

B Supplemental Appendix

What You Don't Know May Be Good For You

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B.1 Inequalities For Proof Of Theorem 1 and Section A.5

We start with parameters $0 < \theta < q^0 < 1$, $r > 0$, $\lambda > 0$. Note that, for all useful purposes (in particular, the inequalities below), r and λ only enter via the ratio λ/r , so one can normalize one of these. Thus, throughout, we normalize r to 1. Also, let $\phi := \theta/q^0 < 1$. Recall (19):

$$q^0 \leq \bar{q}^0 := \frac{\sqrt{8r\lambda\theta + (r - \lambda(1 + \theta))^2} - (r - \lambda(1 + \theta))}{4\lambda}. \quad (44)$$

Also, recall (22):

$$q^0 - \frac{r + \lambda q^0}{r + \lambda} \theta < \lambda q^0 (1 - q^0) \int_0^\infty \frac{e^{-(r+\lambda)s}}{q^0 + (1 - q^0)e^{-\lambda s}} ds. \quad (45)$$

This appendix establishes the following two claims:

Claim 1 *Let (45) hold. Then*

[1.1] $q^0 \leq \bar{q}^0$, and

[1.2] $f(\cdot)$, the derivative of $F(\cdot)$, defined by (8), is positive for all $t \in [0, T]$.

Claim 2 *Let $\alpha_t \in [0, 1]$, where α_t is given by (39), and $q^0 < \bar{q}^0$. Then*

[2.1] $A \leq 0$, where A is given by (40),

[2.2] $A_2(t)$ (see (41)) crosses 0 at most once from above, for $t \leq T$, and

[2.3] $A_3(t)$ (see (43)) is negative for all $t \leq T$ for which $A_2(t) \leq 0$.

Proof.

[**Claim 1.1**] ((45) $\implies q^0 \leq \bar{q}^0$). The inequality $q^0 \leq \bar{q}^0$ is equivalent to

$$1 - \phi \leq \lambda \frac{1 - q^0}{1 + \lambda q^0},$$

whereas (45) is equivalent to (recalling that we normalize $r = 1$)

$$1 - \phi \leq \lambda \frac{1 - q^0}{1 + \lambda q^0} \left((1 + \lambda) \int_0^\infty \frac{e^{-t}}{1 - q^0 + q^0 e^{\lambda t}} dt - 1 \right),$$

We note that $\lambda \frac{1 - q^0}{1 + \lambda q^0} \leq 1 \Leftrightarrow \lambda - 1 \leq 2\lambda q^0$. Hence, $q^0 \leq \bar{q}^0$ is automatically satisfied unless $\lambda - 1 < 2\lambda q^0$. Also, comparing the last two inequalities, (45) $\implies q^0 \leq \bar{q}^0$

unless $(1 + \lambda) \int_0^\infty \frac{e^{-t}}{1 - q^0 + q^0 e^{\lambda t}} dt > 2$. We now show that it is never the case that both $(1 + \lambda) \int_0^\infty \frac{e^{-t}}{1 - q^0 + q^0 e^{\lambda t}} dt > 2$ and $\lambda - 1 < 2\lambda q^0$. We recall that

$$\int_0^\infty \frac{e^{-t}}{1 - q^0 + q^0 e^{\lambda t}} dt = \frac{{}_2F_1\left(1, 1 + \frac{1}{\lambda}; 2 + \frac{1}{\lambda}; -\frac{1 - q^0}{q^0}\right)}{(1 + \lambda)q^0} = \frac{{}_2F_1\left(1, 1; 2 + \frac{1}{\lambda}; 1 - q^0\right)}{1 + \lambda},$$

where the latter equality is an immediate application of a Pfaff transformation. Hence, we want to show that ${}_2F_1\left(1, 1; 2 + \frac{1}{\lambda}; 1 - q^0\right) \geq 2$ and $\lambda - 1 < 2\lambda q^0$ are mutually inconsistent. If $\lambda < 1$, then

$$\begin{aligned} & {}_2F_1\left(1, 1; 2 + \frac{1}{\lambda}; 1 - q^0\right) \\ & \leq {}_2F_1\left(1, 1; 2 + \frac{1}{\lambda}; 1\right) \\ & < {}_2F_1(1, 1; 2 + 1; 1) = 2, \end{aligned}$$

where the first inequality follows from the monotonicity of ${}_2F_1\left(1, 1; 2 + \frac{1}{\lambda}; 1 - q^0\right)$ in q^0 , and the second inequality from the monotonicity of ${}_2F_1\left(1, 1; 2 + \frac{1}{\lambda}; 1\right)$ in λ .

Similarly, if $\lambda \geq 1$, then because $\lambda - 1 < 2\lambda q^0 \Leftrightarrow q^0 > \frac{\lambda - 1}{2\lambda}$, then

$$\begin{aligned} & {}_2F_1\left(1, 1; 2 + \frac{1}{\lambda}; 1 - q^0\right) \\ & < {}_2F_1\left(1, 1; 2 + \frac{1}{\lambda}; \frac{1 + \lambda}{2\lambda}\right) \\ & \leq {}_2F_1(1, 1; 2 + 1; 1) = 2, \end{aligned}$$

using the monotonicity of the hypergeometric function in its final argument for the strict inequality and the weak inequality following from a straightforward application of the definition of the hypergeometric function. This concludes the proof.

[Claim 1.2] ((45) $\Rightarrow f(t) := F'(t) > 0$ for all $t \in (0, T)$). Computing, we obtain

$$f'(t) \propto \lambda(1 - q^0) - (1 - \phi) \left(1 + \lambda q^0 \left(1 + (1 - \phi) \left(e^{(1 + \lambda q^0)t} - 1 \right) \right) \right),$$

which is clearly a decreasing function of t . Hence, it suffices to show that $f'(T) \geq 0$. Recall that $e^{(1 + \lambda q^0)t} = \frac{q_T - \theta}{q^0 - \theta}$. Inserting into the right-hand side, it reduces to

$$\begin{aligned} & \lambda(1 - q^0) - (1 - \phi) \left(1 + \lambda q^0 \left(1 + (1 - \phi) \left(e^{(1 + \lambda q^0)T} - 1 \right) \right) \right) \\ & = \lambda(1 - q^0) - (1 - \phi)(1 + \lambda q_T). \end{aligned}$$

We then solve for q_T from (45), taking $J := \int_0^\infty \frac{e^{-t}}{1 - q_T + q_T e^{\lambda t}} dt$ as given, plug it into $\lambda(1 - q^0) - (1 - \phi)(1 + \lambda q_T)$, and obtain that this expression is positive if J lies below some

upper bound that is lowest when ϕ is lowest. If $\lambda(1 - q^0)/(1 + \lambda q^0) \leq 1$, we plug in $1 - \phi = \lambda(1 - q^0)/(1 + \lambda q^0)$ and get as requirement that $J \leq 2/(1 + \lambda)$. Otherwise, we plug in $1 - \phi = 1$ and get $J \leq 1/(\lambda(1 - q))$, which given $\lambda(1 - q^0)/(1 + \lambda q^0) \geq 1$, is again lower than $2/(1 + \lambda)$. The conclusion then follows from the inequalities established in the previous point (involving the hypergeometric function).

[Claim 2.1] ($A \leq 0$). Clearly, there is nothing to show if $\lambda q^0 - \lambda + r \leq 0$, so assume this is not the case. Then A is increasing in t .

[Step 1] $t = 0 \Rightarrow A < 0$. Note that $q^0 < \bar{q}^0$, from (44), is equivalent to $\phi + \lambda - \lambda q^0(2 - \phi) \geq 1$, or

$$\phi \geq 1 - \lambda \frac{1 - q^0}{1 + \lambda q^0}, \quad (46)$$

whereas, for $t = 0$, $A \geq 0$ is equivalent to

$$\phi \leq 1 - \frac{1 + \lambda(1 + q^0)}{2(1 + \lambda q^0)}, \quad (47)$$

which is only possible (since $\phi > 0$, and so the term subtracted must be less than 1) if $\lambda(1 - q^0) < 1$. Yet, the two inequalities (46) and (47) are only consistent if $1 + \lambda(1 + q^0) < 2\lambda(1 - q^0)$, which cannot hold if $\lambda(1 - q^0) < 1$. Hence, for $t = 0$, $A < 0$, and α_t is decreasing for small enough t .

[Step 2] $\alpha_0 > 0$. For $t = 0$, we have

$$\alpha_0 = 1 - \frac{(1 - \phi)(1 + \lambda q^0)}{\lambda(1 - q^0)}.$$

Again, it is readily shown that (46) is inconsistent with this expression being negative.

[Step 3] $t > 0 \Rightarrow A < 0$. Let us prove that the value of t such that $A = 0$ is such that $\alpha_t < 0$. (By Step 1, this value of t must be strictly positive, and since without loss $\lambda q^0 - \lambda + r > 0$, it does exist.) Hence, it will follow that $\alpha_t \geq 0 \Rightarrow A < 0$ (and so α decreasing). Evaluating α_t at the time \tilde{t} such that $A = 0$ gives

$$\alpha_{\tilde{t}} = \frac{\phi(q^0)^2(\lambda q^0 + 1) \left\{ 4\lambda\phi(1 - q^0)(\lambda q^0 + 1) - \left((\lambda q^0 + \lambda + 1)(1 - \lambda(1 - q^0)) \left(\frac{\phi(\lambda q^0 + \lambda + 1)}{(1 - \phi)(\lambda(q^0 - 1) + 1)} \right)^{\frac{\lambda}{\lambda q^0 + 1}} \right) \right\}}{(\lambda(1 - q^0) + 1)^2}.$$

The term in curly brackets is first convex then concave in ϕ , and goes from 0 to $-\infty$ as ϕ goes from 0 to 1. The slope at $\phi = 0$ is $-\infty$, and it has at most one critical point (given

that the first term in curly brackets is linear in ϕ), so cannot be positive on $(0, 1)$. Hence, $\alpha_{\tilde{t}} < 0$, concluding the argument.

[Claim 2.2] ($A_2(t)$ crosses 0 at most once from above on $[0, T]$). Given our normalization and our notation, with $x := e^{(\lambda(1+q^0)+r)t}$, $A_2 = (q^0)^2 \tilde{A}_2(x)$ with

$$\tilde{A}_2(x) := \lambda(1 - q^0) \left(\phi + (1 - \phi)x^{\frac{1+\lambda q^0}{1+\lambda(1+q^0)}} \right)^2 - (1 + \lambda q^0)(1 - \phi)x,$$

whose second derivative with respect x is

$$\frac{2(1 - \phi)(1 - q^0)\lambda(1 + \lambda q^0)x^{-\frac{1+\lambda(2+q^0)}{1+\lambda(1+q^0)}}}{(1 + \lambda(1 + q^0))^2} D(x),$$

with

$$D(x) := (1 - \phi)(1 - \lambda(1 - q^0))x^{\frac{1+\lambda q^0}{1+\lambda(1+q^0)}} - \lambda\phi.$$

We note that $\tilde{A}_2(1) = \lambda(1 - q^0) - (1 + \lambda q^0)(1 - \phi) > 0$ given that $1 - \phi < \frac{\lambda(1 - q^0)}{1 + \lambda q^0}$ (cf. (45)), and also, from inspection of the exponent in $\tilde{A}_2(x)$, we have $\lim_{x \rightarrow \infty} \tilde{A}_2(x) = +\infty$ if $\lambda(1 - q^0) < 1$ and $\lim_{x \rightarrow \infty} \tilde{A}_2(x) = -\infty$ if $\lambda(1 - q^0) > 1$. The term $D(x)$ is (possibly) first negative then positive in x , so that \tilde{A}_2 is first concave then convex, and so $\tilde{A}_2(\cdot)$ (and hence $A(t)$) has either 0, 1 or 2 roots, and the first (if any) is from above.

Further, using $\tilde{A}_2(x) = 0$ so simplify $\tilde{A}'_2(x) \Big|_{\tilde{A}_2(x)=0}$ gives

$$\operatorname{sgn} \tilde{A}'_2(x) \Big|_{\tilde{A}_2(x)=0} = \operatorname{sgn} (q_t(1 - \lambda(1 - q^0)) - 2q^0(1 + \lambda q^0)\phi),$$

where $q_t := \theta + (q^0 - \theta)e^{(1+\lambda q^0)t}$. The right side is increasing in q_t , and so it is maximum over $[0, T]$ when $t = T$, and so when $q_t = q_T$, where we recall that q_T , the belief when delay in the equilibrium with delay stops, solves

$$\phi = \frac{\lambda(1 - q^0)}{q^0(1 + \lambda q^0)} \left({}_2F_1 \left(1, 1; 2 + \frac{1}{\lambda}; 1 - q_T \right) q_T - q^0 \right), \quad (48)$$

which implies, with some elementary algebra, that $\tilde{A}'_2(x) \Big|_{\tilde{A}_2(x)=0} < 0$. Hence, to conclude, over the interval $[0, T]$, $A_2(t)$ crosses 0 at most once, and if it does, say, at t^* , it is positive for $t < t^*$, and negative for $t \in (t^*, T]$.

[Claim 2.3] ($A_3(t)$ (see (43)) is negative for all $t \leq T$ for which $A_2(t) \leq 0$). Note that $A_3(0) = 0$, $\lim_{t \rightarrow \infty} A_3(t) = +\infty$. Further, $A'_3(0) = -q^0((1 + \lambda q^0)\phi - 1 - \lambda - 2\lambda q^0)$, a decreasing function of ϕ , equal to zero when $\phi = 1 - \frac{\lambda(1 - q^0)}{1 - \lambda q^0}$; hence, given that $1 - \phi < \frac{\lambda(1 - q^0)}{1 + \lambda q^0}$

(cf. (45)), $A_3'(0) < 0$, and so $A_3(t) < 0$ for all $t > 0$ small enough. Further, if we let $\tilde{A}_3(x) = A_3(t)$, with $x = e^{\lambda t}$, we readily verify that $\tilde{A}_3''(x)$ admits at most one root (the second derivative is the product of some positive terms, multiplied by an affine increasing function of x); hence, $\tilde{A}_3''(x) > 0$ for all $x > x^*$ for some $x^* \geq 0$ (possibly $x^* = 0$, depending on the sign of $1 - \lambda(1 - q^0)$). Hence, as a function of x , it is first concave (up to x^*) and then convex. Since it starts at 0, is first decreasing, and $\lim_{t \rightarrow \infty} \tilde{A}_3(x) = +\infty$, it admits exactly one root, and so does $A_3(t)$. It remains to evaluate $A_3(T)$, where T is the time at which delay stops in the equilibrium with delay. Since $A_3(t)$ is an affine decreasing function of ϕ , it is negative if and only if ϕ is above a threshold (a function of t). Comparing this threshold, evaluated at T , to (48), it is readily verified that $A_3(T) \leq 0$. ■

B.2 Differentiability Properties For Proof Of Theorem 1

Differentiability of $t \mapsto V_t(\tilde{q})$ is immediate, considering that the payoff is the integral of the flow payoff. Differentiability of $V_t(\cdot)$ is also immediate, since the payoff is a continuous and convex function of the belief, by familiar arguments.

For the sake of contradiction, suppose that $t \mapsto \bar{q}_t$ is not non-decreasing over some interval $[t_1, t_2]$, some $t_2 > t_1$. Plainly, this implies that $t \mapsto \tilde{q}_t^*$ is not non-decreasing on this interval either (if \tilde{q}_t^* increases, then $\mathbf{E}[\tilde{q}_t \mid \tilde{q}_t \geq \tilde{q}_t^*]$ increases, as \tilde{q}_t is non-decreasing in t). Let $A = \{t \in [t_1, t_2] : \bar{q}_t < \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \bar{q}_s ds\}$ (in words, those times where \bar{q}_t is lower than its average value). By assumption, we can take t_1, t_2 such that, for some $A' \subset A$, $A'' \subset [t_1, t_2] \setminus A$, both of positive (without loss, equal) measure, $t'' \in A'', t' \in A' \rightarrow t'' < t'$ (“the set of times in A'' arrive before those in A' ”), and, for some types \tilde{q}_{t_1} of positive measure under σ (and some pure strategy within the support of their continuation strategy), $k_t = 0$ for all $t \in A'$, $k_t = 1$ for all $t \in A''$ (“types that take high clients at some times when the belief—and hence the price—is below average, and low clients when it is above”). We can then define an alternative (pure) strategy for this type, σ' , such that (along the history of no failure) $k'_t = k_t$, all $t \notin A', A''$, and $k'_t = 1 - k_t$ for all $t \in A' \cup A''$. Plainly, positive, higher prices (over and above the cost of potential foregone continuation payoff due to failure) are pocketed earlier, over a length of time of equal duration. The straightforward calculation is omitted.

A similar re-arrangement can be done to show that \bar{q}_t cannot have a jump up (the price would jump up, yet that set of types switching to low clients would strictly benefit from taking high clients at the higher price an instant after, rather than at the lower price prevailing an instant before). Hence, \bar{q}_t is continuous. By indifference of the type \tilde{q}_t^* , whenever $\tilde{q}_t^* \in (0, 1)$,

$$\bar{q}_t - \theta = \lambda(1 - \tilde{q}_t^*)X_t^\lambda(\tilde{q}_t^*),$$

where $X_t^\lambda(\tilde{q})$ is the continuation payoff at time t under an optimal strategy of an expert

whose belief is \tilde{q} and whose type is inept—plainly, a continuous function in both \tilde{q} and t , and strictly increasing in \tilde{q} . Hence, its inverse is continuous, and since \bar{q}_t is continuous, so is \tilde{q}_t^* . Since \bar{q}_t is monotone, it is a.e. differentiable. Next, indifference at \tilde{q}_t^* means

$$\bar{q}_t - \theta = \lambda(1 - \tilde{q}_t^*) \int_{\{s: \bar{q}_{t+s} \geq \tilde{q}_{t+s}^*\}} e^{-r(s-t) - \lambda m(\{[t,s] \cap \{u: q_{t+u} \geq \tilde{q}_{t+u}^*\})} (\bar{q}_{t+s} - \theta) ds,$$

where $m(\cdot)$ is Lebesgue measure. Since \tilde{q}^* is continuous, $k_t = 1$ on some set of intervals I_k (and 0 otherwise), and we can rewrite

$$\frac{1}{1 - \tilde{q}_t^*} = \frac{\lambda \sum_k \int_{I_k} e^{-r(s-t) - \lambda m(\{[t,s] \cap \cup_k I_k\})} (\bar{q}_{t+s} - \theta) ds}{\bar{q}_t - \theta}.$$

Since the right-hand side is differentiable a.e., so must \tilde{q}_t^* be. Given q^0 , define \bar{q} as the solution in q to

$$q - \theta = \lambda(1 - q^0) \int_0^\infty e^{-(r+\lambda)s} \left(\frac{q}{q + (1-q)e^{-\lambda s}} - \theta \right) ds.$$

or, in terms of the hypergeometric function ${}_2F_1$,

$$\frac{1}{1 - q^0} = \frac{{}_2F_1\left(1, \frac{r}{\lambda} + 1; \frac{r}{\lambda} + 2; \frac{q-1}{q}\right) - \theta}{\left(1 + \frac{r}{\lambda}\right)(q - \theta)}.$$

The right side is strictly decreasing in q for $q > \theta$, with limit $+\infty$ as $q \downarrow \theta$, and limit $1/\left(1 + \frac{r}{\lambda}\right) < 1$ as $q \uparrow 1$. Hence, given that $\frac{1}{1-q^0} > 1$, this equation has a unique solution q on $[\theta, 1]$; of course, not necessarily higher than q^0 . Since $(1 - q^0)^{-1}$ is increasing in q^0 , \bar{q} is decreasing in q^0 , and so there exists $\bar{q}^0 > \theta$ such that, for all $q^0 \in (\theta, \bar{q}^0)$, $\bar{q} > q^0$.

B.3 Tea Breaks During Delay Are Suboptimal

Consider here an equilibrium in which the agent is indifferent to start between all times up to T , except for the tea breaks, modeled as before as a fraction allocated to high types (without indifference). Let $\phi_t \in [0, 1]$ denote the fraction allocated to high types at time t , $t < T$. The function ϕ is a parameter (in particular, $\phi_t = 1$, all $t > T$). We generalize the formulas for the case without tea breaks ($\phi_t = 1$ all t), by following the same steps as in Appendix A.3.1.

The payoff from starting at time t is

$$V_t = e^{-rt} \int_0^\infty h(s, t) y_{t+s} ds,$$

with

$$h(s, t) := e^{-rs} \left(q^0 + (1 - q^0) e^{-\lambda \int_t^{t+s} \phi(u) du} \right), \quad y_t := \phi_t p_t.$$

Consider times $t \in [0, T]$ over which the expert is indifferent to start. Setting the derivative of V_t equal to 0 gives

$$\begin{aligned} 0 &= \int_0^\infty h(s, t) (\dot{y}_{t+s} - r y_{t+s}) ds - \lambda(1 - q^0) \int_0^\infty e^{-rs} (\phi(t+s) - \phi(t)) e^{-\lambda \int_t^{t+s} \phi(u) du} y_{t+s} ds \\ &= -y_t + \lambda(1 - q^0) \int_0^\infty \phi(t) e^{-rs} e^{-\lambda \int_t^{t+s} \phi(u) du} y_{t+s} ds, \end{aligned}$$

using integration by parts. We can differentiate once more with respect to t , and use integration by parts, so that

$$\begin{aligned} \dot{y}_t &= \lambda(1 - q^0) \left[-y_t - \int_0^\infty \phi(t) \frac{d}{ds} \left(e^{-rs} e^{-\lambda \int_t^{t+s} \phi(u) du} \right) y_{t+s} ds + \int_0^\infty e^{-rs} \frac{d}{dt} \left(\phi(t) e^{-\lambda \int_t^{t+s} \phi(u) du} \right) y_{t+s} ds \right] \\ &= -\lambda(1 - q^0) y_t + r y_t + \lambda(1 - q^0) \int_0^\infty e^{-rs} (\phi'(t) + \lambda \phi(t)^2) e^{-\lambda \int_t^{t+s} \phi(u) du} y_{t+s} ds \\ &= \left(r - \lambda(1 - q^0) + \lambda \phi(t) + \frac{\phi'(t)}{\phi(t)} \right) y_t, \end{aligned}$$

and so

$$\frac{y_t}{y_0} = \frac{\phi_t}{\phi_0} e^{(r - \lambda(1 - q^0))t + \lambda \int_0^t \phi_s ds},$$

or, given that $y_t = \phi_t(q_t - \theta)$,

$$\frac{q_t - \theta}{q^0 - \theta} = e^{(r + \lambda q^0)t - \lambda \int_0^t (1 - \phi_s) ds}. \quad (49)$$

Recognizing that, independent of $\phi(t)$, $t \leq T$, q_T is the same as in the equilibrium with delay,³³ and that the expert is willing to wait until then to start taking on high clients, maximizing his payoff (and hence welfare) is equivalent to minimizing T , which, given (49), solves

$$(r + \lambda q^0)T = \lambda \int_0^T (1 - \phi_t) dt + \ln \frac{q_T - \theta}{q^0 - \theta}.$$

Plainly, $T \geq \frac{1}{r + \lambda q^0} \ln \frac{q_T - \theta}{q^0 - \theta}$ with equality if and only if $\phi_t = 1$ for all $t \leq \frac{1}{r + \lambda q^0} \ln \frac{q_T - \theta}{q^0 - \theta}$. Hence, it is optimal to set $\phi_t = 1$, that is, to not introduce any “breaks:” conditional on starting at any time $t \leq T$, the expert only takes high clients for all $s \geq t$ (until a failure occurs, if ever).

³³More generally, if $\phi_t < 1$ for some $t > T$, it is the same as in the case in which $\phi_t = 1$ for all $t < T$, fixing $\phi(\cdot)$ over (T, ∞) .

B.4 Learning-By-Doing

The agent's value function (as always, conditional on $\omega = B$ and no failure) solves the following optimality equation

$$X_t^\lambda = \max_k \{ (q_t - \theta)kdt + (1 - rdt)(1 - \lambda kdt)(1 - \gamma kdt)(X_t^\lambda + \dot{X}_t^\lambda dt) + \gamma kdt X_t^0 \},$$

where X_t^0 denotes the continuation value when failures no longer occur (state G). As $dt \rightarrow 0$, this becomes

$$0 = \max_k \{ (q_t - \theta)k - (r + \lambda k + \gamma k)X_t^\lambda + \dot{X}_t^\lambda + \gamma k X_t^0 \}. \quad (50)$$

The equilibrium, for low enough priors $q^0 > \theta$, calls for the bad expert to choose $k_t \in (0, 1)$ until some time T , after which $k_t = 1$. For $t \in (0, T)$, this requires

$$q_t - \theta = (\lambda + \gamma)X_t^\lambda - \gamma X_t^0 \quad (51)$$

$$0 = \dot{X}_t^\lambda - rX_t^\lambda. \quad (52)$$

Equation (51) ensures the indifference over k needed to support an interior value of k , while (52) is an accounting relationship obtained by considering the case of $k = 0$.

For $t \geq T$, X_t^λ is given by

$$X_t^\lambda = \int_t^\infty e^{-(r+\lambda+\gamma)(s-t)} (q_s - \theta + \gamma X_s^0) ds, \quad (53)$$

whereas, for all t ,

$$X_t^0 = \int_t^\infty e^{-r(s-t)} (q_s - \theta) ds. \quad (54)$$

Differentiating (53) confirms that (52) also holds for $t > T$, while differentiating (54) yields the counterpart (for all t) for X_t^0 :

$$0 = \dot{X}_t^0 - rX_t^0 + q_t - \theta. \quad (55)$$

We can combine (53) and (54) to obtain the following, subsequently useful relationship:

$$X_t^\lambda = X_t^0 - \frac{\lambda}{\lambda + \gamma} \int_t^\infty e^{-r(s-t)} (q_s - \theta) (1 - e^{-(\lambda+\gamma)(s-t)}) ds. \quad (56)$$

Bayes' rule reads, for all t ,

$$q_{t+dt} = \frac{q_t + (1 - q_t)\gamma k_t dt}{q_t + (1 - q_t)(1 - \lambda k_t dt)}, \text{ or, as } dt \rightarrow 0, \dot{q}_t = (1 - q_t)(\gamma + \lambda q_t)k_t. \quad (57)$$

In particular, for $t \geq T$,

$$q_t = 1 - \frac{(\gamma + \lambda)(1 - q_T)}{\lambda(1 - q_T) + (\gamma + \lambda q_T)e^{(\gamma + \lambda)(t - T)}}. \quad (58)$$

Using (58) to simplify (53) and (54), we obtain

$$X_T^0 = \frac{1 - \theta}{r} - \frac{(\gamma + \lambda)(1 - q_T) {}_2F_1\left(1, \frac{\gamma + \lambda + r}{\gamma + \lambda}; 2 + \frac{r}{\gamma + \lambda}; \frac{\lambda(q_T - 1)}{\gamma + \lambda q_T}\right)}{(\gamma + \lambda + r)(\gamma + \lambda q_T)},$$

and

$$\begin{aligned} X_T^\lambda &= \frac{r + \gamma}{r + \gamma + \lambda} \frac{1 - \theta}{r} - \frac{\gamma(1 - q_T) {}_2F_1\left(1, \frac{\gamma + \lambda + r}{\gamma + \lambda}; 2 + \frac{r}{\gamma + \lambda}; \frac{\lambda(q_T - 1)}{\gamma + \lambda q_T}\right)}{(\gamma + \lambda + r)(\gamma + \lambda q_T)} \\ &\quad - \frac{\lambda(1 - q_T) {}_2F_1\left(1, \frac{r}{\gamma + \lambda} + 2; \frac{r}{\gamma + \lambda} + 3; \frac{\lambda(q_T - 1)}{\gamma + \lambda q_T}\right)}{(2(\gamma + \lambda) + r)(\gamma + \lambda q_T)}. \end{aligned}$$

We may differentiate (51) and use (52) and (55) to simplify, giving

$$\dot{q}_t = (r + \gamma)(q_t - \theta).$$

We then solve for

$$q_t - \theta = e^{(r + \gamma)t}(q_0 - \theta). \quad (59)$$

We then need only identify q_T , which fixes T .

To determine q_T and T , consider $t > T$. There, the counterpart of (51) given by $q_t - \theta \geq (\lambda + \gamma)X_t^\lambda - \gamma X_t^0$ must hold. We may then first solve for the lowest belief q_T for which this inequality holds. Hence, we solve

$$q_t - \theta = (\lambda + \gamma)X_t^\lambda - \gamma X_t^0 = \lambda X_t^0 - (\lambda + \gamma)(X_t^0 - X_t^\lambda).$$

Using (54) and (56), we can write this as

$$q_t - \theta + \lambda \int_t^\infty e^{-r(s-t)} (q_s - \theta) (1 - e^{-(\lambda + \gamma)(s-t)}) ds = \lambda \int_t^\infty e^{-r(s-t)} (q_s - \theta) ds,$$

Simplifying, we obtain

$$q_T - \theta = \lambda \int_T^\infty e^{-(r + \gamma + \lambda)(t - T)} (q_t - \theta) dt,$$

where we recall that q_t is given by (58). Simplifying,

$$q_T - \theta + \frac{\lambda}{r + \lambda + \gamma}(\theta - 1) = -\frac{\lambda(\gamma + \lambda)(1 - q_T) {}_2F_1\left(1, \frac{r}{\gamma + \lambda} + 2; \frac{r}{\gamma + \lambda} + 3; \frac{\lambda(q_T - 1)}{\gamma + \lambda q_T}\right)}{(2(\gamma + \lambda) + r)(\gamma + \lambda q_T)}.$$

One readily shows that this uniquely determines q_T . If $q^0 \geq q_T$, it follows that the expert always picks high clients. If instead $q^0 < q_T$, the bad expert takes the fraction k_t of high clients given by (57) (plugging in q_t from (59)) until time T . Given q^0 and q_T , we may use (59) to explicitly solve for T .