Supplemental Appendix for "Innovation and Competition on a Rugged Technological Landscape"

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Abstract

This supplemental appendix describes the model variants in Section 5.3 of the main text and derives optimal innovation by the firms. All section references refer to the main text.

1 A Preference for Risk

We augment the model in the main text by allowing firms to have a preference for risk. The objective function of firm t is now given by

$$\pi_t^R(l_t) \equiv \pi_t(l_t) + \alpha_r \text{Var}[v(l_t)] \text{ for } r = f, n,$$

where $\pi_t(l_t)$ are profits (Equation 6 of the main text), and the parameters $\alpha_f \geq 0$ and $\alpha_n \geq 0$ capture how much firms value risk when they engage in frontier and niche innovation respectively. The superscript R stands for "risk." We assume that when consumers are indifferent between buying a new product and not, at least some of them do, so that the product's quality becomes known.

A property of the model in the main text is that the market is always covered: if consumer $s' \in \mathcal{P}$ buys a product, all consumers $s \leq s'$ also buy a product. To ensure that the market continues to be covered when firms have a preference for risk, we assume that fixed costs are

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not too small, specifically,

$$F \geq \underline{F} \equiv \frac{\tau\left(1+2c\tau\right)}{\left(2c\tau-1\right)^2}\alpha_f^2\sigma^4. \tag{1}$$

The rest of the model is the same as in the main text. Below we provide an analytical characterization of the firms' behavior conditional on engaging in frontier or niche innovation by adapting Propositions 1 to 3 in the main text. We also show that condition (1) ensures that the market is covered. We then use these results in the simulations of the firms' optimal innovation strategies reported in the main text.

Proposition R1 *There is a threshold*

$$\underline{v}_1^R \geq \frac{2\alpha_f\sigma^2}{2c\tau-1}$$

such that, in period one,
(i.) if $v\left(l_0\right) \leq \underline{v}_1^R$*, there is no profitable innovation.*
(ii.) if $v\left(l_0\right) > \underline{v}_1^R$*, the optimal innovation is located at*

$$l_1^{R*} = \frac{2\tau}{2c\tau+1}\left(v\left(l_0\right)+\alpha_f\sigma^2\right)$$

and generates benefits

$$\pi_t^R\left(l_1^{R*}, F\left|\mathcal{E}_t\right.\right) = \frac{\tau}{2c\tau+1}\left(v\left(l_0\right)+\alpha_f\sigma^2\right)^2 - F > 0.$$

Proof: In the first period, the firm solves

$$\max_{l_1 \geq l_0} \pi_1^R\left(l_1\right), \tag{2}$$

where

$$\pi_1^R\left(l_1\right) = \pi_1\left(l_1\right) + \alpha_f \mathrm{Var}\left[v\left(l_1\right)\right]. \tag{3}$$

We know from Equation 3 of the main text that

$$\mathrm{Var}\left[v\left(l_1\right)\right] = \left(l_1-l_0\right)\sigma^2$$

and from Equation 7 in the proof of Proposition 1 that

$$\pi_1\left(l_1\right) = \begin{cases} 0 & \text{if } l_1 = l_0 \\ \tau v\left(l_0\right)^2 - \frac{1}{4\tau}\left(2\tau v\left(l_0\right)-\left(l_1-l_0\right)\right)^2 - \frac{1}{2}c\left(l_1-l_0\right)^2 - F & \text{if } l_1 \in \left(l_0, l_0+2\tau v\left(l_0\right)\right] \\ \tau v\left(l_0\right)^2 - \frac{1}{2}c\left(l_1-l_0\right)^2 - F & \text{if } l_1 \in \left[l_0+2\tau v\left(l_0\right), \infty\right). \end{cases}$$

Substituting into (3) we have

$$\pi_1^R(l_1) = \begin{cases} 0 & \text{if } l_1 = l_0 \\ \tau v(l_0)^2 - \frac{1}{4\tau}\left(2\tau v(l_0) - (l_1 - l_0)\right)^2 \\ \quad + (l_1 - l_0)\alpha_f \sigma^2 - \frac{1}{2}c(l_1 - l_0)^2 - F & \text{if } l_1 \in (l_0, l_0 + 2\tau v(l_0)] \\ \tau v(l_0)^2 + (l_1 - l_0)\alpha_f\sigma^2 - \frac{1}{2}c(l_1 - l_0)^2 - F & \text{if } l_1 \in [l_0 + 2\tau v(l_0), \infty). \end{cases} \tag{4}$$

Let $\widehat{l}_1^R(v(l_0))$ denote the solution to the firm's problem (2) if $F = 0$. It is routine to show that

$$\widehat{l}_1^R(v(l_0)) = \begin{cases} \frac{1}{c}\alpha_f\sigma^2 & \text{if } v(l_0) \in \left[0, \frac{\alpha_f\sigma^2}{2c\tau}\right] \\ \frac{2\tau}{2c\tau+1}\left(v(l_0) + \alpha_f\sigma^2\right) & \text{if } v(l_0) \in \left[\frac{\alpha_f\sigma^2}{2c\tau}, \infty\right) \end{cases}$$

and

$$\pi_1^R\left(\widehat{l}_1^R(v(l_0))\right) = \begin{cases} \frac{1}{2c}\left(2c\tau v(l_0)^2 + \alpha_f^2\sigma^4\right) - F & \text{if } v(l_0) \in \left[0, \frac{\alpha_f\sigma^2}{2c\tau}\right] \\ \frac{\tau}{2c\tau+1}\left(v(l_0) + \alpha_f\sigma^2\right)^2 - F & \text{if } v(l_0) \in \left[\frac{\alpha_f\sigma^2}{2c\tau}, \infty\right). \end{cases}$$

Notice that $\pi_1^R\left(\widehat{l}_1^R(v(l_0))\right)$ is increasing and convex in $v(l_0)$ and that

$$\pi_1^R\left(\widehat{l}_1^R\left(\frac{2\alpha_f\sigma^2}{2c\tau - 1}\right)\right) = 0 \text{ if } F = \underline{F}.$$

This implies that, for any $F \geq \underline{F}$, there exists a threshold value $\underline{v}_1^R \geq \frac{2\alpha_f\sigma^2}{2c\tau-1}$ such that $\pi_1^R\left(\widehat{l}_1^R\left(\underline{v}_0^R\right)\right) = 0$, $\pi_1^R\left(\widehat{l}_1^R(v(l_0))\right) < 0$ if $v(l_0) < \underline{v}_1^R$, and $\pi_1^R\left(\widehat{l}_1^R(v(l_0))\right) > 0$ if $v(l_0) > \underline{v}_1^R$. We can then state the solution to (2) as

$$l_1^{R*} = \begin{cases} l_0 & \text{if } v(l_0) \in \left[0, \underline{v}_1^R\right] \\ \frac{2\tau}{2c\tau+1}\left(v(l_0) + \alpha_f\sigma^2\right) & \text{if } v(l_0) \in \left(\underline{v}_1^R, \infty\right) \end{cases}$$

and

$$\pi_1^R\left(l_1^{R*}\right) = \begin{cases} 0 & \text{if } v(l_0) \in \left[0, \underline{v}_1^R\right] \\ \frac{\tau}{2c\tau+1}\left(v(l_0) + \alpha_f\sigma^2\right)^2 - F > 0 & \text{if } v(l_0) \in \left(\underline{v}_1^R, \infty\right). \end{cases}$$

The statement in the proposition follows from this solution. ■

Proposition R2 *There is a threshold*

$$\underline{v}_t^R \geq \frac{2\alpha_f\sigma^2}{2c\tau - 1}$$

such that, in any period $t \geq 2$,

(i.) if $v\left(l_t^f\right) \leq \underline{v}_t^R$*, there is no profitable frontier innovation.*

(ii.) if $v\left(l_t^f\right) > \underline{v}_t^R$ *and* $u\left(l_t^f, l_t^a\right) = v\left(l_t^f\right)$*, the optimal frontier innovation is located at*

$$l_t^{R*} = l_t^f + \frac{2\tau}{2c\tau + 1}\left(v\left(l_t^f\right) + \alpha_f \sigma^2\right)$$

and generates benefits

$$\pi_t^R\left(l_t^{R*} | \mathcal{E}_t\right) = \frac{\tau}{2c\tau + 1}\left(v\left(l_t^f\right) + \alpha_f \sigma^2\right)^2 - F > 0.$$

(iii.) if $v\left(l_t^f\right) > \underline{v}_t^R$ *and* $u\left(l_t^f, l_t^a\right) > v\left(l_t^f\right)$*, the optimal frontier innovation is located at*

$$\begin{aligned} l_1^{R*} &= l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right) \\ &\quad + \frac{2\tau(1 + c\tau)}{2c\tau + 1}\max\left(0, v\left(l_t^f\right) - \frac{c\tau}{(1 + c\tau)}\left(u\left(l_t^f, l_t^a\right) - \frac{\sigma^2 \alpha_f}{c\tau}\right)\right) \end{aligned}$$

and generates benefits

$$\begin{aligned} \pi_t^R\left(l_t^{R*} | \mathcal{E}_t\right) &= \frac{1}{2}\tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right)\left(2\sigma^2 \alpha_f - c\tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right)\right) \\ &\quad + \frac{\tau(1 + c\tau)^2}{2c\tau + 1}\max\left(0, v\left(l_t^f\right) - \frac{c\tau}{(1 + c\tau)}\left(u\left(l_t^f, l_t^a\right) - \frac{\sigma^2 \alpha_f}{c\tau}\right)\right)^2 - F > 0. \end{aligned}$$

Proof: Suppose firm t locates to the right of the frontier product l_t^f. Its optimal location solves

$$\max_{l_t \geq l_t^f} \pi_t^R\left(l_t\right), \tag{5}$$

where

$$\pi_t^R\left(l_t\right) = \pi_t\left(l_t\right) + \alpha_f \mathrm{Var}\left[v\left(l_t\right)\right]. \tag{6}$$

The quality of product $l_t > l_t^f$ is only realized if at least one consumer buys it, that is, if $l_t \geq l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right)$. We, therefore, have

$$\mathrm{Var}\left[v\left(l_t\right)\right] = \begin{cases} 0 & \text{if } l_t \in \left[l_t^f, l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right)\right) \\ \left(l_t - l_t^f\right)\sigma^2 & \text{if } l_t \in \left[l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right), \infty\right). \end{cases}$$

It is immediate that if $v\left(l_t^f\right) < 0$ the solution to the firm's problem (5) is to locate at l_t^f and make zero profits. For the remainder of this proof suppose then that $v\left(l_t^f\right) \geq 0$. Using the expression for profits (Equation 9 in the proof of Proposition 2) we then have that benefits are given by

$$\pi_t^R(l_t) = 0 \tag{7}$$

if $l_t = l_t^f$,

$$\pi_t^R(l_t) = -\frac{1}{2}c\left(l_t - l_t^f\right)^2 - F \tag{8}$$

if $l_t \in \left(l_t^f, l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right)\right)$,

$$\begin{aligned}\pi_t^R(l_t) &= \tau v\left(l_t^f\right)^2 - \frac{\tau}{4}\left(u\left(l_t^f, l_t^a\right) + v\left(l_t^f\right) - \frac{1}{\tau}\left(l_t - l_t^f\right)\right)^2 &(9)\\ &\quad +\alpha_f\left(l_t - l_t^f\right)\sigma^2 - \frac{1}{2}c\left(l - l_t^f\right)^2 - F &(10)\end{aligned}$$

if $l_t \in \left[l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right), l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) + v\left(l_t^f\right)\right)\right]$, and

$$\pi_t^R(l_t) = \tau v\left(l_t^f\right)^2 + \alpha_f\left(l_t - l_t^f\right)\sigma^2 - \frac{1}{2}c\left(l_t - l_t^f\right)^2 - F \tag{11}$$

if $l_t \in \left[l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) + v\left(l_t^f\right)\right), \infty\right)$.

Case 1 (active frontier product) Suppose first that the frontier product is active, $u\left(l_t^f, l_t^a\right) = v\left(l_t^f\right)$. Let $\widehat{l}_t^{RA}\left(v\left(l_t^f\right)\right)$ denote the solution to the firm's problem (5) if $F = 0$, where the superscript A stands for "active." It is routine to show that

$$\widehat{l}_t^{RA}\left(v\left(l_t^f\right)\right) = \begin{cases} l_t^f + \frac{\alpha_f\sigma^2}{c} & \text{if } v\left(l_t^f\right) \in \left[0, \frac{\alpha_f\sigma^2}{2c\tau}\right] \\ l_t^f + \frac{2\tau}{2c\tau+1}\left(v\left(l_t^f\right) + \alpha_f\sigma^2\right) & \text{if } v\left(l_t^f\right) \in \left[\frac{\alpha_f\sigma^2}{2c\tau}, \infty\right). \end{cases}$$

Substituting into (7) to (11) we have

$$\pi_t^R\left(\widehat{l}_t^{RA}\left(v\left(l_t^f\right)\right)\right) = \begin{cases} \tau v\left(l_t^f\right)^2 + \frac{\alpha_f^2\sigma^4}{2c} - F & \text{if } v\left(l_t^f\right) \in \left[0, \frac{\alpha_f\sigma^2}{2c\tau}\right] \\ \frac{\tau}{2c\tau+1}\left(v\left(l_t^f\right) + \alpha_f\sigma^2\right)^2 - F & \text{if } v\left(l_t^f\right) \in \left[\frac{\alpha_f\sigma^2}{2c\tau}, \infty\right). \end{cases}$$

Notice that $\pi_t^R(0) < 0$ if $F = \underline{F}$ and that, for all $v\left(l_t^f\right) \in [0, \infty)$,

$$\frac{\mathrm{d}\pi_t^R\left(\widehat{l}_t^{RA}\left(v\left(l_t^f\right)\right)\right)}{\mathrm{d}v\left(l_t^f\right)} > 0 \text{ and } \frac{\mathrm{d}^2\pi_t^{RA}\left(\widehat{l}_t^{R}\left(v\left(l_t^f\right)\right)\right)}{\mathrm{d}v\left(l_t^f\right)^2} \geq 0.$$

This implies that, for any $F \geq \underline{F}$, there exists a threshold value $\underline{v}_t^{RA} > 0$ such that

$$\pi_t^R\left(\widehat{l}_t^R\left(\underline{v}_t^{RA}\right)\right) = 0.$$

Next, notice that if $F = \underline{F}$ we have

$$\pi_t^R\left(\widehat{l}_t^{RA}\left(\frac{2\alpha_f\sigma^2}{2c\tau - 1}\right)\right) = \pi_t^R\left(l_t^f + \tau v\left(l_t^f\right)\right) = 0,$$

so that

$$\underline{v}_t^{RA} \geq \frac{2\alpha_f\sigma^2}{2c\tau - 1} > \frac{\alpha_f\sigma^2}{2c\tau}\ .$$

If the frontier product is active, the solution to the firm's problem (5) is then given by

$$l_t^{RA*} = l_t^f \text{ and } \pi_t^R\left(l_t^{RA*}\right) = 0$$

if $v\left(l_t^f\right) \in \left[0, \underline{v}_t^R\right]$ and

$$l_1^{RA*} = l_t^f + \frac{2\tau}{2c\tau + 1}\left(v\left(l_t^f\right) + \alpha_f\sigma^2\right)$$

and

$$\pi_t^R\left(l_t^{RA*}\right) = \frac{\tau}{2c\tau + 1}\left(v\left(l_t^f\right) + \alpha_f\sigma^2\right)^2 - F > 0$$

if $v\left(l_t^f\right) \in \left(\underline{v}_t^R, \infty\right)$.

Case 2 (inactive frontier product) Suppose next that the frontier product is inactive, $u\left(l_t^f, l_t^a\right) > v\left(l_t^f\right)$. Frontier innovation is less profitable if the frontier product is inactive than if it is active. Accordingly, the expressions for $\pi_t^R\left(l_t\right)$ in (7) to (11) are decreasing in $u\left(l_t^f, l_t^a\right)$. We know from Case 1 that even if the frontier product is active, frontier innovation generates negative benefits if $v\left(l_t^f\right) < \underline{v}_t^{RA}$. If the frontier product is inactive, frontier innovation must, therefore, also generate negative benefits if $v\left(l_t^f\right) < \underline{v}_t^{RA}$.

Suppose then that $v\left(l_t^f\right) \geq \underline{v}_t^{RA}$. Let $\widehat{l}_t^{RI}\left(v\left(l_t^f\right)\right)$ denote the solution to the problem

$$\max_{l_t} \pi_t^R\left(l_t\right) \text{ subject to } l_t \geq l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right).$$

This is the firm's optimal location if it is forced to located far enough from the frontier that at least some customers buy the new product. The superscript I stands for "inactive." It is routine to show that

$$\widehat{l}_t^{RI}\left(v\left(l_t^f\right)\right) = l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right) + \frac{2\tau(1+c\tau)}{2c\tau+1}\max\left(0, v\left(l_t^f\right) - \frac{c\tau}{(1+c\tau)}\left(u\left(l_t^f, l_t^a\right) - \frac{\alpha_f\sigma^2}{c\tau}\right)\right).$$

Substituting into (7) to (11) we then obtain

$$\pi_t^R\left(\widehat{l}_t^{RI}\left(v\left(l_t^f\right)\right)\right) = \frac{1}{2}\tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right)\left(2\sigma^2\alpha_f - c\tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right)\right) + \frac{\tau(1+c\tau)^2}{2c\tau+1}\max\left(0, v\left(l_t^f\right) - \frac{c\tau}{(1+c\tau)}\left(u\left(l_t^f, l_t^a\right) - \frac{\sigma^2\alpha_f}{c\tau}\right)\right)^2 - F.$$

Note that $\pi_t^R\left(\widehat{l}_t^{RI}\left(v\left(l_t^f\right)\right)\right)$ is strictly increasing in $v\left(l_t^f\right) \in \left[\underline{v}_t^{RA}, u\left(l_t^f, l_t^a\right)\right]$. Moreover,

$$\pi_t^R\left(\widehat{l}_t^{RI}\left(\underline{v}_t^{RA}\right)\right) < 0 \text{ and } \pi_t^P\left(\widehat{l}_t^{RI}\left(u\left(l_t^f, l_t^a\right)\right)\right) \geq 0.$$

The former inequality holds because the firm generates zero benefits when $v\left(l_t^f\right) = \underline{v}_t^{RA}$ and the frontier product is active. It must then generate strictly negative benefits when $v\left(l_t^f\right) = \underline{v}_t^{RA}$ and the frontier product is inactive. The latter inequality, in turn, holds because (i.) the frontier product is active when $v\left(l_t^f\right) = u\left(l_t^f, l_t^a\right)$ and (ii.) we know from Case 1 that benefits are positive when the frontier product is active and $v\left(l_t^f\right) \geq \underline{v}_t^{RA}$.

It follows that for any $F \geq \underline{F}$, there exists a threshold $\underline{v}_t^{RI} > \underline{v}_t^{RA}$ such that $\pi_t^R\left(\widehat{l}_t^{RI}\left(\underline{v}_t^{RI}\right)\right) = 0$, $\pi_t^R\left(\widehat{l}_t^{RI}\left(v\left(l_t^f\right)\right)\right) < 0$ if $v\left(l_t^f\right) < \underline{v}_t^{RI}$, and $\pi_t^R\left(\widehat{l}_t^{RI}\left(v\left(l_t^f\right)\right)\right) > 0$ if $v\left(l_t^f\right) > \underline{v}_t^{RI}$. If the frontier product is inactive, we can then state the solution to the firm's problem as

$$l_t^{RI*} = l_t^f \text{ and } \pi_t^R\left(l_t^{RI*}\right) = 0,$$

if $v\left(l_t^f\right) \in \left[0, \underline{v}_t^{RI}\right]$ and

$$l_1^{RI*} = l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right) + \frac{2\tau(1+c\tau)}{2c\tau+1}\max\left(0, v\left(l_t^f\right) - \frac{c\tau}{(1+c\tau)}\left(u\left(l_t^f, l_t^a\right) - \frac{\sigma^2\alpha_f}{c\tau}\right)\right)$$

and

$$\pi_t^R\left(l_t^{RI*}\right)=\begin{array}{c}\frac{1}{2}\tau\left(u\left(l_t^f,l_t^a\right)-v\left(l_t^f\right)\right)\left(2\sigma^2\alpha_f-c\tau\left(u\left(l_t^f,l_t^a\right)-v\left(l_t^f\right)\right)\right)\\+\frac{\tau(1+c\tau)^2}{2c\tau+1}\max\left(0,v\left(l_t^f\right)-\frac{c\tau}{(1+c\tau)}\left(u\left(l_t^f,l_t^a\right)-\frac{\sigma^2\alpha_f}{c\tau}\right)\right)^2-F>0\end{array}$$

if $v\left(l_t^f\right)\in\left(\underline{v}_t^{RI},\infty\right)$. The statement in the proposition follows by defining $\underline{v}_t^R\equiv\underline{v}_t^{RA}$ if the frontier product is active and $\underline{v}_t^R\equiv\underline{v}_t^{RI}$ if it is not. ■

Proposition R3. *Suppose firm t locates in viable niche $[a,b]$. Let l_a denote the closest product to the left of a and l_b the closest product to the right of b. The firm's optimal location is then given by*

$$l_t^{Rn*}(a,b)=\begin{cases}a & \text{if } (b-a)\leq -X\\ \frac{1}{2}(a+b+X) & \text{if } (b-a)\geq |X|\\ b & \text{if } (b-a)\leq X,\end{cases}$$

where

$$X\equiv\frac{2\alpha_n\sigma^2\tau\left((l_b-b)-(a-l_a)\right)}{\left(1-\beta^2\tau^2\right)(l_b-l_a)+2\alpha_n\sigma^2\tau},$$

and its benefits are given by

$$\pi_t^R\left(l_t^{Rn*}(a,b)\right)=\begin{cases}\frac{\alpha_n\sigma^2}{l_b-l_a}(a-l_a)(l_b-a) & \text{if } (b-a)\leq -X\\ \begin{array}{c}\frac{1}{8\tau}\left(1-(\beta\tau)^2\right)\left((b-a)^2-X^2\right)\\+\frac{\alpha_n\sigma^2}{l_b-l_a}\left(\frac{1}{2}(a+b+X)-l_a\right)\left(l_b-\frac{1}{2}(a+b+X)\right)\end{array} & \text{if } (b-a)\geq |X|\\ \frac{\alpha_n\sigma^2}{l_b-l_a}(b-l_a)(l_b-b) & \text{if } (b-a)\leq X.\end{cases}$$

Proof: Firm t solves

$$\max_{l_t\in[a,b]}\pi_t^R(l_t), \tag{12}$$

where

$$\pi_t^R(l_t)=\pi_t(l_t)+\alpha_n\mathrm{Var}\left[v(l_t)\right]. \tag{13}$$

We know from Equation 5 of the main text that

$$\mathrm{Var}\left[v(l_t)\right]=\frac{(l_t-l_a)(l_b-l_t)}{l_b-l_a}\sigma^2$$

and from Equation 13 in the proof of Proposition 3 that

$$\pi_t(l_t)=\frac{1}{2\tau}(l_t-a)(b-l_t)\left(1-(\tau\beta(a,b))^2\right),$$

where

$$\beta(a,b) \equiv \frac{\mathrm{E}[v(b)|\,\mathcal{E}_t] - \mathrm{E}[v(a)|\,\mathcal{E}_t]}{b-a}.$$

Substituting this expression into (13) we have

$$\pi_t^R(l_t) = \frac{1}{2\tau}(l_t - a)(b - l_t)\left(1 - (\tau\beta(a,b))^2\right) + \alpha_n \frac{(l_t - l_a)(l_b - l_t)}{l_b - l_a}\sigma^2. \tag{14}$$

It is the routine to show that the solution to the firm's problem (12) is given by

$$l_t^{Rn*}(a,b) = \begin{cases} a & \text{if } (b-a) \leq -X \\ \frac{1}{2}(a+b+X) & \text{if } (b-a) \geq |X| \\ b & \text{if } (b-a) \leq X. \end{cases}$$

Substituting $l_t^{Rn*}(a,b)$ into (14) delivers the expression for $\pi_t^R\left(l_t^{Rn*}(a,b)\right)$ in the statement of the proposition. ■

Proposition R4. *The market is always covered, that is,*

$$l_t^{R*} \leq l_t^f + \tau u\left(l_t^f, l_t^a\right) \text{ for all } l_t^f \geq l_t^a \geq 0 \text{ and } t \geq 1.$$

Proof: Consider first the first period. We know from Proposition R1 that the first firm only innovates if

$$v(l_0) \geq \frac{\alpha_f \sigma^2}{2c\tau - 1},$$

in which case it locates at

$$l_1^{R*} = l_t^f + \frac{2\tau\left(v(l_0) + \alpha_f\sigma^2\right)}{2c\tau + 1},$$

or, equivalently,

$$l_1^{R*} = l_t^f + \tau v(l_0) - \frac{\tau(2c\tau - 1)}{2c\tau + 1}\left(v(l_0) - \frac{\alpha_f\sigma^2}{2c\tau - 1}\right).$$

The last term is positive, so that the market is covered in this case.

Consider next any period $t > 1$ in which the frontier product is active, $u\left(l_t^f, l_t^a\right) = v\left(l_t^f\right)$. We know from Proposition R2 that firm t only engages in frontier innovation if

$$v\left(l_t^f\right) \geq \frac{2\alpha_f\sigma^2}{2c\tau - 1},$$

in which case it locates at

$$l_t^{R*} = l_t^f + \frac{2\tau}{2c\tau + 1}\left(v\left(l_t^f\right) + \alpha_f\sigma^2\right),$$

or, equivalently,

$$l_t^{R*} = l_t^f + \tau v\left(l_t^f\right) - \frac{\tau\left(2c\tau - 1\right)}{2c\tau + 1}\left(v\left(l_t^f\right) - \frac{\alpha_f \sigma^2}{2c\tau - 1}\right).$$

The last term is positive, so that that market is covered in this case.

Finally, consider any period $t > 0$ in which the frontier product is inactive, $u\left(l_t^f, l_t^a\right) > v\left(l_t^f\right)$. We know from Proposition R2 that firm t only engages in frontier innovation if

$$v\left(l_t^f\right) \geq \frac{2\alpha_f \sigma^2}{2c\tau - 1},$$

in which case it locates at

$$\begin{aligned} l_t^{R*} &= l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right) \\ &\quad + \frac{2\tau\left(1 + c\tau\right)}{2c\tau + 1}\max\left(0, v\left(l_t^f\right) - \frac{c\tau}{\left(1 + c\tau\right)}\left(u\left(l_t^f, l_t^a\right) - \frac{\sigma^2 \alpha_f}{c\tau}\right)\right). \end{aligned} \tag{15}$$

Suppose first that the second term is zero. We then have

$$l_t^{R*} = l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right) \leq l_t^f + \tau u\left(l_t^f, l_t^a\right)$$

where the inequality follows from $v\left(l_t^f\right) \geq 0$.

Suppose next that the second term on the right-hand side of (15) is positive. The market is then covered if

$$l_t^{R*} = l_t^f + \frac{\tau}{2c\tau + 1}\left(u\left(l_t^f, l_t^a\right) + v\left(l_t^f\right) + 2\alpha_f \sigma^2\right) \leq l_t^f + \tau u\left(l_t^f, l_t^a\right).$$

We can rewrite this inequality as

$$u\left(l_t^f, l_t^a\right) \geq v\left(l_t^f\right) - \frac{2c\tau - 1}{2c\tau}\left(v\left(l_t^f\right) - \frac{2\alpha_f \sigma^2}{2c\tau - 1}\right) \geq 0,$$

where the sign follows from the fact that the second term on the right-hand side is positive and that $u\left(l_t^f, l_t^a\right) > v\left(l_t^f\right)$. ■

2 Patent Protection

The model in the main text assumes that if firm t develops a new product in period t, the competitive fringe can only imitate it in periods $t' > t$. As such, the new product enjoys complete patent protection in period t. We now relax this assumption and allow for imperfect patent protection. In particular, if firm $t \geq 1$ develops a new product in period t, it enjoys patent protection in period t with probability $p \in [0,1]$, and faces competition from the competitive fringe with complementary probability. The rest of the model is as in the main text.

Below we provide an analytical characterization of the firms' behavior conditional on engaging in frontier or niche innovation by adapting Propositions 1 to 3 in the main text. We also show that, as in the model in the main text, the market is always covered: in any period $t \geq 1$, if consumer $s' \in \mathcal{P}$ buys a product, all consumers $s \leq s'$ also buy a product. We then use these results in the simulations of the firms' optimal innovation strategies reported in the main text.

Proposition P1. *There is a threshold* $\underline{v}_0 \geq 0$ *such that, in period one,*
(i.) if $v(l_0) \leq \underline{v}_1$*, there is no profitable innovation.*
(ii.) if $v(l_0) > \underline{v}_1$*, the optimal innovation is located at*

$$l_1^{P*} = \frac{2\tau p}{p + 2c\tau} v(l_0)$$

and generates profits

$$\pi_1^P \left(l_1^{P*}\right) = \tau p v(l_0)^2 - \frac{2c\tau^2}{p + 2c\tau} p v(l_0)^2 - F > 0.$$

Proof: We know from the proof of Proposition 1 that if patent protection is perfect, profits are given by $\pi_1(l_1) = 0$ if $l_1 = l_0$ and by

$$\pi_1(l_1) = \tau v(l_0)^2 - \max\left[\frac{1}{4\tau}\left(2\tau v(l_0) - (l_1 - l_0)\right)^2, 0\right] - \frac{1}{2}c(l_1 - l_0)^2 - F$$

if $l_1 > l_0$. If there is no patent protection, price competition between the firm and the competitive fringe results in zero prices. Profits are then given by $\pi_1^P(l_1) = 0$ if $l_1 = l_0$ and by

$$\pi_1^P(l_1) = -\frac{1}{2}c(l_1 - l_0)^2 - F$$

if $l_1 > l_0$, where the superscript P stands for "patent protection."

The firm's problem is given by

$$\max_{l_1 \geq 0} \pi_1^P(l_1). \tag{16}$$

Let $\widehat{l}_1^P(v(l_0))$ denote the solution to this problem if $F = 0$. It is routine to show that

$$\widehat{l}_1^P(v(l_0)) = \frac{2\tau p v(l_0)}{p + 2c\tau}$$

and

$$\pi_1^P\left(\widehat{l}_1^P(v(l_0))\right) = \tau p v(l_0)^2 - \frac{2c\tau^2}{p + 2c\tau} p v(l_0)^2 - F.$$

Notice that $\pi_1^P\left(\widehat{l}_1^P(v(l_0))\right)$ is increasing and convex in $v(l_0)$ and that $\pi_1^P\left(\widehat{l}_1^P\left(\underline{v}_1^P\right)\right) = 0$, where

$$\underline{v}_1^P \equiv \sqrt{F(p + 2c\tau)/(p^2\tau)}.$$

We can then state the solution to (16) as

$$l_1^{P*} = \begin{cases} l_0 & \text{if } v(l_0) \in \left[0, \underline{v}_1^P\right] \\ \frac{2\tau p}{p+2c\tau} v(l_0) & \text{if } v(l_0) \in \left(\underline{v}_1^P, \infty\right) \end{cases}$$

and

$$\pi_1^P\left(l_1^{P*}\right) = \begin{cases} 0 & \text{if } v(l_0) \in \left[0, \underline{v}_1^P\right] \\ \tau p v(l_0)^2 - \frac{2c\tau^2}{p+2c\tau} p v(l_0)^2 - F > 0 & \text{if } v(l_0) \in \left(\underline{v}_0^P, \infty\right). \end{cases}$$

The statement in the proposition follows from this solution. ■

Proposition P2. *There is a threshold* $\underline{v}_t^P \geq 0$*, such that, in any period* $t \geq 2$*,*
(i.) if $v\left(l_t^f\right) \leq \underline{v}_t^P$*, there is no profitable froniter innovation.*
(ii.) if $v\left(l_t^f\right) > \underline{v}_t^P$*, the optimal frontier innovation is located at*

$$l_t^{P*} = l_t^f + \frac{p\tau}{p + 2c\tau}\left(v\left(l_t^f\right) + u\left(l_t^f, l_t^a\right)\right)$$

and generates profits

$$\pi_t\left(l_t^{P*}\right) = \tau p v\left(l_t^f\right)^2 - \frac{1}{2}\frac{pc\tau^2}{p + 2c\tau}\left(v\left(l_t^f\right) + u\left(l_t^f, l_t^a\right)\right)^2 - F > 0.$$

Proof: Suppose firm t locates to the right of the frontier product l_t^f. Its optimal location solves

$$\max_{l_t \geq l_t^f} \pi_t^P(l_t), \tag{17}$$

where $\pi_t^P(l_t)$ are profits with imperfect patent protection. If firm t's patent is protected, which happens with probability p, these profits are the same as in the main model. If, instead, firm t's patent is not protected, price competition between firm t and the competitive fringe results in zero prices. Profits are then given by zero if firm t locates at the frontier l_t^f, and they are given by

$$-\frac{1}{2}c\left(l_t - l_t^f\right)^2 - F \tag{18}$$

if it locates strictly to its right.

It is immediate that if $v\left(l_t^f\right) \leq 0$ the solution to the firm's problem (17) is to locate at l_t^f and make zero profits. For the remainder of this proof suppose then that $v\left(l_t^f\right) > 0$. Using the expressions of profits under perfect patent protection in the proof of Proposition 2, profits under imperfect patent protection are given by

$$\pi_t^P\left(l_t\right) = 0 \tag{19}$$

if $l_t = l_t^f$,

$$\pi_t^P\left(l_t\right) = -\frac{1}{2}c\left(l_t - l_t^f\right)^2 - F \tag{20}$$

if $l_t \in \left(l_t^f, l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right)\right)$

$$\pi_t^P\left(l_t\right) = p\tau v\left(l_t^f\right)^2 - \frac{p\tau}{4}\left(u\left(l_t^f, l_t^a\right) + v\left(l_t^f\right) - \frac{1}{\tau}\left(l_t - l_t^f\right)\right)^2 - \frac{1}{2}c\left(l - l_t^f\right)^2 - F \tag{21}$$

if $l_t \in \left[l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right), l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) + v\left(l_t^f\right)\right)\right]$, and

$$\pi_t^P\left(l_t\right) = p\tau v\left(l_t^f\right)^2 - \frac{1}{2}c\left(l_t - l_t^f\right)^2 - F \tag{22}$$

if $l_t \in \left[l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) + v\left(l_t^f\right)\right), \infty\right)$.

Case 1 (active frontier product) Suppose first that the frontier product is active, $u\left(l_t^f, l_t^a\right) = v\left(l_t^f\right)$. Firm t's problem is then analogous to firm 1's. Given the solution to firm 1's problem in Proposition P1, we can state the solution to firm t's problem (17) as

$$l_t^{PA*} = \begin{cases} l_t^f & \text{if } v\left(l_t^f\right) \in \left[0, \underline{v}_t^{PA}\right] \\ \frac{2\tau p}{p+2c\tau} v\left(l_t^f\right) & \text{if } v\left(l_t^f\right) \in \left(\underline{v}_t^{PA}, \infty\right) \end{cases}$$

and

$$\pi_t^P\left(l_t^{PA*}\right) = \begin{cases} 0 & \text{if } v\left(l_t^f\right) \in \left[0, \underline{v}_t^{PA}\right] \\ \tau p v\left(l_t^f\right)^2 - \frac{2c\tau^2}{p+2c\tau} p v\left(l_t^f\right)^2 - F > 0 & \text{if } v\left(l_t^f\right) \in \left(\underline{v}_t^{PA}, \infty\right), \end{cases}$$

where

$$\underline{v}_t^{PA} \equiv \sqrt{F\left(p+2c\tau\right)/\left(p^2\tau\right)},$$

and where the superscript A stands for "active."

Case 1 (inactive frontier product) Suppose next that the frontier product is inactive, $u\left(l_t^f, l_t^a\right) > v\left(l_t^f\right)$. Frontier innovation is less profitable if the frontier product is inactive than if it is active. Accordingly, the expressions for $\pi_t^R\left(l_t\right)$ in (19) to (22) are decreasing in $u\left(l_t^f, l_t^a\right)$. We know from Case 1 that even if the frontier product is active, frontier innovation generates negative profits if $v\left(l_t^f\right) < \underline{v}_t^{PA}$. If the frontier product is inactive, frontier innovation must, therefore, also generate negative profits if $v\left(l_t^f\right) < \underline{v}_t^{PA}$.

Suppose then that $v\left(l_t^f\right) \geq \underline{v}_t^{PA}$. Let $\widehat{l}_t^P\left(v\left(l_t^f\right)\right)$ denote the solution to the problem

$$\max_{l_t} \pi_t^P\left(l_t\right) \text{ subject to } l_t \geq l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right).$$

This is firm t's optimal location if it is forced to locate sufficiently far to the right of the frontier that at least some consumers buy its product. It is routine to show that the solution to this problem is given by

$$\begin{aligned}\widehat{l}_t^P\left(v\left(l_t^f\right)\right) &= l_t^f + \tau\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right) \\ &\quad + \frac{2\tau\left(p+c\tau\right)}{p+2c\tau}\max\left[\left(v_f - \frac{c\tau}{\left(p+c\tau\right)}u\left(l_t^f, l_t^a\right)\right), 0\right].\end{aligned} \tag{23}$$

Substituting into (21) and (22) we have

$$\begin{aligned}\pi_t^P\left(\widehat{l}_t^P\left(v\left(l_t^f\right)\right)\right) &= -\frac{1}{2}\tau^2 c\left(u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right)^2 \\ &\quad + \frac{\tau\left(p+c\tau\right)^2}{p+2c\tau}\max\left(0, v\left(l_t^f\right) - \frac{c\tau}{p+c\tau}u\left(l_t^f, l_t^a\right)\right)^2 - F.\end{aligned} \tag{24}$$

Note that $\pi_t^P\left(\widehat{l}_t^P\left(v\left(l_t^f\right)\right)\right)$ is strictly increasing in $v\left(l_t^f\right) \in \left[\underline{v}_t^{PA}, u\left(l_t^f, l_t^a\right)\right]$. Moreover,

$$\pi_t^P\left(\widehat{l}_t^P\left(\underline{v}_t^{PA}\right)\right) < 0 \text{ and } \pi_t^P\left(\widehat{l}_t^P\left(u\left(l_t^f, l_t^a\right)\right)\right) \geq 0.$$

The former inequality holds because $\pi_t^P\left(\widehat{l}_t^P\left(\underline{v}_t^{PA}\right)\right) = 0$ if the frontier product is active. The latter inequality holds because (i.) the frontier product is active when $v\left(l_t^f\right) = u\left(l_t^f, l_t^a\right)$ and (ii.) we know from Case 1 that profits are positive when the frontier product is active and $v\left(l_t^f\right) \geq \underline{v}_t^{PA}$. There, therefore, exists a threshold $\underline{v}_t^{PI} > \underline{v}_t^{PA}$ such that $\pi_t^P\left(\widehat{l}_t^P\left(\underline{v}_t^{PI}\right)\right) = 0$, $\pi_t^P\left(\widehat{l}_t^P\left(v\left(l_t^f\right)\right)\right) < 0$ if $v\left(l_t^f\right) < \underline{v}_t^{PI}$, and $\pi_t^P\left(\widehat{l}_t^P\left(v\left(l_t^f\right)\right)\right) > 0$ if $v\left(l_t^f\right) > \underline{v}_t^{PI}$, where the superscript I stands for "inactive."

If the frontier product is inactive, the solution to the firm's problem (17) is then given by: (i.) if $v\left(l_t^f\right) \in \left[0, \underline{v}_t^{PI}\right]$,

$$l_t^{PI*} = l_t^f \text{ and } \pi_t^{PI}\left(l_t^{PI*}\right) = 0,$$

and (ii.) if $v\left(l_t^f\right) \in \left(\underline{v}_t^{PI}, \infty\right)$,

$$l_1^{PI*} = l_t^f + \frac{p\tau}{p + 2c\tau}\left(u\left(l_t^f, l_t^a\right) + v\left(l_t^f\right)\right)$$

and

$$\pi_t^{PI}\left(l_t^{PI*}\right) = \tau p v\left(l_t^f\right) - \frac{c\tau^2}{2\left(p + 2c\tau\right)} p\left(u\left(l_t^f, l_t^a\right) + v\left(l_t^f\right)\right)^2 - F > 0.$$

Note that the expressions for l_1^{PI*} and $\pi_t^{PI}\left(l_t^{PI*}\right)$ follow from (23) and (24) because

$$\pi_t^P\left(\widehat{l}_t^P\left(\frac{c\tau}{p + c\tau} u\left(l_t^f, l_t^a\right)\right)\right) < 0.$$

The statement in the proposition follows by defining $\underline{v}_t^P \equiv \underline{v}_t^{PA}$ if the frontier product is active and $\underline{v}_t^P \equiv \underline{v}_t^{PI}$ if it is not. ■

Proposition P3. *Suppose firm t locates in viable niche $[a, b]$. Its optimal location is then given by*

$$l_t^{Pn*}\left(a, b\right) = \frac{1}{2}\left(a + b\right)$$

and its profits are given by

$$\pi_t^R\left(l_t^{Rn*}\left(a, b\right)\right) = \frac{p}{8\tau}\left(b - a\right)^2\left(1 - \tau^2 \beta\left(a, b\right)^2\right),$$

where $\beta\left(a, b\right)$ is the slope of expected quality in the niche. Innovation in a non-viable niche is not profitable.

Proof: We know from the proof of Proposition 3 that if patent protection is perfect, profits are given by

$$\pi_t\left(l_t\right) = \frac{1}{2\tau}\left(l_t - a\right)\left(b - l_t\right)\left(1 - \tau^2 \beta\left(a, b\right)^2\right),$$

where

$$\beta(a,b) \equiv \frac{\mathrm{E}\left[v(b)|\,\mathcal{E}_t\right] - \mathrm{E}\left[v(a)|\,\mathcal{E}_t\right]}{b-a}.$$

If there is no patent protection, price competition between the firm and the competitive fringe results in zero prices. Profits are then zero for any location in the niche. Under imperfect patent protection, profits are, therefore, given by

$$\pi_t^{Pn}(l_t) = \frac{p}{2\tau}(l_t - a)(b - l_t)\left(1 - \tau^2\beta(a,b)^2\right).$$

Firm t's problem is to solve

$$\max_{l_t \in [a,b]} \pi_t^{Pn}(l_t).$$

It is routine to show that the solution to this problem is given by

$$l_t^{Pn*}(a,b) = \frac{1}{2}(a+b)$$

and

$$\pi_t^R\left(l_t^{Pn*}(a,b)\right) = \frac{p}{8\tau}(b-a)^2\left(1 - \tau^2\beta(a,b)^2\right). \blacksquare$$

Proposition P4. *The market is always covered, that is,*

$$l_t^{P*} \leq l_t^f + \tau u\left(l_t^f, l_t^a\right) \text{ for all } l_t^f \geq l_t^a \geq 0 \text{ and } t \geq 1.$$

Proof: We know from Propositions P1 and P2 that for any $t \geq 1$,

$$l_t^{P*} = l_t^f + \frac{\tau}{1+2c\tau}\left(v\left(l_t^f\right) + u\left(l_t^f, l_t^a\right)\right).$$

The market is then covered if

$$l_t^f + \frac{\tau}{1+2c\tau}\left(v\left(l_t^f\right) + u\left(l_t^f, l_t^a\right)\right) \leq l_t^f + \tau u\left(l_t^f, l_t^a\right)$$

or, equivalently,

$$l_t^f + \tau u\left(l_t^f, l_t^a\right) - \frac{\tau}{1+2c\tau}\left(2c\tau u\left(l_t^f, l_t^a\right) - v\left(l_t^f\right)\right) \leq l_t^f + \tau u\left(l_t^f, l_t^a\right).$$

This inequality holds because $c\tau \geq 5/6$ and $u\left(l_t^f, l_t^a\right) \geq v\left(l_t^f\right)$, so that the third term on the left-hand side is positive. $\blacksquare$