Supplemental Materials for "A Theory of Supply Function Choice and Aggregate Supply"

by Joel P. Flynn, George Nikolakoudis, and Karthik A. Sastry

A Omitted Proofs

A.1 Proof of Theorem 1

Proof. Fix a supply function f. The realized price of the firm in state z solves $f(\hat{p}(z), z\hat{p}(z)^{-\eta}) = 0$. As we placed no restrictions on f, it is equivalent to think of the firm as choosing $\hat{p}$ directly. For a given choice of $\hat{p}$, the firm's payoff is given by:

$$J(\hat{p}) = \int_{\mathbb{R}^4_{++}} \Lambda \left(\frac{\hat{p}(z)}{P} - \mathcal{M} \right) z\hat{p}(z)^{-\eta} \, \mathrm{d}G(\Lambda, P, \mathcal{M}, z) \tag{38}$$

where G is the cumulative distribution function representing the firm's beliefs. We therefore study the problem:

$$\sup_{\hat{p}:\mathbb{R}_+ \to \mathbb{R}_{++}} J(\hat{p}) \tag{39}$$

Given a solution $\hat{p}$ for how firms optimally adapt their prices to demand, we will recover the optimal plan f for how firms optimally set a supply function.

We first derive Equation 8 using variational methods. Consider a variation $\tilde{p}(z) = p(z) + \varepsilon h(z)$. The expected payoff under this variation is:

$$J(\varepsilon; h) = \int_{\mathbb{R}^4_{++}} \Lambda \left(\frac{p(z) + \varepsilon h(z)}{P} - \mathcal{M} \right) z \left(p(z) + \varepsilon h(z) \right)^{-\eta} \mathrm{d}G(\Lambda, P, \mathcal{M}, z) \tag{40}$$

A necessary condition for the optimality of a function p is that $J_\varepsilon(0; h) = 0$ for all $G-$measurable h. Taking this derivative and setting $\varepsilon = 0$, we obtain:

$$0 = \int_{\mathbb{R}^4_{++}} \left[\Lambda \frac{h(z)}{P} z p(z)^{-\eta} - \eta \Lambda h(z) \left(\frac{p(z)}{P} - \mathcal{M} \right) z p(z)^{-\eta - 1} \right] \mathrm{d}G(\Lambda, P, \mathcal{M}, z) \tag{41}$$

Consider h functions given by the Dirac delta functions on each z, $h(z) = \delta_z$. This condition becomes:

$$0 = \int_{\mathbb{R}^3_{++}} \left[\Lambda \frac{1}{P} t p(t)^{-\eta} - \eta \Lambda \left(\frac{p(t)}{P} - \mathcal{M} \right) t p(t)^{-\eta - 1} \right] g(\Lambda, P, \mathcal{M}, t) \, \mathrm{d}\Lambda \, \mathrm{d}P \, \mathrm{d}\mathcal{M} \tag{42}$$

for all $t \in \mathbb{R}_{++}$. This is equivalent to:

$$\begin{aligned}
0 &= \int_{\mathbb{R}^3_{++}} \left[\Lambda \frac{1}{P} tp(t)^{-\eta} - \eta\Lambda \left(\frac{p(t)}{P} - \mathcal{M}\right) tp(t)^{-\eta-1}\right] g(\Lambda, P, \mathcal{M}|t)\, \mathrm{d}\Lambda\, \mathrm{d}P\, \mathrm{d}\mathcal{M} \\
&= (1-\eta)\mathbb{E}\left[\Lambda \frac{1}{P} | z = t\right] tp(t)^{-\eta} + \eta \mathbb{E}\left[\Lambda\mathcal{M}|z=t\right] tp(t)^{-\eta-1}
\end{aligned} \tag{43}$$

Thus, we have that an optimal solution necessarily follows:

$$p(t) = \frac{\eta}{\eta - 1} \frac{\mathbb{E}[\Lambda\mathcal{M}|z=t]}{\mathbb{E}[\Lambda P^{-1}|z=t]} \tag{44}$$

as claimed in Equation 8.

We now evaluate the expectations. Using log-normality,

$$\begin{aligned}
\mathbb{E}[\Lambda\mathcal{M}|z=t] &= \exp\left\{\mu_{\Lambda|z}(t) + \mu_{\mathcal{M}|z}(t) + \frac{1}{2}\sigma^2_{\Lambda|z} + \frac{1}{2}\sigma^2_{\mathcal{M}|z} + \sigma_{\Lambda,\mathcal{M}|z}\right\} \\
\mathbb{E}[\Lambda P^{-1}|z=t] &= \exp\left\{\mu_{\Lambda|z}(t) - \mu_{P|z}(t) + \frac{1}{2}\sigma^2_{\Lambda|z} + \frac{1}{2}\sigma^2_{P|z} - \sigma_{\Lambda,P|z}\right\}
\end{aligned} \tag{45}$$

where $\mu_{X|z} = \mathbb{E}[\log X \,|\, \log z]$ and $\sigma_{X,Y|z} = \mathrm{Cov}[\log X, \log Y \,|\, \log z]$. Thus,

$$\frac{\mathbb{E}[\Lambda\mathcal{M}|z=t]}{\mathbb{E}[\Lambda P^{-1}|z=t]} = \exp\left\{\mu_{\mathcal{M}|z}(t) + \mu_{P|z}(t) + \frac{1}{2}\sigma^2_{\mathcal{M}|z} - \frac{1}{2}\sigma^2_{P|z} + \sigma_{\Lambda,\mathcal{M}|z} + \sigma_{\Lambda,P|z}\right\} \tag{46}$$

Using standard formulae for Gaussian conditional expectations,

$$\begin{aligned}
\mu_{\mathcal{M}|z}(t) &= \mu_{\mathcal{M}} + \frac{\sigma_{\mathcal{M},z}}{\sigma^2_z}(\log t - \mu_z) & \mu_{P|z}(t) &= \mu_P + \frac{\sigma_{P,z}}{\sigma^2_z}(\log t - \mu_z) \\
\sigma^2_{\mathcal{M}|z} &= \sigma^2_{\mathcal{M}} - \frac{\sigma^2_{\mathcal{M},z}}{\sigma^2_z} & \sigma^2_{P|z} &= \sigma^2_P - \frac{\sigma^2_{P,z}}{\sigma^2_z} \\
\sigma_{\Lambda,\mathcal{M}|z} &= \sigma_{\Lambda,\mathcal{M}} - \frac{\sigma_{\Lambda,z}\sigma_{\mathcal{M},z}}{\sigma^2_z} & \sigma_{\Lambda,P|z} &= \sigma_{\Lambda,P} - \frac{\sigma_{\Lambda,z}\sigma_{P,z}}{\sigma^2_z}
\end{aligned} \tag{47}$$

where:

$$\begin{aligned}
\sigma^2_z &= \sigma^2_\Psi + \eta^2\sigma^2_P + 2\eta\sigma_{\Psi,P} & \sigma_{P,z} &= \sigma_{P,\Psi} + \eta\sigma^2_P \\
\sigma_{\mathcal{M},z} &= \sigma_{\mathcal{M},\Psi} + \eta\sigma_{\mathcal{M},P} & \sigma_{\Lambda,z} &= \sigma_{\Lambda,\Psi} + \eta\sigma_{\Lambda,P}
\end{aligned} \tag{48}$$

We now combine these expressions with Equation 44 to derive the optimal supply function. We first observe that

$$\log p = \omega_0 + \omega_1 \log t \tag{49}$$

where:

$$\begin{aligned}\omega_0 &= \log\frac{\eta}{\eta-1} + \mu_{\mathcal{M}} + \mu_P - \omega_1\mu_z + \frac{1}{2}\sigma^2_{\mathcal{M}|z} - \frac{1}{2}\sigma^2_{P|z} + \sigma_{\Lambda,\mathcal{M}|z} + \sigma_{\Lambda,P|z} \\ \omega_1 &= \frac{\sigma_{\mathcal{M},z}+\sigma_{P,z}}{\sigma^2_z} = \frac{\sigma_{\mathcal{M},\Psi}+\eta\sigma_{\mathcal{M},P}+\sigma_{P,\Psi}+\eta\sigma^2_P}{\sigma^2_\Psi+\eta^2\sigma^2_P+2\eta\sigma_{\Psi,P}}\end{aligned} \tag{50}$$

Next, using the demand curve, we observe that $z = qp^\eta$. Therefore, $\log t = \log q + \eta \log p$. Substituting this into Equation 49, and re-arranging, we obtain

$$\log p = \alpha_0 + \alpha_1 \log q \tag{51}$$

where:

$$\alpha_0 = \frac{\omega_0}{1-\eta\omega_1}, \qquad \alpha_1 = \frac{\omega_1}{1-\eta\omega_1} \tag{52}$$

We finally derive the claimed expression for α_1,

$$\alpha_1 = \frac{\frac{\sigma_{\mathcal{M},\Psi}+\eta\sigma_{\mathcal{M},P}+\sigma_{P,\Psi}+\eta\sigma^2_P}{\sigma^2_\Psi+\eta^2\sigma^2_P+2\eta\sigma_{\Psi,P}}}{1-\eta\frac{\sigma_{\mathcal{M},\Psi}+\eta\sigma_{\mathcal{M},P}+\sigma_{P,\Psi}+\eta\sigma^2_P}{\sigma^2_\Psi+\eta^2\sigma^2_P+2\eta\sigma_{\Psi,P}}} = \frac{\sigma_{\mathcal{M},\Psi}+\eta\sigma_{\mathcal{M},P}+\sigma_{P,\Psi}+\eta\sigma^2_P}{\sigma^2_\Psi+\eta\sigma_{\Psi,P}-\eta\sigma_{\mathcal{M},\Psi}-\eta^2\sigma_{\mathcal{M},P}} \tag{53}$$

Completing the proof.

□

A.2 Proof of Corollary 1

Proof. If $2\eta\sigma_{\mathcal{M},P} + \sigma_{\mathcal{M},\Psi} \geq \sigma_{P,\Psi}$, then the denominator of Equation 6 is decreasing in η. Moreover, if $\sigma_{\mathcal{M},P} \geq 0$, the numerator is increasing in η. Hence, α_1 is increasing in η whenever $\alpha_1 > 0$. □

A.3 Proof of Proposition 1

Proof. From the household's choice among varieties, the demand curve for each variety i is

$$\frac{p_{it}}{P_t} = \left(\frac{c_{it}}{\vartheta_{it}C_t}\right)^{-\frac{1}{\eta}} \tag{54}$$

From the intratemporal Euler equation for consumption demand *vs.* labor supply, the household equates the marginal benefit of supplying additional labor $w_{it}C_t^{-\gamma}P_t^{-1}$ with its marginal cost ϕ_{it}. Thus, variety-specific wages are given by

$$w_{it} = \phi_{it}P_tC_t^\gamma \tag{55}$$

From the intertemporal Euler equation between consumption and money today, the cost of holding an additional dollar today equals the benefit of holding an additional dollar today plus the value of an additional dollar tomorrow:

$$C_t^{-\gamma}\frac{1}{P_t} = \frac{1}{M_t} + \beta\mathbb{E}_t\left[C_{t+1}^{-\gamma}\frac{1}{P_{t+1}}\right] \tag{56}$$

Further, from the intertemporal choice between bonds, the cost of saving an additional dollar today equals the nominal interest rate $1+i_t$ times the value of an additional dollar tomorrow:

$$C_t^{-\gamma}\frac{1}{P_t} = \beta(1+i_t)\mathbb{E}_t\left[C_{t+1}^{-\gamma}\frac{1}{P_{t+1}}\right] \tag{57}$$

From Equations 56 and 57, we obtain:

$$\frac{1}{M_t} + \beta\mathbb{E}_t\left[C_{t+1}^{-\gamma}\frac{1}{P_{t+1}}\right] = \beta(1+i_t)\mathbb{E}_t\left[C_{t+1}^{-\gamma}\frac{1}{P_{t+1}}\right] \tag{58}$$

It follows that:

$$\frac{1}{M_t} = \beta i_t\mathbb{E}_t\left[C_{t+1}^{-\gamma}\frac{1}{P_{t+1}}\right] = \frac{i_t}{1+i_t}C_t^{-\gamma}\frac{1}{P_t} \tag{59}$$

where the second equality uses Equation 57 once again. This rearranges to:

$$C_t = \left(\frac{i_t}{1+i_t}\right)^{\frac{1}{\gamma}}\left(\frac{M_t}{P_t}\right)^{\frac{1}{\gamma}} \tag{60}$$

We next derive the interest rate. Substituting equation 60 into Equation 57, we obtain:

$$\frac{1+i_t}{i_t}\frac{1}{M_t} = \beta(1+i_t)\mathbb{E}_t\left[\frac{1+i_{t+1}}{i_{t+1}}\frac{1}{M_{t+1}}\right] \tag{61}$$

Dividing both sides by $(1+i_t)$, multiplying by M_t, and then adding one, we obtain:

$$\frac{1+i_t}{i_t} = 1 + \beta\mathbb{E}_t\left[\frac{1+i_{t+1}}{i_{t+1}}\frac{M_t}{M_{t+1}}\right] = 1 + \beta\mathbb{E}_t\left[\exp\{-\mu_M - \sigma_{t+1}^M\varepsilon_{t+1}^M\}\frac{1+i_{t+1}}{i_{t+1}}\right] \tag{62}$$

where the second equality exploits the fact that M_t follows a random walk with drift. If we guess that i_t is deterministic and define $x_t = \frac{1+i_t}{i_t}$, then we obtain that:

$$x_t = 1 + \delta_t x_{t+1} \tag{63}$$

where:

$$\delta_t = \beta \exp\left\{-\mu_M + \frac{1}{2}(\sigma^M_{t+1})^2\right\} \tag{64}$$

We observe that $\delta_t \in [0, \beta]$ for all t due to the assumption that $\frac{1}{2}(\sigma^M_t)^2 \leq \mu_M$. Solving this equation forward, we obtain that for $T \geq 2$:

$$x_t = 1 + \delta_t \left(1 + \sum_{i=1}^{T-1} \prod_{j=1}^{i} \delta_{t+j}\right) + \delta_t \left(\prod_{j=1}^{T} \delta_{t+j}\right) x_{t+T+1} \tag{65}$$

Taking the limit $T \to \infty$, this becomes:

$$x_t = 1 + \delta_t \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \delta_{t+j}\right) + \delta_t \lim_{T\to\infty} \left(\prod_{j=1}^{T} \delta_{t+j}\right) x_{t+T+1} \tag{66}$$

where the final term can be bounded using the fact that $\delta_t \in [0, \beta]$:

$$0 \leq \delta_t \lim_{T\to\infty} \left(\prod_{j=1}^{T} \delta_{t+j}\right) x_{t+T+1} \leq \lim_{T\to\infty} \beta^{T+1} x_{t+T+1} \tag{67}$$

The household's transversality condition ensures that this upper bound is zero. Formally, the transversality condition (necessary for the optimality of the household's choices) is that:

$$\lim_{T\to\infty} \beta^T \frac{C_T^{-\gamma}}{P_T}(M_T + (1 + i_T)B_T) = 0 \tag{68}$$

Moreover, as $B_t = 0$ for all $t \in \mathbb{N}$, this reduces to $\lim_{T\to\infty} \beta^T \frac{C_T^{-\gamma}}{P_T} M_T = 0$. By Equation 59, we have that $\frac{x_t}{M_t} = \frac{C_t^{-\gamma}}{P_t}$. Thus, the transversality condition reduces to $\lim_{T\to\infty} \beta^T x_T = 0$. Combining this with Equation 67, we have that $\lim_{T\to\infty} \left(\prod_{j=1}^{T} \delta_{t+j}\right) x_{t+T+1} = 0$. An explicit formula for the interest rate follows:

$$\frac{1 + i_t}{i_t} = 1 + \beta \exp\left\{-\mu_M + \frac{1}{2}(\sigma^M_{t+1})^2\right\} \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \beta \exp\left\{-\mu_M + \frac{1}{2}(\sigma^M_{t+j+1})^2\right\}\right) \tag{69}$$

The formulae in Equation 20 then follow. In particular, $\Psi_{it} = \vartheta_{it} C_t$ follows from comparing Equations 2 and 54. $P_t = \frac{i_t}{1+i_t} C_t^{-\gamma} M_t$ follows from Equation 60. $\Lambda_t = C_t^{-\gamma}$ is the households marginal utility from consumption. Finally, $\mathcal{M}_{it} = \frac{1}{\zeta_{it} A_t} \frac{w_{it}}{P_t} = \frac{\phi_{it} C_t^{\gamma}}{\zeta_{it} A_t}$ follows from Equation 55. □

A.4 Proof of Theorem 2

Proof. We begin by characterizing log-linear equilibria, which is achieved by the following Lemma:

Lemma 1 (Macroeconomic Dynamics with Supply Functions)**.** *If all firms use log-linear supply functions of the form in Equation 21, output in the unique log-linear temporary equilibrium follows:*

$$\log C_t = \tilde{\chi}_{0,t} + \frac{1}{\gamma}\frac{\kappa_t^A}{1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^A)}\log A_t + \frac{1}{\gamma}\frac{(1-\kappa_t^M)(1-\eta\omega_{1,t})}{1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^M)}\log M_t \quad (70)$$

and the aggregate price in the unique log-linear temporary equilibrium is given by:

$$\log P_t = \chi_{0,t} - \frac{\kappa_t^A}{1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^A)}\log A_t + \frac{\kappa_t^M + \frac{\omega_{1,t}}{\gamma}(1-\kappa_t^M)}{1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^M)}\log M_t \quad (71)$$

where $\chi_{0,t}$ and $\tilde{\chi}_{0,t}$ are constants that depend only on parameters (including $\alpha_{1,t}$) and past shocks to the economy.

Proof. We suppress dependence on t for ease of notation. Consider a plan:

$$\log p_i = \log \tilde{\alpha}_{0,i} + \alpha_1 \log q_i \quad (72)$$

where $\tilde{\alpha}_{0,i} = e^{\alpha_{0,i}}$. The demand-supply relationship that the firm faces is:

$$\log p_i = -\frac{1}{\eta}(\log q_i - \log \Psi) + \log P \quad (73)$$

The realized quantity therefore is:

$$\log q_i = \frac{-\eta}{1+\eta\alpha_1}\log \tilde{\alpha}_{0,i} + \frac{1}{1+\eta\alpha_1}\log \Psi_i P^\eta \quad (74)$$

and the realized price is:

$$\log p_i = \frac{1}{1+\eta\alpha_1}\log \tilde{\alpha}_{0,i} + \frac{\alpha_1}{1+\eta\alpha_1}\log \Psi_i P^\eta \quad (75)$$

It is useful to make the change of variables $\omega_1 = \frac{\alpha_1}{1+\eta\alpha_1}$ to write

$$\log p_i = (1-\eta\omega_1)\log \tilde{\alpha}_{0,i} + \omega_1 \log \Psi_i P^\eta \quad (76)$$

Our goal is to express dynamics only as a function of ω_1. We first find the optimal $\tilde{\alpha}_{0,i}$ in terms of ω_1. The firm therefore solves:

$$\max_{\tilde{\alpha}_{0,i}} \mathbb{E}_i \left[\Lambda \left(\frac{p_i}{P} - \mathcal{M}_i \right) \left(\frac{p_i}{P} \right)^{-\eta} \Psi_i \right] \tag{77}$$

Substituting for the realized price using the demand-supply relationship yields:

$$\max_{\tilde{\alpha}_{0,i}} \mathbb{E} \left[\Lambda \left(\frac{\tilde{\alpha}_{0,i}^{1-\eta\omega_1}}{P} \left(\Psi_i P^\eta \right)^{\omega_1} - \mathcal{M}_i \right) \tilde{\alpha}_{0,i}^{\eta^2\omega_1 - \eta} \left(\Psi_i P^\eta \right)^{1-\eta\omega_1} \right] \tag{78}$$

The optimal $\tilde{\alpha}_{0,i}$ is:

$$\tilde{\alpha}_{0,i}^{1-\eta\omega_1} = \frac{\eta}{\eta - 1} \frac{\mathbb{E}_i[\Lambda \mathcal{M}_i \left(\Psi_i P^\eta \right)^{1-\eta\omega_1}]}{\mathbb{E}_i[\frac{\Lambda}{P} \left(\Psi_i P^\eta \right)^{1-\eta\omega_1+\omega_1}]} \tag{79}$$

Substituting back into the realized price yields:

$$p_i = \frac{\eta}{\eta - 1} \frac{\mathbb{E}_i[\Lambda \mathcal{M}_i \left(\Psi_i P^\eta \right)^{1-\eta\omega_1}]}{\mathbb{E}_i[\frac{\Lambda}{P} \left(\Psi_i P^\eta \right)^{1-\eta\omega_1+\omega_1}]} \left(\Psi_i P^\eta \right)^{\omega_1} \tag{80}$$

We may express this only in terms of P by using Proposition 1, where we let $I = \frac{1+i}{i}$ for ease of notation:

$$p_i = \frac{\eta}{\eta - 1} \frac{\mathbb{E}_i \left[\phi(\zeta_i A)^{-1} \left(\vartheta_i I^{-\frac{1}{\gamma}} P^{-\frac{1}{\gamma}} M^{\frac{1}{\gamma}} P^\eta \right)^{1-\eta\omega_1} \right]}{\mathbb{E}_i \left[I^{1-\frac{1}{\gamma}(1+\omega_1-\eta\omega_1)} M^{\frac{1}{\gamma}(1+\omega_1-\eta\omega_1)-1} \vartheta^{1+\omega_1-\eta\omega_1} P^{\left(\eta - \frac{1}{\gamma}\right)(1+\omega_1-\eta\omega_1)} \right]} \times \left(\vartheta_i I^{-\frac{1}{\gamma}} M^{\frac{1}{\gamma}} P^{\eta - \frac{1}{\gamma}} \right)^{\omega_1} \tag{81}$$

Given the ideal price index formula (Equation 14), P must satisfy the aggregation:

$$P^{1-\eta} = \mathbb{E} \left[\vartheta_i p_i^{1-\eta} \right] \tag{82}$$

where the expectation is over the cross-section of firms. We guess and verify that the aggregate price is log-linear in aggregates

$$\log P = \chi_0 + \chi_A \log A + \chi_M \log M \tag{83}$$

Moreover, if the p_i are log-normally distributed (we will verify this below), then:

$$\log P = \mathbb{E}[\log p_i] + \frac{1}{2(1-\eta)} \mathrm{Var}((1-\eta) \log p_i) + \mathrm{const} \tag{84}$$

We first simplify the numerator of the first term by collecting all the terms involving s_i^A and s_i^M:

$$\begin{aligned}\log \mathbb{E}_i\left[\phi_i(\zeta_i A)^{-1}\left(\vartheta I^{-\frac{1}{\gamma}} P^{-\frac{1}{\gamma}} M^{\frac{1}{\gamma}} P^{\eta}\right)^{1-\eta\omega_1}\right] &= \left[-\kappa^A + \kappa^A\left(\eta - \frac{1}{\gamma}\right)\chi_A(1-\eta\omega_1)\right] s_i^A \\ + \left[\chi_M\left(\eta-\frac{1}{\gamma}\right)(1-\eta\omega_1)\kappa^M + \frac{1}{\gamma}(1-\eta\omega_1)\kappa^M\right] s_i^M + \text{const}\end{aligned} \tag{85}$$

where the constants are independent of signals. We similarly simplify the denominator of the second term:

$$\begin{aligned}&\log \mathbb{E}_i\left[I^{1-\frac{1}{\gamma}(1+\omega_1-\eta\omega_1)} M^{\frac{1}{\gamma}(1+\omega_1-\eta\omega_1)-1} \vartheta^{1+\omega_1-\eta\omega_1} P^{\left(\eta-\frac{1}{\gamma}\right)(1+\omega_1-\eta\omega_1)}\right] = \\ &\left[\chi_A\left(\eta-\frac{1}{\gamma}\right)(1+\omega_1-\eta\omega_1)\kappa^A\right] s_i^A \\ &+ \left[\left[\frac{1}{\gamma}(1+\omega_1-\eta\omega_1)-1\right]\left(\kappa^M\right) + \chi_M\left(\eta-\frac{1}{\gamma}\right)(1+\omega_1-\eta\omega_1)\left(\kappa^M\right)\right] s_i^M \\ &+ \text{const}\end{aligned} \tag{86}$$

where the constants are again independent of signals. Finally, we can simplify the last term:

$$\log\left(\vartheta_i I^{-\frac{1}{\gamma}} M^{\frac{1}{\gamma}} P^{\eta-\frac{1}{\gamma}}\right)^{\omega_1} = \omega_1\chi_A\left(\eta-\frac{1}{\gamma}\right)\log A + \omega_1\left[\chi_M\left(\eta-\frac{1}{\gamma}\right)+\frac{1}{\gamma}\right]\log M + \text{const} \tag{87}$$

where the constants are independent of the aggregate shocks. Hence, $\log p_i$ is indeed normally distributed and its variance is independent of the realization of aggregate shocks. We can now collect terms to verify our log-linear guess. Substituting the resulting expression for $\log p_i$ and our guess for $\log P$ from Equation 83 into Equation 84, and solving for χ_A by collecting coefficients on $\log A$ yields:

$$\chi_A = -\frac{\kappa^A}{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)(1-\kappa^A)} \tag{88}$$

We may similarly solve for χ_M:

$$\chi_M = \frac{\kappa^M + \frac{\omega_1}{\gamma}(1-\kappa^M)}{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)(1-\kappa^M)} \tag{89}$$

This proves the dynamics for the price level. The dynamics for consumption then follow from Proposition 1. □

With this characterization in hand, by Equation 70 and market clearing $C_t = Y_t$, we have:

$$\log M_t = \frac{1}{\tilde{\chi}_{M,t}} (\log Y_t - \tilde{\chi}_{A,t} \log A_t - \tilde{\chi}_{0,t}) \tag{90}$$

Substituting for $\log M_t$ in Equation 71 and defining $\log \bar{P}_t = \chi_{0,t} - \epsilon_t^S \tilde{\chi}_{0,t}$ and $\delta_t = \chi_{A,t} - \epsilon_t^S \tilde{\chi}_{A,t}$ then yields Equation AS:

$$\log P_t = \log \bar{P}_t + \epsilon_t^S \log Y_t + \delta_t \log A_t \tag{91}$$

Doing a similar substitution for $\log A_t$ in Equation 70 then yields Equation AD:

$$\log P_t = \log \left(\frac{i_t}{1 + i_t} \right) - \epsilon_t^D \log Y_t + \log M_t \tag{92}$$

Completing the proof. □

A.5 Proof of Theorem 3

Proof. We suppress dependence on t for ease of notation. We have χ_M and χ_A as a function of ω_1 from Lemma 1. We also know that:

$$\omega_1 = \frac{\sigma_{\mathcal{M}_i,z} + \sigma_{P,z}}{\sigma_z^2} \tag{93}$$

from Equation 50. As $z_i = \vartheta_i \left(\frac{i}{1+i} \right)^{\frac{1}{\gamma}} M^{\frac{1}{\gamma}} P^{\eta - \frac{1}{\gamma}}$ and $\mathcal{M}_i = \phi_i (\zeta_i A)^{-1} \frac{i}{1+i} \frac{M}{P}$, we have that:

$$\begin{aligned}
\sigma_{\mathcal{M}_i,z} &= \text{Cov} \left(-(1 + \chi_A) \log A + (1 - \chi_M) \log M, \left(\eta - \frac{1}{\gamma} \right) \chi_A \log A + \left(\frac{1}{\gamma} + \left(\eta - \frac{1}{\gamma} \right) \chi_M \right) \log M \right) \\
&= - \left(\eta - \frac{1}{\gamma} \right) \chi_A (1 + \chi_A) \sigma_A^2 + (1 - \chi_M) \left(\frac{1}{\gamma} + \left(\eta - \frac{1}{\gamma} \right) \chi_M \right) \sigma_M^2 \\
\sigma_{P,z} &= \text{Cov} \left(\chi_A \log A + \chi_M \log M, \left(\eta - \frac{1}{\gamma} \right) \chi_A \log A + \left(\frac{1}{\gamma} + \left(\eta - \frac{1}{\gamma} \right) \chi_M \right) \log M \right) \\
&= \left(\eta - \frac{1}{\gamma} \right) \chi_A^2 \sigma_A^2 + \chi_M \left(\frac{1}{\gamma} + \left(\eta - \frac{1}{\gamma} \right) \chi_M \right) \sigma_M^2 \\
\sigma_z^2 &= \sigma_\vartheta^2 + \left(\eta - \frac{1}{\gamma} \right)^2 \chi_A^2 \sigma_A^2 + \left(\frac{1}{\gamma} + \left(\eta - \frac{1}{\gamma} \right) \chi_M \right)^2 \sigma_M^2
\end{aligned} \tag{94}$$

Thus:

$$\omega_1 = \frac{-(\eta - \frac{1}{\gamma}) \chi_A \sigma_A^2 + (\frac{1}{\gamma} + (\eta - \frac{1}{\gamma}) \chi_M) \sigma_M^2}{\sigma_\vartheta^2 + (\eta - \frac{1}{\gamma})^2 \chi_A^2 \sigma_A^2 + (\frac{1}{\gamma} + (\eta - \frac{1}{\gamma}) \chi_M)^2 \sigma_M^2} \tag{95}$$

Note that the optimal ω_1 is common across all firms i. We may express this in fully reduced form as:

$$\omega_1 = T(\omega_1) = \frac{(\eta - \frac{1}{\gamma})\frac{\kappa_A}{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_A)}\sigma_A^2 + (\frac{1}{\gamma} + (\eta - \frac{1}{\gamma})\frac{\kappa_M + \frac{\omega_1}{\gamma}(1-\kappa_M)}{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_M)})\sigma_M^2}{\sigma_\vartheta^2 + (\eta - \frac{1}{\gamma})^2\left(\frac{\kappa_A}{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_A)}\right)^2\sigma_A^2 + (\frac{1}{\gamma} + (\eta - \frac{1}{\gamma})\frac{\kappa_M + \frac{\omega_1}{\gamma}(1-\kappa_M)}{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_M)})^2\sigma_M^2} \tag{96}$$

or

$$\omega_1 = T(\omega_1) = \frac{\frac{\left(\eta-\frac{1}{\gamma}\right)\kappa_A}{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_A)}\sigma_A^2 + \frac{\frac{1}{\gamma}+\left(\eta-\frac{1}{\gamma}\right)\kappa_M}{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_M)}\sigma_M^2}{\sigma_\vartheta^2 + \left(\frac{\left(\eta-\frac{1}{\gamma}\right)\kappa_A}{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_A)}\right)^2\sigma_A^2 + \left(\frac{\frac{1}{\gamma}+\left(\eta-\frac{1}{\gamma}\right)\kappa_M}{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_M)}\right)^2\sigma_M^2} \tag{97}$$

□

A.6 Proof of Proposition 2

Proof. We first establish equilibrium existence. First, we observe that T_t is a continuous function. The only possible points of discontinuity are: $\omega_{1,t}^M = \frac{1}{(\eta-\frac{1}{\gamma})(1-\kappa_t^M)}$ and $\omega_{1,t}^A = \frac{1}{(\eta-\frac{1}{\gamma})(1-\kappa_t^A)}$. However, at these points $\lim_{\omega_{1,t}\to\omega_{1,t}^M} T_t(\omega_{1,t}) = \lim_{\omega_{1,t}\to\omega_{1,t}^A} T_t(\omega_{1,t}) = T_t(\omega_{1,t}^M) = T_t(\omega_{1,t}^A) = 0$. Second, we observe that $\lim_{\omega_{1,t}\to-\infty} T_t(\omega_{1,t}) = \lim_{\omega_{1,t}\to\infty} T_t(\omega_{1,t}) = 0$. Consider now the function $W_t(\omega_{1,t}) = \omega_{1,t} - T_t(\omega_{1,t})$. This is a continuous function, $\lim_{\omega_{1,t}\to-\infty} W_t(\omega_{1,t}) = -\infty$, and $\lim_{\omega_{1,t}\to\infty} W_t(\omega_{1,t}) = \infty$. Thus, by the intermediate value theorem, there exists an $\omega_{1,t}^*$ such that $W_t(\omega_{1,t}^*) = 0$. By Theorem 3, $\omega_{1,t}^*$ defines a log-linear equilibrium.

We now show that there are at most five log-linear equilibria. For $\omega_{1,t} \neq \omega_{1,t}^A, \omega_{1,t}^M$ (neither of which can be a fixed point), we can rewrite Equation 29 as:

$$\begin{aligned}
&\omega_{1,t}\Bigg[\sigma_{\vartheta,t}^2\left(1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^A)\right)^2\left(1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^M)\right)^2 \\
&\quad + \left(\sigma_{t|s}^A\right)^2\left(\eta-\frac{1}{\gamma}\right)\kappa_t^A\left(1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^M)\right)^2 \\
&\quad + \left(\sigma_{t|s}^M\right)^2\left(\frac{1}{\gamma}+\left(\eta-\frac{1}{\gamma}\right)\kappa_t^M\right)\left(1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^A)\right)^2\Bigg] \\
&= \left(\sigma_{t|s}^A\right)^2\left(\eta-\frac{1}{\gamma}\right)\kappa_t^A\left(1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^M)\right)^2\left(1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^A)\right) \\
&\quad + \left(\sigma_{t|s}^M\right)^2\left(\frac{1}{\gamma}+\left(\eta-\frac{1}{\gamma}\right)\kappa_t^M\right)\left(1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^A)\right)^2\left(1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^M)\right)
\end{aligned} \tag{98}$$

This is a quintic polynomial in $\omega_{1,t}$, which has at most five real roots. Thus, by Theorem 3, there are at most five log-linear equilibria. ☐

A.7 Proof of Corollary 5

Proof. We drop time subscripts for ease of notation. Substituting $\eta = \frac{1}{\gamma}$ in Equation 29 yields:

$$\omega_1 = \frac{\frac{1}{\gamma}}{\rho^2 + \left(\frac{1}{\gamma}\right)^2} \tag{99}$$

Substituting this into Equation 24 yields:

$$\epsilon_t^S = \gamma \frac{\kappa_t^M}{(1-\kappa_t^M)} + \frac{1}{\gamma\rho^2(1-\kappa_t^M)} \tag{100}$$

☐

A.8 Proof of Corollary 6

We drop time subscripts for ease of notation. The first statement follows directly from Equation 29. Furthermore, using Equation 29, as $\sigma_{t|s}^M \to \infty$, ω_1 must solve:

$$\begin{aligned}\omega_1 &= \frac{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)\left(1-\kappa^M\right)}{\frac{1}{\gamma}+\left(\eta-\frac{1}{\gamma}\right)\kappa^M} = \frac{\gamma}{1+(\eta\gamma-1)\,\kappa^M} + \left(1-\frac{\eta\gamma}{1+(\eta\gamma-1)\kappa^M}\right)\omega_1 \\ &= \frac{1}{\eta}\end{aligned} \tag{101}$$

This proves the second statement. As $\sigma_{t|s}^A \to \infty$ and $\eta\gamma \neq 1$, ω_1 must solve:

$$\begin{aligned}\omega_1 &= \frac{1-\omega_1\left(\eta-\frac{1}{\gamma}\right)\left(1-\kappa^A\right)}{\left(\eta-\frac{1}{\gamma}\right)\kappa^A} = \frac{\gamma}{(\eta\gamma-1)\,\kappa^A} + \left(1-\frac{1}{\kappa^A}\right)\omega_1 \\ &= \frac{1}{\eta-\frac{1}{\gamma}}\end{aligned} \tag{102}$$

This proves the third statement.

A.9 Proof of Proposition 3

Proof. By Theorem 3, The map describing equilibrium $\omega_{1,t}$ is invariant to λ for $\lambda > 0$. Thus, $\mathcal{E}_t^S(\lambda)$ is constant for $\lambda > 0$. If $\lambda = 0$, there are potentially many equilibria in supply functions. Nevertheless, from the proof of Theorem 1, we have that firms set $p_{it}/P_t = \frac{\eta}{\eta-1}\mathcal{M}_{it} = \frac{\eta}{\eta-1}(\phi_{it}C_t^{\gamma})/(\zeta_{it}A_t)$ under any optimal supply function. This implies that $\frac{\eta}{\eta-1}C_t^{\gamma}/A_t = \text{const}$, and so money has no real effects, which implies that $\epsilon_t^S = \infty$. $\square$

B Supply Functions in Richer Economic Settings

In this appendix, we generalize the firm's partial-equilibrium supply schedule problem in four ways. First, we enrich both the firm's technology and input space by allowing for many inputs, decreasing returns to scale, and monopsony power. Second, we enrich the demand the firm faces by decoupling the own-price elasticity and the cross-price elasticity and allowing for non-isoelastic demand curves that feature endogenous markups (allowing for Marshall's Second and Third laws of demand). Third, we enrich the firm's decisionmaking by allowing the firm to choose additional non-price and non-quantity variables at a cost. This allows, for example, the firm to invest in improving the quality of its product. Finally, we enrich the firm's problem by introducing Calvo price stickiness. In all four cases, we characterize firms' optimal supply functions, show that our core insights generalize, and highlight the new economic features that each of these extensions introduces. In the interest of brevity, we leave embedding these generalizations in general equilibrium to future research, though it is clear to see how one could do this by embedding these characterizations in our general equilibrium model and leveraging the techniques from our main analysis.[20]

B.1 Multiple Inputs, Decreasing Returns, and Monopsony

In this section, we generalize our baseline model of supply function choice to allow for multiple inputs, decreasing returns, and monopsony. We find that: (i) supply functions remain endogenously log-linear and (ii) decreasing returns and monopsony flatten the optimal supply schedule.

Primitives. Consider the baseline model from Section 2 with two modifications. First, the production function uses multiple inputs with different input shares and possibly features decreasing returns-to-scale:

$$q = \Theta \prod_{i=1}^{I} x_i^{a_i} \tag{103}$$

where $x_i \in \mathbb{R}_+$, $a_i \geq 0$, and $\sum_{i=1}^{I} a_i \leq 1$. Moreover, suppose that the producer potentially has monopsony power and faces an upward-sloping factor price curve such that the price of acquiring any input i when the firm demands x_i units is given by $\tilde{p}_i(x_i) = p_{xi} x_i^{b_i - 1}$, where $p_{xi} \in \mathbb{R}_{++}$ and $b_i \geq 1$. The case of no monopsony, or price-taking in the input market, occurs

[20] The only complication with endogenous markups would be the endogenous non-log-linearity of the optimal supply curve. This would have to be dealt with via either approximation arguments similar to those we adopt in our extension to allow for price stickiness or numerical methods, or both.

when $b_i = 1$. Thus, the cost of acquiring each type of input is given by:

$$c_i(x_i) = p_{xi} x_i^{b_i} \tag{104}$$

The firm believes that $(\Psi, P, \Lambda, \Theta, p_x)$ is jointly log-normal.

The Firm's Problem. We begin by solving the firm's cost minimization problem:

$$K(q; \Theta, p_x) = \min_x \sum_{i=1}^{I} p_{x_i} x_i^{b_i} \quad \text{s.t.} \quad q = \Theta \prod_{i=1}^{I} x_i^{a_i} \tag{105}$$

This has first-order condition given by:

$$\lambda = \frac{b_i p_{xi}}{a_i} x_i^{b_i} q^{-1} \tag{106}$$

Which implies that:

$$K(q; \Theta, p_x) = \lambda q \sum_{i=1}^{I} \frac{a_i}{b_i} \tag{107}$$

Moreover, fixing i, the FOC implies that we may write for all $j \neq i$:

$$x_j = \left(\frac{\frac{b_i p_{x_i}}{a_i}}{\frac{b_j p_{x_j}}{a_j}} \right)^{\frac{1}{b_j}} x_i^{\frac{b_i}{b_j}} \tag{108}$$

By substituting this into the production function we have that:

$$q = \Theta x_i^{a_i + b_i \sum_{j \neq i} \frac{a_j}{b_j}} \prod_{j \neq i} \left(\frac{\frac{b_i p_{x_i}}{a_i}}{\frac{b_j p_{x_j}}{a_j}} \right)^{\frac{a_j}{b_j}} \tag{109}$$

which implies that:

$$x_i = \left(\frac{q}{\Theta \prod_{j \neq i} \left(\frac{\frac{b_i p_{x_i}}{a_i}}{\frac{b_j p_{x_j}}{a_j}} \right)^{\frac{a_j}{b_j}}} \right)^{\frac{1}{a_i + b_i \sum_{j \neq i} \frac{a_j}{b_j}}} \tag{110}$$

Returning to the FOC, we have that the Lagrange multiplier is given by:

$$\lambda = q^{-1+\frac{1}{\sum_{i=1}^{I}\frac{a_i}{b_i}}} \frac{b_i p_{xi}}{a_i} \left(\Theta \prod_{j \neq i} \left(\frac{\frac{b_i p_{x_i}}{a_i}}{\frac{b_j p_{x_j}}{a_j}} \right)^{\frac{a_j}{b_j}} \right)^{\frac{-1}{\sum_{i=1}^{I}\frac{a_j}{b_j}}} \tag{111}$$

Which then yields the cost function:

$$K(q; \Theta, p_x) = \mathcal{M} P q^{\frac{1}{\delta}} \tag{112}$$

where:

$$\delta = \sum_{i=1}^{I} \frac{a_i}{b_i} \quad \text{and} \quad \mathcal{M} = P^{-1} \left(\Theta \prod_{i=1}^{I} \left(\frac{b_i p_{x_i}}{a_i} \right)^{\frac{a_i}{b_i}} \right)^{\frac{1}{\sum_{i=1}^{I}\frac{a_i}{b_i}}} \sum_{i=1}^{I} \frac{a_i}{b_i} \tag{113}$$

and we observe that $\mathcal{M}$ is log-normal given the joint log-normality of (Θ, p_x).

Turning to the firm's payoff function, we therefore have:

$$\mathbb{E}\left[\Lambda \left(\frac{p}{P} q - \mathcal{M} q^{\frac{1}{\delta}} \right) \right] \tag{114}$$

Thus, the problem with multiple inputs, monopsony, and decreasing returns modifies the firms' original payoff by only introducing the parameter δ. Helpfully, observe that $\delta = 1$ when: (i) there are constant returns to scale $\sum_{i=1}^{I} a_i = 1$ and (ii) there is no monopsony $b_i = 1$ for all i.

Given this, we can write the firm's objective as:

$$J(\hat{p}) = \int_{\mathbb{R}^4_{++}} \Lambda \left(\frac{\hat{p}(z)^{1-\eta}}{P} z - \mathcal{M} z^{\frac{1}{\delta}} \hat{p}(z)^{-\frac{\eta}{\delta}} \right) \mathrm{d}G(\Lambda, P, \mathcal{M}, z) \tag{115}$$

And, as before, we study the problem:

$$\sup_{\hat{p}: \mathbb{R}_+ \to \mathbb{R}_{++}} J(\hat{p}) \tag{116}$$

By doing this, we obtain a modified formula for the optimal supply function:

Proposition 4 (Optimal Supply Schedule With Multiple Inputs, Decreasing Returns, and Monopsony)**.** *Any optimal supply schedule is almost everywhere given by:*

$$f(p, q) = \log p - \frac{\omega_0 - \log \delta}{1 - \eta\omega_1} - \frac{\omega_1 + \frac{1-\delta}{\delta}}{1 - \eta\omega_1} \log q \tag{117}$$

where ω_0 and ω_1 are the same as those derived in Theorem 1. Thus, the optimal inverse *supply elasticity is given by:*

$$\hat{\alpha}_1 = \frac{\eta\sigma_P^2 + \sigma_{\mathcal{M},\Psi} + \sigma_{P,\Psi} + \eta\sigma_{\mathcal{M},P}}{\sigma_\Psi^2 - \eta\sigma_{\mathcal{M},\Psi} + \eta\sigma_{P,\Psi} - \eta^2\sigma_{\mathcal{M},P}} \left(1 + \frac{1-\delta}{\delta} \frac{\sigma_\Psi^2 + \eta^2\sigma_P^2 + 2\eta\sigma_{\Psi,P}}{\sigma_{\mathcal{M},\Psi} + \eta\sigma_{\mathcal{M},P} + \sigma_{P,\Psi} + \eta\sigma_P^2}\right) \tag{118}$$

Proof. Applying the same variational arguments as in the Proof of Theorem 1, we obtain that $\hat{p}(t)$ must solve:

$$(\eta - 1)\mathbb{E}[\Lambda P^{-1}|z = t]t\hat{p}(t)^{-\eta} = \frac{\eta}{\delta}\mathbb{E}[\Lambda\mathcal{M}|z = t]t^{\frac{1}{\delta}}\hat{p}(z)^{-\frac{\eta}{\delta}-1} \tag{119}$$

Which yields:

$$\hat{p}(t) = \left(\delta^{-1}\frac{\eta}{\eta-1}\frac{\mathbb{E}[\Lambda\mathcal{M}|z=t]}{\mathbb{E}[\Lambda P^{-1}|z=t]}\right)^{\frac{1}{1+\eta\left(\frac{1-\delta}{\delta}\right)}} t^{\frac{\frac{1-\delta}{\delta}}{1+\eta\left(\frac{1-\delta}{\delta}\right)}} \tag{120}$$

Thus, we have that:

$$\log p = \frac{1}{1+\eta\left(\frac{1-\delta}{\delta}\right)}(\omega_0 - \log\delta) + \frac{1}{1+\eta\left(\frac{1-\delta}{\delta}\right)}\left(\omega_1 + \frac{1-\delta}{\delta}\right)\log z \tag{121}$$

where ω_0 and ω_1 are as in Theorem 1. Rewriting as a supply function, we obtain:

$$\log p = \frac{\frac{1}{1+\eta\left(\frac{1-\delta}{\delta}\right)}(\omega_0 - \log\delta)}{1 - \frac{\eta}{1+\eta\left(\frac{1-\delta}{\delta}\right)}\left(\omega_1 + \frac{1-\delta}{\delta}\right)} + \frac{\frac{1}{1+\eta\left(\frac{1-\delta}{\delta}\right)}\left(\omega_1 + \frac{1-\delta}{\delta}\right)}{1 - \frac{\eta}{1+\eta\left(\frac{1-\delta}{\delta}\right)}\left(\omega_1 + \frac{1-\delta}{\delta}\right)}\log q \tag{122}$$

Which reduces to the claimed formula. □

Thus, when the supply curve is initially upward-sloping ($\omega_1 \in [0, \eta^{-1}]$), the introduction of decreasing returns and/or monopsony unambiguously increases the supply elasticity and makes firms closer to quantity-setting.

B.2 Beyond Isoelastic Demand

Isoelastic demand imposes both that the firm's own price elasticity of demand and its cross-price elasticity of demand are constant. In this appendix, we show how to derive optimal supply functions in closed form when the firm's own price elasticity of demand varies. This allows the demand curve to satisfy Marshall's second law of demand that the price elasticity of demand is increasing in the price as well as Marshall's third law of demand that the rate of increase of the price elasticity goes down with the price. We show that uncertainty about demand, prices, and marginal costs continue to operate in a very similar fashion. However, due to endogeneity of the optimal markup, the optimal supply schedule now ceases to be log-linear.

To capture these features, suppose that demand is *multiplicatively separable*: $d(p, \Psi, P) = z(\Psi, P)\phi(p)$ for some function ϕ such that $p\phi''(p)/\phi'(p) < -2$. This latter condition is satisfied by isoelastic demand exactly under the familiar condition that $\eta > 1$ and ensures the existence of a unique optimal price. We further assume that $z(\Psi, P) = \nu_0 \Psi^{\nu_1} P^{\nu_2}$ for $\nu_0, \nu_1, \nu_2 \in \mathbb{R} \setminus \{0\}$. This makes firms' uncertainty about the location of their demand curve log-normal. This assumption does rule out non-separable demand, such as the demand system proposed by Kimball (1995). However, it is important to note that this demand system is motivated by evidence on the firm's own price elasticity, which is governed by ϕ, and not the cross-price elasticity, which is governed by ν_2. Thus, our proposed demand system is equally able to capture facts about the firms' own price elasticity as the one proposed in Kimball (1995), or the richer structures proposed by Fujiwara and Matsuyama (2022) and Wang and Werning (2022).

Under this demand system, we can derive a modified formula for the optimal supply curve which is now no longer log-linear, but continues to be governed by similar forces:

Proposition 5. *If demand is multiplicatively separable, then any optimal supply function is almost everywhere given by:*

$$f(p, q) = \log q + \hat{\alpha}_0 - \log\left(\phi(p)\left\{p\left[1 + \frac{\phi(p)}{p\phi'(p)}\right]\right\}^{\frac{1}{\hat{\omega}_1}}\right) \tag{123}$$

where:

$$\hat{\omega}_1 = \frac{\nu_1(\sigma_{\mathcal{M},\Psi} + \sigma_{P,\Psi}) + \nu_2(\sigma_P^2 + \sigma_{\mathcal{M},P})}{\nu_1^2\sigma_\Psi^2 + \nu_2^2\sigma_P^2 + 2\nu_1\nu_2\sigma_{\Psi,P}} \tag{124}$$

Proof. Applying the same variational arguments as in Theorem 1, we obtain that:

$$\hat{p}(z) + \frac{\phi(\hat{p}(z))}{\phi'(\hat{p}(z))} = \frac{\mathbb{E}[\Lambda\mathcal{M}|z]}{\mathbb{E}[\Lambda P^{-1}|z]} \tag{125}$$

where the condition $p\phi''(p)/\phi'(p) < -2$ yields strict concavity of the objective and makes $\hat{p}(z)$ the unique maximizer. Taking logarithms of both sides and evaluating the conditional expectations as per Theorem 1, we obtain that:

$$\log\left(\hat{p}(z)\left[1 + \frac{\phi(\hat{p}(z))}{\hat{p}(z)\phi'(\hat{p}(z))}\right]\right) = \hat{\omega}_0 + \hat{\omega}_1 \log z \tag{126}$$

where $\hat{\omega}_1 = \frac{\sigma_{\mathcal{M},z} + \sigma_{P,z}}{\sigma_z^2}$, which yields Equation 124. Using $\log z = \log q - \log \phi(p)$ and rearranging yields Equation 123. □

Demand uncertainty and price uncertainty enter the same way as before, via $\hat{\omega}_1$, and the intuition is the same. However, there are now two distinct notions of market power and they therefore operate in a more subtle way. First, consider the role of the cross-price elasticity of demand ν_2. When ν_2 is higher, the firm's price is *ex post* more responsive to changes in others' prices. Second, consider the role of the own-price elasticity of demand $\left(\frac{p\phi'(p)}{\phi(p)}\right)^{-1}$. This induces non-linearity of the optimal supply schedule to the extent that it is not constant. This is because the firm's optimal markup changes as it moves along its demand curve.

B.3 Additional Choice Variables

Our approach of studying firms' supply functions has thus far focused on firms that choose prices and quantities. However, it is natural to imagine that firms can make richer choices, such as deciding what quality or type of product they will sell. In this appendix, we generalize our characterization of firms' optimal supply functions to incorporate additional choice margins. We find that supply functions remain log-linear conditional on these other choices. We also show how to characterize the optimal values of these other choices given this fact.

To model additional choice margins, suppose that the firm, in addition to its price and quantity decisions, chooses a vector of non-quantity decisions $x \in X \subseteq \mathbb{R}^n$. These decisions are made at the beginning of the period and potentially affect the joint distribution of $(\Lambda, P, \mathcal{M}, \Psi)$ via the map $G : X \to \Delta(\mathbb{R}^4_+)$. We suppose that choices of $x \in X$ lead to a dollar cost to the firm of $C(x)$. To see how this framework accommodates quantity investments, suppose that $X \subseteq \mathbb{R}$ and $x \in X$ represents the quality of the good. Investing in different qualities comes at a cost. Moreover, higher quality might increase both the mean of firms' demand Ψ and the mean of firms' marginal costs $\mathcal{M}$.

We now characterize firms' optimal supply function decisions in this framework. We let $H(f, x)$ denote the joint distribution over $(\Lambda, P, \mathcal{M}, \Psi, p, q)$ induced by a supply function $f : \mathbb{R}^2_{++} \to \mathbb{R}$ and other decisions x. With this, the firm's problem of optimal supply function and other decisions is given by:

$$\sup_{x \in X, f:\mathbb{R}^2_{++} \to \mathbb{R}} \mathbb{E}_{H(f,x)} \left[\Lambda \left(\frac{p}{P} - \mathcal{M} \right) q \right] - \mathbb{E}_{H(f,x)}[\Lambda] C(x) \tag{127}$$

This can be split into two optimization problems. First, for every choice of $x \in X$, we solve for the optimal supply function f_x:

$$V(x) = \sup_{f:\mathbb{R}^2_{++} \to \mathbb{R}} \mathbb{E}_{H(f,x)} \left[\Lambda \left(\frac{p}{P} - \mathcal{M} \right) q \right] \tag{128}$$

Second, we can compute the optimal choice of $x \in X$ by solving:

$$\sup_{x \in X} V(x) - \mathbb{E}_{H(f,x)}[\Lambda] C(x) \tag{129}$$

By identical arguments to those of Theorem 1 (simply index G by x up to Equation 44), we immediately obtain that under the optimal prices in demand state $z = t$ must be given by:

$$p_x(t) = \frac{\eta}{\eta - 1} \frac{\mathbb{E}_{G(x)}[\Lambda \mathcal{M} | z = t]}{\mathbb{E}_{G(x)}[\Lambda P^{-1} | z = t]} \tag{130}$$

If we further assume that $G(x)$ is a multivariate log-normal distribution with mean μ_x and variance-covariance matrix Σ_x, then we obtain (by identical arguments to those in Theorem 1) that the optimal supply function for a fixed choice of $x \in X$ obeys the following Proposition, the proof of which follows immediately from that of Theorem 1.

Proposition 6 (Supply Function Choice When Firms Choose More Than Prices and Quantities)**.** *If for $x \in X$ the distribution $G(x)$ is multivariate normal, then the optimal supply function is given by:*

$$f_x(p, q) = \log p - \alpha_{0,x} - \alpha_{1,x} \log q \tag{131}$$

where $\alpha_{0,x}$ and $\alpha_{1,x}$ follow exactly the formulae derived in Theorem 1, where all appropriate means and variances are computed under the distribution $G(x)$.

From this, we observe that Theorem 1 carries as written in this extended setting. In particular, supply functions remain log-linear and the same variances and covariances govern their elasticity. The new feature here is that the choice of x can affect both the intercept and the slope of the optimal supply function. In this way, the choice of x can have a non-trivial effect on firms' optimal pricing and production decisions.

With this, we can now explicitly characterize the value of any choice of x and thereby solve for the optimal choice of x. Concretely, we have that:

$$V(x) = \int_{\mathbb{R}^4_{++}} \Lambda \frac{z}{P} p_x(z)^{1-\eta} \, \mathrm{d}G_x(\Lambda, P, \mathcal{M}, \Psi) - \int_{\mathbb{R}^4_{++}} \Lambda z \mathcal{M} p_x(z)^{-\eta} \, \mathrm{d}G_x(\Lambda, P, \mathcal{M}, \Psi) \tag{132}$$

Substituting Equation 130, this becomes:

$$\begin{aligned} V(x) = &\int_{\mathbb{R}^4_{++}} \Lambda \frac{z}{P} \exp\{(1-\eta)\omega_{0,x}\} z^{(1-\eta)\omega_{1,x}} \, \mathrm{d}G_x(\Lambda, P, \mathcal{M}, \Psi) \\ &- \int_{\mathbb{R}^4_{++}} \Lambda z \mathcal{M} \exp\{-\eta\omega_{0,x}\} z^{-\eta\omega_{1,x}} \, \mathrm{d}G_x(\Lambda, P, \mathcal{M}, \Psi) \end{aligned} \tag{133}$$

where $\omega_{0,x}$ and $\omega_{1,x}$ have the same formulae as those in the proof of Theorem 1 (with all means, variances, and covariances indexed by x). Exploiting joint log-normality of G_x, we can evaluate these integrals to obtain:

$$\begin{aligned} V(x) = \exp\left\{(1-\eta)\omega_{0,x} + \mu_{\Lambda,x} - \mu_{P,x} + (1+\omega_{1,x}(1-\eta))\mu_{z,x} + \frac{1}{2}\sigma^2_{R,x}\right\} \\ - \exp\left\{-\eta\omega_{0,x} + \mu_{\Lambda,x} + \mu_{\mathcal{M},x} + (1-\eta\omega_{1,x})\mu_{z,x} + \frac{1}{2}\sigma^2_{C,x}\right\} \end{aligned} \tag{134}$$

where:

$$\begin{aligned} \sigma^2_{R,x} &= \mathbb{V}_x\left[\log\Lambda - \log P + (1+\omega_{1,x}(1-\eta))\log z\right] \\ \sigma^2_{C,x} &= \mathbb{V}_x\left[\log\Lambda + \log\mathcal{M} + (1-\eta\omega_{1,x})\log z\right] \end{aligned} \tag{135}$$

With this, solving for the optimal choice of $x \in X$ reduces to solving Equation 129 using this V and given the exogenous function C.

We conclude by characterizing the optimal x in a simple example.

Example 1. *Suppose that quality can be improved at some ex ante cost and that quality affects how much consumers demand the product and nothing else. Formally, suppose that* $C(x) = \frac{\zeta}{2}x^2$, $\mu_{\Psi,x} = \mu_\Psi + \log x$, $\sigma_x \equiv \sigma$ *and* μ_x *is invariant to* x *except for* $\mu_{\Psi,x}$. *In the previous formulae, observe that* $(\omega_{1,x}, \sigma^2_{C,x}.\sigma^2_{R,x}, \mu_{\Lambda,x}, \mu_{P,x}, \mu_{\mathcal{M},x})$ *are invariant to* x. *Thus, observing that* $\omega_{0,x}$ *is affine in* $\log x$, *we obtain that* V *is linear in* x, *i.e.,* $V(x) = Kx$ *for some* $K > 0$. *It follows that the optimal choice is given by* $x^* = \frac{K}{\zeta}$.

This example shows that the approach followed in this appendix can be practically useful in extending the supply function approach to consider firms that can choose additional variables.

B.4 Supply Functions with Sticky Prices

In our main analysis, we allowed firms to change their prices every period to emphasize the new economic features that supply functions generate. At the same time, our approach can be augmented to include price stickiness. In this appendix, we show how to solve for the optimal supply function when firms are subject to Calvo pricing.

Firms are as in our main analysis, except their prices are sticky each period with probability $\theta \in [0,1]$. For this appendix, we apply the standard second-order approximation to firms' profits and write the flow profit of the firm as:

$$-B(\log p - \log p^{**})^2 \tag{136}$$

where we recall that $p^{**} = \frac{\eta}{\eta-1}\mathcal{M}P$ and $B > 0$ is the curvature of the profit function. Under this approximation, the firm's lifetime loss from setting price p_t at date t is given by:

$$\mathcal{L}(p_t) = B\sum_{j=0}^{\infty}(\beta\theta)^j(\log p_t - \log p^{**}_{t+j})^2 \tag{137}$$

As in our main analysis, at date t, a price-resetting firm chooses a supply function f_t and they will produce at the price and quantity such that the $f_t(p_t, q_t) = 0$ locus intersects the demand curve $\log z_t = \log q_t + \eta \log p_t$. By applying similar arguments to those of Theorem 1, we obtain the following characterization of the optimal supply function:

Proposition 7 (Optimal Supply Function with Price Stickiness)**.** *For a firm with Calvo stickiness parameter $\theta \in [0, 1]$ and discount factor $\beta \in [0, 1)$, any optimal supply curve is almost everywhere given by:*

$$f_t(p_t, q_t) = \log p_t - \alpha_{0,t} - \alpha_{1,t}\log q_t \tag{138}$$

where the slope of the optimal price-quantity locus, $\alpha_{1,t} \in \overline{\mathbb{R}}$, is given by:

$$\alpha_{1,t} = \frac{\hat{\omega}_{1,t}}{1 - \eta\hat{\omega}_{1,t}} \tag{139}$$

where:

$$\hat{\omega}_{1,t} = (1 - \beta\theta)\sum_{j=0}^{\infty}(\beta\theta)^j\omega_{1,t,j} \tag{140}$$

and:

$$\omega_{1,t,j} = \frac{\sigma_{\mathcal{M}_{t+j},z_t} + \sigma_{P_{t+j},z_t}}{\sigma^2_{z_t}} \tag{141}$$

Proof. We first characterize the optimal z_t-measurable price, $\hat{p}_t(z_t)$. Taking the first-order condition of the firm's expected loss, we have that:

$$\log \hat{p}_t(z_t) = (1 - \beta\theta)\mathbb{E}_t\left[\sum_{j=0}^{\infty}(\beta\theta)^j \log p^{**}_{t+j} \mid z_t\right] \tag{142}$$

We moreover have that:

$$\mathbb{E}_t[\log p^{**}_{t+j}|z_t] = \mathbb{E}_t\left[\log\frac{\eta}{\eta-1} + \log P_{t+j} + \log\mathcal{M}_{t+j} \mid z_t\right] = \omega_{0,t,j} + \omega_{1,t,j}z_t \tag{143}$$

where:

$$\begin{aligned}\omega_{1,t,j} &= \frac{\sigma_{\mathcal{M}_{t+j},z_t} + \sigma_{P_{t+j},z_t}}{\sigma^2_{z_t}} \\ \omega_{0,t,j} &= \log\frac{\eta}{\eta-1} + \mu_{\mathcal{M}_{t+j}} + \mu_{P_{t+j}} - \omega_{1,t,j}\mu_{z_t}\end{aligned} \tag{144}$$

which are both deterministic functions of t and j. Substituting this into the formula for the firm's optimal z_t-measureable price, we obtain that:

$$\begin{aligned}\log\hat{p}_t(z_t) &= (1-\beta\theta)\sum_{j=0}^{\infty}(\beta\theta)^j\omega_{0,t,j} + \left[(1-\beta\theta)\sum_{j=0}^{\infty}(\beta\theta)^j\omega_{1,t,j}\right]z_t \\ &= \hat{\omega}_{0,t} + \hat{\omega}_{1,t}z_t\end{aligned} \tag{145}$$

Using the fact that the firm's demand curve is $\log z_t = \log q_t + \eta\log p_t$, we obtain that $\log p_t = \alpha_{0,t} + \alpha_{1,t}\log q_t$ with $\alpha_{0,t} = \frac{\hat{\omega}_{0,t}}{1-\eta\omega_{\hat{1},t}}$ and $\alpha_{1,t} = \frac{\hat{\omega}_{1,t}}{1-\eta\hat{\omega}_{1,t}}$, completing the proof. □

From this, we observe that price stickiness modifies the slope of the firm's optimal supply function, but it remains optimally log-linear (at least under the quadratic approximation to the firm's flow profit that is standard in dynamic Calvo pricing models). The firm's optimal supply elasticity now incorporates how much the firm learns from its demand today about the whole sequence of its current and future nominal marginal costs. The inference that it performs about its date $t+j$ marginal costs from today's demand is captured by $\omega_{1,t,j}$, which is precisely the least-squares regression coefficient that one obtains from regressing nominal marginal costs at date $t+j$ on demand at date t. In deciding its optimal price today, the firm then must weigh its inference about future nominal marginal costs by how much it cares about the future j periods from now (β^j) and how likely its price today is to prevail in j periods (θ^j). This weighting yields $\hat{\omega}_{1,t}$, which captures the overall responsiveness of the price today to demand today. Once this has been obtained, we can convert this into the slope of the optimal supply curve as we did in our main analysis via the transformation $\hat{\omega}_{1,t} \mapsto \frac{\hat{\omega}_{1,t}}{1-\eta\hat{\omega}_{1,t}} \equiv \alpha_{1,t}$.

This analysis highlights that the supply function approach is not a replacement for sticky price models, but rather represents a different approach to modelling how firms that can reset their prices do so. While we abstract from sticky prices in our main analysis to make plain the new implications of supply functions, the analysis of this appendix demonstrates that it is practically simple to combine our supply function approach with canonical approaches to modelling sticky prices.

C Allowing for Correlated Aggregate Shocks

In this extension, we allow for the shocks to the money supply and aggregate productivity to be correlated. Specifically, we assume that, conditional on outcomes in period $t-1$ that $(\log A_t, \log M_t)$ is jointly normally distributed. Our main analysis assumes that $\log A_t$ and $\log M_t$ are uncorrelated. Allowing for correlation modifies firms' conditional expectations of the aggregate shocks to the following:

$$\begin{aligned} \mathbb{E}_{i,t}[\log A_t] &= \text{cons}_t + \kappa_t^A s_{it}^A + \tilde{\kappa}_t^A s_{it}^M \\ \mathbb{E}_{i,t}[\log M_t] &= \text{cons}_t + \kappa_t^M s_{it}^M + \tilde{\kappa}_t^M s_{it}^A \end{aligned} \tag{146}$$

where cons_t are terms independent of the realized shocks at date t, and $(\kappa_t^A, \tilde{\kappa}_t^A, \kappa_t^M, \tilde{\kappa}_t^M)$ are the Kalman gains.

In this extended setting, Theorem 1 on firms' optimal supply functions holds as written. Theorem 2 on the AS/AD representation holds with modified formulae for the slopes of the aggregate demand and aggregate supply curves as the guess and verify argument must be modified to account for the new formulae for firms' expectations of aggregate shocks. Performing this modification, we obtain the following:

Proposition 8. *There exists a unique log-linear temporary equilibrium that is described by an "Aggregate Demand/Aggregate Supply" model in which the slope of the aggregate supply curve is given by:*

$$\epsilon_t^S = \gamma \frac{\chi_{M,t}}{1-\chi_{M,t}} \tag{147}$$

where:

$$\chi_{M,t} = \frac{\kappa_t^M + (1-\kappa_t^M)\frac{1}{\gamma}\omega_{1,t} + \tilde{\kappa}_t^A \left(-1 - \left(\eta - \frac{1}{\gamma}\omega_{1,t}\frac{-\kappa_t^A + \left(1-\frac{1}{\gamma}\omega_{1,t}\right)\tilde{\kappa}_t^M}{1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^A)}\right)\right)}{1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)\left((1-\kappa_t^M) - \frac{\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)\tilde{\kappa}_t^M}{1-\omega_{1,t}\left(\eta-\frac{1}{\gamma}\right)(1-\kappa_t^A)}\tilde{\kappa}_t^A\right)} \tag{148}$$

Proof. As in the proof of Theorem 2, we will guess and verify that (dropping t subscripts for notational simplicity):

$$\log P = \chi_0 + \chi_A \log A + \chi_M \log M \tag{149}$$

The same arguments as Theorem 2 imply that we must compute:

$$p_i = \frac{\eta}{\eta - 1} \frac{\mathbb{E}_i\left[\phi(\zeta_i A)^{-1}\left(\vartheta_i I^{-\frac{1}{\gamma}} P^{-\frac{1}{\gamma}} M^{\frac{1}{\gamma}} P^{\eta}\right)^{1-\eta\omega_1}\right]}{\mathbb{E}_i\left[I^{1-\frac{1}{\gamma}(1+\omega_1-\eta\omega_1)} M^{\frac{1}{\gamma}(1+\omega_1-\eta\omega_1)-1} \vartheta^{1+\omega_1-\eta\omega_1} P^{\left(\eta-\frac{1}{\gamma}\right)(1+\omega_1-\eta\omega_1)}\right]} \times \left(\vartheta_i I^{-\frac{1}{\gamma}} M^{\frac{1}{\gamma}} P^{\eta-\frac{1}{\gamma}}\right)^{\omega_1} \tag{150}$$

Moreover, the same arguments as Theorem 2 imply that:

$$\log P = \mathbb{E}[\log p_i] + \frac{1}{2(1-\eta)} \mathrm{Var}((1-\eta)\log p_i) + \text{cons} \tag{151}$$

We now compute the numerator, denominator and multiplicative terms in the firm's pricing equation that obtain under their chosen supply function:

$$\begin{aligned} &\log \mathbb{E}_i\left[\phi(\zeta_i A)^{-1}\left(\vartheta_i I^{-\frac{1}{\gamma}} P^{-\frac{1}{\gamma}} M^{\frac{1}{\gamma}} P^{\eta}\right)^{1-\eta\omega_1}\right] = \text{cons} \\ &+ \left(-1 + \left(\eta - \frac{1}{\gamma}\right)(1-\eta\omega_1)\chi_A\right)\mathbb{E}_i[\log A] \\ &+ \left(\frac{1}{\gamma}(1-\eta\omega_1) + \left(\eta - \frac{1}{\gamma}\right)(1-\eta\omega_1)\chi_M\right)\mathbb{E}_i[\log M] \end{aligned} \tag{152}$$

$$\begin{aligned} &\log \mathbb{E}_i\left[I^{1-\frac{1}{\gamma}(1+\omega_1-\eta\omega_1)} M^{\frac{1}{\gamma}(1+\omega_1-\eta\omega_1)-1} \vartheta^{1+\omega_1-\eta\omega_1} P^{\left(\eta-\frac{1}{\gamma}\right)(1+\omega_1-\eta\omega_1)}\right] = \text{cons} \\ &+ \left(\eta - \frac{1}{\gamma}\right)(1-\eta\omega_1+\omega_1)\chi_A \mathbb{E}_i[\log A] \\ &+ \left(-1 + \frac{1}{\gamma}(1-\eta\omega_1+\omega_1) + \left(\eta - \frac{1}{\gamma}\right)(1-\eta\omega_1+\omega_1)\chi_M\right)\mathbb{E}_i[\log M] \end{aligned} \tag{153}$$

$$\begin{aligned} &\log\left(\vartheta_i I^{-\frac{1}{\gamma}} M^{\frac{1}{\gamma}} P^{\eta-\frac{1}{\gamma}}\right)^{\omega_1} = \text{cons} \\ &+ \omega_1\left(\eta - \frac{1}{\gamma}\right)\chi_A \log A + \omega_1\left(\frac{1}{\gamma} + \left(\eta - \frac{1}{\gamma}\right)\chi_M\right)\log M \end{aligned} \tag{154}$$

From this, we have that:

$$\begin{aligned} &\log p_i = \text{cons} \\ &+ \left(-1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_A\right)\mathbb{E}_i[\log A] + \omega_1\left(\eta - \frac{1}{\gamma}\right)\chi_A \log A \\ &+ \left(1 - \frac{1}{\gamma}\omega_1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_M\right)\mathbb{E}_i[\log M] + \omega_1\left(\frac{1}{\gamma} + \left(\eta - \frac{1}{\gamma}\right)\chi_M\right)\log M \end{aligned} \tag{155}$$

Aggregating this according to the aggregation formula, we obtain:

$$\begin{aligned}
&\log P = \text{cons} \\
&+ \left(-1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_A\right)\overline{\mathbb{E}}[\log A] + \omega_1\left(\eta - \frac{1}{\gamma}\right)\chi_A \log A \\
&+ \left(1 - \frac{1}{\gamma}\omega_1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_M\right)\overline{\mathbb{E}}[\log M] + \omega_1\left(\frac{1}{\gamma} + \left(\eta - \frac{1}{\gamma}\right)\chi_M\right)\log M
\end{aligned} \tag{156}$$

Up to this point, everything is the same as Theorem 2. The presence of correlated aggregate shocks now changes the formulae for $(\overline{\mathbb{E}}[\log A], \overline{\mathbb{E}}[\log M])$. These are now given by:

$$\begin{aligned}
\overline{\mathbb{E}}[\log A] &= \text{cons} + \kappa^A \log A + \tilde{\kappa}^A \log M \\
\overline{\mathbb{E}}[\log M] &= \text{cons} + \kappa^M \log M + \tilde{\kappa}^M \log A
\end{aligned} \tag{157}$$

Plugging these into the formula for the aggregate price level and collecting terms:

$$\begin{aligned}
&\log P = \text{cons} \\
&+ \left(\omega_1\left(\eta - \frac{1}{\gamma}\right)\chi_A + \left(-1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_A\right)\kappa^A + \left(1 - \frac{1}{\gamma}\omega_1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_M\right)\tilde{\kappa}^M\right)\log A \\
&+ \left(\omega_1\left(\frac{1}{\gamma} + \left(\eta - \frac{1}{\gamma}\right)\chi_M\right) + \left(1 - \frac{1}{\gamma}\omega_1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_M\right)\kappa^M + \left(-1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_A\right)\tilde{\kappa}^A\right)\log M
\end{aligned} \tag{158}$$

Thus, by matching coefficients and simplifying, we have that:

$$\begin{aligned}
\chi_A &= -\kappa^A + (1-\kappa^A)\left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_A + \left(1 - \frac{1}{\gamma}\omega_1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_M\right)\tilde{\kappa}^M \\
\chi_M &= \kappa^M + (1-\kappa^M)\frac{1}{\gamma}\omega_1 + (1-\kappa^M)\left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_M + \left(-1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_A\right)\tilde{\kappa}^A
\end{aligned} \tag{159}$$

We can now solve this linear system of equations in (χ^A, χ^M). To do this, we first solve for χ_A as a function of χ_M:

$$\chi_A = \frac{-\kappa^A + \left(1 - \frac{1}{\gamma}\omega_1 - \left(\eta - \frac{1}{\gamma}\right)\omega_1\chi_M\right)\tilde{\kappa}^M}{1 - \omega_1\left(\eta - \frac{1}{\gamma}\right)(1-\kappa^A)} \equiv a - b\chi_M \tag{160}$$

where:

$$
\begin{aligned}
a &= \frac{-\kappa_A + \left(1 - \frac{1}{\gamma}\omega_1\right)\tilde{\kappa}^M}{1 - \omega_1\left(\eta - \frac{1}{\gamma}\right)(1 - \kappa^A)} \\
b &= \frac{\omega_1\left(\eta - \frac{1}{\gamma}\right)\tilde{\kappa}^M}{1 - \omega_1\left(\eta - \frac{1}{\gamma}\right)(1 - \kappa^A)}
\end{aligned}
\tag{161}
$$

Substituting this into the equation for χ_M, we obtain that:

$$
\chi_M = \frac{\kappa^M + (1 - \kappa^M)\frac{1}{\gamma}\omega_1 + \tilde{\kappa}^A\left(-1 - \left(\eta - \frac{1}{\gamma}\omega_1 a\right)\right)}{1 - \omega_1\left(\eta - \frac{1}{\gamma}\right)((1 - \kappa^M) - b\tilde{\kappa}^A)} \tag{162}
$$

Completing the solution. Using Proposition 1, which establishes that $\epsilon_t^S = \gamma\frac{\chi_{M,t}}{1-\chi_{M,t}}$, we obtain the result. $\square$

D Additional Quantitative and Empirical Analysis

This Appendix provides additional details for the analysis in Section 5.

D.1 Methods and Estimation

Data. We use quarterly-frequency data from the United States from 1960Q1 to 2024Q4. We measure real GDP and the price level using data from the US BEA. From these variables, we construct GDP growth $\Delta \log Y_t$ and inflation $\Delta \log P_t$ in log differences. We measure TFP growth using the dataset of Fernald (2025), based on the work of Fernald (2014). Specifically, we take raw data on the annualized growth rate in capacity-utilization adjusted TFP and divide by 400 to obtain a comparable quarter-to-quarter growth rate $\Delta \log A_t$. Finally, as described in the main text, we construct a variable corresponding to aggregate marginal cost growth as

$$\Delta \log \mathcal{M}_t = \gamma \cdot \Delta \log Y_t - \Delta \log A_t \tag{163}$$

where we calibrate $\gamma = 0.11$ based on the findings of Gagliardone et al. (2023), who use micro-data from Belgian manufacturers to calculate the implied pass-through from the output gap to real marginal costs. This calibration is also consistent with evidence of substantial wage rigidity over the business cycle in the United States (Grigsby et al., 2021), and comparable to what one would estimate by directly looking at the relationship between detrended real wages and output in the US.[21]

Time-Varying Volatility from a GARCH Model. We estimate time-varying uncertainties regarding inflation, real output, and real marginal costs using a multivariate GARCH model. In particular, letting X_t denote the vector $(\Delta \log P_t, \Delta \log Y_t, \Delta \log \mathcal{M}_t)$, we model

$$X_t = A + BX_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \Sigma_t), \qquad \Sigma_t = D_t^{\frac{1}{2}} C D_t^{\frac{1}{2}} \tag{164}$$

where A is a 3×1 vector of constants, B is a 3×3 matrix of AR(1) coefficients, D_t is a diagonal matrix of time-varying variances (and $D_t^{\frac{1}{2}}$ is a diagonal matrix of standard deviations), and C is a static matrix of correlations. We assume that each diagonal element of D_t, denoted as $\sigma_{i,t}^2$, evolves according to:

$$\sigma_{i,t}^2 = s_i + \alpha_i \varepsilon_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2 \tag{165}$$

with unknown constant s_i and coefficients (α_i, β_i). Formally, this is a GARCH (1,1) model with constant conditional correlations (Bollerslev, 1990). We estimate all of the parameters

[21] For example, using this latter method, Flynn and Sastry (2022) calibrate $\gamma = 0.095$.

Table A1: Testing the GARCH Model Against Alternatives

	(1)	(2)	(3)
Model	VAR	GARCH (CCC)	GARCH (VCC)
Likelihood ratio	—	194.82	0.10
Degrees of freedom	—	6	1
p-value (χ^2(df))	—	0.000	0.746

Notes: This table presents specification tests of the GARCH model used for analysis. The data are quarterly-frequency GDP growth, GDP deflator inflation, and real marginal cost growth in the US from 1960Q1 to 2024Q4. The models are, respectively, a vector auto-regression in first differences (column 1); the same model plus a residual GARCH (1,1) with constant conditional correlations (column 2; see also Equations 164 and 165); and the same model plus varying conditional correlations (column 3). The second row gives the likelihood ratio for the model in question versus the nested model in the previous column. The third row gives the degrees of freedom of the likelihood ratio test, equal to the number of additional free parameters. The fourth row gives the p-value from evaluating the test statistic at the χ^2 distribution with the corresponding degrees of freedom.

via joint maximum likelihood.

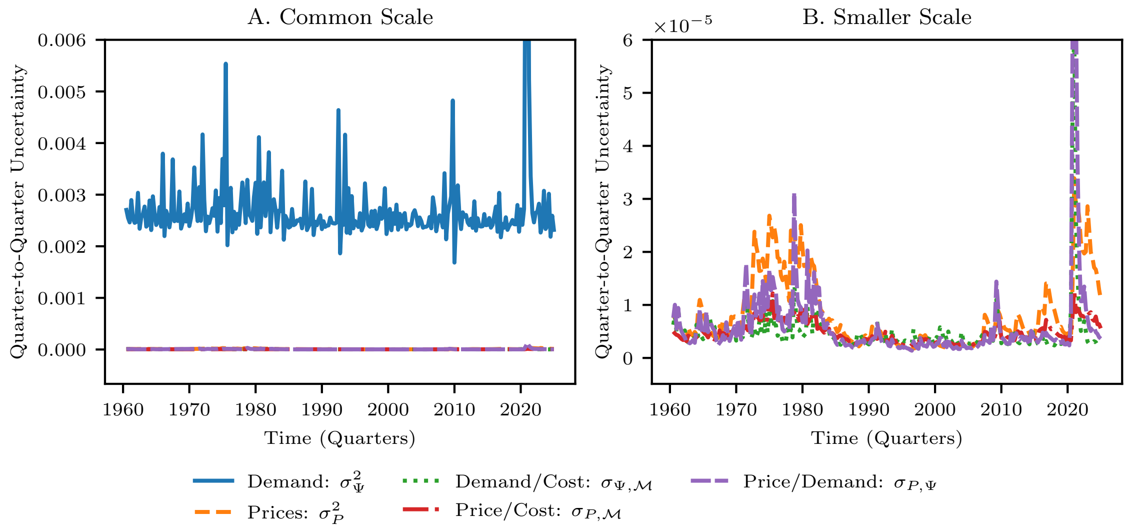
In calibrating the model, we use volatilities dated at time t to stand in for economic agents' uncertainty about making decisions at time t. As is apparent from Equation 165, these volatilities are measurable in macroeconomic history up to period $t-1$. Thus, this timing convention is consistent with our notion in the model that economic agents observe all macroeconomic history up to time $t-1$ and their priors are informed by these observations. All in all, for each quarter t, we set

$$\begin{aligned} \hat{\sigma}^2_{\Psi,t} &= \hat{\Sigma}_{Y,Y,t} + R^2\hat{\Sigma}_{A,A,t} & \hat{\sigma}_{\Psi,P,t} &= \hat{\Sigma}_{Y,P,t} \\ \hat{\sigma}_{\mathcal{M},\Psi,t} &= \hat{\Sigma}_{\mathcal{M},\mathcal{M},t} & \hat{\sigma}_{\mathcal{M},P,t} &= \hat{\Sigma}_{\mathcal{M},P,t} \end{aligned} \tag{166}$$

where the $\hat{\Sigma}_{\cdot,\cdot,t}$ are the elements of the residual covariance matrix and $R = 6.5$ from the quantitative estimates of Bloom et al. (2018).

Our estimation procedure allows us to naturally test the specified model against nested alternatives (Table A1). In column 2, we compare our GARCH model with the nested model with constant volatility: a vector auto-regression (VAR) in first differences for the variable X_t. This model has six fewer parameters, corresponding to the ARCH and GARCH parameter in each residual's equation. The likelihood ratio of 194.82 comfortably rejects the nested VAR model. In column 3, we compare the constant conditional correlations GARCH model (our baseline) with an expanded model that allows for varying conditional correlations (Tse and Tsui, 2002). In particular, in this model, the covariance matrix of residuals is now

Figure A1: Estimates of Time-Varying Uncertainty



Notes: Both panels plot our quarterly time-series estimates of uncertainty, estimated as described in this appendix. All lines are computed from one-quarter-ahead volatility predictions from a constant conditional correlations (CCC) GARCH model. The left plot shows all series on a common scale, and the right plot zooms in on the series other than demand. Both plots feature spikes that are off the scale of the graph during the Covid-19 lockdown.

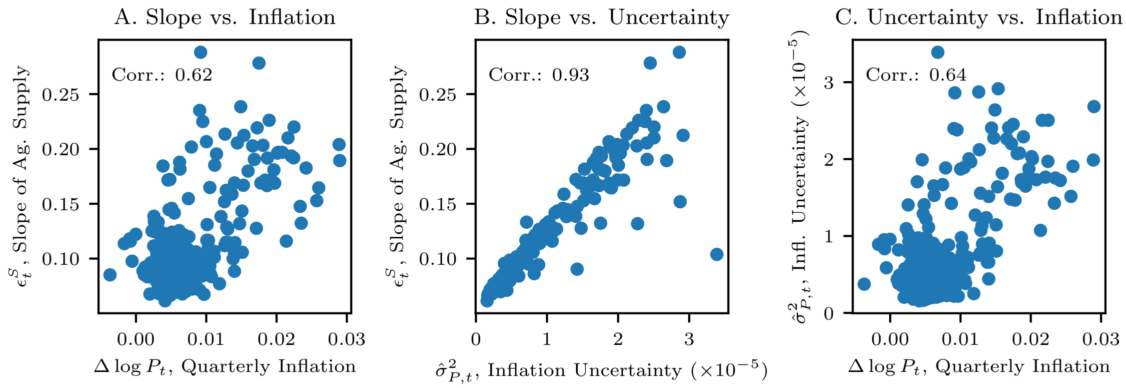
$\Sigma_t = D_t^{\frac{1}{2}} C_t D_t^{\frac{1}{2}}$ (cf. Equation 164) where

$$C_t = (1 - \lambda_1 - \lambda_2)C + \lambda_1 \Psi_t + \lambda_2 C_{t-1} \tag{167}$$

where $\lambda_1, \lambda_2 \geq 0$ are parameters governing the dynamics of the correlations, which satisfy the restriction $0 \leq \lambda_1 + \lambda_2 < 1$; C is a long-run mean of the correlations; and Ψ_t is a 4-period (number of variables plus one) rolling estimator of the standardized residuals $\tilde{\varepsilon}_t = D_t^{-\frac{1}{2}} \varepsilon_t$. Due to the additional restriction on λ_1 and λ_2, this model has only one more free parameter than the nested constant conditional correlations model. The likelihood ratio of 0.10 demonstrates an only marginal and statistically insignificant improvement in fit. Thus, the data suggest that a model with time-varying volatility, but constant conditional correlations, is a good fit for recent US history.

Estimates of Time-Varying Uncertainty. In Figure A1, we plot the raw time series for each of our uncertainty measures. We observe that our estimates of demand uncertainty are an order of magnitude larger than our estimates of other uncertainties. This is natural given our large assumed value of R, the (square root of the) ratio between idiosyncratic demand uncertainty and aggregate real marginal cost uncertainty. But this does not necessarily imply

Figure A2: Inflation, Inflation Uncertainty, and the Slope of Aggregate Supply

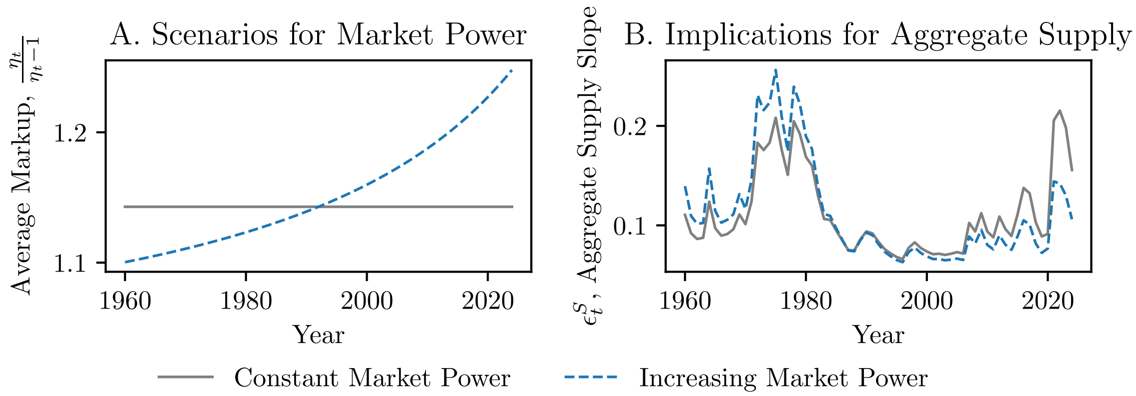


Notes: This figure shows the bivariate relationships between the estimated slope of aggregate supply (see Section 5.1 and Appendix D.1), the level of inflation (quarterly log difference in GDP deflator), and uncertainty about inflation (estimated one-quarter-ahead from a constant conditional correlations GARCH model; see Section 5.1 and Appendix D.1) in US data. Each observation corresponds to one quarter. The numbers in the top left indicate the correlations for each pair of variables.

that demand uncertainty is the only influential force shaping the slope of microeconomic or macroeconomic supply, since uncertainties enter our formulae in interaction with the elasticity of demand η. This is apparent from our results—the fluctuations in the slope of aggregate supply in Figure 4 clearly reflect significant fluctuations in the other components of uncertainty that are plotted in the second panel of Figure A1.

Inflation Levels, Inflation Uncertainty, and the Estimated Slope. Figure A2 shows the correlations between our estimated slope, the level of inflation, and uncertainty regarding inflation. Broadly speaking, we estimate the slope of aggregate supply to be high when the level and uncertainty regarding inflation are high (panels A and B). Moreover, the level of inflation and inflation uncertainty are highly correlated with one another (panel C). This finding echoes the observation of Ball et al. (1988) that it is difficult, empirically, to find circumstances in which levels and volatilities of inflation are decoupled from one another, posing a difficulty for testing different models of state-dependent aggregate supply against one another. However, as observed in Section 5.3, our model based on *relative* uncertainty gives quite different predictions than simple models based on the level of or one-dimensional uncertainty regarding inflation when confronted with global data.

Figure A3: Rising Market Power and Flattening Aggregate Supply



Notes: This figure plots the slope of aggregate supply under different scenarios of declining market power and shows how trends in market power affect the slope of aggregate supply. We calibrate the model under two scenarios: a fixed value of $\eta_t \equiv 8$ (grey line, "Constant Market Power") and a linear trend over the sample from an initial value of $\eta_{1960Q1} = 11$ to a final value of $\eta_{2024Q4} = 5$ (blue dashed line, "Rising Market Power"). All other parameters, including the measured uncertainties, are exactly as in our baseline calculations (see Section 5.1 and Appendix D.1). Panel A shows the time series behavior of average markups, $\frac{\eta_t}{\eta_t - 1}$, implied by our different assumptions about the elasticity of demand. Panel B shows the resulting calculations for the slope of aggregate supply, averaged over years.

D.2 Market Power and Aggregate Supply

A recent literature has suggested that market power, as measured by rising markups, has risen throughout time (De Loecker et al., 2020; Demirer, 2020; Edmond et al., 2023). Combined with our theoretical finding that increased market power flattens aggregate supply under plausible parameter values, this suggests another potentially relevant culprit for the long-run flattening of supply.

To study this possibility, we consider alternative calibrations of the slope of aggregate supply in which we allow a secular downward trend in the elasticity of demand. Specifically, we consider a scenario in which η linearly declines from 11 to 5 between 1960 and 2024. This implies an increase in average markups from $11/10 = 1.10$ to $5/4 = 1.25$. These exercises are *not* counterfactuals, which would require fully estimating the model and accounting for the effects of market power on macroeconomic uncertainty. Instead, they are alternative calibrations that would be more appropriate than our baseline if the elasticity of demand has truly fallen over time.

Introducing a decline in market power increases the slope of aggregate supply in the 1970s and decreases the slope in modern times (Figure A3). Calibrating to this different scenario implies that the slope of aggregate supply flattens by 41% from 1978-1990 to 1991-2018, compared to an estimate of 28% in our baseline model and an empirical estimate of 51%

from Hazell et al. (2022). Thus, allowing for an increase in market power allows the model to more closely match empirical estimates for the flattening of aggregate supply from the 1970s to the 2010s. These calculations provide suggestive evidence that market power interacts in a quantitatively relevant way with the slope of aggregate supply in our model. We leave further analysis of this interaction to future work.

D.3 International Evidence

Data. We take annual data from 1960-2019 from the most recent edition of the Penn World Tables (Feenstra et al., 2015; Zeileis, 2023). In particular, we measure real GDP, GDP deflator (expressed in local currency), total hours, and the real value of the capital stock. We construct real GDP growth and inflation as log differences (annual) in the corresponding variables. We calculate TFP at the level of countries c and years t based on a constant labor share of 2/3 as

$$\log A_{ct} = \log \text{RealGDP}_t - \frac{1}{3} \log \text{RealCapitalStock}_{ct} - \frac{2}{3} \log \text{LaborHours}_{ct} \tag{168}$$

Finally, we construct growth in real marginal costs as described in Equation 163, using the same calibration for γ. To calculate the slope of aggregate supply in each country, we also carry over our calibration of $\eta = 8$, $R = 6.5$, and $\kappa^M = 0.29$.

Volatility from a VAR Model. Because our interest is cross-sectional differences, we estimate a VAR model with time-*in*variant volatility for each country, rather than a model of time-varying volatility (e.g., a GARCH model). In particular, letting X_t again denote the vector $(\Delta \log P_t, \Delta \log Y_t, \Delta \log \mathcal{M}_t)$, we model

$$X_{ct} = A_c + B_c X_{c,t-1} + \varepsilon_{ct}, \qquad \varepsilon_{ct} \sim N(0, \Sigma_c), \tag{169}$$

where (A_c, B_c) are country-specific coefficients and Σ_c is a country-specific covariance matrix. We map the covariances from the VAR to the model using the same method described in Equation 166, but with an estimate for Σ that depends on countries rather than time periods.

Finally, we drop three outliers from our calculations, Greece, Iceland, and Sweden, for which we calculate a slope of aggregate supply and/or inflation-output relationship more than 3 standard deviations away from the median.

Empirical Proxies for the Slope of Aggregate Supply. We calculate two country-level proxies for the slope of aggregate supply. The first is the country-level, reduced-form relationship between inflation and real output growth. That is, the coefficient β_c^S from the regression

$$\Delta \log P_{ct} = \alpha_c + \beta_c^S \cdot \Delta \log Y_{ct} + \varepsilon_{ct} \tag{170}$$

estimated by ordinary least squares for each country c, using variation across time periods. The coefficient

$$\beta_c^S = \frac{\text{Cov}[\Delta \log P_{ct}, \Delta \log Y_{ct}]}{\text{Var}[\Delta \log Y_{ct}]} \tag{171}$$

measures the strength of the reduced-form relationship between real output growth and inflation. This is in the spirit of the reduced-form tests of Lucas (1973) and Ball et al. (1988), who similarly look at covariances of real and nominal components of GDP. To understand the structural interpretation of β_c^S, we observe from Theorem 2 that, in the equilibrium of the model,

$$\Delta \log P_t = \epsilon^S \Delta \log Y_t + \underbrace{\left(\delta_t \Delta \log A_t + \Delta \log \bar{P}_t\right)}_{=\tilde{\varepsilon}_{ct}} \tag{172}$$

where the term in parenthesis can be interpreted as the structural residual of Equation 170. Intuitively, the structural residual of the reduced-form relationship between aggregate prices and aggregate quantities can be thought of as the "shock to aggregate supply," and the reduced-form relationship traces out the "aggregate supply curve" if and only if all variation in real GDP growth is induced by "aggregate demand" shocks (*i.e.*, money supply shocks). If this does not hold (*i.e.*, if some variation in real GDP growth, in deviation from the mean, is driven by productivity), then we expect $\text{Cov}[\Delta \log Y_{ct}, \tilde{\varepsilon}_{ct}] < 0$ and a downward bias in the ordinary least squares estimate, or plim $\hat{\beta}_c^{S,OLS} < \epsilon^S$

As a second strategy, we construct a model-based instrument for money supply growth. Using the money demand equation (the second equation in Proposition 1), we observe that, in equilibrium,

$$M_t = Y_t^{\gamma} P_t \frac{1 + i_t}{i_t} \tag{173}$$

Abstracting from nominal interest rate changes, which is what our model implies under the imposed simplification of time-invariant volatility (and time-invariant ϵ^S), the model implies

$$\Delta \log M_t = \gamma \Delta \log Y_t + \Delta \log P_t \tag{174}$$

and moreover, due to the random-walk behavior of the money supply, that these increments are idiosyncratic across time and uncorrelated with shocks to productivity. Therefore, we construct the money growth instrument $\Delta \log \tilde{M}_{ct} = \gamma \Delta \log Y_{ct} + \Delta \log P_{ct}$ and use it as an instrument for real GDP growth. The first-stage equation is

$$\Delta \log Y_{ct} = \zeta_c + \beta_c^F \cdot \Delta \log \tilde{M}_{ct} + \nu_{ct} \tag{175}$$

Table A2: Predicting the Slope of Aggregate Supply

	(1)	(2)	(3)	(4)	(5)	(6)
	Outcome is $\hat{\beta}_c^S$ (Reduced-form)			Outcome is $\hat{\beta}_c^{S,IV}$ (Structural)		
$\hat{\epsilon}_c^S$, Slope of ag. supply	0.261	0.132	0.0174	7.226	7.618	8.823
	(0.0379)	(0.0392)	(0.0565)	(2.410)	(3.406)	(4.996)
Mean inflation		-7.542			22.86	
		(1.583)			(137.7)	
Inflation uncertainty			-84.96			556.1
			(17.14)			(1516.1)
Observations	29	29	29	29	29	29
R^2	0.638	0.807	0.814	0.250	0.251	0.254

Notes: This table reports the cross-country relationship between empirical and theoretical proxies for the slope of aggregate supply. All estimates are from linear regressions where the unit of observation is an OECD country. In columns 1-3, the outcome is the "reduced-form" slope of aggregate supply defined in Equation 171. In columns 4-6, the outcome is the "structural" slope of aggregate supply defined in Equation 176. The independent variables are the model-implied slope of aggregate supply, calculated based on a macroeconomic calibration and measurements of relative uncertainty in each country; the mean value of GDP deflator inflation from 1960-2019; and the one-step-ahead forecast variance of inflation from a three-variable VAR model (see Equation 169) over the same period. Standard errors are in parentheses.

and the structural equation remains Equation 170. The population two-stage least squares coefficient of the slope of supply is

$$\beta_c^{S,IV} = \frac{\text{Cov}[\Delta \log P_{ct}, \Delta \log \tilde{M}_{ct}]}{\text{Cov}[\Delta \log Y_{ct}, \Delta \log \tilde{M}_{ct}]} = \frac{\gamma \text{Cov}[\Delta \log P_{ct}, \Delta \log Y_{ct}] + \text{Var}[\Delta \log P_{ct}]}{\text{Cov}[\Delta \log P_{ct}, \Delta \log Y_{ct}] + \gamma \text{Var}[\Delta \log Y_{ct}]} \tag{176}$$

Cross-Country Evidence. Table A2 summarizes the relationship between our empirical proxies and model-based calculations for the slope of aggregate supply. Column 1 shows the positive relationship between the reduced-form slope and model-based slope that is visualized in the left panel of Figure 5. This relationship is robust to controlling for the level of inflation (column 2). The relationship becomes statistically insignificant when controlling for one-step-ahead inflation uncertainty (column 3), although the coefficient on the latter is inconsistent with the theoretical prediction. Turning to the structural estimates (columns 4-6), we estimate a large and quantitatively stable relationship between the data-based and model-based estimates. The larger magnitudes in columns 4-6 versus 1-3 are consistent with the hypothesis that the reduced-form coefficients are biased toward zero by spurious correlation with aggregate supply shocks. The coefficients on mean inflation and inflation uncertainty in columns 5 and 6 are consistent with theory, but imprecisely estimated and of marginal consequence to the R^2 of the model.

While these results are to be interpreted with caution, given the limited sample size and abundance of confounding factors in cross-country analysis, they offer suggestive evidence that the model-based slope of aggregate supply helps predict cross-country variation in the inflation-output relationship. Moreover, our model's prediction based on relative variance has predictive power over and above the mean and one-step-ahead uncertainty regarding inflation, which are the main factors influencing the slope of aggregate supply in other theories of state-dependent firm adjustment (Ball et al., 1988). Further investigation of the differences between these models may be possible by incorporating both time-series and cross-sectional variation in an international panel or by turning to micro data. We leave these investigations to future work.

D.4 Counterfactual Analyses and Equilibrium Multiplicity

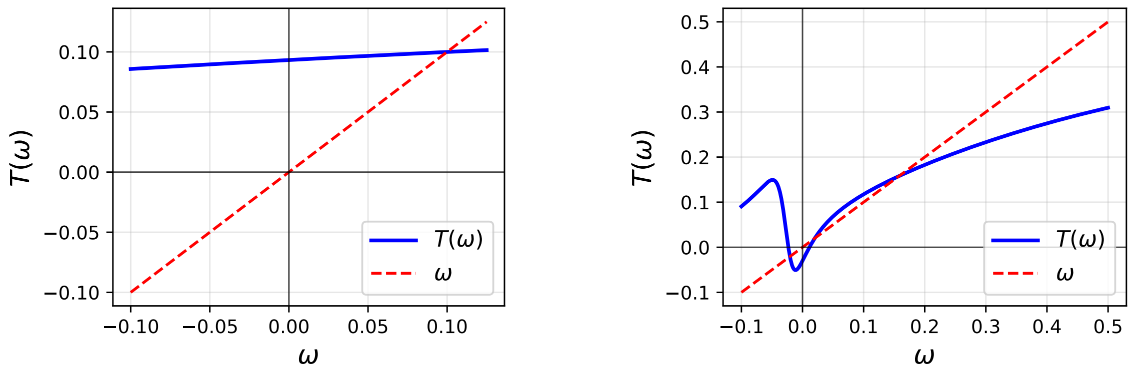
The analysis in the main text leveraged the "reduced form" uncertainties relevant to the firm in Theorem 1 and did not estimate the structural uncertainties that mediate firms' supply functions slopes via the fixed point in Theorem 3. As noted in the main text, an advantage of this approach is that the economic analyst can measure the slope of firms' supply functions without taking a stance on the general equilibrium features of the economy. Moreover, by using such observational data, the analyst can bypass issues of equilibrium selection.

However, a limitation of this approach is that this method precludes conducting counterfactuals which would be relevant when model parameters endogenously respond to policy. In this section, we outline how one can use our theory to conduct counterfactual exercises and demonstrate that a unique equilibrium exists for a reasonable calibration of the US economy.

Methodology and Calibration. Solving the fixed point in Theorem 3 requires values for the preference parameters (η, γ) and uncertainties $(\sigma_{\vartheta,t}, \sigma_t^A, \sigma_t^M, \kappa_t^A, \kappa_t^M)$. To simplify the analysis, we assume that these parameters are time-invariant. We set $\eta = 8$ and $\gamma = 0.11$ as in the main text. Moreover, we set $\sigma_{\vartheta,t}^2 = 0.0026$ to match the unconditional mean of our GARCH estimates in Section 5 over our sample period. Next, we back out the latent aggregate demand shock using the observation that $M_t = \frac{1+i_t}{i_t} C_t^{\gamma} P_t$ from Proposition 1. We calibrate σ_t^M to match the mean unconditional variance in $\Delta \log M_t$ following Equation 16. We also calibrate σ_t^A to match the mean unconditional variance in $\Delta \log A_t$, where we measure TFP growth A_t using the dataset of Fernald (2025). Finally, we set κ_t^M and κ_t^A to zero. We do so to keep our analysis consistent with the methodology of Golosov and Lucas (2007), which directly estimates firms' uncertainty using realized inflation rates. Nevertheless, the basic message of equilibrium uniqueness is not sensitive to these parameter choices.

Figure A4: Fixed Point of Firms' Supply Function Slopes

A. US Parameters: Unique Equilibrium **B.** Alternative Parameters: Multiple Equilibria



Notes: Panel A plots the fixed point from Theorem 3 under a parameterization with a unique equilibrium when parameters are calibrated to the US economy: $(\kappa^A, \kappa^M, \sigma_M^2, \sigma_A^2, \eta, \gamma, \sigma_\vartheta^2) = (0, 0, 0.00017, 0.000068, 8, 0.11, 0.0026)$. Panel B plots a parameterization with three equilibria: $(\kappa^A, \kappa^M, \sigma_M^2, \sigma_A^2, \eta, \gamma, \sigma_\vartheta^2) = (0.1, 0.9, 5, 10, 2, 0.02, 5)$.

Results. Figure A4 plots the fixed point in Theorem 3 for various parameter values. The left panel depicts the fixed point for a parameterization of the US economy, described above. The right panel depicts the fixed for an alternative parameterization which features multiple equilibria.

Observe that the US parameterization features a unique equilibrium (panel A). The intuition for this result is that idiosyncratic demand uncertainty is large in our estimation relative to other sources of uncertainty. For this reason, the fixed point for firms' microeconomic supply elasticities is well approximated by a linear function. To obtain equilibrium multiplicity in computational experiments, we had to increase firms' relative uncertainty about aggregate *vs.* idiosyncratic demand conditions by more than ten-fold, as well as fix a particularly low value of γ (panel B). To provide some intuition for multiplicity in this environment, observe that the dynamics of real GDP are described by the following Equation (see Lemma 1):

$$\log C_t = \tilde{\chi}_{0,t} + \frac{1}{\gamma} \frac{\kappa_t^A}{1 - \omega_{1,t}\left(\eta - \frac{1}{\gamma}\right)(1 - \kappa_t^A)} \log A_t + \frac{1}{\gamma} \frac{(1 - \kappa_t^M)(1 - \eta\omega_{1,t})}{1 - \omega_{1,t}\left(\eta - \frac{1}{\gamma}\right)(1 - \kappa_t^M)} \log M_t \quad (177)$$

In particular, higher values of $\omega_{1,t}$ reduce the volatility of aggregate consumption that arises through productivity shocks. But this in itself is a force that favors steeper supply functions, since even small shifts in demand are likely to imply large changes in marginal costs. Consequently, this economy can feature multiple equilibria in firms' supply function elasticities, through a general equilibrium feedback loop that arises between supply function choice and firms' endogenous uncertainties, as we described in Section 4.4. Nevertheless, we have found it challenging to construct examples with multiple equilibria and quantitatively reasonable parameter values for the US. Consequently, we believe that our framework is also amenable to counterfactual analyses for the US economy.

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