = 0$. It is straightforward to include it.

Each trader i now holds portfolio, z_i, in inventory, which was acquired at price $p_z \in \mathbb{R}^+$.[27] The inventory is drawn iid across traders from some continuous distribution on bounded support and strictly positive density. It is private information of the trader, and may include a variety of security types. In this case, price p_z represents the average per-unit price of a security in inventory; it may be a function of other model primitives: for example, the distribution of asset supply, or the number of traders.

The inventory position affects the trader's payoff, as well as the capital constraint. Trader

[27] If the trader's inventory position only consists of the asset that is for sale in the upcoming auction, we could replace p_z with the auction clearing price, $\boldsymbol{P^c}$.

i's ex-post gross utility is:

$$V(a_i^c, z_i) = \mu a_i^c + \mu_z z_i - \frac{\rho\sigma^2}{2}[a_i^c]^2 - \frac{\rho\sigma_z^2}{2}[z_i]^2 - \rho\iota\sigma\sigma_z a_i^c z_i, \tag{14}$$

with $\mu, \mu_z, \sigma, \sigma_z > 0$, and $\iota \in [-1, 1]$. When $\mu = \mu_z, \sigma = \sigma_z, \iota = 1$ the asset in inventory is identical to the asset at auction. The capital constraint is:

$$\mathbb{E}[\boldsymbol{\theta_i} - \kappa \boldsymbol{P^c a_i^c} - \kappa p_z \boldsymbol{z_i}] \geq 0. \tag{15}$$

Proposition 1 generalizes to this setting. The only difference is that $v(a, \epsilon_i) = \mu + \epsilon_i$ is replaced by $v(a, z_i) = \mu - \rho\iota\sigma_z z_i$. That is, trader i submits the same function as in the benchmark model with $\mu - \rho\iota\sigma_z z_i$ replacing $\mu + \epsilon_i$, and an adjusted Lagrange multiplier, λ.

A.2 Trade setting

In line with the empirical application our benchmark model features a one-sided auction. Here we illustrate how to adjust the setting to accommodate two-sided markets.

The easiest adjustment is to simply let demand curves in the benchmark model, $p_i(\cdot, \epsilon_i)$, map from $\mathbb{R}$ to $\mathbb{R}$ and represent net-demand. Since all traders are ex-ante identical in our benchmark model, all of them either expect to buy from noise traders, or sell to noise traders in this model adjustment. When traders expect to be net-buyers, all model predictions generalize. When they expect to be net-sellers, the capital constraint of the benchmark model needs to be adjusted to reflect the fact that taking short positions doesn't help fulfill capital requirements.

Next, we consider two additional environments, one with symmetric market conditions for buyers and sellers, and one with asymmetric conditions. For other environments, equilibria and predictions about how capital constraints affect prices and price impact, can be derived analogously.

Our trade framework is similar to the one presented in Appendix A.1, but with two groups of traders: N buyers, indexed by B, and N sellers, indexed by S. They seek to trade units of an asset of zero aggregate supply, $A = 0$. Each trader draws an iid inventory position of the asset, $\boldsymbol{z_i^G}$, from a group-specific distribution with $\mathbb{E}[\boldsymbol{z_i^B}] \leq 0$, $\mathbb{E}[\boldsymbol{z_i^S}] > 0$, and

$\mathbb{E}[\boldsymbol{z_i^B}] + \mathbb{E}[\boldsymbol{z_i^S}] > 0$. In addition, we could assume that the support of the seller's distribution is sufficiently negative to make them want to sell with certainty, and vice versa for buyers.

Ex-post, trader i of group G's gross utility is given by (14) with $\mu_z = \mu, \sigma_z = \sigma, \iota = 1$. Each trader face the same capital constraint (15), i.e.,

$$\mathbb{E}[\boldsymbol{\theta_i^G} - \kappa \boldsymbol{P^c} \boldsymbol{a_i^G} - \kappa p_z \boldsymbol{z_i^G}] \geq 0 \text{ for } G \in \{B, S\}, \tag{16}$$

with equity capital, $\boldsymbol{\theta_i^G}$, which is drawn iid from a group-specific distribution.

First, let us consider a symmetric environment, in which the capital constraint is binding in the same way for buyers and sellers in the symmetric equilibrium. For this to be the case, the distribution of capital positions and inventory positions must be such that: $\mathbb{E}[\boldsymbol{\theta_i^B} + \boldsymbol{\theta_i^S}] = \kappa p_z \mathbb{E}[\boldsymbol{z_i^B} + \boldsymbol{z_i^S}] > 0$.

Proposition 4. *Consider a symmetric setting where* $\mathbb{E}[\boldsymbol{\theta_i^B} + \boldsymbol{\theta_i^S}] = \kappa p_z \mathbb{E}[\boldsymbol{z_i^B} + \boldsymbol{z_i^S}] > 0$. *There is a symmetric equilibrium in which traders submit the following net-demand curve:*

$$p(a, z_i^G) = \frac{1}{1 + \lambda\kappa} \left(\mu - \rho\sigma^2 z_i^G - \rho\sigma^2 \left(\frac{2N-1}{2N-2} \right) a \right) \text{ and} \tag{17}$$

$$\lambda = \begin{cases} 0 & \text{if } \mathbb{E}[\boldsymbol{\theta_i^B} - \kappa p_z \boldsymbol{z_i^B}] \geq \tilde{\theta} \\ \frac{1}{\kappa} \left(\frac{\tilde{\theta}}{\mathbb{E}[\boldsymbol{\theta_i^B} - \kappa p_z \boldsymbol{z_i^B}]} - 1 \right) & \text{otherwise.} \end{cases} \tag{18}$$

with $\tilde{\theta} = \frac{\kappa(N-1)}{2N(2N-1)} \left[2N\mu(\mathbb{E}[\boldsymbol{z_i^S}] - \mathbb{E}[\boldsymbol{z_i^B}]) - \rho\sigma^2 \Big(\mathbb{E}[(\boldsymbol{z_i^S})^2] - \mathbb{E}[(\boldsymbol{z_i^B})^2] + (N-1)[\mathbb{E}[\boldsymbol{z_i^S}]^2 - \mathbb{E}[\boldsymbol{z_i^B}]^2] \Big) \right]$.
In this equilibrium, buyer i wins $a_i^{B*} = \frac{N-1}{N(2N-1)} \left(\sum_i z_i^B + \sum_i z_i^S - 2N z_i^B \right)$ and seller i wins $a_i^{S*} = \frac{N-1}{N(2N-1)} \left(\sum_i z_i^B + \sum_i z_i^S - 2N z_i^S \right)$.

The market clears at $P^* = \frac{1}{1+\lambda\kappa} \left(2N\mu - \rho\sigma^2 \left(\sum_i z_i^B + \sum_i z_i^S \right) \right)$, and each trader's price impact is $\Lambda = \frac{\rho\sigma^2}{1+\lambda\kappa} \frac{1}{2N-2}$. Therefore, it is still the case that the price and price impact increases when constraints are relaxed—Corollary 1.

Proposition 5. *Consider an asymmetric setting where* $\mathbb{E}[\boldsymbol{\theta_i^B} + \boldsymbol{\theta_i^S}] \neq \kappa p_z \mathbb{E}[\boldsymbol{z_i^B} + \boldsymbol{z_i^S}]$.
(i) In a group-symmetric equilibrium, in which buyers and sellers submit the same net-demand schedule, respectively, a trader i of group G chooses:

$$p^G(a, \epsilon_i^G) = \frac{1}{\beta^G} \left(\alpha^G + \gamma^G z_i^G - a \right) \text{ for } G \in \{B, S\}. \tag{19}$$

Parameter β^B is the root of the following polynomial:

$$
\begin{aligned}
0 =& \big(2(1+\lambda^B\kappa)^2(1+\lambda^S\kappa)2N(N-1)\big) \\
&+ \big((1+\lambda^B\kappa)((1+\lambda^B\kappa)(N-2)^2 + 2(1-2\lambda^B\kappa+3\lambda^S\kappa+(\lambda^B\kappa-2\lambda^S\kappa-1)N)N - 3(1+\lambda^S\kappa)N^2)\big)(\rho\sigma^2\beta^B)^1 \\
&+ \big((-2(1+\lambda^B\kappa)(N-2)(N-1) + (2+\lambda^B\kappa(4-3N)+2\lambda^S\kappa(N-1)-N)N + (1+\lambda^S\kappa)N^2)\big)(\rho\sigma^2\beta^B)^2 \\
&+ \big((N-1)(2N-1)\big)(\rho\sigma^2\beta^B)^3. \qquad (20)
\end{aligned}
$$

The other parameters can be expressed as functions of β^B:

$$\beta^S = \frac{\beta^B}{N}\left((N-1) + \frac{1+\lambda^B\kappa}{1+\lambda^B\kappa-\beta^B\rho\sigma^2}\right), \qquad (21)$$

$$\alpha^G = \frac{\beta^G(N-1)+\beta^{-G}N}{1+\lambda^G\kappa+\beta^G(N-1)\rho\sigma^2+\beta^{-G}N\rho\sigma^2}\mu, \qquad (22)$$

$$\gamma^G = -\frac{\beta^G(N-1)+\beta^{-G}N}{1+\lambda^G\kappa+\beta^G(N-1)\rho\sigma^2+\beta^{-G}N\rho\sigma^2}\rho\sigma^2 < 0 \text{ for } G \in \{B,S\} \text{ and } G \neq G. \qquad (23)$$

(ii) This equilibrium exists if the exogenous model parameters are such that $\beta^B > 0, \beta^S > 0$, and the Lagrange multipliers that solve the binding capital constraints at market clearing are non-negative.

In this equilibrium, the market clears at $P^* = \frac{1}{\beta^B+\beta^S}\left(\sum_G \alpha^G + \frac{1}{N}\sum_G \gamma^G \sum_i z_i^G\right)$, and a trader's price impact is $\Lambda^G = (N\beta^{-G} + (N-1)\beta^G)^{-1}$ for $G \in \{B,S\}$ and $G \neq G$. While it is challenging to provide general statements about how constraints affect the price and price impact, it is straightforward to solve for an equilibrium numerically, and verify the price and price impact effect.

For example, let $N = 3, \sigma = \rho = \kappa = \mu = p_z = \mathbb{E}[\boldsymbol{z^S}] = \mathbb{E}[(\boldsymbol{z_i^B})^2] = \mathbb{E}[(\boldsymbol{z_i^S})^2] = 1$, $\mathbb{E}[\boldsymbol{z_i^B}] = 0$, $\mathbb{E}[\boldsymbol{\theta_i^B}] \approx 0.239$ and $\mathbb{E}[\boldsymbol{\theta_i^S}]$ sufficiently high that $\lambda^S = 0$. Then, there exists an equilibrium with $\alpha^B = 0.791, \alpha^S = 0.808, \beta^B \approx 0.870, \beta^S \approx 0.808, \gamma^B \approx -0.791, \gamma^S \approx -0.808$ and $\lambda^B \approx 0.1$, where we write $\approx$ instead of $=$ to highlight the fact that we are rounding numbers. Appendix Figure A3 shows how the price and price impact of buyers and sellers increase when the constraint for buyers is relaxed in this example.

A.3 Other constraints

Although our focus lies on a particular type of constraint, motivated by Basel III, our framework extends to a wider set of constraints.

Alternative ex-ante constraints. Following the same approach as in the proof of Proposition 1, we can characterize necessary conditions in the presence of the following constraint: $\mathbb{E}[h(\boldsymbol{P^c}, \boldsymbol{a_i^c})] \geq 0$, for any $h(\cdot,\cdot)$ that is differentiable in both input arguments.[28] To solve for an equilibrium in explicit form and prove equilibrium existence, as in Proposition 2, function $h(\cdot,\cdot)$ must take a linear form: $h(P,a) = \Gamma + \Xi a + \Omega p + \Upsilon Pa$, with $\Gamma, \Xi, \Omega, \Upsilon \in \mathbb{R}$. This class of constraints includes wealth or budget constraints ($\Gamma > 0, \Xi = \Omega = 0, \Upsilon > 0$), quantity constraints that might come from bidding limits ($\Gamma > 0, \Xi \neq 0, \Omega = \Upsilon = 0$), or constraints on the clearing price which might arise from arbitrage in an outside market ($\Gamma > 0, \Xi = 0, \Omega \neq 0, \Upsilon = 0$) .

Imposing a linear functional form on $h(\cdot,\cdot)$ ensures that there are linear equilibria. In absence of linearity, it becomes challenging to prove that a demand function that fulfills necessary equilibrium conditions is a global optimum. The reason for this is that many sufficient conditions one can derive when maximizing over functions only hold locally (see, for instance, Elsgolts (1977), Chapter 8). To overcome this challenge, the proof of Proposition 2 relies on the property that the objective functional is for any demand function (not just the equilibrium candidate) strictly concave. This must not be the case when the equilibrium candidate is non-linear.

Ex-post capital constraints. In some real-world applications constraints must hold with certainty, i.e., in all states of the world. Here, we characterize equilibrium conditions and illustrate how changes in the constraint can affect the price and price impact when the constraint must hold ex-post. For tractability, we let all traders have the same position, $\theta_i = \theta$ for all i. This allows us to solve for symmetric equilibria.

The trader's maximization problem—the analogue to problem (4)— reads as follows:

$$\max_{p_i(\cdot,\cdot)\in\mathcal{B}} \mathbb{E}\big[V(\boldsymbol{a_i^c}, \boldsymbol{\epsilon_i}) - \boldsymbol{P^c}\boldsymbol{a_i^c}\big] \text{ subject to}$$

$$\begin{aligned} &\text{the capital constraint:} && \theta - \kappa P^c a_i^c \geq 0, \\ &\text{and market clearing:} && P^c = p_i(a_i^c, \boldsymbol{\epsilon_i}) \text{ for all } a_i^c. \end{aligned} \tag{24}$$

[28]It suffices to replace $H(p(a,\epsilon), p_a(a,\epsilon), a, \epsilon)$ in expression (37) in the proof of Proposition 1 with $\left(\frac{\partial h(p(a,\epsilon),a)}{\partial p(a,\epsilon)} p_a(a,\epsilon) + \frac{\partial h(p(a,\epsilon),a)}{\partial a}\right)[1 - G(a, p(a,\epsilon)|\epsilon)]\psi(\epsilon)$, where $p_a(a,\epsilon) = \frac{\partial p(a,\epsilon)}{\partial a}$.

Proposition 6. *Let gross utility $V(a, \epsilon_i)$ be any measurable and bounded function that is continuously differentiable and strictly concave in a for all ϵ_i, and bounded in ϵ_i for all a.*

In any symmetric equilibrium, trader i submits demand function, $p^*(\cdot, \cdot)$, that for all p, a, and ϵ_i satisfies

$$p = v(a, \epsilon_i) + \lambda a \left[\left(\frac{\partial G(a, p|\epsilon_i)}{\partial p} \right) \psi(\epsilon_i) \right]^{-1} - \underbrace{a \left(\frac{\partial G(a, p|\epsilon_i)}{\partial a} \Big/ \frac{\partial G(a, p|\epsilon_i)}{\partial p} \right) (-1)}_{shading}, \tag{25}$$

where $\lambda \geq 0$ is the Lagrange multiplier of the capital constraint, and $G(a, p|\epsilon_i) = \Pr(\boldsymbol{a_i^c} \leq a|\epsilon_i)$ with $\boldsymbol{a_i^c} = \boldsymbol{A} - \sum_{j \neq i} a^*(p, \boldsymbol{\epsilon_j})$ is the probability that trader i, who bids price $p = p^*(a, \epsilon_i)$, wins less than a at market clearance, given that the other traders play the equilibrium strategy.

With ex-post capital constraints it is no longer true that traders bid as if they were bidding in an auction without constraints but with a discounted value (as in Proposition 1).

To build an intuition for the equilibrium condition, we consider a simplified auction environment in which all bidders observe the same signal, and thus do not have private information. Without loss of generality, we normalize this signal to zero, i.e., $\underline{\epsilon} = \overline{\epsilon} = 0$.

Corollary 3. *Let $\underline{\epsilon} = \overline{\epsilon} = 0$, so that $\epsilon_i = 0$ for all traders i, and omit the dependence on ϵ_i in all functions. In any symmetric equilibrium, each trader submits a demand function that satisfies the following condition:*

$$[v(a) - p^*(a)]\phi(aN) = \Lambda(a) \left[\phi(aN) + \lambda(a)\kappa\right] a \text{ for all } a \in \left[0, \frac{\overline{A}}{N}\right], \tag{26}$$

$$\text{where } \Lambda(a) = -\frac{1}{N-1} \frac{p^*(a)}{\partial a}, \tag{27}$$

and $v(a) = \mu - \rho\sigma^2 a$. The unique solution to this differential equation is:

$$p^*(a) = e^{\mathcal{M}(a)} \left(c_1 + \int_1^a \frac{-e^{-\mathcal{M}(y)}(N-1)(\mu - \rho\sigma^2 x)\phi(Nx)}{y(\kappa\lambda(y) + \phi(Ny))} dy \right), \tag{28}$$

with $\mathcal{M}(z) = \int_1^z \frac{(N-1)\phi(Nx)}{x(\kappa\lambda(x)+\phi(Nx))} dx$ for $z = a, y$, $c_1 \in \mathbb{R}$, and $\lambda(a) = \lambda(A/N)$ given by the capital constraint for all $A \in (0, \overline{A}]$. Note that with $c_1 = \mu - \rho\sigma^2 \left(\frac{N-1}{N-2}\right)$ the demand function simplifies to the known solution in an unconstraint auction for $\kappa = 0$.

Intuitively, condition (26) says that the bidder trades off the expected marginal surplus when winning at price p with the expected marginal cost. The expected marginal surplus is the difference between the bidder's value and the price they pay, weighted by the probability density that the market clears at price p and the bidder wins a. The expected marginal cost has two parts. The first part is the increased payments in all states with higher realizations of supply, which is equal to the price impact times the amount won. This is weighted by the probability density that the market clears at (p, a). The second part, $\Lambda(a)\lambda(a)\kappa a$, reflects the marginal cost of the constraint.[29]

Corollary 4. *(i) In the symmetric equilibrium, price impact at amount a is:*

$$\Lambda(a) = \frac{\phi(Na)}{\phi(aN) + \kappa\lambda(a)} \frac{v(a) - p^*(a)}{a} \geq 0. \tag{29}$$

(ii) Fix an amount a. When $\lambda(a)$ increases, price impact at amount a increases as long as $\frac{\partial p^*(a)}{\partial \lambda(a)} < \frac{\phi(Na)}{(\phi(aN)+\kappa\lambda(a))^2}(v(a) - p^*(a))$.

Both the shadow cost of the constraint, $\lambda(a)$, and price impact, as shown in equation (29), are functions of quantity, since the constraint must hold for all states of the world, i.e., for all quantities that can be obtained. When $\lambda(a)$ at a fixed amount a increases, price impact at that point changes due to two opposing effects. A direct effect pushes price impact upwards, while an indirect effect , coming from a change in $p^*(a)$ in response to the changing shadow cost, pushes price downwards. The total change in price impact is positive if the direct effect dominates the indirect effect. This is the case when $\frac{\partial p^*(a)}{\partial \lambda(a)} < \frac{\phi(Na)}{(\phi(aN)+\kappa\lambda(a))^2}(v(a) - p^*(a))$— summarized in Corollary 4 (*ii*).

A.4 Discriminatory price auction

Here we adjust our benchmark model to the case of discriminatory price auctions, in which bidders pay the prices they offered to pay for all units won, rather than the market clearing

[29]To see why $\Lambda(a)\lambda(a)\kappa a$ is the marginal cost of the constraint, note that, at any fixed $\lambda \geq 0$, the total cost from the constraint is $-\lambda(\theta - \kappa pa)$ at all (a, p) at which the market clears. By market clearing, $p_i^{RS}(a) = p$, where $p_i^{RS}(\cdot)$ is the inverse of trader i's residual supply curve $RS_i(\cdot)$. Taking the derivative of $-\lambda(\theta - \kappa p_i^{RS}(a)a)$ with respect to a, and using the fact that $\frac{\partial p_i^{RS}(a)}{\partial a} = (\frac{\partial RS_i(p)}{\partial p})^{-1} = \Lambda(a)$, we obtain the marginal cost of the constraint.

price. The differences in the payment does not only affect strategic incentives when bidding, but also the capital constraint. Since bidders pay the prices they offer, and winning bids are weakly higher than the auction clearing price, the nominal value of the amount bidders win at auction is the total amount bidders pay, i.e., $\int_0^{\boldsymbol{a_i^c}} p_i(a, \epsilon_i)da$. This is higher than the auction clearing price times the amount won, $\boldsymbol{P^c a_i^c}$. With this, the capital constraint, which is analogue to the benchmark constraint in (4), is:

$$\mathbb{E}\left[\boldsymbol{\theta_i} - \kappa \int_0^{\boldsymbol{a_i^c}} p_i(a, \boldsymbol{\epsilon_i})da\right] \geq 0. \tag{30}$$

In practice, this constraint approximates two scenarios. First, traders buy bonds at auction to keep them on their own balance sheet. In this case, bonds are evaluated according to their book-value, that is, according to the prices paid in the acquisition. Second, traders buy bonds to later sell them to clients. These bonds must be kept on the trading book, and evaluated according to their market-value. The market-value could deviate from the prices traders paid at auction. The deviation is small if traders charge clients prices that are close to their own transaction costs. To account for potential markups, we could multiply the auction payment by a scaler.

We consider two settings: one with continuous demand functions, and one with step functions.

(i) Continuous demand. In the first setting, traders submit continuous demand functions, as in our benchmark model.

Proposition 7. *Let gross utility $V(a, \epsilon_i)$ be any measurable and bounded function that is continuously differentiable and strictly concave in a for all ϵ_i, and bounded in ϵ_i for all a.*

In any symmetric equilibrium, traders behave as if they were bidding in an auction without capital constraints, in which their willingness to pay is $\tilde{v}(a, \epsilon_i)$—given in expression (5)—instead of $v(a, \epsilon_i)$, where $\lambda \geq 0$ is the Lagrange multiplier of the capital constraint. Trader i submits demand function, $p^*(\cdot, \cdot)$, that for all p, a, and ϵ_i satisfies

$$p = \tilde{v}(a, \epsilon_i) - \underbrace{(1 - G(a, p|\epsilon_i)) / \frac{\partial G(a, p|\epsilon_i)}{\partial p} (-1)}_{shading}, \tag{6'}$$

where $G(a, p|\epsilon_i) = \Pr(\boldsymbol{a_i^c} \leq a|\epsilon_i)$ *with* $\boldsymbol{a_i^c} = \boldsymbol{A} - \sum_{j \neq i} a^*(p, \boldsymbol{\epsilon_j})$ *is the probability that trader* i*, who bids price* $p = p^*(a, \epsilon_i)$*, wins less than* a *at market clearance, given that the other traders play the equilibrium strategy.*

Proposition 7 is the analogue to Proposition 1; only the shading factor is computed differently due to differences in the payment rules of the auction formats. To show the similarities across auction formats, we use the same notation for $p^*(\cdot, \epsilon_i)$, $G(a, p|\epsilon_i)$, and the Lagrange multiplier, λ, even though all of these are auction-format specific.

Solving for equilibrium demand functions in discriminatory price auctions, and proving equilibrium existence, is much more challenging than in uniform price auctions, even without constraints. This might be one of the reasons for which the literature has focussed on uniform price auctions. Only recently, Pycia and Woodward (2023) have proved pure-strategy equilibrium existence and uniqueness in an auction environment with identical bidders without private information. Earlier contributions that make similar informational restrictions, including Wang and Zender (2002), Holmberg (2009), Ewerhart et al. (2010), and Ausubel et al. (2014), impose additional distributional assumptions. For example, Ausubel et al. (2014) show that there exists a linear equilibrium if and only if supply follows a Generalized Pareto Distribution (GPD). Wittwer (2018) extends their ideas to the case with private information.

Providing novel equilibrium existence results for discriminatory price auctions without capital constraints is beyond the scope of our paper, which focusses on capital constraints. To nevertheless make progress, we follow Wittwer (2018) and impose conditions on equilibrium allocations, which is not ideal. Characterizing conditions on the underlying exogenous model primitives—i.e., the distribution of supply and signals—is challenging when traders have private information. Without private information (when $\bar{\epsilon} = \underline{\epsilon} = \epsilon_i = 0$ for all traders i) we can derive a sharper result: Equilibrium (7') of Proposition 8 exists when supply, $\boldsymbol{A}$, follows a GPD with CDF (31) and $\xi \leq -1$.

Proposition 8. *Let gross utility* $V(a, \epsilon_i)$ *be given by expression (1). Suppose there is a symmetric linear equilibrium in which each trader submits demand function*

$$p^*(a, \epsilon_i) = \frac{1}{1 + \lambda\kappa}\left(\mu + \epsilon_i - \rho\sigma^2\left(\frac{N-1}{N(1-\xi)-1}\right)[a + N\nu(\epsilon_i)]\right), \text{ with} \tag{7'}$$

$$\lambda = \begin{cases} 0 & \textit{if } \mathbb{E}[\boldsymbol{\theta_i}] \geq \tilde{\theta} \\ \frac{1}{\kappa}\left(\frac{\tilde{\theta}}{\mathbb{E}[\boldsymbol{\theta_i}]} - 1\right) > 0 & \textit{otherwise,} \end{cases}$$

where

$$\tilde{\theta} = \frac{\kappa}{N}\Big[\mathbb{E}[\boldsymbol{A}]\left(\mu + \frac{\mathbb{E}[\boldsymbol{\epsilon_i}](1-(1-\sigma^2)\xi)}{1-\xi}\right) + \mathbb{E}[\boldsymbol{A}]\left(\frac{\sigma^2(\underline{\epsilon} + \underline{\epsilon}N(\xi-1) - \overline{A}\rho(\xi-1))\xi}{(1+N(\xi-1)(\xi-1)}\right) + \mathbb{E}[\boldsymbol{A^2}]\left(\frac{(N-1)\rho\sigma^2}{N(1+N(\xi-1))}\right)\Big].$$

This equilibrium exists if the joint distribution of supply and private signals is such that, given all traders submit the demand function (7'), each trader's allocation at the market-clearing price follows a GPD with CDF

$$\Psi_i(a) = 1 - \left(\frac{\nu(\epsilon_i) + \xi a}{\nu(\epsilon_i)}\right)^{-\frac{1}{\xi}}, \tag{31}$$

where $\nu(\epsilon_i) = -\xi\frac{N(1-\xi)-1}{N(1-\xi)\rho}(\epsilon_i - \underline{\epsilon}) - \xi\frac{\overline{A}}{N}$, *and* $\xi \leq -1$.

Propositions 7 and 8 are analogues to Propositions 1 and 2. Therefore, Corollaries 1 and 2 generalize to discriminatory price auctions when equilibria are linear. Price impact, defined as the inverse slope of a bidder's residual supply curve, which is equivalent to $\Lambda^{DPA} = \frac{1}{N-1}\frac{\partial p^*(a,\epsilon_i)}{\partial a}$, is now given by expression (10').

(ii) Step function demand. In the second setting, bidders submit step functions. This is the case in many real-world applications, including Canadian Treasury auctions. It is therefore useful to characterize equilibrium conditions for this environment. For this, we adopt Kastl (2012)'s framework—i.e., Assumptions 1-6 and the equilibrium definition of his paper—with three adjustments. First, we introduce capital constraint, (30). Second, we follow the timing of events of our benchmark model, which implies that each bidder chooses a bidding function for all possible signals prior to observing the signal. Third, for simplicity, we assume that the value function, $v(\cdot, \epsilon_i)$, is strictly decreasing for all ϵ_i, while Assumption 2 in Kastl (2012) allows it to be weakly decreasing. Different to Kastl (2012), we refer to the private signal by ϵ_i, rather than s_i, and let a price bid at step k be $p_k(\epsilon_i)$ instead of b_{ik}, in line with the rest of our paper.

Consider a bidder who observes signal ϵ_i and let all other bidders play the equilibrium strategy. Bidder i obtains the following expected payoff when submitting step function

$\{p_k(\epsilon_i), a_k(\epsilon_i)\}_{k=1}^{K(\epsilon_i)}$ conditional on observing ϵ_i:

$$EU(\epsilon_i) = EV(\epsilon_i) - EP(\epsilon_i), \tag{32}$$

with

$$\begin{aligned} EV(\epsilon_i) = & \sum_{k=1}^{K(\epsilon_i)} \Pr(p_k(\epsilon_i) > \boldsymbol{P^c} > p_{k+1}(\epsilon_i)|\epsilon_i) V(a_k(\epsilon_i), \epsilon_i) \\ & + \sum_{k=1}^{K(\epsilon_i)} \Pr(p_k(\epsilon_i) = \boldsymbol{P^c}|\epsilon_i) \mathbb{E}[V(\boldsymbol{a_i^c}, \epsilon_i)|p_k(\epsilon_i) = \boldsymbol{P^c}, \epsilon_i], \text{ and} \end{aligned} \tag{33}$$

$$\begin{aligned} EP(\epsilon_i) = & \sum_{k=1}^{K(\epsilon_i)} \Pr(p_k(\epsilon_i) > \boldsymbol{P^c}|\epsilon_i) p_k(\epsilon_i)(a_k(\epsilon_i) - a_{k-1}(\epsilon_i)) \\ & + \sum_{k=1}^{K(\epsilon_i)} \Pr(p_k(\epsilon_i) = \boldsymbol{P^c}|\epsilon_i) \mathbb{E}[p_k(\epsilon_i)(\boldsymbol{a_i^c} - a_{ik-1})|p_k(\epsilon_i) = \boldsymbol{P^c}, \epsilon_i]. \end{aligned} \tag{34}$$

As before $\boldsymbol{P^c}$ is the market clearing price, and $\boldsymbol{a_i^c}$ is the amount bidder i wins at market clearing. The bidder faces capital constraint (30). With step functions, it looks as follows:

$$\begin{aligned} \mathbb{E}\Big[\boldsymbol{\theta_i} - \kappa\Big[& \sum_{k=1}^{K(\boldsymbol{\epsilon_i})} \Pr(p_k(\boldsymbol{\epsilon_i}) > \boldsymbol{P^c}|\boldsymbol{\epsilon_i}) p_k(\boldsymbol{\epsilon_i})(a_k(\boldsymbol{\epsilon_i}) - a_{k-1}(\boldsymbol{\epsilon_i})) \\ & + \sum_{k=1}^{K(\boldsymbol{\epsilon_i})} \Pr(p_k(\boldsymbol{\epsilon_i}) = \boldsymbol{P^c}|\boldsymbol{\epsilon_i}) \mathbb{E}\big[p_k(\boldsymbol{\epsilon_i})(\boldsymbol{a_i^c} - a_{ik-1})|p_k(\boldsymbol{\epsilon_i}) = \boldsymbol{P^c}, \boldsymbol{\epsilon_i}\big]\Big]\Big] \geq 0. \end{aligned} \tag{35}$$

The bidder chooses their bidding function to maximize the expectation of (32) subject to (35); market clearing is already guaranteed. Simplifying the objective functional, we obtain the following maximization problem:

$$\max_{\{p_k(\cdot), a_k(\cdot)\}_{k=1}^{K(\cdot)}} \mathbb{E}\Big[EV(\boldsymbol{\epsilon_i}) - (1+\lambda)EP(\boldsymbol{\epsilon_i})\Big], \text{ with } \lambda \geq 0. \tag{36}$$

Similar to the benchmark model, choosing $\{p_k(\epsilon_i), a_k(\epsilon_i)\}_{k=1}^{K(\epsilon_i)}$ to maximize $EV(\epsilon_i) - (1 + \lambda)EP(\epsilon_i)$ for all ϵ_i point-wise is equivalent to choosing $\{p_k(\cdot), a_k(\cdot)\}_{k=1}^{K(\cdot)}$ to maximize $\mathbb{E}[EV(\boldsymbol{\epsilon_i}) - (1+\lambda)EP(\boldsymbol{\epsilon_i})]$ for any fixed $\lambda \geq 0$. Therefore, the only substantial difference to Kastl (2012)'s framework without capital constraints is that the bidder's marginal cost from winning a step is no longer just the price they need to pay, $p_k(\epsilon_i)$, but it is the price plus the shadow cost of the constraint, which is $\lambda\kappa p_k(\epsilon_i)$.

B Additional empirical tests and findings

In Appendix B.1, we explain how we test that the slope coefficients of Figure 4 are higher during the exemption period than during regular time. In Appendix B.2, we summarize our robustness analysis.

B.1 Testing for steepness in the willingness to pay

We conduct a t-test to compare the median slope, β_t, of regression (13) during versus outside the exemption period, by relying on our 100 bootstrapped value estimates, and the assumption that the error in regression (13) comes entirely from measurement noise in the value estimates. For each bootstrap draw d, we estimate regression (13) and collect all β_t^d estimates. We compute the median of these slope parameters across auctions during versus outside the exemption period and take the difference, $\Delta\beta_d$. We repeat this exercise for all bootstrapped values to obtain 100 $\Delta\beta_d$ differences—Appendix Figure A4 shows the distribution of $\Delta\beta_d$'s across draws. With these differences, we compute the t-statistic as the ratio of $\Delta\beta_d$'s mean and their standard deviation, which is divided by the square root of the number of bootstrap draws. The t-statistic is 81.65, so that we can conclude that the difference in slope estimates in Figure 4 is statistically significant.

B.2 Robustness analysis

We conduct a series of robustness checks to validate our risk aversion and shadow cost estimates.[30] We report the median and mean of the risk aversion estimates, in addition to the median and mean of shadow cost estimates, including the respective mean and median of standard errors of the estimates for all regression specifications we estimate in Appendix Table A2. Generally, the magnitude of the risk aversion parameters is affected when changing the units of σ_t^2 or a_{tik} in regression (13), while the magnitude of the shadow cost estimates changes depending on the cross-auction distribution of σ_t^2 during regular times. The higher

[30]A robustness analysis of the value estimates is provided in Allen et al. (2024).

the variance of σ_t^2 across auctions, the more disperse the shadow cost estimates, and the larger the mean due to a few outlying observations.

We start by analyzing the sensitivity of our parameter estimates to the number of steps included in the values functions. In our benchmark specification, we include all functions with at least two steps (which are essentially all functions) to avoid a potential bias coming from omitting functions. Given that we linearly interpolate between steps using our model, we might be concerned about doing this when there are few steps. Our results, however, are qualitatively robust to using value functions with more steps—three to six, where we do not include robustness for seven steps since not all dealers use the maximum allowable number of steps in all auctions.

Next, we estimate equation (13) with quantities expressed in million C$. In our benchmark specification, we normalize quantities by the auction supply to avoid our estimates being affected by the fact that the Bank of Canada issued larger amounts of debt during the exemption period than in regular times. Given that dealers have an obligation to actively participate in the auctions, the increased supply implies that dealers demanded larger amounts during the exemption period (see Appendix Figure A5). Further, since dealers are supposed to bid competitively, and are given a price range when bidding, increasing the total demand decreases the slope in the dealer's bidding function and willingness to pay during the exemption period (relative to the case in which we normalize demand by the supply). The model rationalizes smaller slopes by smaller risk aversion and shadow cost parameters.

Third, we verify robustness with respect to our measure of volatility. We start by smoothing outliers of the return volatility by winsorizing the distribution of volatilities by 2.5% and 5%, respectively. Decreasing the cross-auction variance of the volatility measure results in lower average shadow costs. Similarly, there are fewer large shadow cost outliers, leading to a lower average shadow cost than in our benchmark, when we use the inter-quantile range of yields at which dealers sell prior to the auction instead of the variance of these yields.

Next, we vary the number of trading days we include to measure volatility. In our benchmark specification, we construct volatility using trades where we observe dealers selling the to-be-auctioned security in a five trading day window prior to auction. This is natural given that most pre-auction trading occurs in the one week between the tender open call and the auction close. The more days we include, the larger the volatility. Since this effect is stronger during the exemption period than during regular times, the average shadow cost is higher than in our benchmark. When we include fewer days to construct the volatility this effect goes in the opposite direction, pushing the average shadow cost downward. However, there is an additional effect. The fewer days we include, the more likely it becomes that the volatility index is missing for some auctions. To avoid dropping these auctions entirely, we use the average volatility of same maturity-type auctions within the year—in our benchmark specification there is no need to do this. This interpolation pushes the average shadow cost estimates upwards.

In addition, we can estimate our model using different volatility indices. One alternative is to use the Implied Volatility Index for Canadian Treasuries, which measures the expected volatility in the Treasury market over the next 30 days (Chang and Feunou (2014)). Given that this volatility drops more strongly during the exemption period than our volatility index, shadow cost estimates are higher when relying on the implied volatility.

Another alternative is to construct return volatility using post-auction trades. We refrain from doing so, because dealers do not know what happens after the auction at the time they bid. Further, post-auction prices likely depend on the realization of the dealer's private information, and with that their willingness to pay, in the auction. This implies that the post-auction volatility—an independent variable in equation (13)—is a function of the dependent variable and would lead to a simultaneous equation bias.

C Proofs

We first present the proofs of all propositions, and then of all corollaries.

Proof of Proposition 1. Let $V(a, \epsilon_i)$ be any measurable and bounded function that is continuously differentiable and strictly concave in a for all ϵ_i, and bounded in ϵ_i for all a.

Consider trader i, and fix all other demand schedules at the equilibrium. To determine the best response, trader i solves maximization problem (4). To simplify this problem, we drop the i-subscript, let $v(a, \epsilon) = \frac{\partial V(a,\epsilon)}{\partial a}$, denote $p_a(a, \epsilon) = \frac{\partial p(a,\epsilon)}{\partial a}$, and abbreviate all functions, for instance, $p(\cdot, \cdot)$ by p when useful. Further, we let $\overline{a}^*(\epsilon)$ be the largest amount the bidder wins when playing the equilibrium strategy, and recall that $\psi(\cdot)$ is the density function of signals on support $[\underline{\epsilon}, \overline{\epsilon}]$. With this, and auxiliary distribution $G(a, p|\epsilon)$, which is defined in Proposition 1, the trader's maximization problem becomes:

$$\max_{p \in \mathcal{B}} I(p) \text{ subject to } L(p) \geq 0, \text{ with} \tag{37}$$

$$I(p) = \int_{\underline{\epsilon}}^{\overline{\epsilon}} \int_0^{\overline{A}} F(p(a,\epsilon), p_a(a,\epsilon), a, \epsilon) da d\epsilon \tag{38}$$

$$\text{with } F(p(a,\epsilon), p_a(a,\epsilon), a, \epsilon) = [v(a,\epsilon) - p(a,\epsilon) - a p_a(a,\epsilon)][1 - G(a, p(a,\epsilon)|\epsilon)]\psi(\epsilon),$$

$$L(p) = \int_{\underline{\epsilon}}^{\overline{\epsilon}} \int_0^{\overline{A}} H(p(a,\epsilon), p_a(a,\epsilon), a, \epsilon) da d\epsilon \tag{39}$$

$$\text{with } H(p(a,\epsilon), p_a(a,\epsilon), a, \epsilon) = [\mathbb{E}[\boldsymbol{\theta}] - \kappa[p(a,\epsilon) + p_a(a,\epsilon)a][1 - G(a, p(a,\epsilon)|\epsilon)]]\psi(\epsilon).$$

Here we have integrated the inner integral by parts to obtain $I(p)$ and $L(p)$.[31]

A function p^* is optimal if the following conditions are satisfied:

$$\frac{\partial(F + \lambda H)}{\partial p}(p^*(a,\epsilon), p_a^*(a,\epsilon), a, \epsilon) - \frac{d}{da}\left(\frac{\partial(F + \lambda H)}{\partial p_a}(p^*(a,\epsilon), p_a^*(a,\epsilon), a, \epsilon)\right) = 0 \tag{40}$$

for all $a \in [0, \overline{a}^*(\epsilon)]$, and all ϵ. In addition, we need:

$$L(p^*) \geq 0 \text{ and } \lambda \geq 0, \tag{41}$$

$$\frac{\partial(F + \lambda H)}{\partial p_a}(p^*(0,\epsilon), p_a^*(0,\epsilon), 0, \epsilon) = \frac{\partial(F + \lambda H)}{\partial p_a}(p^*(\overline{a}^*(\epsilon),\epsilon), p_a^*(\overline{a}^*(\epsilon),\epsilon), \overline{a}^*, \epsilon) = 0 \text{ for all } \epsilon. \tag{42}$$

Conditions (42) are the natural boundary conditions. They hold automatically given that $\frac{\partial(F+\lambda H)}{\partial p_a}(p^*(a,\epsilon), p_a^*(a,\epsilon), a, \epsilon) = -(1 + \lambda\kappa)a[1 - G(a, p^*(a,\epsilon)|\epsilon)]\psi(\epsilon)$, and $G(0, p^*(0,\epsilon)|\epsilon) = 0$, and $G(\overline{a}^*(\epsilon), p^*(\overline{a}^*(\epsilon),\epsilon)|\epsilon) = 1$ for all ϵ by definition of G.

[31] To simplify the objective function $I(p) = \int_{\underline{\epsilon}}^{\overline{\epsilon}} \int_0^{\overline{A}} [V(a,\epsilon) - p(a,\epsilon)a] g(a, p(a,\epsilon)|\epsilon) da \psi(\epsilon) d\epsilon$ we integrate the inner integral by parts, as follows: $\int_0^{\overline{A}} [V(a,\epsilon) - p(a,\epsilon)a] g(a, p(a,\epsilon)|\epsilon) da = [v(a,\epsilon) - p(a,\epsilon) - a p_a(a,\epsilon)] G(a, p(a,\epsilon)|\epsilon)\big|_0^{\overline{A}} - \int_0^{\overline{A}} [v(a,\epsilon) - p(a,\epsilon) - a p_a(a,\epsilon)] G(a, p(a,\epsilon)|\epsilon) da$. Since $G(\overline{A}, p(\overline{A},\epsilon)|\epsilon) = 1$ and $G(0, p(0,\epsilon)|\epsilon)) = 0$ for any function p by definition of G, this simplifies to $\int_0^{\overline{A}} [v(a,\epsilon) - p(a,\epsilon) - a p_a(a,\epsilon)][1 - G(a, p(a,\epsilon)|\epsilon)] da$. Similarly, for the constraint $L(p)$ we use $\int_0^{\overline{A}} [\mathbb{E}[\boldsymbol{\theta}] - \kappa p(a,\epsilon)]\, da = [\mathbb{E}[\boldsymbol{\theta}] - \kappa[p(a,\epsilon) + a p_a(a,\epsilon)]] G(a, p(a,\epsilon)|\epsilon)\big|_0^{\overline{A}} - \int_0^{\overline{A}} [\mathbb{E}[\boldsymbol{\theta}] - \kappa[p(a,\epsilon) + a p_a(a,\epsilon)]] G(a, p(a,\epsilon)|\epsilon) da = \int_0^{\overline{A}} [\mathbb{E}[\boldsymbol{\theta}] - \kappa[p(a,\epsilon) + a p_a(a,\epsilon)]][1 - G(a, p(a,\epsilon)|\epsilon)] da$.

Dividing (40) by $\psi(\epsilon)$ since $\psi(\epsilon) > 0$ for any ϵ, and simplifying gives:

$$-(1+\lambda\kappa)[1 - G(a, p^*(a,\epsilon)|\epsilon)] - [v(a,\epsilon) - (1+\lambda\kappa)(p^*_a(a,\epsilon)a + p^*(a,\epsilon))]\frac{\partial G(a, p^*(a,\epsilon)|\epsilon)}{\partial p} + \frac{d}{da}\Big((1+\lambda\kappa)a[1 - G(a, p^*(a,\epsilon)|\epsilon)]\Big) = 0,$$

where $\frac{d}{da}\big(a[1-G(a,p^*(a,\epsilon)|\epsilon)]\big) = [1-G(a,p^*(a,\epsilon)|\epsilon)] - a\big[\frac{\partial G(a,p^*(a,\epsilon)|\epsilon))}{\partial a} + \frac{\partial G(a,p(a,\epsilon)|\epsilon)}{\partial p}p^*_a(a,\epsilon)\big]$. This rearranges to condition (6), when setting $p = p^*(a,\epsilon)$. □

Proof of Proposition 2. Guess that there is a linear equilibrium of the following form: $a_i(p,\epsilon_i) = \alpha - \beta p + \gamma\epsilon_i$, and let all traders but trader i play this equilibrium. For convenience we drop the trader i supscript. The residual supply trader i faces is $RS(p, \boldsymbol{Z}) = \boldsymbol{Z} - (N-1)\alpha + (N-1)\beta p$, where $\boldsymbol{Z} = \boldsymbol{A} - \gamma\sum_{j\neq i}\boldsymbol{\epsilon_j}$. Following Wittwer (2021), we will maximize over a function $b(Z,\epsilon)$ that maps from Z into prices rather than from quantities into prices directly. This allows us to show that a function that fulfills the necessary conditions of the maximization problem also fulfills sufficient conditions, and is therefore indeed optimal. Maximization problem (4), becomes:

$$\max_{b(\cdot,\cdot)\in\mathcal{B}} \mathbb{E}[V(RS(b(\boldsymbol{Z},\boldsymbol{\epsilon}),\boldsymbol{Z}),\boldsymbol{\epsilon}) - b(\boldsymbol{Z},\boldsymbol{\epsilon})RS(b(\boldsymbol{Z},\boldsymbol{\epsilon}),\boldsymbol{Z})] \quad (43)$$
$$\text{subject to: } \mathbb{E}[\boldsymbol{\theta} - \kappa b(\boldsymbol{Z},\boldsymbol{\epsilon})RS(b(\boldsymbol{Z},\boldsymbol{\epsilon}),\boldsymbol{Z})] \geq 0.$$

Abbreviating $b(\cdot,\cdot)$ by b, this problem is equivalent to

$$\max_{b\in\mathcal{B}} I(b) \text{ subject to } L(b) \geq 0, \text{ with} \quad (44)$$
$$I(b) = \int_{\underline{Z}}^{\overline{Z}}\int_{\underline{\epsilon}}^{\overline{\epsilon}} F(b(\epsilon,Z),Z,\epsilon)\psi(\epsilon)\phi_Z(Z)d\epsilon dZ,$$
$$\text{where } F(b(Z,\epsilon),Z,\epsilon) = V(RS(b(Z,\epsilon),Z),\epsilon) - b(Z,\epsilon)RS(b(\epsilon,Z),Z)$$
$$L(b) = \int_{\underline{Z}}^{\overline{Z}}\int_{\underline{\epsilon}}^{\overline{\epsilon}} H(b(Z,\epsilon),Z,\epsilon)\psi(\epsilon)\phi_Z(Z)d\epsilon dZ,$$
$$\text{where } H(b(Z,\epsilon),Z,\epsilon) = \mathbb{E}[\boldsymbol{\theta}] - \kappa b(Z,\epsilon)RS(b(Z,\epsilon),Z).$$

Here, $\phi_Z(\cdot)$ is the density function of $\boldsymbol{Z}$ which has support $[\underline{Z},\overline{Z}]$, and $\psi(\cdot)$ is the density function of $\boldsymbol{\epsilon}$ on support $[\underline{\epsilon},\overline{\epsilon}]$.

Function b^* is optimal if $L(b^*) \geq 0, \lambda L(b^*) = 0, \lambda \geq 0$, $\frac{\partial(F+\lambda H)}{\partial b}$ evaluated at the optimum is 0 for all Z, ϵ:

$$(\mu + \epsilon)RS' - \rho\sigma^2 RSRS' - (1 + \lambda\kappa)(RS + bRS') = 0, \tag{45}$$

where we abbreviate $RS = RS(b^*, Z)$ and $RS' = \frac{\partial RS(b^*,Z)}{\partial b}$. Rearranging we obtain:

$$\mu + \epsilon - \rho\sigma^2 RS(b^*, Z) = (1 + \lambda\kappa)\Big[b^* + RS(b^*, Z)\Big(\frac{\partial RS(b^*, Z)}{\partial b}\Big)^{-1}\Big]. \tag{46}$$

In addition, natural boundary conditions,

$$\frac{\partial(F + \lambda H)}{\partial b_Z}(b(\underline{Z}, \epsilon), \underline{Z}, \epsilon) = \frac{\partial(F + \lambda H)}{\partial b_Z}(b(\overline{Z}, \epsilon), \overline{Z}, \epsilon) = 0, \tag{47}$$

where b_Z denotes the partial derivative of $b(Z, \epsilon)$ with respect to Z, must be satisfied. This is the case because $F + \lambda H$ is independent of $\frac{\partial b(Z,\epsilon)}{\partial Z}$ and $\frac{\partial b(Z,\epsilon)}{\partial \epsilon}$.

To show that a function b^* that fulfills the necessary conditions is indeed optimal, we rely on the fact that $F(b, Z)$ and $K(b, Z) = F(b, Z) + \lambda H(b, Z)$ are for any Z, ϵ, and $\lambda \geq 0$, strictly concave as functions of b, since

$$-\rho\sigma^2(RS'(b, Z))^2 - 2(1 + \lambda\kappa)RS'(b, Z) < 0 \text{ and } RS''(b, Z) = 0.$$

Therefore, $K(b(Z, \epsilon), Z) - K(b^*(Z, \epsilon), Z) < \frac{\partial K(b(Z,\epsilon),Z)}{\partial b}(b(Z, \epsilon) - b^*(Z, \epsilon)) \leq 0$ for any $b(Z, \epsilon)$ and any Z, ϵ. Multiplying both sides with $\phi_Z(Z)\psi(\epsilon)$ and integrating over Z, ϵ, we see that $\int\int K(b(Z, \epsilon), Z)\phi_Z(Z)\psi(\epsilon)dZd\epsilon < \int\int K(b^*(Z, \epsilon), Z)\phi_Z(Z)\psi(\epsilon)dZd\epsilon$, and similarly for $F(b(Z, \epsilon), Z)$. Thus, $b^*(\cdot, \cdot)$ is indeed optimal.

From here it is straightforward to solve for an equilibrium and show that it is unique within the class of symmetric linear equilibria. For this we rely on the property that function $b^*(\cdot, Z)$ implies a unique demand function $p^*(\cdot, \epsilon)$ for all ϵ. Then we match coefficients of the trader's best reply in (46) with the equilibrium guess and show that these coefficients are unique.

When everyone plays the equilibrium demand function (7), the market clears at

$$P^* = \frac{1}{1 + \lambda\kappa}\left(\mu + \frac{1}{N}\sum_i \epsilon_i - \left(\frac{N-1}{N-2}\right)\rho\sigma^2\frac{A}{N}\right), \tag{48}$$

and trader i wins $a_i^* = \frac{A}{N} - \left(\frac{N-2}{N-1}\right)\frac{1}{\rho\sigma^2}\left(N\epsilon_i - \sum_i \epsilon_i\right)$. The binding Lagrange multiplier is pinned down by the binding constraint: $\mathbb{E}[\boldsymbol{\theta_i} - \kappa \boldsymbol{P^*}\boldsymbol{a_i^*}] = 0$. This completes the proof that equilibrium (7) is the unique linear symmetric equilibrium in the general case in which bidders have private information. □

Proof of Proposition 3. Consider bidder i, and assume that all other bidders play the equilibrium strategy. Fix some $\lambda \geq 0$, and note that for any given λ, choosing $\{p_k(\epsilon_i), a_k(\epsilon_i)\}_{k=1}^{K(\epsilon_i)}$ to maximize $EV(\epsilon_i) - (1+\lambda)EP(\epsilon_i)$ for all ϵ_i point-wise is equivalent to choosing $\{p_k(\cdot), a_k(\cdot)\}_{k=1}^{K(\cdot)}$ to maximize $\mathbb{E}[EV(\boldsymbol{\epsilon_i}) - (1+\lambda)EP(\boldsymbol{\epsilon_i})]$. Therefore, we can determine the optimal bids for the event that the bidder observes ϵ_i point-wise for all ϵ_i.

For this, we can follow the proof of Proposition 1 in Kastl (2012) one-by-one with one difference. The marginal expected payment (A.3) in Kastl (2012)'s proof must be multiplied by $(1 + \lambda\kappa)$ to account for the shadow cost of the capital constraint. Similarly, the marginal cost of winning a bid at step k is $(1+\lambda\kappa)p_k(\epsilon_i)$ instead of $p_k(\epsilon_i)$. This adjustment also needs to be made in the two auxiliary lemmas that the proof of the proposition relies on. For example, with capital constraints, a bidder places a bid such that $(1 + \lambda\kappa)p_k(\epsilon_i) \leq v_i(a_k(\epsilon_i), \epsilon_i)$ when there is a positive probability of a tie at any step.

In the end, we determine the correct λ. This is the λ for which $\lambda\mathbb{E}\Big[\boldsymbol{\theta_i} - \kappa\Big[\sum_{k=1}^{K(\boldsymbol{\epsilon_i})} \Pr(p_k(\boldsymbol{\epsilon_i})) > \boldsymbol{P^c}|\boldsymbol{\epsilon_i})p_k(\boldsymbol{\epsilon_i})(a_k(\boldsymbol{\epsilon_i}) - a_{k-1}(\boldsymbol{\epsilon_i})) + \sum_{k=1}^{K(\boldsymbol{\epsilon_i})} \Pr(p_k = \boldsymbol{P^c}|\boldsymbol{\epsilon_i})\mathbb{E}\big[p_k(\boldsymbol{\epsilon_i})(\boldsymbol{a_i^c} - a_{k-1}(\boldsymbol{\epsilon_i}))|p_k(\boldsymbol{\epsilon_i}) = \boldsymbol{P^c}, \boldsymbol{\epsilon_i}\big]\Big]\Big] = 0$, when all bidders submit step function that satisfy the necessary equilibrium condition for all ϵ_i. □

Proof of Proposition 4. We derive a symmetric equilibrium by simplifying the equilibrium of Proposition 5 under the assumption that $\mathbb{E}[\boldsymbol{\theta_i^B} + \boldsymbol{\theta_i^S}] = \kappa p_z \mathbb{E}[\boldsymbol{z_i^B} + \boldsymbol{z_i^S}] > 0$.

We start by guessing that there is an equilibrium in which the constraint is binding with equal strength for both buyers and sellers, i.e., $\lambda^B = \lambda^S = \lambda \geq 0$. Under this conjecture, we simplify the demand coefficients provided in Proposition 5 to obtain candidate equilibrium demand function (17). This candidate is indeed an equilibrium if the Lagrange multipliers that ensure that the capital constraints hold for both buyers and sellers are non-negative and common across trader groups, i.e., $\lambda^B = \lambda^S = \lambda \geq 0$.

To show that this is the case, and derive the functional form for λ, we determine the λ at which the constraint binds, simultaneously for both groups of traders. The solution is given in equation (18). Since $\mathbb{E}[\boldsymbol{\theta_i^B}+\boldsymbol{\theta_i^S}]=\kappa p_z\mathbb{E}[\boldsymbol{z_i^B}+\boldsymbol{z_i^S}]>0$ by assumption, the Lagrange multiplier is identical for buyers and sellers. □

Proof of Propositions 5. We guess that there is a group-symmetric BNE, in which a trader of group G submits the following net-demand curve:

$$a^G(p,z_i^G)=\alpha^G-\beta^G p+\gamma^G z_i^G \text{ with } \beta^G>0,\ \text{for } G\in\{B,S\}.$$

When everyone submits the equilibrium guess, the market clears when total demand equals total supply: $\sum_{i=1}^N a^B(p,z_i^B)=-\sum_{i=1}^N a^S(p,z_i^S)$. Thus, ex-post, buyers and sellers face different types of residual supply curves, and price impact:

$$RS^B(p,z_i^B)=-\sum_{i=1}^N a^S(p,z_i^S)-\sum_{j\neq i}^N a^B(p,z_j^B)\quad,\ RS^S(p,z_i^B)=-\sum_{j\neq i}^N a^S(p,z_j^S)-\sum_{i=1}^N a^B(p,z_i^B)$$
$$\frac{\partial RS^B(p,z_i^B)}{\partial p}=N\beta^S+(N-1)\beta^B=\frac{1}{\Lambda^B}\quad,\ \frac{\partial RS^S(p,z_i^S)}{\partial p}=N\beta^B+(N-1)\beta^S=\frac{1}{\Lambda^S}.$$

Following the same steps as the proof of Proposition 2, we obtain the following necessary and sufficient equilibrium conditions for the buyers and sellers best response, respectively:

$$\mu-\rho\sigma^2(a^B+z_i^B)=(1+\lambda^B\kappa)\Big[p+a^B\Big(N\beta^S+(N-1)\beta^B\Big)^{-1}\Big], \tag{49}$$
$$\mu-\rho\sigma^2(a^S+z_i^S)=(1+\lambda^S\kappa)\Big[p+a^S\Big(N\beta^B+(N-1)\beta^S\Big)^{-1}\Big]. \tag{50}$$

In the group-symmetric equilibrium, all buyers and sellers must choose the equilibrium guess. Matching the β coefficients from the best replies to the equilibrium, we can express β^S as a function of β^B, shown in equation (21), and characterize β^B as the root of polynomial (20). For these coefficients to be valid in equilibrium they must be strictly positive. Similarly, we back out the α^G and γ^G coefficients as functions of β^B and β^S from the best responses (49) and (50).

The equilibrium Lagrange multiplier is pinned down by the binding constraint (16) for buyers and sellers, whenever it is binding, similar to the benchmark model. However, now, we can no longer solve for $\lambda^S,\lambda^B\geq 0$ in explicit form, since β^B and β^S are complicated functions of λ^S,λ^B. □

Proof of Proposition 6. The proof is analogous to the proof of Proposition 1, but the constraint is different. Using the same notation as before, the bidder's maximization problem is:

$$\max_{p \in \mathcal{B}} I(p) \text{ subject to } L(p(a,\epsilon),a) = \theta - \kappa p(a,\epsilon)a \geq 0 \tag{51}$$

for all a, ϵ, with $I(p)$ given by (38).

A function p^* is optimal if the following conditions are satisfied:

$$\frac{\partial F}{\partial p}(p^*(a,\epsilon), p_a^*(a,\epsilon), a, \epsilon) - \frac{d}{da}\left(\frac{\partial F}{\partial p_a}(p^*(a,\epsilon), p_a^*(a,\epsilon), a, \epsilon)\right) - \lambda\kappa a = 0, \tag{52}$$

and $L(p^*(a,\epsilon),a) \geq 0, \lambda \geq 0$ for all $a \in [0, \overline{a}^*(\epsilon)]$, and all ϵ. As before, natural boundary conditions,

$$\frac{\partial F}{\partial p_a}(p^*(0,\epsilon), p_a^*(0,\epsilon), 0, \epsilon) = \frac{\partial F}{\partial p_a}(p^*(\overline{a}^*(\epsilon),\epsilon), p_a^*(\overline{a}^*(\epsilon),\epsilon), \overline{a}^*(\epsilon), \epsilon) = 0, \tag{53}$$

are satisfied. Inserting the expressions for the partial derivatives of F like in the proof for Proposition 1, and simplifying gives:

$$(p^*(a,\epsilon) - v(a,\epsilon))\frac{\partial G(a,p|\epsilon)}{\partial p}\psi(\epsilon) = \lambda a + a\frac{\partial G(a,p|\epsilon)}{\partial a}\psi(\epsilon), \tag{54}$$

which rearranges to condition (25). □

Proof of Proposition 7. The proof is analogous to the proof of Proposition 1. There is only one difference, which comes from the fact that bidders pay the prices they bid for all units that they win instead of the market clearing price. This implies that $F(p(a,\epsilon), p_a(a,\epsilon), a, \epsilon)$, and $H(p(a,\epsilon), p_a(a,\epsilon), a, \epsilon)$ in maximization problem (37) are

$$F(p(a,\epsilon), p_a(a,\epsilon), a, \epsilon) = [v(a,\epsilon) - p(a,\epsilon)][1 - G(a, p(a,\epsilon)|\epsilon)]\psi(\epsilon), \tag{55}$$

and

$$H(p(a,\epsilon), p_a(a,\epsilon), a, \epsilon) = [\mathbb{E}[\boldsymbol{\theta}] - \kappa p(a,\epsilon)[1 - G(a, p(a,\epsilon)|\epsilon)]]\psi(\epsilon), \tag{56}$$

respectively. With slight abuse of notation we are using the same notation as in the uniform-price auction. □

Proof of Proposition 8. When equilibrium quantities, $\boldsymbol{a_i^*}$, follow a Generalized Pareto distribution, we can solve for a function that fulfills condition (6') of Proposition 7. For this, we combine the insight that a trader bids as if their true willingness to pay was $\frac{v(a,\epsilon_i)}{1+\lambda\kappa}$ for any given $\lambda \geq 0$, with a known result from the literature on equilibrium existence (e.g., Proposition 7 of Ausubel et al. (2014), Theorem 2 of Wittwer (2018))). In equilibrium, $\lambda > 0$ is pinned down by the capital constraint if the constraint binds, and is zero otherwise. □

Proof of Corollary 1. To show that price impact increases when the constraint is relaxed, which decreases λ, we take the derivative of price impact (8) with respect to λ. It is negative. Similarly, the derivative of the clearing price (48) with respect to λ is negative. □

Proof of Corollary 2. To prove the corollary, we compute the following elasticity:

$$\eta_P = \frac{\partial P^*}{\partial(\lambda\kappa)} \frac{\lambda\kappa}{P^*} \text{ with } \Lambda \text{ defined in (8) and } P^* \text{ defined in (48),}$$
$$\eta_\Lambda = \frac{\partial \Lambda}{\partial(\lambda\kappa)} \frac{\lambda\kappa}{\Lambda} \text{ with } \Lambda \text{ defined in (8).}$$

Simplifying terms, we obtain that $\eta_P = \eta_\Lambda = \eta = \frac{1}{1+\lambda\kappa} - 1$. □

Proof of Corollary 3. Let $\underline{\epsilon} = \overline{\epsilon} = 0$, so that $\epsilon_i = 0$ for all traders i, and omit the dependence on ϵ_i in all functions. In this case, we can simplify the condition for a symmetric equilibrium (in which each trader wins $\frac{A}{N}$) of Proposition 6. For this, we replace $\psi(\epsilon_i) = 1$ and insert:

$$\frac{\partial G(a,p|\theta_i)}{\partial a} = \phi(Na) \tag{57}$$
$$\frac{\partial G(a,p|\theta_i)}{\partial p} = (N-1)\phi(Na)\left(\frac{\partial a^*(p)}{\partial p}\right) = (N-1)\phi(Na)\left(\frac{\partial p^*(a)}{\partial a}\right)^{-1}. \tag{58}$$

We obtain differential equation (26). This differential equation has unique solution (28). □

Proof of Corollary 4. Statement (i) follows from condition (26) of Corollary 3. Price impact is non-negative since any valid equilibrium demand function must be decreasing, which implies $p^*(a) \leq v(a)$. To prove statement (ii), we fix some point a, and take the partial derivative of price impact (29) with respect to $\lambda(a)$ for a fixed point a. This derivative is positive if $\frac{\partial p^*(a)}{\partial \lambda(a)} < \frac{\phi(Na)}{(\phi(aN)+\kappa\lambda(a))^2}(v(a) - p^*(a))$. □

Appendix Table A1: Bid functions are approximately linear

	mean	median	sd
β_t	0.21	0.16	0.13
R_t^2	0.72	0.74	0.11
Adj. R_t^2	0.64	0.67	0.15
Within R_t^2	0.54	0.56	0.15

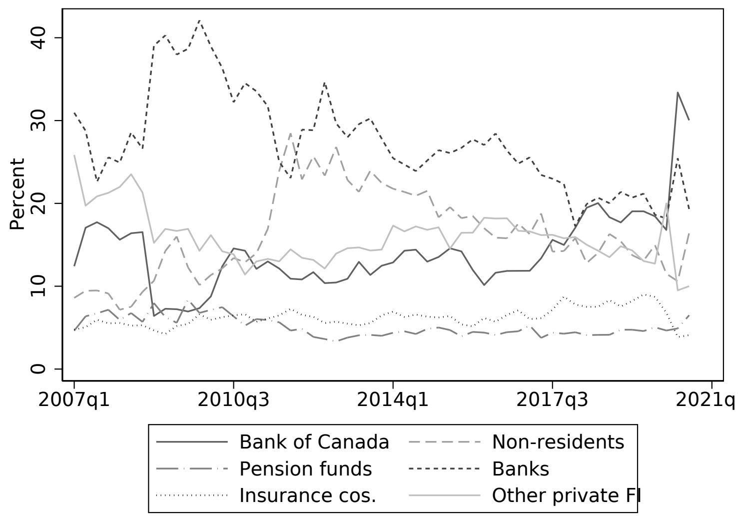
Appendix Table A1 shows the point estimate and R^2 from regressing bids on quantities and an auction-dealer fixed effect in each auction, $b_{tik} = \zeta_{ti} + \beta_t a_{tik} + \epsilon_{tik}$, using bidding functions with at least two steps. Bids are in bps of yields and quantities in percentage of supply.

Appendix Table A2: Estimates for all regression specifications

	Median ρ		Mean ρ		Median $\lambda\kappa$		Mean $\lambda\kappa$	
Benchmark	0.0053	(4.438e-05)	0.0048	(5.272e-03)	1.518e-11	(6.193e-03)	0.5292	(1.347e-02)
3 steps	0.0053	(3.379e-05)	0.0049	(4.534e-05)	1.518e-11	(6.835e-03)	0.5571	(1.416e-02)
4 steps	0.0052	(3.378e-05)	0.0048	(4.516e-05)	1.634e-11	(9.320e-03)	0.5098	(1.529e-02)
5 steps	0.0052	(3.340e-05)	0.0047	(3.340e-05)	5.540e-12	(1.539e-02)	0.5494	(2.097e-02)
6 steps	0.0049	(3.494e-05)	0.0049	(4.869e-05)	1.682e-12	(3.403e-11)	0.3389	(1.627e-02)
In mil C$	0.0001	(8.749e-07)	0.0001	(9.970e-05)	3.638e-12	(1.378e-03)	0.3138	(8.806e-03)
2.5%-winsorized	0.0050	(3.214e-05)	0.0048	(4.307e-05)	3.014e-10	(6.193e-03)	0.4115	(1.191e-02)
5.0%-winsorized	0.0050	(3.153e-05)	0.0048	(4.270e-05)	2.405e-10	(6.194e-03)	0.3530	(1.133e-02)
IQR	0.0043	(3.615e-05)	0.0044	(3.857e-05)	1.968e-09	(7.657e-03)	0.4658	(1.293e-02)
2 days	0.0050	(3.881e-05)	0.0045	(4.003e-05)	2.249e-14	(1.421e-04)	0.6188	(1.127e-02)
3 days	0.0047	(3.847e-05)	0.0048	(4.196e-05)	3.513e-13	(4.719e-04)	0.6285	(1.202e-02)
4 days	0.0048	(3.766e-05)	0.0048	(4.360e-05)	2.217e-11	(4.802e-03)	0.5559	(1.200e-02)
6 days	0.0054	(4.386e-05)	0.0049	(3.414e-05)	1.489e-11	(4.898e-0.3)	0.6159	(1.504e-02)
7 days	0.0054	(3.666e-05)	0.0047	(4.067e-05)	1.153e-10	(1.175e-02)	0.6597	(1.607e-02)

Appendix Table A2 shows the median and mean of the risk aversion estimates and the shadow costs, respectively for different regression specifications. The mean and median standard errors are in parentheses. "Benchmark" reports the estimates of our benchmark specification. In columns "N steps" for $N \in \{2, 3, 4, 5, 6\}$ we use only values that correspond to bidding step functions with weakly more than N steps. In column "In mil C$" we use quantities in absolute values, expressed in million C$. In columns "$x$%-winsorized" for $x \in \{2.5, 5\}$ we winsorize the distribution of return volatilities, σ_t^2, by x% to exclude outliers which lead to large shadow costs, and a higher average of these costs. In column IQR we use the inter-quantile range of yields at which dealers trade up to five days prior to the auction as our volatility measure, which also reduces across-auction heterogeneity in volatility. In columns "T days" for $T \in \{2, 3, 4, 6, 7\}$ we construct our volatility measure using trades on T days prior to the auction. We exclude $T = 5$ since this is used in our benchmark specification, and $T = 1$, because there aren't sufficiently many trades for each auction.

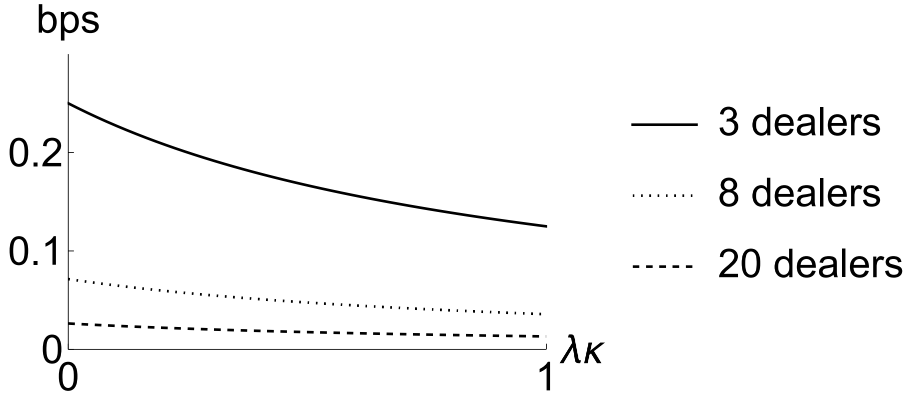
Appendix Figure A1: Holders of Canadian government bonds



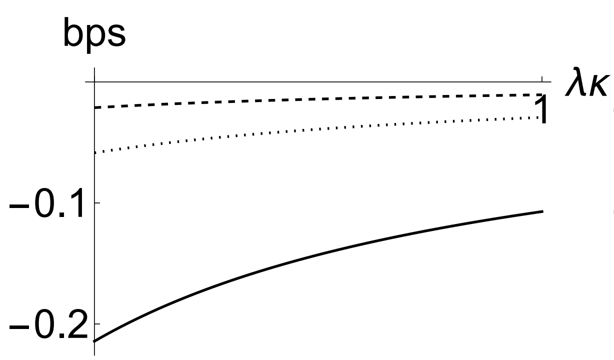
Appendix Figure A1 shows who holds Canadian government bonds and bills from 2007 until 2021 in percentage of par value outstanding: Bank of Canada, Non-residents, Canadian pension funds, Canadian banks, Canadian insurance companies, and other private firms. The bank category holdings are mostly driven by the eight banks we focus on, as they hold over 80% of the assets of all banks.

Appendix Figure A2: How price impact varies in the number of bidders and shadow costs

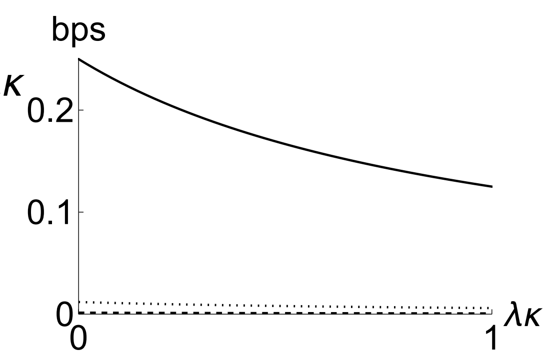
(a) Direct price impact



(b) Strategic effect in DPA



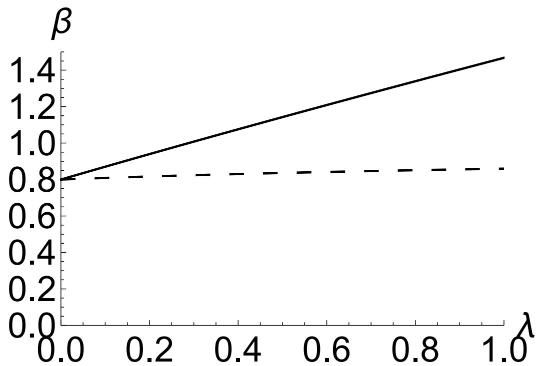
(c) Strategic effect in UPA



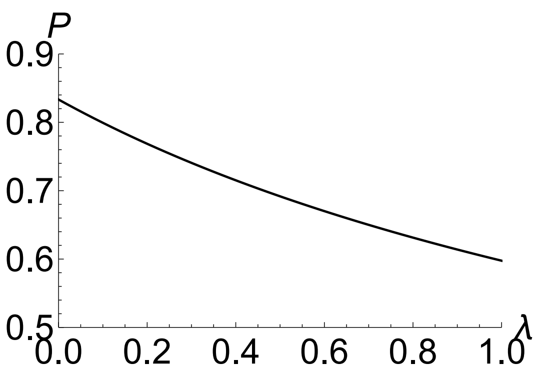
Appendix Figure A2a hows how the direct price impact, $\frac{1}{N-1}\frac{\rho\sigma^2}{1+\lambda\kappa}$, varies in the shadow cost for 3, 8, and 20 bidders, presented by the black, the dotted, and dashed line, respectively, using the average risk aversion estimate, and the average volatility. Figure A2b shows the price impact that arises due to demand reduction, according to equation (10'), in a typical discriminatory price auction (DPA). For illustration, we use the median ξ estimate, which is negative, so that the strategic effect is negative. Figure A2c shows the strategic price impact effect for uniform price auctions (UPA) using equation (10).

Appendix Figure A3: Equilibrium example in an asymmetric trade setting

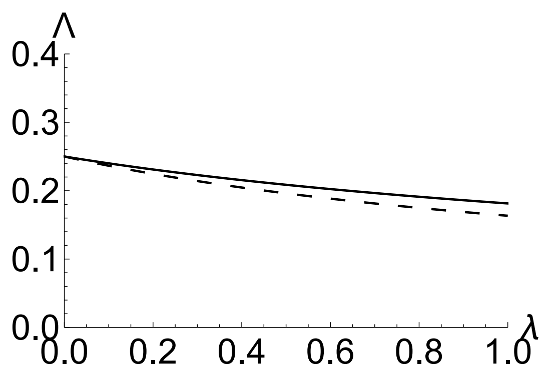
(a) Effect on slopes: β^B, β^S



(b) Effect on price: P^*

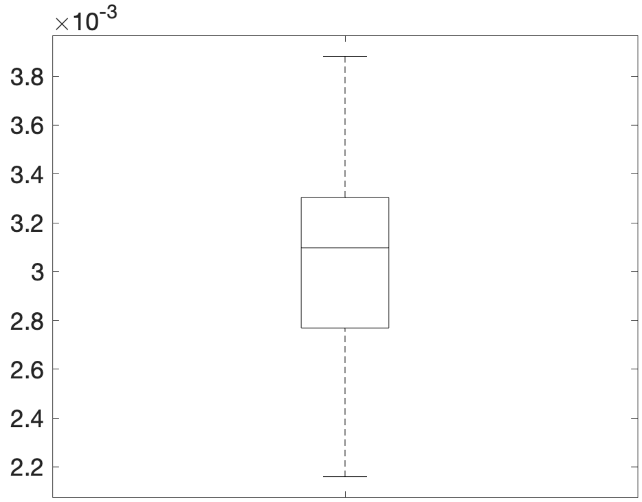


(c) Effect on price impact: Λ^B, Λ^S



Appendix Figure A3 visualizes the effect of tightening the capital constraint for buyers, i.e., increasing their λ, on the slope coefficients in (a), on the market clearing price in (c), and the price impact of buyers and sellers in (c) for buyers (the solid line) and for sellers (the dashed line). To compute the price, we assume $\sum_i z_i^B = 2$ and $\sum_i = z^S = -1$.

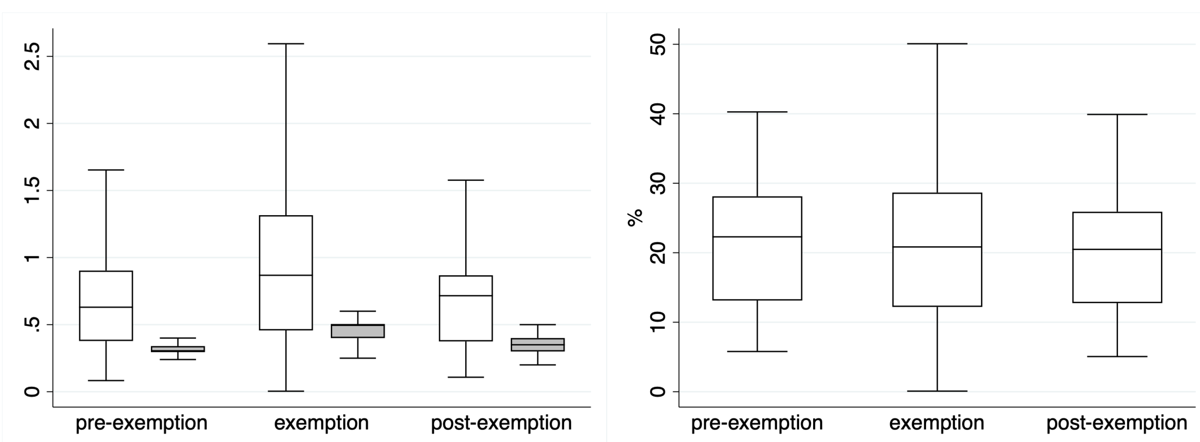
Appendix Figure A4: Difference of slope estimates during and outside the exemption period



Appendix Figure A4 shows the distribution of the difference in the median slope in all auctions that took place during the exemption period versus the median slope in all auctions that took place during regular times, $\Delta\beta_d$, across bootstraps d.

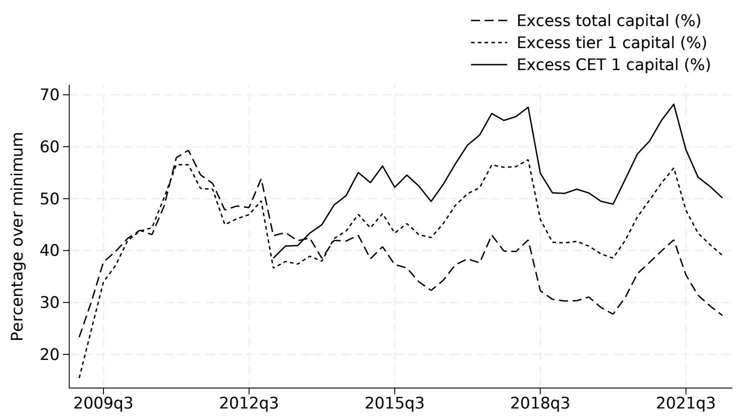
Appendix Figure A5: Variation in quantities

(a) Supply and total demand (b) Total demand in percentage of supply



Appendix Figure A5a shows the distribution of the total amount a dealer demands in an auction before, during, and after the exemption period (in white) and the distribution of the supply (in gray). Demand is expressed in million C$, and supply is in 10 million C$ to make the two comparable. Appendix Figure A5b shows the distribution of the total amount demanded as percentage of supply across periods.

Appendix Figure A6: Excess capital holdings



Appendix Figure A6 shows the average excess capital holdings for three different regulatory capital constraints: (i) Total risk-weighted capital, (ii) Tier 1 capital, and (iii) Common equity Tier 1 (CET1) capital. The main take-away is that excess capital holdings increased during the COVID crisis, moving banks away from regulatory thresholds (other than the LR constraint).