Supplemental Appendix: “Bias and Sensitivity under Ambiguity”

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A Proofs of Main Results

In this appendix, we present the proofs of the main results from Section II. We start by proving Proposition 3, which yields the fixed point conditions that characterize the equilibrium. We proceed by proving the general equivalence result, Proposition 4, based on which we can prove the existence of equilibrium, Proposition 2, as well as the comparative statics of sensitivity $\mathcal{S}$ and bias $\mathcal{B}$ with respect to the coordination motive α, Proposition 6.

Proof of Proposition 3. The equilibrium concept from Definition 1 is equivalent to the notion of ex-ante equilibrium from Hanany, Klibanoff, and Mukerji (2020). It is equivalent to the characterization of sequential equilibria with ambiguity (SEA) when conditional preferences are updated using the smooth rule of updating proposed in Hanany and Klibanoff (2009). The key for the equilibrium refinement of SEA is to ensure dynamic consistency, in the sense that ex-ante contingent plans are respected ex-post with the arrival of new information. Specifically, conditional on the realization of any possible history of private information, x_i^t, the optimal strategy of agent i maximizes their conditional preference, given by

$$\phi^{-1}\left(\int_{\mu^t}\phi\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it},K_t,\xi_t\right)\mid x_i^t\right]\right)\tilde{p}\left(\mu^t\mid x_i^t\right)\mathrm{d}\mu^t\right), \tag{A.1}$$

where $\mathbb{E}^{\mu^t}\left[u\left(k_{it},K_t,\xi_t\right)\mid x_i^t\right]$ denotes the expected utility conditional on x_i^t under a particular model μ^t. The interim belief system is characterized by a posterior belief $\tilde{p}\left(\mu^t\mid x_i^t\right)$ that follows the smooth rule of updating:

$$\tilde{p}\left(\mu^t\mid x_i^t\right)\propto\underbrace{\frac{\phi'\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}^*,K_t^*,\xi_t\right)\right]\right)}{\phi'\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}^*,K_t^*,\xi_t\right)\mid x_i^t\right]\right)}}_{\text{Weights}}\underbrace{p\left(x_i^t\mid\mu^t\right)p\left(\mu^t\right)}_{\text{Bayesian Kernel}},$$

where $\{k_{it}^*\left(x_i^t\right)\}_{x_i^t,i}$ denotes the equilibrium strategy profiles in the cross-section of the economy and $K_t^*\equiv\int_i k_{it}^*\mathrm{d}i$ denotes the equilibrium aggregate action.

The first-order condition of maximizing (A.1) with respect to k_{it} yields

$$\int_{\mu^t}\phi'\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it},K_t,\xi_t\right)\mid x_i^t\right]\right)\frac{\partial\mathbb{E}^{\mu^t}\left[u\left(k_{it},K_t,\xi_t\right)\mid x_i^t\right]}{\partial k_{it}}\tilde{p}\left(\mu^t\mid x_i^t\right)\mathrm{d}\mu^t=0.$$

Since

$$\frac{\partial\mathbb{E}^{\mu^t}\left[u\left(k_{it},K_t,\xi_t\right)\mid x_i^t\right]}{\partial k_{it}}=k_{it}-\left(1-\alpha\right)\mathbb{E}^{\mu^t}\left[\xi_t\mid x_i^t\right]-\alpha\mathbb{E}^{\mu^t}\left[K_t\mid x_i^t\right],$$

the first-order condition can be used to solve for the optimal strategies $\{k_{it}^*\left(x_i^t\right)\}_{x_i^t,i}$,

$$k_{it}^*(x_i^t)=\int_{\mu^t}\left(\left(1-\alpha\right)\mathbb{E}^{\mu^t}\left[\xi_t\mid x_i^t\right]+\alpha\mathbb{E}^{\mu^t}\left[K_t^*\mid x_i^t\right]\right)\hat{p}\left(\mu^t\mid x_i^t\right)\mathrm{d}\mu^t,$$

with

$$\hat{p}\left(\mu^t\mid x_i^t\right)\equiv\frac{\phi'\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}^*,K_t^*,\xi_t\right)\right]\right)p\left(x_i^t\mid\mu^t\right)p\left(\mu^t\right)}{\int_{\mu^t}\phi'\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}^*,K_t^*,\xi_t\right)\right]\right)p\left(x_i^t\mid\mu^t\right)p\left(\mu^t\right)\mathrm{d}\mu^t},$$

which completes the proof. □

Proof of Proposition 4. Following Huo and Pedroni (2020), we first consider a truncated version of our model. After solving this truncated version, the appropriate limits yield the desired result.[1]

Fix t and define

$$\vartheta \equiv \xi_t = \sum_{k=0}^{\infty} a_k \eta_{t-k}.$$

Let ϑ_q denote the MA(q) truncation of ϑ, such that

$$\vartheta_q = \sum_{k=0}^{q} a_k \eta_{t-k},$$

and let $x_{p,i}^N \equiv \{x_{p,it}, \cdots, x_{p,it-N}\}$, with $x_{p,it-k}$ denoting the MA(p) truncation of x_{it-k}.

Consider the truncated problem of forecasting the the fundamental ϑ_q given $x_{p,i}^N$. To further ease notation, define

$$\eta \equiv \begin{bmatrix} \eta_t \\ \vdots \\ \eta_{t-T} \end{bmatrix}, \quad \mu \equiv \begin{bmatrix} \mu_t \\ \vdots \\ \mu_{t-T} \end{bmatrix}, \quad \epsilon_i \equiv \begin{bmatrix} \epsilon_{it} \\ \vdots \\ \epsilon_{it-T} \end{bmatrix}, \quad \text{and} \quad \nu_i \equiv \begin{bmatrix} \eta \\ \epsilon_i \end{bmatrix}$$

Let R denote the length of $x_{p,i}^N$, and N the length of ϵ_{it}. It follows that, there exists a vector a with length $u \equiv T+1$, and a matrix B with dimensions $n \times m$, where $n \equiv R(T+1)$ and $m \equiv (1+N)(T+1)$, such that the truncated fundamental and the private signals are given by

$$\theta \equiv \vartheta_q = A\nu_i, \quad \text{with} \quad A \equiv \left[a', 0'_{m-u,1}\right], \quad \text{and} \quad x_i \equiv x_{p,i}^N = B\nu_i,$$

where $0_{m-u,1}$ is an $(m-u) \times 1$ vector of zeros. In the objective environment, ν_i is normally distributed,

$$\nu_i \sim \mathcal{N}(0, \Omega), \quad \text{with} \quad \Omega = \begin{bmatrix} \sigma_\eta^2 \mathrm{I}_u & 0 \\ 0 & \Xi \end{bmatrix},$$

where I_u denotes the identity matrix of size u and Ξ denotes the variance-covariance matrix of the $(m-u) \times 1$ vector of idiosyncratic shocks, ϵ_i. Subjectively, agents believe that η is drawn from a Gaussian distribution with variance-covariance matrix $\sigma_\eta^2 \mathrm{I}_u$ but there is uncertainty about its prior mean, denoted by μ. Ambiguity is then captured by the perception that

$$\eta \sim \mathcal{N}\left(\mu, \sigma_\eta^2 \mathrm{I}_u\right), \quad \text{and} \quad \mu \sim \mathcal{N}(0, \Omega_\mu), \quad \text{with} \quad \Omega_\mu \equiv \sigma_u^2 \mathrm{I}_u.$$

From Proposition 3, we know that the best response of agent i satisfies

$$k_i = \int_\mu \big((1-\alpha)\,\mathbb{E}^\mu[\theta \mid x_i] + \alpha \mathbb{E}^\mu[K \mid x_i]\big)\hat{p}(\mu \mid x_i)\,\mathrm{d}\mu, \tag{A.2}$$

[1] See Online Appendix A.1 of Huo and Pedroni (2020) for detailed proofs.

with

$$\hat{p}(\mu \mid x_i) \propto \exp\left(-\lambda \mathbb{E}^\mu\left[u(k_i, K, \theta)\right]\right) p(x_i \mid \mu) p(\mu).$$

We proceed by using a guess-and-verify strategy. First, we guess a symmetric linear equilibrium that

$$k_i = h' B \nu_i + h_0 \qquad \forall i.$$

We can show that ex-ante expected utility, under a particular model μ, is such that

$$\begin{aligned} \mathbb{E}^\mu\left[u(k_i, K, \theta)\right] = & -\mu'\left[\frac{1}{2}(1-\alpha)\mathcal{K}(A' - B'h)(A - h'B)\mathcal{K}' + \frac{1}{2}\gamma\mathcal{K}A'A\mathcal{K}'\right]\mu \\ & + \left[\frac{1}{2}(1-\alpha)h_0(A - h'B)\mathcal{K}' + \frac{1}{2}\chi A\mathcal{K}'\right]\mu + \mu'\left[\frac{1}{2}(1-\alpha)h_0\mathcal{K}(A' - B'h) + \frac{1}{2}\chi\mathcal{K}A'\right] \\ & \underbrace{-\frac{1}{2}(1-\alpha)(A - h'B)\Omega(A' - B'h) - \frac{1}{2}(1-\alpha)h_0^2 - \frac{1}{2}\alpha h'B(\mathrm{I}_m - \Lambda)\Omega B'h - \frac{1}{2}\gamma A\Omega A}_{\text{independent of } \mu}, \end{aligned} \tag{A.3}$$

where matrices $\mathcal{K}$ and Λ are such that

$$\mathcal{K} \equiv \left[\mathrm{I}_u, 0_{u,m-u}\right], \quad \text{and} \quad \Lambda \equiv \mathcal{K}'\mathcal{K}.$$

At the same time, we have that

$$p(\mu \mid x_i) \propto \exp\left(-\frac{1}{2}\mu'\left(\mathcal{K}B'(B\Omega B')^{-1}B\mathcal{K}'\right)^{-1}\mu + \frac{1}{2}\mu'\mathcal{K}(B\Omega B')^{-1}x_i + \frac{1}{2}x_i'(B\Omega B')^{-1}\mathcal{K}'\mu\right).$$

It follows that

$$\hat{p}(\mu \mid x_i) \propto \exp\left(-\frac{1}{2}\mu' S^{-1}\mu + \frac{1}{2}\mu' S^{-1}(Mx_i + \pi) + \frac{1}{2}(Mx_i + \pi)' S^{-1}\mu\right),$$

where matrices M, π, and S are such that

$$M \equiv S\mathcal{K}(B\Omega B')^{-1}, \quad \pi \equiv S\left[-\lambda(1-\alpha)h_0\mathcal{K}(A' - B'h) + \lambda\chi\mathcal{K}A'\right],$$

and

$$S \equiv \left(\mathcal{K}B'(B\Omega B')^{-1}B\mathcal{K}' + \Omega_\mu^{-1} - \lambda\left[(1-\alpha)\mathcal{K}(A' - B'h)(A - h'B)\mathcal{K}' + \gamma\mathcal{K}A'A\mathcal{K}'\right]\right)^{-1}.$$

Accordingly, we can show that the subjective expectations are such that

$$\int_\mu \mathbb{E}^\mu\left[\theta \mid x_i\right]\hat{p}(\mu \mid x_i)\,\mathrm{d}\mu = \mathrm{T}\,x_i + (A - \mathrm{T}B)\mathcal{K}'\left[S\mathcal{K}B'(B\Omega B')^{-1}x_i + \pi\right],$$

and

$$\int_\mu \mathbb{E}^\mu\left[K \mid x_i\right]\hat{p}(\mu \mid x_i)\,\mathrm{d}\mu = \mathrm{H}\,x_i + h'(B\Lambda - \mathrm{H}B)\mathcal{K}'\left[S\mathcal{K}B'(B\Omega B')^{-1}x_i + \pi\right] + h_0,$$

where matrices T and H are given by

$$\mathrm{T} \equiv A\Omega B' \left(B\Omega B'\right)^{-1}, \quad \text{and} \quad \mathrm{H} \equiv B\Lambda\Omega B' \left(B\Omega B'\right)^{-1}.$$

Therefore, matching coefficients leads to the following equilibrium conditions for h and h_0,

$$h' = (1-\alpha)\,\mathrm{T} + \alpha h'\mathrm{H} + \left[(1-\alpha)\,(A - \mathrm{T}B) + \alpha h'\,(B\Lambda - \mathrm{H}B)\right]\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}, \tag{A.4}$$

and

$$(1-\alpha)\,h_0 = \left[(1-\alpha)\,(A - \mathrm{T}B) + \alpha h'\,(B\Lambda - \mathrm{H}B)\right]\mathcal{K}'\pi. \tag{A.5}$$

In what follows, we first focus on equation (A.4). Through a sequence of lemmas, we show that this fixed-point problem for h can be recast as the solution of a pure forecasting problem. We then proceed to characterize h_0 using equation (A.5).

The next lemmas are organized as follows. Lemma A.1 rewrites the equilibrium condition for h described above as a beauty-contest problem with a modified variance-covariance matrix. Lemma A.2 establishes that h can be obtained by solving a forecasting problem with a modified variance-covariance matrix. Lemma A.3 simplifies the variance-covariance matrix of the forecasting problem, and Lemma A.4 further simplifies it yielding a symmetric variance-covariance matrix. After the lemmas we take the limits of the truncated forecasting problem as $T \to \infty$.

Lemma A.1. *Define*

$$\hat{\Omega} \equiv \Omega + \mathcal{K}'W\mathcal{K}, \quad \hat{\mathrm{T}} \equiv A\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}, \quad \hat{\mathrm{H}} \equiv B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1},$$

and

$$W \equiv \left(\Omega_\mu^{-1} - \lambda\left[(1-\alpha)\,\mathcal{K}\,(A' - B'h)\,(A - h'B)\,\mathcal{K}' + \gamma\mathcal{K}A'A\mathcal{K}'\right]\right)^{-1}.$$

Then, the equilibrium h solves the following fixed-point problem

$$h' = (1-\alpha)\,\hat{\mathrm{T}} + \alpha h'\hat{\mathrm{H}}.$$

Proof. Using the Woodbury matrix identity, we have that

$$\begin{aligned}\left(B\hat{\Omega}B'\right)^{-1} &= \left(B\Omega B' + B\mathcal{K}'W\mathcal{K}B'\right)^{-1}\\ &= \left(B\Omega B'\right)^{-1} - \left(B\Omega B'\right)^{-1}B\mathcal{K}'\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}' + W^{-1}\right)^{-1}\mathcal{K}B'\left(B\Omega B'\right)^{-1}\\ &= \left(B\Omega B'\right)^{-1} - \left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1},\end{aligned} \tag{A.6}$$

Then, if $\hat{h}$ is such that $\hat{h}' = (1-\alpha)\,\hat{\mathrm{T}} + \alpha\hat{h}'\hat{\mathrm{H}}$, we have that

$$\begin{aligned}
\hat{h}' =&\ (1-\alpha)\,A\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} + \alpha\hat{h}'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} \\
=&\ (1-\alpha)\,A\left(\Omega+\mathcal{K}'W\mathcal{K}\right)B'\left(B\hat{\Omega}B'\right)^{-1} + \alpha\hat{h}'B\Lambda\left(\Omega+\mathcal{K}W\mathcal{K}'\right)B'\left(B\hat{\Omega}B'\right)^{-1} \\
=&\ (1-\alpha)\,A\Omega B'\left(B\hat{\Omega}B'\right)^{-1} + (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\hat{\Omega}B'\right)^{-1} \\
&+ \alpha\hat{h}'B\Lambda\Omega B'\left(B\hat{\Omega}B'\right)^{-1} + \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\hat{\Omega}B'\right)^{-1}.
\end{aligned}$$

Using equation (A.6), it follows that

$$\begin{aligned}
\hat{h}' =&\ (1-\alpha)\,A\Omega B'\left(B\Omega B'\right)^{-1} - (1-\alpha)\,A\Omega B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&+ (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&+ \alpha\hat{h}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1} - \alpha\hat{h}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&+ \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
=&\ \underbrace{(1-\alpha)\,A\Omega B'\left(B\Omega B'\right)^{-1}}_{(1-\alpha)\mathrm{T}} + \underbrace{\alpha\hat{h}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1}}_{\alpha\hat{h}'\mathrm{H}} \\
&- \underbrace{(1-\alpha)\,A\Omega B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}}_{(1-\alpha)\mathrm{T}B\mathcal{K}'S\mathcal{K}B'(B\Omega B')^{-1}} - \underbrace{\alpha\hat{h}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}}_{\alpha\hat{h}'\mathrm{H}B\mathcal{K}'S\mathcal{K}B'(B\Omega B')^{-1}} \\
&+ (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&+ \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}.
\end{aligned}$$

Further, notice that the terms in the second-to-last line can be rewritten as

$$\begin{aligned}
&(1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
=&\ (1-\alpha)\,A\mathcal{K}'W\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'+W^{-1}\right)\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'+W^{-1}\right)^{-1}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&- (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'+W^{-1}\right)^{-1}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
=&\ (1-\alpha)\,A\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1},
\end{aligned}$$

and, similarly, the terms in the last line can be rewritten as

$$\begin{aligned}
&\alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&= \alpha\hat{h}'B\Lambda\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}.
\end{aligned}$$

Therefore, we have that

$$\hat{h}' = (1-\alpha)\,\mathrm{T} + \alpha\hat{h}'\mathrm{H} + \left[(1-\alpha)\left(A-\mathrm{T}B\right) + \alpha\hat{h}'\left(B\Lambda-\mathrm{H}B\right)\right]\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1},$$

which is equivalent to the expression for h in equation (A.4). □

Lemma A.2. *Define*

$$\Omega_\Gamma \equiv \Gamma\hat{\Omega}, \quad \textit{with} \quad \Gamma \equiv \begin{bmatrix} \mathrm{I}_u & 0_{u,m-u} \\ 0_{m-u,u} & \frac{\mathrm{I}_{m-u}}{1-\alpha} \end{bmatrix}.$$

Then, the equilibrium h satisfies

$$h' = A\Omega_\Gamma B' \left(B\Omega_\Gamma B'\right)^{-1}.$$

Proof. Follows directly from Lemma A.1 and Theorem 1 in Huo and Pedroni (2020). □

Lemma A.3. *Define*

$$\Delta \equiv \Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}, \quad \textit{and} \quad \tilde{\Omega}_\mu \equiv \left(\Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1},$$

with the scalar $\hat{w}$ given by

$$\hat{w} \equiv \frac{\tau_\mu}{1 - \lambda(1-\alpha)\left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)}.$$

Then, the equilibrium h satisfies

$$h' = A\Delta B' \left(B\Delta B'\right)^{-1}.$$

Proof. It follows from Lemma A.2 that

$$\left(A - h'B\right)\Omega_\Gamma B' = 0,$$

and from the definition of Ω_Γ and $\tilde{\Omega}_\mu$ we have that

$$\Omega_\Gamma = \Gamma\Omega + \mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} - \lambda(1-\alpha)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}.$$

It is then sufficient to show that

$$\left(A - h'B\right)\left(\Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\right) = \left(A - h'B\right)\left(\Gamma\Omega + \mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} - \lambda\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}\right),$$

or, equivalently,

$$\begin{aligned}\hat{w}\tau_\mu^{-1}\left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K} &= \left(A - h'B\right)\mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} - \lambda\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1}\mathcal{K} \\ &= \left(A - h'B\right)\mathcal{K}'\left(\mathrm{I}_u - \lambda\left(1-\alpha\right)\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1}\tilde{\Omega}_\mu\mathcal{K}.\end{aligned}$$

Thus, it is sufficient to establish that

$$\hat{w}\tau_\mu^{-1}\left(A - h'B\right)\mathcal{K}' = \left(A - h'B\right)\mathcal{K}'\left(\mathrm{I}_u - \lambda\left(1-\alpha\right)\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1},$$

or

$$\hat{w}\tau_\mu^{-1}\left(A - h'B\right)\mathcal{K}'\left(\mathrm{I}_u - \lambda\left(1-\alpha\right)\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right) = \left(A - h'B\right)\mathcal{K}',$$

which can be rewritten as

$$\hat{w}\tau_\mu^{-1}\left(1-\lambda\left(1-\alpha\right)\left(A-h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(A'-B'h\right)\right)\left(A-h'B\right)\mathcal{K}'=\left(A-h'B\right)\mathcal{K}'.$$

The definition of $\hat{w}$ then yields the result. □

Lemma A.4. *Define*

$$\bar{\Delta}\equiv\Gamma\Omega+\hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu\mathcal{K},$$

with the scalar $\hat{w}$ given by

$$\hat{w}=\frac{\tau_\mu}{1-\lambda\left(1-\alpha\right)\left(A-h'B\right)\mathcal{K}'\left(\Omega_\mu+\frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\mathcal{K}\left(A'-B'h\right)}.$$

Also, let the scalar $\hat{r}$ be given by

$$\hat{r}\equiv\frac{\hat{w}}{1+\hat{w}}\left(\frac{\lambda\gamma}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\left(A-h'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A'.$$

Then, the equilibrium h satisfies

$$h'=(1+\hat{r})A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1}.$$

Proof. From the definition of $\tilde{\Omega}_\mu$ and Δ in Lemma A.3, we have that

$$\tilde{\Omega}_\mu\equiv\left(\Omega_\mu^{-1}-\lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1}=\Omega_\mu+\frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'},$$

and

$$\Delta\equiv\Gamma\Omega+\hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}=\bar{\Delta}+\hat{w}\tau_\mu^{-1}\mathcal{K}'\left(\frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\mathcal{K}=\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K},$$

with $s\equiv\lambda\gamma\tau_\mu^{-1}/(1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A')$. Hence, it follows from the result in Lemma A.3 that

$$h'=A\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'\left[B\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'\right]^{-1},$$

and, therefore,

$$h'\left[B\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'\right]=A\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'.$$

Rearranging, we get

$$h'B\bar{\Delta}B'+s\hat{w}h'B\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}B'=A\bar{\Delta}B'+s\hat{w}A\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}B',$$

and right-multiplying both sides by $\left(B\bar{\Delta}B'\right)^{-1}$ yields

$$\begin{aligned} h' &= A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + s\hat{w}\left(A - h'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1} \\ &= A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + (1+\hat{w})\hat{r}\tau_\mu^{-1}A\mathcal{K}'\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1}. \end{aligned}$$

Then, from the definition of $\bar{\Delta}$ and using the fact that $\Omega_\mu = \tau_\mu\mathcal{K}\Omega\mathcal{K}'$ and $A\Gamma\Omega = \tau_\mu^{-1}A\mathcal{K}'\Omega_\mu\mathcal{K}$, it follows that

$$A\bar{\Delta} = A\left(\Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu\mathcal{K}\right) = (1+\hat{w})\,\tau_\mu^{-1}A\mathcal{K}'\Omega_\mu\mathcal{K}.$$

Plugging this back into the equation for h' we obtain the desired result,

$$h' = (1+\hat{r})\,A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1}.$$

□

Parts 1 and 2 of Proposition 4. Given the result in Lemma A.4, we are left with taking the limit as $T \to \infty$ of the truncated problem. In particular, we have that

$$\lim_{T\to\infty} A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} = p\left(L; w, \alpha\right), \quad \lim_{T\to\infty} A\mathcal{K}'\Omega_\eta\mathcal{K}A' = \mathbb{V}(\xi_t),$$

$$\lim_{T\to\infty} \left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}(A' - Bh') = \mathbb{V}(\xi_t - K_t), \quad \lim_{T\to\infty} \left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}A' = \mathbb{COV}\left(\xi_t - K_t, \xi_t\right),$$

$$\lim_{T\to\infty} \frac{\left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}A'}{A\mathcal{K}'\Omega_\eta\mathcal{K}A'} = 1 - \mathcal{S}.$$

Let $w \equiv \lim_{T\to\infty}\hat{w}$, and $r \equiv \lim_{T\to\infty}\hat{r}$. Then, we can show that

$$\begin{aligned} r &= \lim_{T\to\infty} \frac{\hat{w}}{1+\hat{w}}\frac{\lambda\gamma\tau_\mu A\mathcal{K}'\Omega_\eta\mathcal{K}A'}{1-\lambda\gamma\tau_\mu A\mathcal{K}'\Omega_\eta\mathcal{K}A'}\frac{\left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}A'}{A\mathcal{K}'\Omega_\eta\mathcal{K}A'} \\ &= \frac{w}{1+w}\frac{\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}{1-\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}\left(1-\mathcal{S}\right), \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} w &= \lim_{T\to\infty} \frac{\tau_\mu}{1 - \lambda\left(1-\alpha\right)\left(A - h'B\right)\mathcal{K}'\left(\Omega_\mu + \frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\mathcal{K}\left(A' - B'h\right)} \\ &= \lim_{T\to\infty} \frac{\tau_\mu}{1 - \lambda\left(1-\alpha\right)\tau_\mu\left(\left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}\left(A' - B'h\right) + \hat{r}\frac{1+\hat{w}}{\hat{w}}A\mathcal{K}'\Omega_\eta\mathcal{K}\left(A' - B'h\right)\right)} \\ &= \frac{\tau_\mu}{1 - \lambda\left(1-\alpha\right)\tau_\mu\left(\mathbb{V}\left(\xi_t - K_t\right) + r\frac{1+w}{w}\left(1-\mathcal{S}\right)\mathbb{V}\left(\xi_t\right)\right)}. \end{aligned}$$

Solving for w, we obtain

$$w = \frac{\tau_\mu}{1 - \lambda\left(1-\alpha\right)\tau_\mu\left(\mathbb{V}\left(\xi_t - K_t\right) + \frac{\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2}{1-\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}\right)}. \tag{A.8}$$

Lemma A.5 below establishes that $w \geq \tau_\mu$ and $r \geq 0$, which completes the proof of parts 1 and 2 of Proposition 4.

Lemma A.5. *If w and r satisfy equations (A.7) and (A.8), then $w \geq \tau_\mu$ and $r \geq 0$.*

Proof. The ex-ante objective of an agent i must obtain finite values under an equilibrium strategy $k_i = h'B\nu_i + h_0$. The ex-ante objective is given by

$$\begin{aligned}\mathcal{V} = & -\frac{1}{\lambda}\ln\left(\int_\mu \exp\left(-\lambda\mathbb{E}^\mu\left[u\left(-k_i, K, \theta\right)\right]\right)p\left(\mu\right)\mathrm{d}\mu\right) \\ = & \text{ constant} - \frac{1}{\lambda}\ln\left(\int_\mu \exp\left(-\frac{1}{2}\mu'\bar{S}\mu + \mu'\bar{\pi}' + \bar{\pi}\mu\right)\mathrm{d}\mu\right),\end{aligned}$$

with the matrix $\bar{S}$ and the vector $\bar{\pi}$ given by

$$\begin{aligned}\bar{S} \equiv & \ \Omega_\mu^{-1} - \lambda\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}' - \lambda\gamma\mathcal{K}A'A\mathcal{K}', \\ \bar{\pi} \equiv & -\lambda\frac{1}{2}\left(1-\alpha\right)h_0\left(A - h'B\right)\mathcal{K}' - \lambda\frac{1}{2}\chi A\mathcal{K}',\end{aligned}$$

where we used the fact that $\mathbb{E}^\mu\left[u\left(k_i, K, \theta\right)\right]$ is given by equation (A.3) and

$$p(\mu) = \left(2\pi\right)^{-u/2}\det\left(\Omega_\mu\right)^{-1/2}\exp\left(-\frac{1}{2}\mu'\Omega_\mu^{-1}\mu\right).$$

Thus, a necessary condition for $\mathcal{V}$ to be finite in equilibrium is for $\bar{S}$ to be positive definite; otherwise, the integral would become explosive.[2] Since

$$\tilde{\Omega}_\mu^{-1} = \Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}',$$

it must be that

$$\tilde{\Omega}_\mu^{-1} - \lambda\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}' \quad \text{is positive definite.}$$

Defining the vector $\digamma \equiv \left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu$, it follows that

$$\begin{aligned}0 \leq & \ \digamma\left(\tilde{\Omega}_\mu^{-1} - 2\lambda\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)\digamma' \\ = & \ \left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)\left(1 - \lambda\left(1-\alpha\right)\left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)\right).\end{aligned}$$

[2] The same argument applies to how Assumption 2 ensures the problem is well defined. Specifically, a well-defined problem requires the choice set to be non-empty, which is equivalent to requiring $\bar{S}$ to be positive definite for at least one h. The necessary and sufficient condition for the existence of an h that makes $\bar{S}$ positive definite is that $\tilde{\Omega}_\mu$ is positive definite. Notice that $\tilde{\Omega}_\mu = \Omega_\mu + \frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}$. It is then straightforward to see that $1 - \lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A' > 0$ is the sufficient condition to ensure that $\tilde{\Omega}_\mu$ is positive definite. Taking the limit as $T \to \infty$, this is equivalent to Assumption 2.

Let $x \equiv (A - h'B)\,\mathcal{K}'\tilde{\Omega}_\mu \mathcal{K}\,(A' - B'h)$, then we have that

$$x(1 - \lambda(1-\alpha)x) \geq 0 \quad \text{or} \quad x \geq \lambda(1-\alpha)x^2 \geq 0.$$

Hence, we have that $x \geq 0$, and $1 - \lambda(1-\alpha)x \geq 0$, which implies that

$$\hat{w} = \frac{\tau_\mu}{1 - \lambda(1-\alpha)x} \geq \tau_\mu,$$

and, since $w = \lim_{T\to\infty} \hat{w}$, it follows that $w \geq \tau_\mu$.

Next, for a contradiction, suppose that $r < 0$. Then, it follows from equation (A.7) and Assumption 2 that $\mathbb{COV}\left(\xi_t - K_t, \xi_t\right) < 0$. Further, we have that

$$\mathbb{COV}\left(\xi_t - K_t, \xi_t\right) = \mathbb{V}\left(\xi_t\right) - (1+r)\,\mathbb{COV}\left(\hat{K}_t, \xi_t\right),$$

where $\hat{K}_t \equiv K_t / (1+r)$ is the average optimal forecast of the fundamental ξ_t under the (w,α)-modified signal process (net of the bias $\mathcal{B}$, which is uncorrelated with ξ_t),[3] so that it must be that

$$0 \leq \mathbb{COV}\left(\hat{K}_t, \xi_t\right) \leq \mathbb{V}\left(\xi_t\right).$$

Hence, $\mathbb{COV}\left(\xi_t - K_t, \xi_t\right) < 0$ implies $r > 0$ and we have a contradiction. Therefore, $r \geq 0$. □

Part 3 of Proposition 4. Next, we switch focus to the level of the $\mathcal{B} \equiv \lim_{T\to\infty} h_0$. From equation (A.5) and the definition of π, we have that

$$(1-\alpha)\,h_0 = \left[(1-\alpha)\,(A - \mathrm{T}B) + \alpha h'\,(B\Lambda - \mathrm{H}B)\right]\mathcal{K}'S\left[-\lambda\,(1-\alpha)\,h_0\mathcal{K}\,(A' - B'h) + \lambda\chi\mathcal{K}A'\right].$$

It is straightforward to see there exists a unique h_0 that satisfies this equation. We postulate that there exists $\tilde{\mu}$ such that

$$(1-\alpha)\,h_0 = \left[(1-\alpha)\,A + \alpha h'B\Lambda - h'B\right]\mathcal{K}'\tilde{\mu},$$

so that solving for $\tilde{\mu}$ pins down the unique h_0. To proceed, first replace the guess for h_0 on the RHS of equation (A.5),

$$\begin{aligned} \text{RHS} \equiv\ & \left[(1-\alpha)\,(A - \mathrm{T}B) + \alpha h'\,(B\Lambda - \mathrm{H}B)\right]\mathcal{K}'S\left[-\lambda\,(1-\alpha)\,h_0\mathcal{K}\,(A' - B'h) + \lambda\chi\mathcal{K}A'\right] \\ =\ & \left[(1-\alpha)\,(A - \mathrm{T}B) + \alpha h'\,(B\Lambda - \mathrm{H}B)\right]\mathcal{K}'S \\ & \times \left\{-\lambda\mathcal{K}\,(A' - B'h)\left[(1-\alpha)\,(A - h'B) + \alpha h'B\,(\Lambda - \mathrm{I}_m)\right]\mathcal{K}'\tilde{\mu} + \lambda\chi\mathcal{K}A'\right\} \end{aligned}$$

[3] More precisely, notice that $\hat{K}_t = p(L; w, \alpha)\int x_{it} - \mathcal{B}/(1+r)$, and that it follows from Definition 2 that $\int \tilde{x}_{it} = \sqrt{1 + w\tau_\mu}\int x_{it}$ and $\tilde{\xi}_t = \sqrt{1 + w\tau_\mu}\,\xi_t$. Therefore, $\hat{K}_t = \int \tilde{\mathbb{E}}_{it}[\xi_t] - \mathcal{B}/(1+r)$ and $\mathbb{COV}\left(\hat{K}_t, \xi_t\right) = \mathbb{COV}\left(\int \tilde{\mathbb{E}}_{it}[\xi_t], \xi_t\right)$.

Next, for the LHS of the equation, we have that

$$\text{LHS} \equiv (1-\alpha)\, h_0 = \left[(1-\alpha)\, A + \alpha h' B\Lambda - h'B\right] \mathcal{K}'\tilde{\mu},$$

and, substituting the last h using equation (A.4), it follows that

$$\begin{aligned}
\text{LHS} =\ & \left[(1-\alpha)\left(A - \text{TB}\right) + \alpha h'\left(B\Lambda - \text{H}B\right)\right]\left[\text{I}_\text{m} - \mathcal{K}'\text{S}\mathcal{K}\text{B}'\left(\text{B}\Omega\text{B}'\right)^{-1}\text{B}\right]\mathcal{K}'\tilde{\mu} \\
=\ & \left[(1-\alpha)\left(A - \text{T}B\right) + \alpha h'\left(B\Lambda - \text{H}B\right)\right]\mathcal{K}'S\left[S^{-1} - \mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'\right]\tilde{\mu} \\
=\ & \left[(1-\alpha)\left(A - \text{T}B\right) + \alpha h'\left(B\Lambda - \text{H}B\right)\right]\mathcal{K}'S \\
& \times\left\{\Omega_\mu^{-1} - \lambda\left[(1-\alpha)\,\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}' + \gamma\mathcal{K}A'A\mathcal{K}'\right]\right\}\tilde{\mu},
\end{aligned}$$

where the last equality uses the definition of S. Putting these results together, we have that

$$\begin{aligned}
\text{LHS} - \text{RHS} =\ & \left[(1-\alpha)\left(A - \text{TB}\right) + \alpha h'\left(B\Lambda - \text{HB}\right)\right]\mathcal{K}'S \\
& \times\left[\Omega_\mu^{-1}\tilde{\mu} + \alpha\lambda\mathcal{K}\left(A' - B'h\right)h'B\left(\Lambda - \text{I}_\text{m}\right)\mathcal{K}'\tilde{\mu} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\tilde{\mu} - \lambda\chi\mathcal{K}A'\right].
\end{aligned}$$

Since $\alpha\lambda\mathcal{K}\left(A' - B'h\right)h'B\left(\Lambda - \text{I}_\text{m}\right)\mathcal{K}' = 0$, a sufficient condition for $\text{LHS} - \text{RHS} = 0$ is

$$\Omega_\mu^{-1}\tilde{\mu} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\tilde{\mu} - \lambda\chi\mathcal{K}A' = 0,$$

which, using the Sherman-Morrison formula, implies that

$$\tilde{\mu} = \chi\lambda\left(\Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1}\mathcal{K}A' = \chi\lambda\left(\text{I}_\text{u} + \frac{\lambda\gamma\Omega_\mu\mathcal{K}\text{A}'\text{A}\mathcal{K}'}{1 - \lambda\gamma\text{A}\mathcal{K}'\Omega_\mu\mathcal{K}\text{A}'}\right)\Omega_\mu\mathcal{K}A'.$$

Therefore, we have that

$$\begin{aligned}
h_0 =\ & (1-\alpha)^{-1}\left[(1-\alpha)\,A + \alpha h'B\Lambda - h'B\right]\mathcal{K}'\tilde{\mu} \\
=\ & \left(A - h'B\right)\mathcal{K}'\tilde{\mu} \\
=\ & \left(A - h'B\right)\mathcal{K}'\chi\lambda\left(\text{I}_\text{u} + \frac{\lambda\gamma\Omega_\mu\mathcal{K}\text{A}'\text{A}\mathcal{K}'}{1 - \lambda\gamma\text{A}\mathcal{K}'\Omega_\mu\mathcal{K}\text{A}'}\right)\Omega_\mu\mathcal{K}A' \\
=\ & \chi\lambda\tau_\mu\left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}A'\left(1 + \frac{\lambda\gamma\tau_\mu A\mathcal{K}'\Omega_\eta\mathcal{K}A'}{1 - \lambda\gamma\tau_\mu A\mathcal{K}'\Omega_\eta\mathcal{K}A'}\right).
\end{aligned}$$

Taking the limit we get

$$\mathcal{B} = \lim_{T\to\infty} h_0 = \chi\lambda\tau_\mu\mathbb{COV}(\xi_t - K_t, \xi_t)\left(1 + \frac{\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}{1 - \lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}\right) = \chi\frac{\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)}{1 - \lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}\left(1 - \mathcal{S}\right),$$

which completes the proof of part 3 of the proposition. □

Proof of Proposition 2. Using the equivalence result from Proposition 4, establishing existence of an equilibrium reduces to showing that there exists a (w, r) pair that satisfies equations (A.7) and (A.8).

We start by using the intermediate value theorem to prove that there exists $w \in [\tau_\mu, \infty)$ that satisfies equation (A.8). Define

$$F(w) \equiv w\left[1-\lambda\left(1-\alpha\right)\tau_\mu\left(\mathbb{V}\left(\xi_t-K_t\right)+\frac{\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)^2\left(1-\mathcal{S}\right)^2}{1-\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}\right)\right]-\tau_\mu,$$

such that $F(w)=0$ implies equation (A.8). Next, notice that as $w\to\infty$, private information becomes infinitely precise and, therefore, $p\left(L;w,\alpha\right)\to a(L)$, or $K_t\to\xi_t$. It follows that $\mathcal{S}\to 1$ and $\mathbb{V}\left(\xi_t-K_t\right)\to 0$, so that $\lim_{w\to\infty}F(w)=\infty$ and there must exist some finite $\bar{w}\geq\tau_\mu$ large enough such that $F(\bar{w})>0$. Next, notice that when $w=\tau_\mu$,

$$F\left(\tau_\mu\right)=-\lambda\left(1-\alpha\right)\tau_\mu^2\left(\mathbb{V}\left(\xi_t-K_t\right)+\frac{\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)^2\left(1-\mathcal{S}\right)^2}{1-\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}\right)<0.$$

Thus, since $F(\cdot)$ is continuous, $F(\tau_\mu)<0$, and $F(\bar{w})>0$, there must exist some finite $w\in[\tau_\mu,\bar{w}]$ such that $F(w)=0$.

Further, from the definition of $\mathcal{S}$ we have that (see footnote 3)

$$1-\mathcal{S}=\frac{\mathbb{COV}\left(\xi_t-K_t,\xi_t\right)}{\mathbb{V}\left(\xi_t\right)}\quad\Rightarrow\quad 1-\mathcal{S}=1-(1+r)\frac{\mathbb{COV}(\hat{K}_t,\xi_t)}{\mathbb{V}\left(\xi_t\right)}.$$

Therefore, equation (A.7) becomes

$$r=\frac{w}{1+w}\frac{\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}{1-\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}\left(1-(1+r)\frac{\mathbb{COV}(\hat{K}_t,\xi_t)}{\mathbb{V}\left(\xi_t\right)}\right).$$

Since $\mathbb{COV}(\hat{K}_t,\xi_t)$ does not depend on r, the existence of w directly implies the existence of r. □

Proof of Proposition 6. According to equation (25), α affects the bias, $\mathcal{B}$, only through $1-\mathcal{S}$. It is, then, sufficient to prove that the sensitivity, $\mathcal{S}$, is decreasing in α. Further, since $\gamma=0$ implies $r=0$, α affects $\mathcal{S}$ only through the endogenous scalar w. To facilitate the proof, define an alternative signal process such that

$$\xi_t=a(L)\eta_t,\qquad\text{with}\quad\eta_t\sim\mathcal{N}(0,\sigma_\eta^2),\tag{A.9}$$

$$\hat{x}_{it}=m(L)\eta_t+n(L)\hat{\epsilon}_{it},\qquad\text{with}\quad\hat{\epsilon}_{it}\sim\mathcal{N}(0,(1-\alpha)^{-1}(1+w)^{-1}\Sigma),\tag{A.10}$$

and let the corresponding optimal Bayesian forecast be given by

$$\hat{\mathbb{E}}_{it}[\xi_t]=\hat{p}(L;w,\alpha)\hat{x}_{it}.$$

It is straightforward to show that this signal process is equivalent to the (w,α)-modified signal process for Definition 2, that is

$$\hat{p}(L;w,\alpha)=p(L;w,\alpha).$$

For the current proof, this signal process is more helpful. Notice that $\mathcal{S}$ is affected by α only through $p(L;w,\alpha)$, since it is defined on the basis of the objective signal process.

In what follows, we first show that

$$\lim_{\alpha \to 1^-} \frac{\mathrm{d}\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)}{\mathrm{d}\alpha} > 0.$$

We then prove, by contradiction, that there does not exist $\alpha \in [0,1)$ such that

$$\frac{\mathrm{d}\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)}{\mathrm{d}\alpha} < 0.$$

Then, the result follows by continuity of $\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)$ with respect to α.

Step 1: $\lim_{\alpha \to 1^-} \frac{\mathrm{d}\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathrm{d}\alpha} > 0$:

It follows from equation (23) that $\lim_{\alpha \to 1^-} w = \tau_\mu$. So, as $\alpha \to 1^-$, the signals $\hat{x}_{it}$ become useless and, as a result,

$$\mathbb{COV}\left(K_t, \xi_t\right) = \mathbb{V}\left(K_t\right) = 0.$$

Further, since $w \geq \tau_\mu$, we have that

$$\lim_{\alpha \to 1^-} \frac{\mathrm{d}w}{\mathrm{d}\alpha} \leq 0 \quad \Rightarrow \quad \lim_{\alpha \to 1^-} \frac{\mathrm{d}\left(1-\alpha\right)\left(1+w\right)}{\mathrm{d}\alpha} < 0.$$

Therefore, at the limit of $\alpha \to 1^-$, an increase in α is akin to an increase in the variance of every idiosyncratic noise, which implies that (see Lemma D.2 in the Online Appendix D of Huo and Pedroni (2020)),

$$\lim_{\alpha \to 1^-} \frac{\mathrm{d}\,\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)}{\mathrm{d}\alpha} > 0.$$

Step 2: $\frac{\mathrm{d}\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathrm{d}\alpha} > 0$ for all $\alpha \in [0,1)$:

Suppose the there exists $\alpha \in [0,1)$ such that $\frac{\mathrm{d}\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathrm{d}\alpha} < 0$. Then, by the intermediate value theorem and continuity of $\frac{\mathrm{d}\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathrm{d}\alpha}$, there must exist some $\alpha_\dagger$ such that

$$\left.\frac{\mathrm{d}\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} = 0 \quad \Rightarrow \quad \left.\frac{\mathrm{d}\left(1-\alpha\right)\left(1+w\right)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} = 0 \quad \Rightarrow \quad \left.\frac{\mathrm{d}\mathbb{V}\left(\xi_t - K_t\right)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} = 0,$$

since, for $\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)$ not to change with α, it must be that the variance of the noise, $\left(1-\alpha\right)\left(1+w\right)$, is unchanged. Since

$$\frac{\mathrm{d}\left(1-\alpha\right)\left(1+w\right)}{\mathrm{d}\alpha} = -\left(1 + w\tau_\mu\right) + \left(1-\alpha\right)\frac{\mathrm{d}w}{\mathrm{d}\alpha},$$

it follows that

$$\left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} > 0.$$

However, since $\mathbb{COV}(\xi_t - K_t, \xi_t)$ and $\mathbb{V}(\xi_t - K_t)$ do not vary with α, it follows from equation (23) that

$$\left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} = -\frac{\lambda\tau_\mu\left(\mathbb{V}(\xi_t - K_t) + \frac{\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2}{1-\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}\right)}{\left[1-\lambda(1-\alpha_\dagger)\,\tau_\mu\left(\mathbb{V}(\xi_t - K_t) + \frac{\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2}{1-\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}\right)\right]^2} < 0.$$

Thus, we have a contradiction, and we can conclude that

$$\frac{\mathrm{d}\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathrm{d}\alpha} < 0 \;\;\Rightarrow\;\; \frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\alpha} < 0.$$

□

B Extensions

In this section, we consider three extensions to the baseline model setup. The first is the multiple-actions extension discussed in Section II.E; here we simply provide a proof of the results presented there. The second extension allows for a more general utility specification, which covers economies with different forms of inefficiencies. The third extension is to the information structure, allowing the fundamental to depend on multiple aggregate shocks.

B.1 Multiple actions

In this section, we extend the baseline setup to allow for multiple actions instead of just a single one. Each agent i takes J actions, so that $k_{it} \in \mathrm{R}^J$. In what follows, we first demonstrate that the utility specification with multiple actions introduced in Section II.E, equation (26), represents an efficient economy under both complete and incomplete information, provided there is no concern for ambiguity. We then proceed to present the proof of Proposition 7, which characterizes the equilibrium when there is ambiguity and ambiguity aversion under this multiple actions setup.

B.1.1 An Efficient Economy

Consider the following extension to multiple actions of the generic quadratic utility specification from Angeletos and Pavan (2007):

$$u\left(k_{it}, K_t, \Sigma_t, \xi_t\right) = \frac{1}{2}k_{it}' U_{kk} k_{it} + \frac{1}{2}K_t' U_{KK} K_t + \frac{1}{2}\xi_t' U_{\xi\xi}\xi_t + \frac{1}{2}\Sigma_t' U_{\Sigma\Sigma}\Sigma_t + \xi_t' U_{k\xi}' k_{it} + K_t' U_{kK}' k_{it} + \xi_t' U_{K\xi}' K_t \\ + U_k k_{it} + U_K K_t + U_\xi \xi_t + \text{const.},$$

where K_t and Σ_t denote respectively the cross-sectional mean and dispersion of the J actions,

$$K_t \equiv \int_i k_{it}\,\mathrm{d}i, \quad \text{and} \quad \Sigma_t \equiv \left(\sqrt{\int_i \left(k_{1,it} - K_{1,t}\right)^2 \mathrm{d}i}, \cdots, \sqrt{\int_i \left(k_{j,it} - K_{j,t}\right)^2 \mathrm{d}i}, \cdots, \sqrt{\int_i \left(k_{J,it} - K_{J,t}\right)^2 \mathrm{d}i}\right).$$

The jth elements of k_{it} and K_t are represented by $k_{j,it}$ and $K_{j,t}$, respectively. We assume that $U_{\Sigma\Sigma}$ is diagonal, and that the information structure is the same as in the single-action setup.

Equilibrium. Without any concern for ambiguity, we now define and characterize an equilibrium for this model.

Definition B.1. *In the absence of ambiguity, an equilibrium is a strategy $k\left(x_i^t\right)$ such that*

$$k\left(x_i^t\right) = \operatorname{argmax} k\,\mathbb{E}\left[u\left(k, K\left(\eta^t\right), \Sigma\left(\eta^t\right), \xi\left(\eta^t\right)\right) \mid x_i^t\right],$$

where $K\left(\eta^t\right) \equiv \int_i k\left(x_i^t\right)\mathrm{d}i$ denotes the equilibrium aggregate action, and

$$\Sigma\left(\eta^t\right) \equiv \left(\sigma_1\left(\eta^t\right), \cdots, \sigma_j\left(\eta^t\right), \cdots, \sigma_J\left(\eta^t\right)\right)', \quad \textit{with} \quad \sigma_j\left(\eta^t\right) \equiv \sqrt{\int_i \left(k_{j,i}\left(x_i^t\right) - K_j\left(\eta^t\right)\right)^2 \mathrm{d}i},$$

denotes the equilibrium cross-sectional dispersion of actions.

Proposition B.1. *In the absence of ambiguity, a strategy $k\left(x_i^t\right)$ is an equilibrium under incomplete information if and only if*

$$k\left(x_i^t\right) = \left(\mathrm{I}-\Theta\right)\mathbb{E}\left[\kappa\left(\xi_t\right) \mid x_i^t\right] + \Theta\,\mathbb{E}\left[K\left(\eta^t\right) \mid x_i^t\right],$$

where the equilibrium degree of coordination is captured by the $J \times J$ matrix

$$\Theta \equiv -U_{kk}^{-1}U_{kK},$$

and $\kappa\left(\xi_t\right)$ denotes the equilibrium allocation under complete information, given by

$$\kappa\left(\xi_t\right) \equiv \underbrace{-\left(U_{kk}+U_{kK}\right)^{-1}U_{k\xi}}_{\kappa}\ \xi_t\ \underbrace{-\left(U_{kk}+U_{kK}\right)^{-1}U_k'}_{\kappa_0}.$$

Proof. We first characterize the complete-information benchmark. Let $\mathcal{I}_{it}$ be the information set of agent i in period t. Under complete information, we have that $\xi_t \in \mathcal{I}_{it}$. That is, all agents have perfect information about both the fundamental ξ_t and, consequently, about the aggregate action K_t. The agent's first-order condition is then given by

$$\frac{\partial u\left(k_{it}, K_t, \Sigma_t, \xi_t\right)}{\partial k_{it}} = k_{it}'U_{kk} + \xi_t'U_{k\xi}' + K_t'U_{kK}' + U_k = 0.$$

Using the fact that $k_{it} = K_t$, the equilibrium strategy under complete information is such that

$$k_{it} = \kappa\left(\xi_t\right) \equiv \underbrace{-\left(U_{kk}+U_{kK}\right)^{-1}U_{k\xi}}_{\kappa}\ \xi_t\ \underbrace{-\left(U_{kk}+U_{kK}\right)^{-1}U_k'}_{\kappa_0},$$

where both κ and κ_0 are $J \times 1$ vectors.

When information is incomplete, the agent's first-order condition becomes

$$-U_{kk}\ k_{it} = U_{k\xi}\ \mathbb{E}\left[\xi_t \mid x_i^t\right] + U_{kK}\ \mathbb{E}\left[K_t \mid x_i^t\right] + U_k'.$$

Multiplying $-\left(U_{kk}+U_{kK}\right)^{-1}$ to both sides of the equation implies

$$\left(U_{kk}+U_{kK}\right)^{-1}U_{kk}\ k_{it} = -\left(U_{kk}+U_{kK}\right)^{-1}U_{k\xi}\ \mathbb{E}\left[\xi_t \mid x_i^t\right] - \left(U_{kk}+U_{kK}\right)^{-1}U_{kK}\ \mathbb{E}\left[K_t \mid x_i^t\right] - \left(U_{kk}+U_{kK}\right)^{-1}U_k',$$

and it follows that

$$k_{it} = U_{kk}^{-1}\left(U_{kk}+U_{kK}\right)\ \mathbb{E}\left[\kappa\left(\xi_t\right) \mid x_i^t\right] - U_{kk}^{-1}U_{kK}\ \mathbb{E}\left[K_t \mid x_i^t\right].$$

which completes the proof. □

Efficient allocation. Abstracting from ambiguity concerns, an efficient allocation is the strategy $k\left(x_i^t\right)$ that maximizes ex-ante utility, subject only to the constraint that the private information of any agent cannot be transferred to any other agent.

Definition B.2. *In the absence of ambiguity, an efficient allocation is a strategy* $k\left(x_i^t\right)$ *that maximizes ex-ante expected utility,*

$$\mathbb{E}\left[u\left(k, K\left(\eta^t\right), \Sigma\left(\eta^t\right), \xi\left(\eta^t\right)\right)\right].$$

Proposition B.2. *In the absence of ambiguity, a strategy* $k\left(x_i^t\right)$ *is efficient under incomplete information if and only if*

$$k\left(x_i^t\right) = \left(\mathrm{I} - \Theta^*\right) \int_{\eta^t} \kappa^*(\xi\left(\eta^t\right)) \mathrm{d}P\left(\eta^t \mid x_i^t\right) + \Theta^* \int_{\eta^t} K\left(\eta^t\right) \mathrm{d}P\left(\eta^t \mid x_i^t\right),$$

where $P\left(\eta^t \mid x_i^t\right)$ *denotes the cumulative distribution function of* η_t *conditional on* x_i^t*, the efficient degree of coordination is captured by the* $J \times J$ *matrix*

$$\Theta^* = -\left(U_{kk} + U_{\Sigma\Sigma}\right)^{-1}\left(U_{KK} + U_{kK} + U_{kK}' - U_{\Sigma\Sigma}\right),$$

and $\kappa^*(\xi_t)$ *denotes the efficient allocation under complete information, given by*

$$\kappa^*(\xi_t) \equiv \underbrace{-\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right)^{-1}\left(U_{k\xi} + U_{K\xi}\right)}_{\kappa^*} \xi_t \underbrace{-\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right)^{-1}\left(U_k + U_K\right)'}_{\kappa_0^*}.$$

Proof. We first characterize the first-best allocation, that is, the efficient allocation under complete information. Let $\mathcal{I}_{it}$ be the information set of agent i in period t. Under complete information, we have that $\xi_t \in \mathcal{I}_{it}$. It is, then, straightforward to show that the first-best allocation features $k_{it} = K_t$, which implies that $\Sigma_t = 0$. It follows that the efficient level of K_t must maximize

$$\frac{1}{2}K_t'\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right)K_t + \frac{1}{2}\xi_t' U_{\xi\xi}\xi_t + \xi_t'\left(U_{k\xi} + U_{K\xi}\right)' K_t + \left(U_k + U_K\right)K_t + U_\theta \xi_t + \text{const.},$$

which implies the following first-order condition,

$$K_t'\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right) + \xi_t'\left(U_{k\xi} + U_{K\xi}\right)' + \left(U_k + U_K\right) = 0.$$

It follows that the efficient allocation satisfies

$$k_{it} = K_t = \underbrace{-\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right)^{-1}\left(U_{k\xi} + U_{K\xi}\right)}_{\kappa^*} \xi_t \underbrace{-\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right)^{-1}\left(U_k + U_K\right)'}_{\kappa_0^*},$$

where both κ^* and κ_0^* are $J \times 1$ vectors.

To characterize the efficient allocation under incomplete information, define the Lagrangian of the problem in

Definition B.2 by

$$\Lambda = \int_{\eta^t}\int_{x_i^t} u\left(k\left(x_i^t\right), K\left(\eta^t\right), \Sigma\left(\eta^t\right), \xi\left(\eta^t\right)\right) \mathrm{d}P\left(x_i^t \mid \eta^t\right) \mathrm{d}P\left(\eta^t\right)$$
$$+ \int_{\eta^t} \iota\left(\eta^t\right)\left[K\left(\eta^t\right) - \int_{x_i^t} k\left(x_i^t\right) \mathrm{d}P\left(x_i^t \mid \eta^t\right)\right] \mathrm{d}P\left(\eta^t\right)$$
$$+ \int_{\eta^t} \sum_{j=1}^{J} \varphi_j\left(\eta^t\right)\left[\sigma_j^2\left(\eta^t\right) - \int_{x_i^t}\left(k_{j,i}\left(x_i^t\right) - K_j\left(\eta^t\right)\right)^2 P\left(x_i^t \mid \eta^t\right) \mathrm{d}x_i^t\right] \mathrm{d}P\left(\eta^t\right),$$

where $\iota\left(\eta^t\right)$ and $\varphi_j\left(\eta^t\right)$ denote the multipliers on the definitions of $K\left(\eta^t\right)$ and $\sigma_j\left(\eta^t\right)$, respectively. Further, $P\left(x_i^t \mid \eta^t\right)$ denotes the CDF of x_i^t conditional on η_t, and and $P\left(\eta^t\right)$ denotes the unconditional CDF of η^t.

To ease notation, denote $\varphi\left(\eta^t\right) \equiv \operatorname{diag}\left(\varphi_1\left(\eta^t\right), \cdots, \varphi_j\left(\eta^t\right), \cdots, \varphi_J\left(\eta^t\right)\right)$. Then, the first-order conditions can be written as

$$\int_{x_i^t}\left(\frac{\partial u(\cdot)}{\partial K} + \iota\left(\eta^t\right) + 2\varphi\left(\eta^t\right)\left(k\left(x_i^t\right) - K\left(\eta^t\right)\right)\right) \mathrm{d}P\left(x_i^t \mid \eta^t\right) = 0, \quad \text{for almost all } \eta^t, \tag{B.1}$$

$$\int_{\eta^t}\left(\frac{\partial u(\cdot)}{\partial k} - \iota\left(\eta^t\right) - 2\varphi\left(\eta^t\right)\left(k\left(x_i^t\right) - K\left(\eta^t\right)\right)\right) \mathrm{d}P\left(\eta^t \mid x_i^t\right) = 0, \quad \text{for almost all } x_i^t, \tag{B.2}$$

$$\int_{x_i^t}\left(\frac{\partial u(\cdot)}{\partial \Sigma}\right) \mathrm{d}P\left(x_i^t \mid \eta^t\right) + 2\varphi\left(\eta^t\right)\Sigma\left(\eta^t\right) = 0, \quad \text{for almost all } \eta^t. \tag{B.3}$$

Rearranging equations (B.1) and (B.3), we obtain

$$\int_{x_i^t} \frac{\partial u(\cdot)}{\partial K} \mathrm{d}P\left(x_i^t \mid \eta^t\right) + \iota\left(\eta^t\right) = 0, \quad \text{and} \quad \varphi\left(\eta^t\right) = -\frac{1}{2} U_{\Sigma\Sigma}, \quad \text{for almost all } \eta^t.$$

Further, since

$$\frac{\partial u(\cdot)}{\partial K} = U_{KK} K\left(\eta^t\right) + U_{kK}^{\prime}\, k\left(x_i^t\right) + U_{K\xi}\, \xi\left(\eta^t\right) + U_K,$$

it follows that

$$\iota\left(\eta^t\right) = -\left(U_{KK} + U_{kK}^{\prime}\right) K\left(\eta^t\right) - U_{K\xi}\, \xi\left(\eta^t\right) - U_K.$$

Using these two expressions to replace $\iota\left(\eta^t\right)$ and $\varphi\left(\eta^t\right)$ in equation (B.2), and using the fact that

$$\frac{\partial u(\cdot)}{\partial k} = U_{kk} k\left(x_i^t\right) + U_{kK} K\left(\eta^t\right) + U_{k\xi}\xi\left(\eta^t\right) + U_k,$$

yields

$$k\left(x_i^t\right) = \left(U_{kk} + U_{\Sigma\Sigma}\right)^{-1}\left(U_{kk} + U_{kK} + U_{kK}^{\prime} + U_{KK}\right) \int_{\eta^t} \kappa^*(\xi\left(\eta^t\right)) \mathrm{d}P\left(\eta^t \mid x_i^t\right)$$
$$- \left(U_{kk} + U_{\Sigma\Sigma}\right)^{-1}\left(U_{KK} + U_{kK} + U_{kK}^{\prime} - U_{\Sigma\Sigma}\right) \int_{\eta^t} K\left(\eta^t\right) \mathrm{d}P\left(\eta^t \mid x_i^t\right),$$

which completes the proof. □

By comparing Propositions B.1 and B.2, we arrive at the following corollary.

Corollary B.1. *An economy is efficient if and only if*

$$\kappa(\xi_t) = \kappa^*(\xi_t), \quad \textit{and} \quad \Theta = \Theta^*.$$

Next, notice that the utility specification in equation (26), used in Section II.E,

$$u(k_{it}, K_t, \xi_t) = \frac{1}{2}(k_{it} - \kappa\xi_t)' \Psi_k (k_{it} - \kappa\xi_t) + \frac{1}{2}(k_{it} - K_t)' \Psi_K (k_{it} - K_t) + \chi\xi_t - \frac{1}{2}\gamma\xi_t^2,$$

implies that

$$U_K = 0, \quad U_{\Sigma\Sigma} = U_{K\xi} = 0, \quad \text{and} \quad U_{kK} = U_{kK'} = U_{KK}.$$

These constraints imply the conditions from Corollary B.1, which then leads to following result.

Claim 1. *The economy with utility given by equation (26) is efficient under both complete and incomplete information.*

We conclude this subsection by two additional remarks:

1. We can normalize $U_k = 0$, and thus, $\kappa_0 = 0$ without loss of generality. A nonzero U_k would only add an exogenous vector of constants to the action strategy under complete or incomplete information. This same exogenous vector of constants also applies to the equilibrium action strategy with ambiguity. This vector of constants can be regarded as the deterministic steady state of the economy, which can always be abstracted away by redefining actions as deviations from the deterministic steady state.
2. We demonstrate that economy with the utility specified as in equation (26) is efficient. This statement can be strengthened in the sense that, as long as $U_{\Sigma\Sigma} = 0$, equation (26) is the only utility specification that ensures efficiency under complete and incomplete information.

B.1.2 Equilibrium with Ambiguity

We now proceed to characterize the equilibrium with ambiguity. First notice that the utility specified in equation (26) is equivalent to the generic quadratic utility if we set

$$U_{kk} = \Psi_k + \Psi_K, \quad \text{and} \quad U_{KK} = \Psi_K.$$

From this point forward, we use these conditions to switch to the notation used in the paper, with Ψ_k and Ψ_K.

Analogously to Proposition 3, it can be shown that the optimal strategies for the vector of J actions of all agents are such that

$$k_{it} = (\mathrm{I} - \Theta)\,\mathcal{F}_{it}[\kappa\xi_t] + \Theta\,\mathcal{F}_{it}[K_t], \tag{B.4}$$

where $\mathcal{F}_{it}[\cdot]$ represents agent i's subjective expectation operator, that is,

$$\mathcal{F}_{it}[\cdot] \equiv \int_{\mu^t} \mathbb{E}^{\mu^t}[\,\cdot\, \mid x_i^t]\, \hat{p}\left(\mu^t \mid x_i^t\right) \mathrm{d}\mu^t, \quad \text{with} \quad \hat{p}\left(\mu^t \mid x_i^t\right) \propto \phi'\left(\mathbb{E}^{\mu^t}[u\left(k_{it}, K_t, \xi_t\right)]\right) p\left(\mu^t \mid x_i^t\right).$$

Moreover, the coordination matrix, Θ, is such that

$$\Theta = U_{kk}^{-1} U_{KK} = \left(\Psi_k + \Psi_K\right)^{-1} \Psi_K.$$

B.1.3 Proof of Proposition 7

Next, as in the single action case, we consider a truncated version of the problem using exactly the same notation as in the proof of Proposition 4. We identify a specific form for the equilibrium optimal strategies, which we then use to prove the main equivalence result, Proposition 7.

Define higher-order subjective expectations recursively as follows:

$$\bar{\mathcal{F}}^n[\mathrm{X}] \equiv \begin{cases} \mathrm{X}, & \text{if} \quad n = 0; \\ \int_i \mathcal{F}_i\left[\bar{\mathcal{F}}^{n-1}[\mathrm{X}]\right] \mathrm{d}i, & \text{if} \quad n \geq 1. \end{cases}$$

By iteratively eliminating $\mathcal{F}_{it}[K]$ in the best response (B.4), we obtain

$$k_i = \sum_{m=0}^{\infty} \Theta^m \left(I - \Theta\right) \kappa\, \mathcal{F}_i\left[\bar{\mathcal{F}}^m[\theta]\right].$$

Notice that, as long as subjective expectations are Gaussian, agent i's subjective expectations about any order must be linear in signals, that is,

$$\mathcal{F}_i\left[\bar{\mathcal{F}}^m[\theta]\right] = \tilde{h}_m'\, x_i + \tilde{q}_m,$$

where the $J \times n$ matrix $\tilde{h}_m'$ and $J \times 1$ vector $\tilde{q}_m$ represent the sensitivity and bias of the mth-order subjective expectation. Further, let the eigenvalue decomposition of Θ be given by

$$\Theta \equiv \sum_{j=1}^{J} \alpha_j Q^{-1} e_j e_j'\, Q,$$

where e_j denotes the j-th column of a $J \times J$ identity matrix. It follows that

$$\begin{aligned}
k_i &= \sum_{m=0}^{\infty} \left(\sum_{j=1}^{J} \alpha_j Q^{-1} e_j e_j' \, Q \right)^m \left(\sum_{j=1}^{J} (1-\alpha_j) \, Q^{-1} e_j e_j' \, Q \right) \kappa \left(\tilde{h}_m' \, x_i + \tilde{q}_m \right) \\
&= \sum_{m=0}^{\infty} \left(\sum_{j=1}^{J} \alpha_j^m Q^{-1} e_j e_j' \, Q \right) \left(\sum_{j=1}^{J} (1-\alpha_j) \, Q^{-1} e_j e_j' \, Q \right) \kappa \left(\tilde{h}_m' \, x_i + \tilde{q}_m \right) \\
&= \sum_{m=0}^{\infty} \sum_{j=1}^{J} (1-\alpha_j) \, \alpha_j^m Q^{-1} e_j e_j' \, Q \, \kappa \left(\tilde{h}_m' \, x_i + \tilde{q}_m \right) \\
&= \sum_{j=1}^{J} Q^{-1} e_j e_j' \, Q \, \kappa \left((1-\alpha_j) \sum_{m=0}^{\infty} \alpha_j^m \tilde{h}_m' \, x_i + (1-\alpha_j) \sum_{m=0}^{\infty} \alpha_j^m \tilde{q}_m \right) \\
&= \sum_{j=1}^{J} Q^{-1} e_j e_j' \, Q \, \kappa \left(\hat{h}_j' \, x_i + \hat{q}_j \right),
\end{aligned}$$

where $\hat{h}_j$ and $\hat{q}_j$ are defined as

$$\hat{h}_j \equiv (1-\alpha_j) \sum_{m=0}^{\infty} \alpha_j^m \tilde{h}_m', \quad \text{and} \quad \hat{q}_j \equiv (1-\alpha_j) \sum_{m=0}^{\infty} \alpha_j^m \tilde{q}_m.$$

Interpret $\kappa \left(\hat{h}_j' \, x_i + \hat{q}_j \right)$, for all j, as a set of forecasting rules for the equilibrium allocation under complete information, $\kappa\theta$. Then, the derived expression implies that the optimal strategy for each of the J actions is a linear combination of these forecasting rules. This linear relationship can be "orthogonalized" by transforming the actions k_i and the complete information allocation $\kappa\theta$ using the matrix Q. Specifically, let

$$\hat{k}_i \equiv Q \, k_i, \quad \text{and} \quad \hat{\kappa} \equiv Q \, \kappa.$$

It follows that

$$\hat{k}_i = \sum_{j=1}^{J} e_j e_j' \, \hat{\kappa} \left(\hat{h}_j' x_i + \hat{q}_j \right),$$

so that the j-th transformed action, the j-th row of $\hat{k}_i$, is equal to $e_j' \, \hat{\kappa} \left(\hat{h}_j' \, x_i + \hat{q}_j \right)$.

By defining

$$\mathcal{H} \equiv \left[\begin{array}{cccc} e_1' \hat{\kappa} \hat{h}_1' & e_2' \hat{\kappa} \hat{h}_2' & \ldots & e_J' \hat{\kappa} \hat{h}_J' \end{array} \right], \quad \text{and} \quad \mathcal{Q} \equiv \left[\begin{array}{cccc} e_1' \hat{\kappa} \hat{q}_1 & e_2' \hat{\kappa} \hat{q}_2 & \ldots & e_J' \hat{\kappa} \hat{q}_J \end{array} \right]',$$

the expression for $\hat{k}_i$ can be compactly written as

$$\hat{k}_i = \mathcal{H}' B \nu_i + \mathcal{Q}.$$

Similarly, the Q-transformed version of the complete information solution can be written as

$$\hat{\kappa}(\theta) = \hat{\kappa} A \nu_i = \sum_{j=1}^{J} e_j e_j' \hat{\kappa} A \nu_i = \mathcal{A}\nu_i, \quad \text{with} \quad \mathcal{A} \equiv \begin{bmatrix} e_1'\hat{\kappa}A & e_2'\hat{\kappa}A & \dots & e_J'\hat{\kappa}A \end{bmatrix}' = \hat{\kappa} \otimes A.$$

Further, the utility function can also be transformed in a similar way,

$$u_i = \frac{1}{2}\left(\hat{k}_i - \hat{\kappa}(\theta)\right)' \hat{\Psi}_k \left(\hat{k}_i - \hat{\kappa}(\theta)\right) + \frac{1}{2}\left(\hat{k}_i - \hat{K}\right)' \hat{\Psi}_K \left(\hat{k}_i - \hat{K}\right) + \chi\theta - \frac{1}{2}\gamma\theta^2 + \text{const.},$$

with

$$\hat{\Psi}_k \equiv \left(Q^{-1}\right)' \Psi_k Q^{-1}, \quad \text{and} \quad \hat{\Psi}_K \equiv \left(Q^{-1}\right)' \Psi_K Q^{-1}.$$

It follows that

$$\begin{aligned}\mathbb{E}^{\mu}[u_i] = &\frac{1}{2}\mu'\mathcal{K}\left(\mathcal{H}'B - \mathcal{A}\right)' \hat{\Psi}_k \left(\mathcal{H}'B - \mathcal{A}\right)\mathcal{K}'\mu - \frac{1}{2}\gamma\mu'\mathcal{K}A'A\mathcal{K}'\mu + \\ &\frac{1}{2}\mu'\mathcal{K}\left(\mathcal{H}'B - \mathcal{A}\right)' \hat{\Psi}_K \mathcal{Q} + \frac{1}{2}\mathcal{Q}'\hat{\Psi}_k\left(\mathcal{H}'B - \mathcal{A}\right)\mathcal{K}'\mu + \frac{1}{2}\chi A\mathcal{K}'\mu + \frac{1}{2}\chi\mu'\mathcal{K}A'.\end{aligned}$$

Thus, the distorted subjective belief must satisfy

$$\hat{p}(\mu \mid x_i) \propto \exp\left(-\frac{1}{2}\mu' S^{-1}\mu + \frac{1}{2}\mu' S^{-1}\left(Mx_i + \Pi\right) + \frac{1}{2}\left(Mx_i + \Pi\right)' S^{-1}\mu\right),$$

with matrices S, M, and Π given by

$$S \equiv \left(\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}' + \Omega_{\mu}^{-1} + \lambda\left(\mathcal{K}\left(\mathcal{H}'B - \mathcal{A}\right)'\hat{\Psi}_k\left(\mathcal{H}'B - \mathcal{A}\right)\mathcal{K}' - \gamma\mathcal{K}A'A\mathcal{K}'\right)\right)^{-1}.$$

$$M \equiv S\mathcal{K}\left(B\Omega B'\right)^{-1}, \quad \text{and} \quad \Pi \equiv S\left(-\lambda\mathcal{K}\left(\mathcal{H}'B - \mathcal{A}\right)'\hat{\Psi}_k\mathcal{Q} - \lambda\chi\mathcal{K}A'\right).$$

From agent i's first order condition, equation (B.4), we have that

$$\hat{k}_i = \left(\mathrm{I} - \sum_{j=1}^{J}\alpha_j e_j e_j'\right)\mathcal{A}\mathcal{F}_i[\nu_i] + \left(\sum_{j=1}^{J}\alpha_j e_j e_j'\right)\mathcal{F}_i\left[\hat{K}\right],$$

and, therefore,

$$\mathcal{H}'B\nu_i + \mathcal{Q} = \left(\mathrm{I} - \sum_{j=1}^{J}\alpha_j e_j e_j'\right)\mathcal{A}\mathcal{F}_i[\nu_i] + \left(\sum_{j=1}^{J}\alpha_j e_j e_j'\right)\left(\mathcal{H}'B\Lambda\mathcal{F}_i[\nu_i] + \mathcal{Q}\right).$$

Moreover, the distorted subjective expectations satisfy

$$\begin{aligned}
\mathcal{F}_i[\nu_i] &= \int_\mu \mathbb{E}^\mu[\nu_i|x_i]\,\hat{p}(\mu|x_i)\,\mathrm{d}\mu \\
&= \int_\mu \left(\mathbb{E}^\mu[\nu_i-\mu|x_i]+\mu\right)\hat{p}(\mu|x_i)\,\mathrm{d}\mu \\
&= \int_\mu \Big(\Omega B'(B\Omega B')^{-1}(x_i-\mu)+\mu\Big)\,\hat{p}(\mu|x_i)\,\mathrm{d}\mu \\
&= \Omega B'(B\Omega B')^{-1}x_i + \Big(\mathrm{I}-\Omega B'(B\Omega B')^{-1}B\Big)\mathcal{K}'\int_\mu \mu\hat{p}_i(\mu)\,\mathrm{d}\mu \\
&= \Omega B'(B\Omega B')^{-1}x_i + \Big(\mathrm{I}-\Omega B'(B\Omega B')^{-1}B\Big)\mathcal{K}'S\mathcal{K}B'(B\Omega B')^{-1}x_i \\
&\quad + \Big(\mathrm{I}-\Omega B'(B\Omega B')^{-1}B\Big)\mathcal{K}'S\Big(-\lambda\mathcal{K}(\mathcal{H}'B-\mathcal{A})'\hat{\Psi}_k\mathcal{Q}-\lambda\chi\mathcal{K}A'\Big).
\end{aligned}$$

Matching coefficients then implies that

$$\mathcal{H}' = (\mathrm{I}-\Phi)\,\mathrm{T}+\Phi\mathcal{H}'\mathrm{H}+\left[(\mathrm{I}-\Phi)(\mathcal{A}-\mathrm{T}B)+\Phi\mathcal{H}'(B\Lambda-\mathrm{H}B)\right]\mathcal{K}'S\mathcal{K}B'(B\Omega B')^{-1}, \tag{B.5}$$

and

$$(\mathrm{I}-\Phi)\,\mathcal{Q} = \left[(\mathrm{I}-\Phi)(\mathcal{A}-\mathrm{T}B)+\Phi\mathcal{H}'(B\Lambda-\mathrm{H}B)\right]\mathcal{K}'S\Big(-\lambda\mathcal{K}(\mathcal{H}'B-\mathcal{A})'\hat{\Psi}_k\mathcal{Q}-\lambda\chi\mathcal{K}A'\Big), \tag{B.6}$$

where T, H, and Φ are given by

$$\mathrm{T}\equiv\mathcal{A}\Omega B'(B\Omega B')^{-1},\quad \mathrm{H}\equiv B\Lambda\Omega B'(B\Omega B')^{-1},\quad \text{and}\quad \Phi\equiv\sum_{j=1}^{J}\alpha_j e_j e_j'.$$

In what follows, we first focus on equation (B.5). Through a sequence of lemmas, we show that this fixed-point problem for $\mathcal{H}$ can be recast as the linear combination of pure forecasting problems. We then proceed to characterize $\mathcal{Q}$ using equation (B.6).

Lemma B.1. *Define*

$$\hat{\Omega}\equiv\Omega+\mathcal{K}'\mathcal{W}\mathcal{K},\quad \hat{\mathrm{T}}\equiv\mathcal{A}\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1},\quad \hat{\mathrm{H}}\equiv B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1},$$

and

$$\mathcal{W}\equiv\left(\Omega_\mu^{-1}+\lambda\left(\mathcal{K}(\mathcal{H}'B-\mathcal{A})'\hat{\Psi}_k(\mathcal{H}'B-\mathcal{A})\mathcal{K}'-\gamma\mathcal{K}A'A\mathcal{K}'\right)\right)^{-1}.$$

Then, the equilibrium $\mathcal{H}$ solves the following fixed-point problem

$$\mathcal{H}'=(\mathrm{I}-\Phi)\,\hat{\mathrm{T}}+\Phi\mathcal{H}'\hat{\mathrm{H}}.$$

Proof. Using the Woodbury matrix identity, we have that

$$\begin{aligned}\left(B\hat{\Omega}B'\right)^{-1} &= \left(B\Omega B' + B\mathcal{K}'\mathcal{W}\mathcal{K}B'\right)^{-1} \\ &= \left(B\Omega B'\right)^{-1} - \left(B\Omega B'\right)^{-1} B\mathcal{K}'\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}' + \mathcal{W}^{-1}\right)^{-1}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ &= \left(B\Omega B'\right)^{-1} - \left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}. \end{aligned} \tag{B.7}$$

If some $\tilde{\mathcal{H}}$ is such that $\tilde{\mathcal{H}}' = (\mathrm{I}-\Phi)\,\hat{\mathrm{T}} + \Phi\tilde{\mathcal{H}}'\hat{\mathrm{H}}$, then

$$\begin{aligned}\tilde{\mathcal{H}}' = &\ (\mathrm{I}-\Phi)\,\mathcal{A}\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} + \Phi\tilde{\mathcal{H}}'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} \\ = &\ (\mathrm{I}-\Phi)\,\mathcal{A}\left(\Omega + \mathcal{K}'\mathcal{W}\mathcal{K}\right)B'\left(B\hat{\Omega}B'\right)^{-1} + \Phi\tilde{\mathcal{H}}'B\Lambda\left(\Omega + \mathcal{K}\mathcal{W}\mathcal{K}'\right)B'\left(B\hat{\Omega}B'\right)^{-1} \\ = &\ (\mathrm{I}-\Phi)\,\mathcal{A}\Omega B'\left(B\hat{\Omega}B'\right)^{-1} + (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\hat{\Omega}B'\right)^{-1} \\ &+ \Phi\tilde{\mathcal{H}}'B\Lambda\Omega B'\left(B\hat{\Omega}B'\right)^{-1} + \Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\hat{\Omega}B'\right)^{-1}. \end{aligned}$$

Using equation (B.7), it follows that

$$\begin{aligned}\tilde{\mathcal{H}}' = &\ (\mathrm{I}-\Phi)\,\mathcal{A}\Omega B'\left(B\Omega B'\right)^{-1} - (\mathrm{I}-\Phi)\,\mathcal{A}\Omega B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ &+ (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ &+ \Phi\tilde{\mathcal{H}}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1} - \Phi\tilde{\mathcal{H}}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ &+ \Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} - \Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ = &\ \underbrace{(\mathrm{I}-\Phi)\,\mathcal{A}\Omega B'\left(B\Omega B'\right)^{-1}}_{(\mathrm{I}-\Phi)\mathrm{T}} + \underbrace{\Phi\tilde{\mathcal{H}}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1}}_{\Phi\tilde{\mathcal{H}}'\mathrm{H}} \\ &- \underbrace{(\mathrm{I}-\Phi)\,\mathcal{A}\Omega B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}}_{(\mathrm{I}-\Phi)\mathrm{T}B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}} - \underbrace{\Phi\tilde{\mathcal{H}}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}}_{\Phi\tilde{\mathcal{H}}'\mathrm{H}B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}} \\ &+ (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ &+ \Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} - \Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}. \end{aligned}$$

Further, notice that the terms in the second-to-last line can be rewritten as

$$\begin{aligned}&(\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ = &\ (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}' + \mathcal{W}^{-1}\right)\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}' + \mathcal{W}^{-1}\right)^{-1}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ &- (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}' + \mathcal{W}^{-1}\right)^{-1}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ = &\ (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}, \end{aligned}$$

and, similarly, the terms in the last line can be rewritten as

$$\begin{aligned}
&\Phi\tilde{\mathcal{H}}' B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B' \left(B\Omega B'\right)^{-1} - \Phi\hat{\mathcal{H}}' B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B' \left(B\Omega B'\right)^{-1} B\mathcal{K}'S\mathcal{K}B' \left(B\Omega B'\right)^{-1} \\
&= \Phi\tilde{\mathcal{H}}' B\Lambda\mathcal{K}'S\mathcal{K}B' \left(B\Omega B'\right)^{-1}.
\end{aligned}$$

Therefore, we have that

$$\tilde{\mathcal{H}}' = (\mathrm{I} - \Phi)\,\mathrm{T} + \Phi\tilde{\mathcal{H}}'\mathrm{H} + \Big[(\mathrm{I} - \Phi)\,(\mathcal{A} - \mathrm{T}B) + \Phi\tilde{\mathcal{H}}'\,(B\Lambda - \mathrm{H}B)\Big]\,\mathcal{K}'S\mathcal{K}B'\,\left(B\Omega B'\right)^{-1},$$

which is equivalent to the expression for $\mathcal{H}$ in equation (B.5). □

Lemma B.2. *For any $j \in \{1, \cdots, J\}$, define*

$$\Omega_{\Gamma_j} \equiv \Gamma_j\,\hat{\Omega}, \quad \textit{with} \quad \Gamma_j \equiv \begin{bmatrix} \mathrm{I}_u & 0_{u,m-u} \\ 0_{m-u,u} & \frac{\mathrm{I}_{m-u}}{1-\alpha_j} \end{bmatrix}.$$

Then, the equilibrium $\mathcal{H}$ satisfies

$$e_j'\mathcal{H}' = e_j'\mathcal{A}\Omega_{\Gamma_j}B'\left(B\Omega_{\Gamma_j}B'\right)^{-1}.$$

Proof. It follows from Lemma B.1 that

$$\mathcal{H}' = (\mathrm{I} - \Phi)\,\mathcal{A}\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} + \Phi\mathcal{H}'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}.$$

Right multiplying by $B\hat{\Omega}B'$, we obtain

$$\mathcal{H}'B\hat{\Omega}B' = (\mathrm{I} - \Phi)\,\mathcal{A}\hat{\Omega}B' + \Phi\mathcal{H}'B\Lambda\hat{\Omega}B',$$

or, using $\Phi = \sum_{j=1}^{J} e_j e_j' \alpha_j$,

$$\sum_{j=1}^{J} e_j e_j'\mathcal{H}'B\hat{\Omega}B' - \sum_{j=1}^{J}\alpha_j e_j e_j'\mathcal{H}'B\Lambda\hat{\Omega}B' = (\mathrm{I} - \Phi)\,\mathcal{A}\hat{\Omega}B',$$

which can be rewritten as

$$\sum_{j=1}^{n} e_j e_j'\mathcal{H}'B\,(\mathrm{I} - \alpha_j\Lambda)\,\hat{\Omega}B' = (\mathrm{I} - \Phi)\,\mathcal{A}\hat{\Omega}B'.$$

Since $(\mathrm{I} - \alpha_j\Lambda) = (1 - \alpha_j)\Gamma_j$, it follows that

$$\sum_{j=1}^{n} (1 - \alpha_j)\,e_j e_j'\mathcal{H}'B\Gamma_j\hat{\Omega}B' = (\mathrm{I} - \Phi)\,\mathcal{A}\hat{\Omega}B'.$$

Guessing that

$$e_j'\mathcal{H}' = e_j'\mathcal{A}\Omega_{\Gamma_j}B'\left(B\Omega_{\Gamma_j}B'\right)^{-1},$$

and using $A\Gamma_j = A$, we obtain

$$\sum_{j=1}^{n} (1-\alpha_j)\, e_j e_j' \mathcal{A}\hat{\Omega}B' \left(B\Omega_{\Gamma_j}B'\right)^{-1} B\Omega_{\Gamma_j}B' = (\mathrm{I}-\Phi)\,\mathcal{A}\hat{\Omega}B',$$

or

$$\sum_{j=1}^{n} (1-\alpha_j)\, e_j e_j' \mathcal{A}\hat{\Omega}B' = (\mathrm{I}-\Phi)\,\mathcal{A}\hat{\Omega}B'.$$

The fact that $(\mathrm{I}-\Phi) = \sum_{j=1}^{J} (1-\alpha_j)\, e_j e_j'$ concludes the proof. □

Lemma B.3. *Define*

$$\Delta_j \equiv \Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}, \quad \textit{and} \quad \tilde{\Omega}_\mu \equiv \left(\Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1},$$

with

$$\hat{W} \equiv \mathrm{I}_u - \lambda\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\bar{W}\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}',$$

and

$$\bar{W} \equiv \left(\mathrm{I}_J + \lambda\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\right)^{-1}.$$

Then, the equilibrium $\mathcal{H}$ satisfies

$$\mathcal{H}' = \sum_{i=1}^{J} e_j e_j' \mathcal{A}\Delta_j B' \left(B\Delta_j B'\right)^{-1}.$$

Proof. It follows from Lemma B.2 that

$$\sum_{j=1}^{n} e_j e_j' \left(\mathcal{A}-\mathcal{H}'B\right)\Omega_{\Gamma_j}B' = 0.$$

From the definitions of Ω_{Γ_j} and $\tilde{\Omega}_\mu$, we have that

$$\Omega_{\Gamma_j} = \Gamma_j\Omega + \mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} + \lambda\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}.$$

It is then sufficient to show that

$$\begin{aligned}
&\sum_{j=1}^{n} e_j e_j' \left(\mathcal{A}-\mathcal{H}'B\right)\left(\Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right) \\
&= \sum_{j=1}^{n} e_j e_j' \left(\mathcal{A}-\mathcal{H}'B\right)\left(\Gamma_j\Omega + \mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} + \lambda\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}\right),
\end{aligned}$$

or, equivalently,

$$\left(\mathcal{A}-\mathcal{H}'B\right)\left(\mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right) = \left(\mathcal{A}-\mathcal{H}'B\right)\left(\mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} + \lambda\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}\right).$$

In turn, a sufficient condition for this equation to be satisfied is that

$$\hat{W} = \left(\mathrm{I}_u + \lambda\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A} - \mathcal{H}'B\right)'\hat{\Psi}_k\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\right)^{-1},$$

which, using the Woodbury matrix identity, can be rewritten as

$$\hat{W} = \mathrm{I}_u - \lambda\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A} - \mathcal{H}'B\right)'\hat{\Psi}_k\bar{W}\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}',$$

with

$$\bar{W} = \left(\mathrm{I}_J + \lambda\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A} - \mathcal{H}'B\right)'\hat{\Psi}_k\right)^{-1}.$$

□

Lemma B.4. *Denote the eigenvalue decomposition of* $(\mathrm{I}_J - \Phi)\left(\tau_\mu^{-1}\mathrm{I}_J + \bar{W}\right)$ *by*

$$(\mathrm{I}_J - \Phi)\left(\tau_\mu^{-1}\mathrm{I}_J + \bar{W}\right) = P^{-1}\left(\sum_{j=1}^{J}\omega_j e_j e_j'\right)P.$$

Define

$$\bar{\Delta}_j \equiv \Gamma_j\Omega + \mathcal{K}'\left(\frac{\omega_j}{(1-\alpha_j)}\Omega_\mu - \Omega_\eta\right)\mathcal{K},$$

and let the scalars $\hat{r}_j$ *and* $\hat{x}_j$ *be given by*

$$\hat{r}_j \equiv \frac{\lambda\gamma e_j'\bar{W}\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A'}{(1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A')\hat{\kappa}_j}, \quad \text{and} \quad \hat{x}_j \equiv \sum_{i=1}^{J}P_{ji}\left(1 + \frac{(1-\alpha_i)\,\hat{r}_i}{\omega_j}\right)\hat{\kappa}_i,$$

and let $\mathcal{X}'$ *be such that*

$$e_j'\mathcal{X}' \equiv \hat{x}_j\left(A\bar{\Delta}_jB'\right)\left(B\bar{\Delta}_jB'\right)^{-1}.$$

Then, the equilibrium $\mathcal{H}$ *satisfies*

$$\mathcal{H}' = P^{-1}\mathcal{X}'.$$

Proof. From Lemma B.3, we have that

$$e_j'\mathcal{H}' = e_j'\mathcal{A}\left(\Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right)B'\left(B\left(\Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right)B'\right)^{-1},$$

and, therefore,

$$e_j'\mathcal{H}'B\left(\Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right)B' = e_j'\mathcal{A}\left(\Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right)B'.$$

Rearranging, we get

$$e_j'\mathcal{H}'B\Gamma_j\Omega B' = e_j'\mathcal{A}\Gamma_j\Omega B' + e_j'\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}B'.$$

Since

$$
\begin{aligned}
(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\hat{W} &= (\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}' - \lambda\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\bar{W}\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}' \\
&= \left(\mathrm{I}_J - \lambda\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\right)\bar{W}\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}' \\
&= \bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}',
\end{aligned}
$$

it follows that

$$
e_j'\mathcal{H}'B\Gamma_j\Omega B' = e_j'\mathcal{A}\Gamma_j\Omega B' + e_j'\bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}B'.
$$

From the definition of $\tilde{\Omega}_\mu$, we have that

$$
\tilde{\Omega}_\mu \equiv \left(\Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1} = \Omega_\mu + s\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu,
$$

with

$$
s \equiv \frac{\lambda\gamma}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}.
$$

So that

$$
\begin{aligned}
e_j'\mathcal{H}'B\Gamma_j\Omega B' &= e_j'\mathcal{A}\Gamma_j\Omega B' + e_j'\bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\left(\Omega_\mu + s\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}B' \\
&= e_j'\mathcal{A}\Gamma_j\Omega B' + e_j'\bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}B' + \hat{r}_j e_j'\mathcal{A}\mathcal{K}'\Omega_\mu\mathcal{K}B',
\end{aligned}
$$

where we used the fact that $e_j'\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A'$ and $\hat{\kappa}_j \equiv e_j'\hat{\kappa}$ are scalars, $\hat{\kappa}_j A = e_j'\mathcal{A}$, and

$$
\hat{r}_j \equiv \frac{s\left(e_j'\bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A'\right)}{\hat{\kappa}_j}.
$$

Thus, it follows that

$$
e_j'\mathcal{H}'B\Gamma_j\Omega B' = e_j'\mathcal{A}\left(\Gamma_j\Omega + \hat{r}_j\mathcal{K}'\Omega_\mu\mathcal{K}\right)B' + e_j'\bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}B',
$$

which implies

$$
\sum_{j=1}^{J} e_j e_j'\mathcal{H}'B\Gamma_j\Omega B' = \sum_{j=1}^{J} e_j e_j'\mathcal{A}\left(\Gamma_j\Omega + \hat{r}_j\mathcal{K}'\Omega_\mu\mathcal{K}\right)B' + \bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}B',
$$

and, therefore,

$$
\sum_{j=1}^{J} e_j e_j'\mathcal{H}'B\Gamma_j\Omega B' + \bar{W}\mathcal{H}'B\mathcal{K}'\Omega_\mu\mathcal{K}B' = \sum_{j=1}^{J} e_j e_j'\mathcal{A}\left(\Gamma_j\Omega + \hat{r}_j\mathcal{K}'\Omega_\mu\mathcal{K}\right)B' + \bar{W}\mathcal{A}\mathcal{K}'\Omega_\mu\mathcal{K}B'.
$$

Next, we use this equation to solve for $\mathcal{H}$. Recall that $A = [a', 0]$, where a is of dimension $u \times 1$, and let

$B = [B_1, B_2]$, with B_1 of dimension $n \times u$ and B_2 of dimension $n \times (m-u)$. Then,

$$\sum_{j=1}^{J} e_j e_j' \mathcal{H}' \left(B_1 \Omega_\eta B_1' + \frac{1}{1-\alpha_j} B_2 \Omega_\varepsilon B_2' \right) + \bar{W} \mathcal{H}' \left(B_1 \Omega_\mu B_1' \right) = \sum_{j=1}^{J} e_j \hat{\kappa}_j a' \left(\Omega_\eta + \hat{r}_j \Omega_\mu \right) B_1' + \bar{W} \hat{\kappa} a' \Omega_\mu B_1'.$$

Using the fact that $\tau_\mu^{-1} \Omega_\mu = \Omega_\eta$, it follows that

$$\left(\tau_\mu^{-1} \mathrm{I}_J + \bar{W} \right) \mathcal{H}' \left(B_1 \Omega_\mu B_1' \right) + \left(\mathrm{I}_J - \Phi \right)^{-1} \mathcal{H}' \left(B_2 \Omega_\varepsilon B_2' \right) = \sum_{j=1}^{J} e_j \hat{\kappa}_j a' \left(\hat{r}_j \Omega_\mu \right) B_1' + \left(\tau_\mu^{-1} \mathrm{I}_J + \bar{W} \right) \hat{\kappa} a' \Omega_\mu B_1'.$$

Left multiplying by $(\mathrm{I}_J - \Phi)$, then, implies

$$\left(\mathrm{I}_J - \Phi \right) \left(\tau_\mu^{-1} \mathrm{I}_J + \bar{W} \right) \mathcal{H}' \left(B_1 \Omega_\mu B_1' \right) + \mathcal{H}' \left(B_2 \Omega_\varepsilon B_2' \right) = \left(\mathrm{I}_J - \Phi \right) \left(\sum_{j=1}^{J} e_j \hat{\kappa}_j a' \left(\hat{r}_j \Omega_\mu \right) B_1' + \left(\tau_\mu^{-1} \mathrm{I}_J + \bar{W} \right) \hat{\kappa} a' \Omega_\mu B_1' \right).$$

Since, by definition,

$$\left(\mathrm{I}_J - \Phi \right) \left(\tau_\mu^{-1} \mathrm{I}_J + \bar{W} \right) = P^{-1} D P, \quad \text{with} \quad D \equiv \left(\sum_{j=1}^{J} \omega_j e_j e_j' \right),$$

it follows that

$$P^{-1} D P \mathcal{H}' \left(B_1 \Omega_\mu B_1' \right) + \mathcal{H}' \left(B_2 \Omega_\varepsilon B_2' \right) = \left(\mathrm{I}_J - \Phi \right) \sum_{j=1}^{J} e_j \hat{\kappa}_j a' \left(\hat{r}_j \Omega_\mu \right) B_1' + P^{-1} D P \hat{\kappa} a' \Omega_\mu B_1'.$$

Left multiplying by P, then, implies

$$D P \mathcal{H}' \left(B_1 \Omega_\mu B_1' \right) + P \mathcal{H}' \left(B_2 \Omega_\varepsilon B_2' \right) = P \left(\mathrm{I}_J - \Phi \right) \sum_{j=1}^{J} e_j \hat{\kappa}_j a' \left(\hat{r}_j \Omega_\mu \right) B_1' + D P \hat{\kappa} a' \Omega_\mu B_1'.$$

Next, define

$$\mathcal{X}' \equiv P \mathcal{H}',$$

so that we can rewrite the equation as

$$D \mathcal{X}' \left(B_1 \Omega_\mu B_1' \right) + \mathcal{X}' \left(B_2 \Omega_\varepsilon B_2' \right) = P \sum_{j=1}^{J} e_j \left(1 - \alpha_j \right) \hat{\kappa}_j a' \left(\hat{r}_j \Omega_\mu \right) B_1' + D P \hat{\kappa} a' \Omega_\mu B_1'.$$

Next, using the definition of D, we obtain

$$e_j' \mathcal{X}' \left(B_1 \omega_j \Omega_\mu B_1' + B_2 \Omega_\varepsilon B_2' \right) = e_j' \left(P \sum_{i=1}^{J} e_i \left(1 - \alpha_i \right) \hat{\kappa}_i a' \left(\hat{r}_i \Omega_\mu \right) B_1' + D P \hat{\kappa} a' \Omega_\mu B_1' \right).$$

Right multiplying by $(B_1\omega_j\Omega_\mu B_1' + B_2\Omega_\varepsilon B_2')^{-1}$, then, yields

$$e_j'\mathcal{X}' = e_j'\left(P\sum_{i=1}^{J} e_i\left(1-\alpha_i\right)\hat{\kappa}_i a'\left(\hat{r}_i\Omega_\mu\right)B_1' + DP\hat{\kappa}a'\Omega_\mu B_1'\right)\left(B_1\omega_j\Omega_\mu B_1' + B_2\Omega_\varepsilon B_2'\right)^{-1}.$$

Notice that

$$\begin{aligned} e_j'\left(P\sum_{i=1}^{J} e_i\left(1-\alpha_i\right)\hat{\kappa}_i a'\left(\hat{r}_i\Omega_\mu\right)B_1' + DP\hat{\kappa}a'\Omega_\mu B_1'\right) &= \sum_{i=1}^{J} e_j'Pe_i\left(1-\alpha_i\right)\hat{\kappa}_i a'\left(\hat{r}_i\Omega_\mu\right)B_1' + e_j'DP\hat{\kappa}a'\Omega_\mu B_1' \\ &= \sum_{i=1}^{J} P_{ji}\left(\left(1-\alpha_i\right)\hat{r}_i + \omega_j\right)\hat{\kappa}_i a'\Omega_\mu B_1', \end{aligned}$$

so that we can further rewrite the expression as

$$\begin{aligned} e_j'\mathcal{X}' &= \left(\sum_{i=1}^{J} P_{ji}\left(\frac{\left(1-\alpha_i\right)\hat{r}_i + \omega_j}{\omega_j}\right)\hat{\kappa}_i a'\omega_j\Omega_\mu B_1'\right)\left(B_1\omega_j\Omega_\mu B_1' + B_2\Omega_\varepsilon B_2'\right)^{-1} \\ &= \left(\sum_{i=1}^{J} P_{ji}\left(\frac{\left(1-\alpha_i\right)\hat{r}_i + \omega_j}{\omega_j}\right)\hat{\kappa}_i a'\frac{\omega_j}{\left(1-\alpha_j\right)}\Omega_\mu B_1'\right)\left(B_1\frac{\omega_j}{\left(1-\alpha_j\right)}\Omega_\mu B_1' + B_2\frac{1}{\left(1-\alpha_j\right)}\Omega_\varepsilon B_2'\right)^{-1}. \end{aligned}$$

Finally, using the definition of $\bar{\Delta}_j$, we get

$$e_j'\mathcal{X}' = \sum_{i=1}^{J} P_{ji}\left(1 + \frac{\left(1-\alpha_i\right)\hat{r}_i}{\omega_j}\right)\hat{\kappa}_i\left(A\bar{\Delta}_j B'\right)\left(B\bar{\Delta}_j B'\right)^{-1},$$

and the definition of $\mathcal{X}'$ implies

$$\mathcal{H}' = P^{-1}\mathcal{X}'.$$

□

Parts 1 and 2 of Proposition 7. Given the result in Lemma B.4, we are left with taking the limit, as $T \to \infty$, of the truncated problem. Define $w_j \equiv \frac{\tau_\mu \omega_j}{1-\alpha_j} - 1$, then, in particular, we have that

$$\lim_{T\to\infty} A\bar{\Delta}_j B'\left(B\bar{\Delta}_j B'\right)^{-1} = p\left(L; w_j, \alpha_j\right), \quad \lim_{T\to\infty} A\mathcal{K}'\Omega_\eta\mathcal{K}A' = \mathbb{V}(\xi_t),$$

$$\lim_{T\to\infty}\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}\left(\mathcal{A} - \mathcal{H}'B\right)' = \mathbb{V}(\hat{\kappa}\xi_t - \hat{K}_t), \quad \lim_{T\to\infty}\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A' = \mathbb{COV}\left(\hat{\kappa}\xi_t - \hat{K}_t, \xi_t\right).$$

Next, let $W \equiv \lim_{T\to\infty} \tau_\mu \bar{W}$, $r_j \equiv \lim_{T\to\infty} \frac{\tau_\mu}{1+w_j}\hat{r}_j$, and $x_j \equiv \lim_{T\to\infty} \hat{x}_j$, for $j \in \{1,\ldots,J\}$. Then, it follows that

$$
\begin{aligned}
W &= \lim_{T\to\infty}\left(\tau_\mu^{-1}\mathrm{I}_J + \lambda\tau_\mu^{-1}\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\left(\Omega_\mu + \frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\right)^{-1} \\
&= \lim_{T\to\infty}\left(\tau_\mu^{-1}\mathrm{I}_J + \lambda\left(\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)' + \frac{\lambda\gamma\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\left(\Omega_\eta\mathcal{K}A'A\mathcal{K}'\Omega_\eta\right)\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'}{\tau_\mu^{-1}-\lambda\gamma A\mathcal{K}'\Omega_\eta\mathcal{K}A'}\right)\hat{\Psi}_k\right)^{-1} \\
&= \left(\tau_\mu^{-1}\mathrm{I}_J + \lambda\left(\mathbb{V}\left(\hat{\kappa}_1\xi_t - \hat{K}_t\right) + \frac{\lambda\gamma\mathbb{COV}\left(\hat{\kappa}\xi_t-\hat{K}_t,\xi_t\right)\mathbb{COV}\left(\hat{\kappa}\xi_t-\hat{K}_t,\xi_t\right)'}{\tau_\mu^{-1}-\lambda\gamma\mathbb{V}\left(\xi_t\right)}\right)\hat{\Psi}_k\right)^{-1} \\
W &= \left(\tau_\mu^{-1}\mathrm{I}_J + \lambda Q\left(\mathbb{V}\left(\kappa\xi_t-K_t\right) + \frac{\lambda\gamma\mathbb{COV}\left(\kappa\xi_t-K_t,\xi_t\right)\mathbb{COV}\left(\kappa\xi_t-K_t,\xi_t\right)'}{\tau_\mu^{-1}-\lambda\gamma\mathbb{V}\left(\xi_t\right)}\right)\Psi_k Q^{-1}\right)^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
r_j &= \lim_{T\to\infty}\frac{\tau_\mu}{1+w_j}\frac{\lambda\gamma}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\frac{e_j'\tau_\mu^{-1}\tau_\mu\bar{W}\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A'}{\hat{\kappa}_j} \\
&= \frac{\tau_\mu}{1+w_j}\frac{\lambda\gamma}{1-\lambda\tau_\mu\gamma\mathbb{V}\left(\xi_t\right)}\frac{e_j'W\mathbb{COV}\left(\hat{\kappa}\xi_t-\hat{K}_t,\xi_t\right)}{\hat{\kappa}_j} \\
&= \gamma\frac{\lambda\tau_\mu}{1-\lambda\tau_\mu\gamma\mathbb{V}\left(\xi_t\right)}\frac{e_j'WQ\mathbb{COV}\left(\kappa\xi_t-K_t,\xi_t\right)}{e_j'Q\kappa_j(1+w_j)},
\end{aligned}
$$

and

$$
x_j = \lim_{T\to\infty}\sum_{i=1}^{J}P_{ji}\left(1+\frac{\left(1-\alpha_i\right)\hat{r}_i}{\omega_j}\right)\hat{\kappa}_i = \sum_{i=1}^{J}P_{ji}\left(1+\frac{\left(1-\alpha_i\right)}{\left(1-\alpha_j\right)}r_j\right)Q\kappa_i.
$$

Part 3 of Proposition 7. Next, we characterize the bias term, $\mathcal{B} \equiv \lim_{T\to\infty} Q^{-1}\mathcal{Q}$. From equation (B.6), we have that

$$
\left(\mathrm{I}_J-\Phi\right)\mathcal{Q} = \left[\left(\mathrm{I}-\Phi\right)\left(\mathcal{A}-\mathrm{T}B\right)+\Phi\mathcal{H}'\left(B\Lambda-\mathrm{H}B\right)\right]\mathcal{K}'S\left(-\lambda\mathcal{K}\left(\mathcal{H}'B-\mathcal{A}\right)'\hat{\Psi}_k\mathcal{Q}-\lambda\chi\mathcal{K}A'\right).
$$

There exists a unique $\mathcal{Q}$ that satisfies this equation. We postulate that there exists $\mathcal{Y}$ such that

$$
\left(\mathrm{I}_J-\Phi\right)\mathcal{Q} = \left[\left(\mathrm{I}_J-\Phi\right)\mathcal{A}+\Phi\mathcal{H}'B\Lambda-\mathcal{H}'B\right]\mathcal{K}'\mathcal{Y},
$$

so that solving for $\mathcal{Y}$ pins down the unique $\mathcal{Q}$. To proceed, first replace the guess for $\mathcal{Q}$ on the RHS of equation (B.6),

$$
\begin{aligned}
\mathrm{RHS} \equiv\ & \left[\left(\mathrm{I}_J-\Phi\right)\left(\mathcal{A}-\mathrm{T}B\right)+\Phi\mathcal{H}'\left(B\Lambda-\mathrm{H}B\right)\right]\mathcal{K}'S\left(-\lambda\mathcal{K}\left(\mathcal{H}'B-\mathcal{A}\right)'\hat{\Psi}_k\left(\mathrm{I}_J-\Phi\right)^{-1}\left(\mathrm{I}_J-\Phi\right)\mathcal{Q}-\lambda\chi\mathcal{K}A'\right) \\
=\ & \left[\left(\mathrm{I}_J-\Phi\right)\left(\mathcal{A}-\mathrm{T}B\right)+\Phi\mathcal{H}'\left(B\Lambda-\mathrm{H}B\right)\right]\mathcal{K}'S \\
& \times\left(-\lambda\mathcal{K}\left(\mathcal{H}'B-\mathcal{A}\right)'\hat{\Psi}_k\left(\mathrm{I}_J-\Phi\right)^{-1}\left[\left(\mathrm{I}_J-\Phi\right)\mathcal{A}+\Phi\mathcal{H}'B\Lambda-\mathcal{H}'B\right]\mathcal{K}'\mathcal{Y}-\lambda\chi\mathcal{K}A'\right).
\end{aligned}
$$

Next, for the LHS of the equation, we have that

$$\text{LHS} \equiv (\mathrm{I}_J - \Phi)\, \mathcal{Q} = [(\mathrm{I}_J - \Phi)\, \mathcal{A} + \Phi \mathcal{H}' B \Lambda - \mathcal{H}' B]\, \mathcal{K}' \mathcal{Y},$$

and, substituting the last $\mathcal{H}'$ using equation (B.5), it follows that

$$\begin{aligned}
\text{LHS} = & \ [(\mathrm{I}_J - \Phi)\,(\mathcal{A} - \mathcal{T} B) + \Phi \mathcal{H}'\,(B\Lambda - \mathrm{H}B)] \left[\mathrm{I}_m - \mathcal{K}' S \mathcal{K} B'\,(B\Omega B')^{-1} B\right] \mathcal{K}' \mathcal{Y} \\
= & \ [(\mathrm{I}_J - \Phi)\,(\mathcal{A} - \mathrm{T}B) + \Phi \mathcal{H}'\,(B\Lambda - \mathrm{H}B)]\, \mathcal{K}' S \left[S^{-1} - \mathcal{K} B'\,(B\Omega B')^{-1} B \mathcal{K}'\right] \mathcal{Y} \\
= & \ [(\mathrm{I}_J - \Phi)\,(\mathcal{A} - \mathrm{T}B) + \Phi \mathcal{H}'\,(B\Lambda - \mathrm{H}B)]\, \mathcal{K}' S \\
& \times \left\{\Omega_\mu^{-1} + \lambda \left(\mathcal{K}\,(\mathcal{H}'B - \mathcal{A})'\, \hat{\Psi}_k\,(\mathcal{H}'B - \mathcal{A})\, \mathcal{K}' - \gamma \mathcal{K} A' A \mathcal{K}'\right)\right\} \mathcal{Y},
\end{aligned}$$

where the last equality uses the definition of S. Putting these results together, we have that

$$\begin{aligned}
\text{LHS} - \text{RHS} = & \ [(\mathrm{I}_J - \Phi)\,(\mathcal{A} - \mathrm{T}B) + \Phi \mathcal{H}'\,(B\Lambda - \mathrm{H}B)]\, \mathcal{K}' S \\
& \times \left\{ \begin{array}{c} \lambda \mathcal{K}\,(\mathcal{H}'B - \mathcal{A})'\, \hat{\Psi}_k \left\{(\mathcal{H}'B - \mathcal{A})\, \mathcal{K}' + (\mathrm{I}_J - \Phi)^{-1} [(\mathrm{I}_J - \Phi)\, \mathcal{A} + \Phi \mathcal{H}' B \Lambda - \mathcal{H}' B]\, \mathcal{K}'\right\} \mathcal{Y} \\ \Omega_\mu^{-1} \mathcal{Y} - \lambda \gamma \mathcal{K} A' A \mathcal{K}' \mathcal{Y} - \lambda \chi \mathcal{K} A' \end{array} \right\} \\
= & \ [(\mathrm{I}_J - \Phi)\,(\mathcal{A} - \mathrm{T}B) + \Phi \mathcal{H}'\,(B\Lambda - \mathrm{H}B)]\, \mathcal{K}' S \\
& \times \left\{\Omega_\mu^{-1} \mathcal{Y} + \lambda \mathcal{K}\,(\mathcal{H}'B - \mathcal{A})'\, \hat{\Psi}_k\,(\mathrm{I}_J - \Phi)^{-1}\, \Phi \mathcal{H}' B\,(\Lambda - \mathrm{I}_m)\, \mathcal{K}' \mathcal{Y} - \lambda \gamma \mathcal{K} A' A \mathcal{K}' \mathcal{Y} - \lambda \chi \mathcal{K} A'\right\}.
\end{aligned}$$

Since $(\Lambda - \mathrm{I}_m)\, \mathcal{K}' = 0$, a sufficient condition for $\text{LHS} - \text{RHS} = 0$ is

$$\Omega_\mu^{-1} \mathcal{Y} - \lambda \gamma \mathcal{K} A' A \mathcal{K}' \mathcal{Y} - \lambda \chi \mathcal{K} A' = 0,$$

which, using the Sherman-Morrison formula, implies that

$$\mathcal{Y} = \lambda \chi \left(\Omega_\mu^{-1} - \lambda \gamma \mathcal{K} A' A \mathcal{K}'\right)^{-1} \mathcal{K} A' = \lambda \chi \left(\Omega_\mu + \frac{\lambda \gamma}{1 - \lambda \gamma A \mathcal{K}' \Omega_\mu \mathcal{K} A'} \Omega_\mu \mathcal{K} A' A \mathcal{K}' \Omega_\mu\right) \mathcal{K} A'.$$

Therefore, we have that

$$\begin{aligned}
\mathcal{Q} = & \ (\mathrm{I}_J - \Phi)^{-1} [(\mathrm{I}_J - \Phi)\, \mathcal{A} + \Phi \mathcal{H}' B \Lambda - \mathcal{H}' B]\, \mathcal{K}' \mathcal{Y} \\
= & \ (\mathrm{I}_J - \Phi)^{-1} \left[(\mathrm{I}_J - \Phi)\left(\mathcal{A} - \mathcal{H}' B\right) \mathcal{K}' + \Phi \mathcal{H}' B\,(\Lambda - \mathrm{I}_m)\, \mathcal{K}'\right] \mathcal{Y} \\
= & \ \left(\mathcal{A} - \mathcal{H}' B\right) \mathcal{K}' \mathcal{Y} \\
= & \ \left(\mathcal{A} - \mathcal{H}' B\right) \mathcal{K}' \lambda \chi \left(\Omega_\mu + \frac{\lambda \gamma}{1 - \lambda \gamma A \mathcal{K}' \Omega_\mu \mathcal{K} A'} \Omega_\mu \mathcal{K} A' A \mathcal{K}' \Omega_\mu\right) \mathcal{K} A' \\
= & \ \lambda \tau_\mu \chi \left(1 + \frac{\lambda \gamma \tau_\mu A \mathcal{K}' \Omega_\eta \mathcal{K} A'}{1 - \lambda \gamma \tau_\mu A \mathcal{K}' \Omega_\eta \mathcal{K} A'}\right) \left(\mathcal{A} - \mathcal{H}' B\right) \mathcal{K}' \Omega_\eta \mathcal{K} A'.
\end{aligned}$$

Taking the limit we obtain

$$\mathcal{B} = \lim_{T \to \infty} Q^{-1} \mathcal{Q} = \frac{\lambda \tau_\mu \chi}{1 - \lambda \gamma \tau_\mu \mathbb{V}(\xi_t)} \mathbb{COV}(\kappa \xi_t - K_t, \xi_t),$$

which completes the proof of part 3 of the proposition.

B.2 Inefficient economies

The economy in our baseline setup is assumed to be efficient under both complete and incomplete information. We now consider a generalized utility in the vein of Angeletos and Pavan (2007),

$$u(k_{it}, k_t, \xi_t) = -\frac{1}{2}\left[(1-\alpha)(k_{it}-\xi_t)^2 + \alpha(k_{it}-K_t)^2\right] - \frac{1}{2}\gamma\xi_t^2 - \chi\xi_t - \frac{1}{2}\psi(K_t-\xi_t)^2 - \phi K_t\xi_t - \varphi K_t, \quad \text{(B.8)}$$

which allows inefficiencies under both complete and incomplete information. Specifically, it can be shown that:

- Under complete information, the equilibrium allocation is such that $k_{it} = K_t = \xi_t$, whereas the efficient allocation is such that $k_{it} = K_t = \kappa_1^*\xi_t + \kappa_0^*$ with (κ_1^*, κ_0^*) being given by

$$\kappa_1^* = \frac{1-(\alpha-\psi)-\phi}{1-(\alpha-\psi)}, \quad \text{, and} \quad \kappa_0^* = \frac{\varphi}{1-(\alpha-\psi)}.$$

- Under incomplete information, the equilibrium degree of coordination is α, while the efficient degree of coordination is $\alpha^* = \alpha - \psi$.

The following proposition generalizes our equivalence result to the utility function in equation (B.8). The equilibrium strategy still features the simple form, which results in additional sensitivity and bias.

Proposition B.3. *The linear strategy in equilibrium takes the following form*

$$g(x_i^t) = (1+r)p(L; w, \alpha)x_{it} + \mathcal{B}. \quad \text{(B.9)}$$

1. *The polynomial matrix $p(L; w, \alpha)$ is the Bayesian forecasting rule with the (w, α)-modified signal process and w satisfies*

$$w = \frac{\tau_\mu}{(1+\nu_1) - \lambda(1-\alpha+\psi)\tau_\mu\left(\mathbb{V}(\xi_t-K_t) + \frac{\lambda\gamma\tau_\mu(1+\nu_2)\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2}{1-\lambda\gamma\tau_\mu(1+\nu_3)\mathbb{V}(\xi_t)}\right)};$$

2. *The additional amplification, r, satisfies*

$$r = \frac{\gamma\lambda\tau_\mu\mathbb{V}(\xi_t)(1+\nu_2)}{1-\gamma\lambda\tau_\mu\mathbb{V}(\xi_t)(1+\nu_3)}\frac{w}{1+w}(1-\mathcal{S});$$

3. *The level of bias, $\mathcal{B}$, satisfies*

$$\mathcal{B} = \frac{\chi\lambda\tau_\mu\mathbb{V}(\xi_t)(1-\mathcal{S})+\nu_4}{1-\gamma\lambda\tau_\mu\mathbb{V}(\xi_t)+\nu_5};$$

4. Relative to Proposition 4, the inefficiencies imply the following correction terms

$$\begin{aligned}
\nu_1 &\equiv \frac{\lambda^2\phi^2\tau_\mu^2\left(\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2-\mathbb{V}(\xi_t)\mathbb{V}(\xi_t-K_t)\right)-\lambda\phi\tau_\mu\mathbb{V}(\xi_t)(1-\mathcal{S})}{1-\lambda\tau_\mu\mathbb{V}(\xi_t)(2\gamma-\phi(1+\mathcal{S}))},\\
\nu_2 &\equiv 1-\frac{\phi}{\gamma}\left(2-\frac{\mathbb{V}(\xi_t-K_t)}{\mathbb{V}(\xi_t)(1-\mathcal{S})}\right),\\
\nu_3 &\equiv 1-\frac{\phi}{\gamma}(1+\mathcal{S}),\\
\nu_4 &\equiv \lambda\varphi\tau_\mu\left(\mathbb{V}(\xi_t)(1-\mathcal{S})-\mathbb{V}(\xi_t-K_t)\right)\\
&\quad-\lambda^2\tau_\mu^2\left(\phi(\chi-\varphi)+2\gamma\varphi\right)\left(\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2-\mathbb{V}(\xi_t)\mathbb{V}(\xi_t-K_t)\right)\\
\nu_5 &\equiv \lambda\tau_\mu\mathbb{V}(\xi_t)(2\phi\mathcal{S}-\gamma)+\lambda^2\tau_\mu^2\phi^2\left(\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2-\mathbb{V}(\xi_t)\mathbb{V}(\xi_t-K_t)\right).
\end{aligned}$$

It is easy to see that without inefficiencies, that is if $\psi=\phi=\varphi=0$, we have that $\nu_1=\nu_2=\nu_3=\nu_4=\nu_5=0$, and the formulas reduce to the ones in Proposition 4.

Proof of Proposition B.3. Consider the same truncated version of the model described in the proof of Proposition 4. For the utility in equation (B.8), we have that

$$\hat{p}(\mu|x_i)\propto\exp\left(-\frac{1}{2}\mu'S^{-1}\mu+\frac{1}{2}\mu'S^{-1}(Mx_i+\pi)+\frac{1}{2}(Mx_i+\pi)'S^{-1}\mu\right),$$

where matrices M, π, and S are such that

$$M\equiv S\mathcal{K}(B\Omega B')^{-1},\quad \pi\equiv S\left[-\lambda(1-\alpha^*)h_0\mathcal{K}(A'-B'h)+\lambda\chi\mathcal{K}A'+\lambda\varphi\mathcal{K}B'h\right],$$

and

$$\begin{aligned}
S\equiv\Big(&\mathcal{K}B'(B\Omega B')^{-1}B\mathcal{K}'+\Omega_\mu^{-1}-\lambda\left[(1-\alpha^*)\mathcal{K}(A'-B'h)(A-h'B)\mathcal{K}'+\gamma\mathcal{K}A'A\mathcal{K}'\right]\\
&-\lambda\phi\mathcal{K}(\Lambda B'hA+A'h'B\Lambda)\mathcal{K}'\Big)^{-1},
\end{aligned}$$

which, using $\phi=(1-\alpha^*)(1-\kappa_1^*)$, can be rearranged into

$$S=\left(\mathcal{K}B'(B\Omega B')^{-1}B\mathcal{K}'+\Omega_\mu^{-1}-\lambda\gamma^*\mathcal{K}A'A\mathcal{K}'-\lambda(1-\alpha^*)\mathcal{K}(\kappa_1^*A'-B'h)(\kappa_1^*A-h'B)\mathcal{K}'\right)^{-1},$$

where

$$\gamma^*\equiv\gamma+(1-\alpha^*)(1-(\kappa_1^*)^2).$$

We have the same equilibrium conditions for h and h_0 as in Proposition 4, equations (A.4) and (A.5), and the proof proceeds analogously and we keep the same structure to facilitate comparison.

Lemma B.5. *Define*

$$\hat{\Omega}\equiv\Omega+\mathcal{K}'W\mathcal{K},\quad \hat{\mathrm{T}}\equiv A\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1},\quad \hat{\mathrm{H}}\equiv B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1},$$

and

$$W \equiv \left(\Omega_\mu^{-1} - \lambda\gamma^*\mathcal{K}A'A\mathcal{K}' - \lambda\left(1-\alpha^*\right)\mathcal{K}\left(\kappa_1^* A' - B'h\right)\left(\kappa_1^* A - h'B\right)\mathcal{K}'\right)^{-1}.$$

Then, the equilibrium h solves the following fixed-point problem

$$h' = (1-\alpha)\,\hat{\mathrm{T}} + \alpha h'\hat{\mathrm{H}}.$$

Proof. This proof is exactly analogous to the proof of Lemma A.1. In particular, notice that W and S are still such that

$$S = \left(\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}' + W^{-1}\right)^{-1}.$$

□

Lemma B.6. *Define*

$$\Omega_\Gamma \equiv \Gamma\,\hat{\Omega}, \quad \textit{with} \quad \Gamma \equiv \begin{bmatrix} \mathrm{I}_u & 0_{u,m-u} \\ 0_{m-u,u} & \frac{\mathrm{I}_{m-u}}{1-\alpha} \end{bmatrix}.$$

Then, the equilibrium h satisfies

$$h' = A\Omega_\Gamma B'\left(B\Omega_\Gamma B'\right)^{-1}.$$

Proof. This lemma is exactly the same as Lemma A.2, and is repeated here just for convenience. □

Lemma B.7. *Define*

$$\Delta \equiv \Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K},$$

and

$$\tilde{\Omega}_\mu \equiv \left(\Omega_\mu^{-1} - \lambda\gamma^*\mathcal{K}A'A\mathcal{K}' - \lambda\left(1-\alpha^*\right)\mathcal{K}\left[\left(\kappa_1^*A' - B'h\right)\left(\kappa_1^*A - h'B\right) - \left(A' - B'h\right)\left(A - h'B\right)\right]\mathcal{K}'\right)^{-1},$$

with the scalar $\hat{w}$ given by

$$\hat{w} \equiv \frac{\tau_\mu}{1 - \lambda(1-\alpha^*)\left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)}.$$

Then, the equilibrium h satisfies

$$h' = A\Delta B'\left(B\Delta B'\right)^{-1}.$$

Proof. It follows from Lemma B.6 that

$$\left(A - h'B\right)\Omega_\Gamma B' = 0,$$

and from the definition of Ω_Γ and $\tilde{\Omega}_\mu$ we have that

$$\Omega_\Gamma = \Gamma\Omega + \mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} - \lambda(1-\alpha^*)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}.$$

It is then sufficient to show that

$$(A - h'B)\left(\Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\right) = (A - h'B)\left(\Gamma\Omega + \mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} - \lambda(1-\alpha^*)\mathcal{K}(A' - B'h)(A - h'B)\mathcal{K}'\right)^{-1}\mathcal{K}\right),$$

or, equivalently,

$$\begin{aligned}\hat{w}\tau_\mu^{-1}(A - h'B)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K} &= (A - h'B)\mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} - \lambda(1-\alpha^*)\mathcal{K}(A' - B'h)(A - h'B)\mathcal{K}'\right)^{-1}\mathcal{K}\\ &= (A - h'B)\mathcal{K}'\left(\mathrm{I}_u - \lambda(1-\alpha^*)\tilde{\Omega}_\mu\mathcal{K}(A' - B'h)(A - h'B)\mathcal{K}'\right)^{-1}\tilde{\Omega}_\mu\mathcal{K}.\end{aligned}$$

Thus, it is sufficient to establish that

$$\hat{w}\tau_\mu^{-1}(A - h'B)\mathcal{K}' = (A - h'B)\mathcal{K}'\left(\mathrm{I}_u - \lambda(1-\alpha^*)\tilde{\Omega}_\mu\mathcal{K}(A' - B'h)(A - h'B)\mathcal{K}'\right)^{-1}.$$

It follows that

$$\hat{w}\tau_\mu^{-1}(A - h'B)\mathcal{K}'\left(\mathrm{I}_u - \lambda(1-\alpha^*)\tilde{\Omega}_\mu\mathcal{K}(A' - B'h)(A - h'B)\mathcal{K}'\right) = (A - h'B)\mathcal{K}',$$

which can be rewritten as

$$\hat{w}\tau_\mu^{-1}\left(1 - \lambda(1-\alpha^*)(A - h'B)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}(A' - B'h)\right)(A - h'B)\mathcal{K}' = (A - h'B)\mathcal{K}'.$$

The definition of $\hat{w}$ then yields the result. □

Lemma B.8. *Let*

$$\omega \equiv -\frac{\gamma}{(1-\alpha^*)(1-\kappa_1^*)}, \quad v_1 \equiv -\frac{\gamma\lambda}{\omega}\mathcal{K}A', \quad v_2 \equiv \mathcal{K}(\omega A' - B'h),$$

and

$$c_{ij} \equiv v_i'\Omega_\mu v_j, \quad \text{for } i,j \in \{1,2\}, \quad \text{and} \quad s_i \equiv (A - h'B)\mathcal{K}'\Omega_\mu v_i, \quad \text{for } i \in \{1,2\}.$$

Further, define

$$\tilde{\Delta} \equiv \Gamma\Omega + \tilde{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu\mathcal{K},$$

with the scalar $\tilde{w}$ given by

$$\tilde{w} = \left(1 + \frac{c_{11}s_2 - (1 + c_{12})s_1}{(1 + c_{12})(1 + c_{21}) - c_{11}c_{22}}\right)\hat{w},$$

and let the scalar $\tilde{r}$ be given by

$$\tilde{r} = -\frac{\frac{\lambda\gamma}{\omega}(c_{22}s_1 - (1 + c_{21})s_2) + (1-\omega)(c_{11}s_2 - (1 + c_{12})s_1)}{(1 + c_{12})(1 + c_{21}) - c_{11}c_{22} + c_{11}s_2 - (1 + c_{12})s_1}\frac{\tilde{w}}{1 + \tilde{w}}.$$

Then, the equilibrium h satisfies

$$h' = (1 + \tilde{r})A\tilde{\Delta}B'\left(B\tilde{\Delta}B'\right)^{-1}.$$

Proof. From the definition of $\tilde{\Omega}_\mu$ in Lemma B.7, we have that

$$\tilde{\Omega}_\mu = \left(\Omega_\mu^{-1} + \lambda \left(1-\alpha^*\right)\left(1-\kappa_1^*\right)\mathcal{K}\left[A'\left(\omega A - h'B\right) + \left(\omega A' - B'h\right)A\right]\mathcal{K}'\right)^{-1},$$

with

$$\omega \equiv \frac{\left(1-\alpha^*\right)\left(1-(\kappa_1^*)^2\right)-\gamma^*}{\left(1-\alpha^*\right)\left(1-\kappa_1^*\right)} = -\frac{\gamma}{\left(1-\alpha^*\right)\left(1-\kappa_1^*\right)}.$$

Thus, defining

$$v_1 \equiv -\frac{\gamma\lambda}{\omega}\mathcal{K}A', \quad \text{and} \quad v_2 \equiv \mathcal{K}\left(\omega A' - B'h\right),$$

we can write

$$\tilde{\Omega}_\mu = (\Omega_\mu^{-1} + v_1 v_2' + v_2 v_1')^{-1},$$

and applying the Sherman-Morrison formula twice, we obtain

$$\tilde{\Omega}_\mu = \Omega_\mu + \frac{c_{11}\Omega_\mu v_2 v_2'\Omega_\mu + c_{22}\Omega_\mu v_1 v_1'\Omega_\mu - \left(1+c_{12}\right)\Omega_\mu v_1 v_2'\Omega_\mu - \left(1+c_{21}\right)\Omega_\mu v_2 v_1'\Omega_\mu}{\left(1+c_{12}\right)\left(1+c_{21}\right) - c_{11}c_{22}},$$

with

$$c_{ij} \equiv v_i'\Omega_\mu v_j, \quad \text{for } i,j \in \{1,2\}.$$

Thus, from the definition of Δ in Lemma B.7 and defining

$$\bar{\Delta} \equiv \Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu\mathcal{K},$$

we have that

$$\Delta = \Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K} = \bar{\Delta} + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K},$$

with

$$V \equiv \frac{c_{11}v_2 v_2' + c_{22}v_1 v_1' - \left(1+c_{12}\right)v_1 v_2' - \left(1+c_{21}\right)v_2 v_1'}{\left(1+c_{12}\right)\left(1+c_{21}\right) - c_{11}c_{22}}.$$

Hence, it follows from the result in Lemma B.7 that

$$h' = A\left(\bar{\Delta} + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}\right)B'\left[B\left(\bar{\Delta} + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}\right)B'\right]^{-1},$$

and, therefore,

$$h'\left[B\left(\bar{\Delta} + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}\right)B'\right] = A\left(\bar{\Delta} + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}\right)B'.$$

Rearranging, we get

$$h'B\bar{\Delta}B' + \hat{w}\tau_\mu^{-1}h'B\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}B' = A\bar{\Delta}B' + \hat{w}\tau_\mu^{-1}A\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}B',$$

and right-multiplying both side by $\left(B\bar{\Delta}B'\right)^{-1}$ yields

$$h' = A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{w}\tau_\mu^{-1}\left(A - h'B\right)\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1}.$$

Defining

$$s_i \equiv (A - h'B)\,\mathcal{K}'\Omega_\mu v_i, \quad \text{for } i \in \{1,2\},$$

we obtain

$$\begin{aligned} h' &= A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{w}\tau_\mu^{-1}\frac{\left(c_{22}s_1 - (1+c_{21})\,s_2\right)v_1' + \left(c_{11}s_2 - (1+c_{12})\,s_1\right)v_2'}{(1+c_{12})\,(1+c_{21}) - c_{11}c_{22}}\Omega_\mu \mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1} \\ &= A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{w}\tau_\mu^{-1}\alpha_1 A\mathcal{K}'\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{w}\tau_\mu^{-1}\alpha_2 h'B\mathcal{K}'\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1} \end{aligned}$$

with

$$\alpha_1 \equiv \frac{-\left(c_{22}s_1 - (1+c_{21})\,s_2\right)\frac{\gamma\lambda}{\omega} + \left(c_{11}s_2 - (1+c_{12})\,s_1\right)\omega}{(1+c_{12})\,(1+c_{21}) - c_{11}c_{22}}, \quad \text{and} \quad \alpha_2 \equiv -\frac{\left(c_{11}s_2 - (1+c_{12})\,s_1\right)}{(1+c_{12})\,(1+c_{21}) - c_{11}c_{22}}.$$

Next, notice that

$$\mathcal{K}'\Omega_\mu\mathcal{K} = \tau_\mu\mathcal{K}'\mathcal{K}\Omega,$$

and

$$\bar{\Delta} = \Big((1+\hat{w})\,\mathcal{K}'\mathcal{K} + (1-\alpha)^{-1}\left(\mathrm{I}_m - \mathcal{K}'\mathcal{K}\right)\Big)\,\Omega,$$

so we have

$$\mathcal{K}'\Omega_\mu\mathcal{K} = \frac{\tau_\mu}{1+\hat{w}}\mathcal{K}'\mathcal{K}\bar{\Delta}.$$

Thus, it follows that

$$h' = \left(1 + \alpha_1\frac{\hat{w}}{1+\hat{w}}\right)A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \alpha_2\frac{\hat{w}}{1+\hat{w}}h'B\mathcal{K}'\mathcal{K}\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1},$$

or

$$h' = \beta_1 A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \beta_2 h'B\mathcal{K}'\mathcal{K}\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1}$$

with

$$\beta_1 \equiv 1 + \alpha_1\frac{\hat{w}}{1+\hat{w}}, \quad \text{and} \quad \beta_2 = \alpha_2\frac{\hat{w}}{1+\hat{w}}$$

Define

$$\tilde{\Delta} \equiv \left(\mathrm{I}_m - \beta_2\mathcal{K}'\mathcal{K}\right)\bar{\Delta},$$

and guess that

$$h' = \frac{\beta_1}{1-\beta_2}A\tilde{\Delta}B'\left(B\tilde{\Delta}B'\right)^{-1}.$$

It follows that

$$\begin{aligned} \frac{\beta_1}{1-\beta_2}A\tilde{\Delta}B'\left(B\tilde{\Delta}B'\right)^{-1}B\bar{\Delta}B' &= \beta_1 A\bar{\Delta}B' + \frac{\beta_1}{1-\beta_2}\beta_2 A\tilde{\Delta}B'\left(B\tilde{\Delta}B'\right)^{-1}B\mathcal{K}'\mathcal{K}\bar{\Delta}B' \\ \frac{\beta_1}{1-\beta_2}A\tilde{\Delta}B'\left(B\tilde{\Delta}B'\right)^{-1}B\left(\mathrm{I}_m - \beta_2\mathcal{K}'\mathcal{K}\right)\bar{\Delta}B' &= \beta_1 A\bar{\Delta}B' \\ \beta_1 A\tilde{\Delta}B' &= \beta_1\left(1-\beta_2\right)A\bar{\Delta}B', \end{aligned}$$

which verifies the guess, since $A\tilde{\Delta} = (1-\beta_2)\, A\bar{\Delta}$. Finally, we have that

$$\begin{aligned}\tilde{\Delta} =& \left(\mathrm{I}_m - \beta_2 \mathcal{K}'\mathcal{K}\right)\left((1+\hat{w})\,\mathcal{K}'\mathcal{K} + (1-\alpha)^{-1}\left(\mathrm{I}_m - \mathcal{K}'\mathcal{K}\right)\right)\Omega \\ =& \left((1-\beta_2)\,(1+\hat{w})\,\mathcal{K}'\mathcal{K} + (1-\alpha)^{-1}\left(\mathrm{I}_m - \mathcal{K}'\mathcal{K}\right)\right)\Omega \\ =& \,\Gamma\Omega + \tilde{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu\mathcal{K},\end{aligned}$$

with

$$\tilde{w} = (1-\beta_2)\,(1+\hat{w}) - 1 = \left(1 - \alpha_2 \frac{\hat{w}}{1+\hat{w}}\right)(1+\hat{w}) - 1 = (1-\alpha_2)\hat{w}.$$

and

$$\tilde{r} = \frac{\beta_1}{1-\beta_2} - 1 = \frac{1+\alpha_1 \frac{\hat{w}}{1+\hat{w}}}{1-\alpha_2 \frac{\hat{w}}{1+\hat{w}}} - 1 = \frac{\alpha_1+\alpha_2}{1-\alpha_2}\frac{\tilde{w}}{1+\tilde{w}}.$$

Substituting the definitions of α_1 and α_2 yields the result. □

Parts 1 and 2 of Proposition B.3. Given the result in Lemma B.8, we are left with taking the limit as $T \to \infty$ of the truncated problem. In particular, we have that

$$\begin{aligned}\lim_{T\to\infty} c_{11} &= \tau_\mu \left(\frac{\gamma\lambda}{\omega}\right)^2 \mathbb{V}\left(\xi_t\right), \quad \lim_{T\to\infty} c_{12} = \lim_{T\to\infty} c_{21} = -\tau_\mu \frac{\gamma\lambda}{\omega}\mathbb{COV}\left(\omega\xi_t - K_t, \xi_t\right), \\ \lim_{T\to\infty} c_{22} &= \tau_\mu \mathbb{V}\left(\omega\xi_t - K_t\right), \quad \lim_{T\to\infty} s_1 = -\tau_\mu \frac{\gamma\lambda}{\omega}\mathbb{COV}\left(\xi_t - K_t, \xi_t\right), \quad \text{and} \\ \lim_{T\to\infty} s_2 &= \tau_\mu \mathbb{COV}\left(\omega\xi_t - K_t, \xi_t - K_t\right).\end{aligned}$$

Notice that

$$\hat{w} = \frac{\tau_\mu}{1-\lambda(1-\alpha^*)\left\{(A-h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}\,(A'-B'h) + \frac{c_{11}s_2^2 + c_{22}s_1^2 - (2+c_{12}+c_{21})s_1 s_2}{(1+c_{12})(1+c_{21}) - c_{11}c_{22}}\right\}}.$$

Let $w \equiv \lim_{T\to\infty} \tilde{w}$, and $r \equiv \lim_{T\to\infty} \tilde{r}$. Using equations $\omega = \frac{\gamma}{(1-\alpha^*)(1-\kappa_1^*)}$, $\alpha^* = \alpha - \psi$, and $(1-\alpha^*)(1-\kappa_1^*) = \phi$, in order to return to primitive parameters, it follows that

$$w = \frac{\tau_\mu\left(1+\lambda\left(1-\alpha+\psi\right) r\mathbb{V}\left(\xi_t\right)(1-\mathcal{S})\right)}{1-\lambda\left(1-\alpha+\psi\right)\tau_\mu\left(\mathbb{V}\left(\xi_t - K_t\right) + r\mathbb{V}\left(\xi_t\right)(1-\mathcal{S})\right) + \nu_1},$$

and

$$r = \frac{\gamma\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)(1+\nu_2)}{1-\gamma\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)(1+\nu_3)}\frac{w}{1+w}\left(1-\mathcal{S}\right),$$

with

$$\nu_1 \equiv \frac{\lambda^2\phi^2\tau_\mu^2\left(\mathbb{V}\left(\xi_t\right)^2\left(1-\mathcal{S}\right)^2-\mathbb{V}\left(\xi_t\right)\mathbb{V}\left(\xi_t-K_t\right)\right)-\lambda\phi\tau_\mu\mathbb{V}\left(\xi_t\right)\left(1-\mathcal{S}\right)}{1-\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)\left(2\gamma-\phi\left(1+\mathcal{S}\right)\right)},$$

$$\nu_2 \equiv 1-\frac{\phi}{\gamma}\left(2-\frac{\mathbb{V}\left(\xi_t-K_t\right)}{\mathbb{V}\left(\xi_t\right)\left(1-\mathcal{S}\right)}\right),$$

$$\nu_3 \equiv 1-\frac{\phi}{\gamma}\left(1+\mathcal{S}\right).$$

This completes the proof of parts 1 and 2 of Proposition B.3.

Part 3 of Proposition B.3. Next, we switch focus to the level of the $\mathcal{B} \equiv \lim_{T\to\infty} h_0$. From equation (A.5) and the definition of π, we have that

$$\begin{aligned}(1-\alpha)\,h_0 =\ & \left[(1-\alpha)\left(A-\mathrm{T}B\right)+\alpha h'\left(B\Lambda-\mathrm{H}B\right)\right] \\ & \times \mathcal{K}'S\left[-\lambda\left(1-\alpha^*\right)h_0\mathcal{K}\left(A'-B'h\right)+\lambda\chi\mathcal{K}A'+\lambda\varphi\mathcal{K}B'h\right],\end{aligned}$$

which, using $\varphi = (1-\alpha^*)\kappa_0^*$ and defining $\chi^* \equiv \chi + (1-\alpha^*)\kappa_0^*$, can be rewritten as

$$\begin{aligned}(1-\alpha)\,h_0 =\ & \left[(1-\alpha)\left(A-\mathrm{T}B\right)+\alpha h'\left(B\Lambda-\mathrm{H}B\right)\right] \\ & \times \mathcal{K}'S\left[-\lambda\left(1-\alpha^*\right)\left(h_0+\kappa_0^*\right)\mathcal{K}\left(A'-B'h\right)+\lambda\chi^*\mathcal{K}A'\right].\end{aligned} \tag{B.10}$$

It is straightforward to see there exists a unique h_0 that satisfies this equation. We postulate that there exists $\tilde{\mu}$ such that

$$(1-\alpha)\,h_0 = \left[(1-\alpha)A+\alpha h'B\Lambda-h'B\right]\mathcal{K}'\tilde{\mu},$$

so that solving for $\tilde{\mu}$ pins down the unique h_0. To proceed, first replace the guess for h_0 on the RHS of equation (B.10),

$$\begin{aligned}\mathrm{RHS} \equiv\ & \left[(1-\alpha)\left(A-\mathrm{T}B\right)+\alpha h'\left(B\Lambda-\mathrm{H}B\right)\right]\mathcal{K}'S\left[-\lambda\left(1-\alpha^*\right)\left(h_0+\kappa_0^*\right)\mathcal{K}\left(A'-B'h\right)+\lambda\chi^*\mathcal{K}A'\right] \\ =\ & \left[(1-\alpha)\left(A-\mathrm{T}B\right)+\alpha h'\left(B\Lambda-\mathrm{H}B\right)\right]\mathcal{K}'S \\ & \times\left\{-\lambda\mathcal{K}\left(A'-B'h\right)\left(1-\alpha^*\right)\left\{\frac{\left[(1-\alpha)\left(A-h'B\right)+\alpha h'B\left(\Lambda-\mathrm{I}_m\right)\right]\mathcal{K}'\tilde{\mu}}{1-\alpha}+\kappa_0^*\right\}+\lambda\chi^*\mathcal{K}A'\right\}\end{aligned}$$

Next, for the LHS of the equation, we have that

$$\mathrm{LHS} \equiv (1-\alpha)\,h_0 = \left[(1-\alpha)A+\alpha h'B\Lambda-h'B\right]\mathcal{K}'\tilde{\mu},$$

and, substituting the last h using equation (A.4), it follows that

$$\begin{aligned}\text{LHS} =& \left[(1-\alpha)(A-\text{T}B)+\alpha h'(B\Lambda-\text{H}B)\right]\left[\text{I}_\text{m}-\mathcal{K}'\text{S}\mathcal{K}\text{B}'(\text{B}\Omega\text{B}')^{-1}\text{B}\right]\mathcal{K}'\tilde{\mu}\\ =& \left[(1-\alpha)(A-\text{T}B)+\alpha h'(B\Lambda-\text{H}B)\right]\mathcal{K}'S\left[S^{-1}-\mathcal{K}B'(B\Omega B')^{-1}B\mathcal{K}'\right]\tilde{\mu}\\ =& \left[(1-\alpha)(A-\text{T}B)+\alpha h'(B\Lambda-\text{H}B)\right]\mathcal{K}'S\\ &\times\left\{\Omega_\mu^{-1}-\lambda\left[(1-\alpha^*)\mathcal{K}(\kappa_1^*A'-B'h)(\kappa_1^*A-h'B)\mathcal{K}'+\gamma^*\mathcal{K}A'A\mathcal{K}'\right]\right\}\tilde{\mu},\end{aligned}$$

where the last equality uses the definition of S. Putting these results together, we have that

$$\begin{aligned}\text{LHS}-\text{RHS} =& \left[(1-\alpha)(A-\text{T}B)+\alpha h'(B\Lambda-\text{H}B)\right]\mathcal{K}'S\\ &\times\Big\{\left\{\Omega_\mu^{-1}-\lambda\left[(1-\alpha^*)\mathcal{K}(\kappa_1^*A'-B'h)(\kappa_1^*A-h'B)\mathcal{K}'+\gamma^*\mathcal{K}A'A\mathcal{K}'\right]\right\}\tilde{\mu}\\ &+\lambda(1-\alpha^*)\mathcal{K}(A'-B'h)(A-h'B)\mathcal{K}'\tilde{\mu}+\lambda(1-\alpha^*)\kappa_0^*\mathcal{K}(A'-B'h)-\lambda\chi^*\mathcal{K}A'\Big\},\end{aligned}$$

where we used the fact that $\mathcal{K}(A'-B'h)h'B(\Lambda-\text{I}_\text{m})\mathcal{K}'=0$. Thus, a sufficient condition for $\text{LHS}-\text{RHS}=0$ is

$$\begin{aligned}&\left\{\Omega_\mu^{-1}-\lambda\left[(1-\alpha^*)\mathcal{K}(\kappa_1^*A'-B'h)(\kappa_1^*A-h'B)\mathcal{K}'+\gamma^*\mathcal{K}A'A\mathcal{K}'\right]\right\}\tilde{\mu}\\ &+\lambda(1-\alpha^*)\mathcal{K}(A'-B'h)(A-h'B)\mathcal{K}'\tilde{\mu}+\lambda(1-\alpha^*)\kappa_0^*\mathcal{K}(A'-B'h)-\lambda\chi^*\mathcal{K}A'=0.\end{aligned}$$

Notice that, using the definitions from Lemma B.8, this equation can be rewritten as

$$\left\{\Omega_\mu^{-1}+v_1v_1'+v_2v_2'\right\}\tilde{\mu}=-\lambda(1-\alpha^*)\kappa_0^*\mathcal{K}(A'-B'h)+\lambda\chi^*\mathcal{K}A'.$$

It follows that

$$\tilde{\mu}=\{\Omega_\mu+\Omega_\mu V\Omega_\mu\}\{-\lambda(1-\alpha^*)\kappa_0^*\mathcal{K}(A'-B'h)+\lambda\chi^*\mathcal{K}A'\},$$

and, therefore,

$$\begin{aligned}h_0 &= (1-\alpha)^{-1}\left[(1-\alpha)A+\alpha h'B\Lambda-h'B\right]\mathcal{K}'\tilde{\mu}\\ &= (A-h'B)\mathcal{K}'\tilde{\mu}\\ &= (A-h'B)\mathcal{K}'\{\Omega_\mu+\Omega_\mu V\Omega_\mu\}\{-\lambda(1-\alpha^*)\kappa_0^*\mathcal{K}(A'-B'h)+\lambda\chi^*\mathcal{K}A'\}\\ &= -\lambda(1-\alpha^*)\kappa_0^*\left\{(A-h'B)\mathcal{K}'\Omega_\mu\mathcal{K}(A'-B'h)+\frac{c_{11}s_2^2+c_{22}s_1^2-(2+c_{12}+c_{21})s_1s_2}{(1+c_{12})(1+c_{21})-c_{11}c_{22}}\right\}\\ &+\lambda\chi^*\left\{(A-h'B)\mathcal{K}'\Omega_\mu\mathcal{K}A'+\frac{c_{11}s_2z_2+c_{22}s_1z_1-(1+c_{12})s_1z_2-(1+c_{21})s_2z_1}{(1+c_{12})(1+c_{21})-c_{11}c_{22}}\right\},\end{aligned}$$

with

$$z_i\equiv A\mathcal{K}'\Omega_\mu v_i,\quad \text{for } i\in\{1,2\}.$$

Notice that we have the following limits

$$\lim_{T\to\infty}z_1=-\tau_\mu\frac{\gamma\lambda}{\omega}\mathbb{V}(\xi_t),\quad\text{and}\quad\lim_{T\to\infty}z_2=\tau_\mu\mathbb{COV}(\omega\xi_t-K_t,\xi_t).$$

Therefore, using $\chi^* = \chi + \varphi$, and $(1-\alpha^*)\,\kappa_0^* = \varphi$, we obtain the bias as a function of primitive parameters,

$$\mathcal{B} = \frac{\chi\lambda\tau_\mu \mathbb{V}(\xi_t)(1-\mathcal{S}) + \nu_4}{1 - \gamma\lambda\tau_\mu\mathbb{V}(\xi_t) + \nu_5},$$

with

$$\begin{aligned}
\nu_4 \equiv\ & \lambda\varphi\tau_\mu\left(\mathbb{V}(\xi_t)(1-\mathcal{S}) - \mathbb{V}(\xi_t - K_t)\right) \\
& - \lambda^2\tau_\mu^2\left(\phi(\chi-\varphi) + 2\gamma\varphi\right)\left(\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2 - \mathbb{V}(\xi_t)\mathbb{V}(\xi_t - K_t)\right) \\
\nu_5 \equiv\ & \lambda\tau_\mu\mathbb{V}(\xi_t)(2\phi\mathcal{S} - \gamma) + \lambda^2\tau_\mu^2\phi^2\left(\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2 - \mathbb{V}(\xi_t)\mathbb{V}(\xi_t - K_t)\right),
\end{aligned}$$

which completes the proof of part 3 of the proposition. □

B.3 Multiple aggregate shocks

Consider the same setup described in Section II, but suppose that the common fundamental, ξ_t, is now driven by a $Z \times 1$ vector of shocks, η_t according to the following stochastic process:

$$\xi_t = a(L)\eta_t, \quad \text{with} \quad \eta_t \sim \mathcal{N}(0, \Sigma_\eta),$$

where $a(L)$ is a polynomial in the lag operator L. In the objective environment, η_t is normally distributed with mean zero: $\mu_t = 0$. Subjectively, agents believe that η_t is drawn from a Gaussian distribution with the same covariance matrix, Σ_η, but there is uncertainty about its prior mean, denoted by the $Z \times 1$ vector μ_t. Ambiguity about ξ_t is then captured by the perception that

$$\eta_t \sim \mathcal{N}(\mu_t, \Sigma_\eta), \quad \text{and} \quad \mu_t \sim \mathcal{N}(0, \Sigma_\mu).$$

In Section II, the degree of ambiguity is captured by the σ_μ^2. Here, the covariance matrix Σ_μ plays this role. Without loss of generality, we assume that Σ_η and Σ_μ are diagonal matrices, that is $\Sigma_\eta = \text{diag}(\sigma_{\eta,1}^2, \ldots, \sigma_{\eta,Z}^2)$ and $\Sigma_\mu = \text{diag}(\sigma_{\mu,1}^2, \ldots, \sigma_{\mu,Z}^2)$.

Auxiliary forecasting problem Consider the following pure forecasting problem, which we later link back to the economy with ambiguity.

Definition B.3. *The $(w, \alpha, \{r_i\}_{i=1}^Z)$-modified signal process is given by*

$$\begin{aligned}
\tilde{\xi}_t &= a(L)\,\text{diag}(1+r_1, \ldots, 1+r_q)\tilde{\eta}_t, && \textit{with} \quad \tilde{\eta}_t \sim \mathcal{N}(0, \Sigma_\eta + w\Sigma_\mu), \\
\tilde{x}_{it} &= m(L)\tilde{\eta}_t + n(L)\tilde{\epsilon}_{it}, && \textit{with} \quad \tilde{\epsilon}_{it} \sim \mathcal{N}(0, (1-\alpha)^{-1}\Sigma),
\end{aligned}$$

where w is a non-negative scalar and α is the degree of complementarity. Let the optimal Bayesian forecast be given by

$$\tilde{\mathbb{E}}_{it}[\tilde{\xi}_t] = p(L; w, \alpha, \{r_i\}_{i=1}^Z)\tilde{x}_{it}.$$

This modified signal process is analogous to the baseline. The adjustment w to the volatility of η_t is the counterpart to $\tilde{w} = w\tau_\mu^{-1}$ in the univariate baseline, that is $\Sigma_\eta + w\Sigma_\mu$ is the counterpart of $(1+w)\sigma_\eta^2 = (1+\tilde{w}\tau_\mu)\sigma_\eta^2 = \sigma_\eta^2 + \tilde{w}\sigma_\mu^2$. Further, the amplification factor, $(1+r)$ in the univariate case, has now been incorporated into this modified signal process since in the multivariate case each shock requires a potentially different adjustment, before being put together into a modified fundamental. So, $p(L; w, \alpha, \{r_i\}_{i=1}^Z)$ here is the counterpart of $(1+r)p(L; w, \alpha)$ in the univariate case. To proceed we need the additional following definitions.

Definition B.4. *Define the μ-modified fundamental and (unbiased) aggregate action as*

$$\xi_t^\mu = a(L)\mu_t, \quad \text{and} \quad K_t^\mu = p(L; w, \alpha, \{r_i\}_{i=1}^Z)\mu_t,$$

and the μ-modified aggregate sensitivity to signals as

$$\mathcal{S}^\mu \equiv 1 - \frac{\mathbb{COV}\left(\xi_t^\mu - K_t^\mu, \xi_t^\mu\right)}{\mathbb{V}\left(\xi_t^\mu\right)}.$$

We can then prove the following proposition.

Proposition B.4. *The linear strategy in equilibrium takes the following form*

$$g(x_i^t) = p(L; w, \alpha, \{r_i\}_{i=1}^Z)x_{it} + \mathcal{B}.$$

1. *The polynomial matrix $p(L; w, \alpha, \{r_i\}_{i=1}^Z)$ is the Bayesian forecasting rule with the $(w, \alpha, \{r_i\}_{i=1}^Z)$-modified signal process and w satisfies*

$$w = \frac{1}{1 - \lambda(1-\alpha)\left(\mathbb{V}\left(\xi_t^\mu - K_t^\mu\right) + \frac{\lambda\gamma\mathbb{V}(\xi_t^\mu)^2(1-\mathcal{S}^\mu)^2}{1-\lambda\gamma\mathbb{V}(\xi_t^\mu)}\right)};$$

2. *For all $i \in \{1, \ldots, Z\}$, the additional amplification, r_i, satisfies*

$$r_i = \gamma\frac{\lambda\mathbb{V}\left(\xi_t^\mu\right)}{1-\lambda\gamma\mathbb{V}\left(\xi_t^\mu\right)}\frac{w\tau_{\mu,i}}{1+w\tau_{\mu,i}}(1-\mathcal{S}^\mu);$$

3. *The level of bias, $\mathcal{B}$, satisfies*

$$\mathcal{B} = \chi\frac{\lambda\mathbb{V}\left(\xi_t^\mu\right)}{1-\lambda\gamma\mathbb{V}\left(\xi_t^\mu\right)}\left(1-\mathcal{S}^\mu\right).$$

Proof of Proposition B.4. The truncated version of the problem is analogous to the case with one common shock, with the following adjustments: (1) the size of the vector of aggregate common shocks must be set to $u \equiv Z(T+1)$; (2) the size of the vector of all shocks becomes $m \equiv (Z+N)(T+1)$; (3) instead of $\Omega_\eta = \mathrm{I}_u\,\sigma_\eta^2$ and $\Omega_\mu = \mathrm{I}_u\,\sigma_\mu^2$, we now have $\Omega_\eta = \mathrm{I}_{T+1} \otimes \Sigma_\eta$ and $\Omega_\mu = \mathrm{I}_{T+1} \otimes \Sigma_\mu$. These modifications do not affect in any way the results in Lemmas A.1, A.2, and A.3. However, Lemma A.4 relies on the fact that $\Omega_\eta = \mathrm{I}_u\,\sigma_\eta^2$ and $\Omega_\mu = \mathrm{I}_u\,\sigma_\mu^2$. The following lemma provides the relevant analogous result.

Lemma B.9. *Define*

$$\bar{\Delta} \equiv \Gamma\Omega + \hat{w}\mathcal{K}'\Omega_\mu\mathcal{K},$$

with the scalar $\hat{w}$ given by

$$\hat{w} = \frac{1}{1-\lambda\left(1-\alpha\right)\left(A-h'B\right)\mathcal{K}'\left(\Omega_\mu + \frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\mathcal{K}\left(A'-B'h\right)}.$$

Also, let the diagonal matrix $\hat{R}$ be given by

$$\hat{R} \equiv \mathrm{I}_{T+1} \otimes \mathrm{diag}\left(\hat{r}_1, \ldots, \hat{r}_Z\right),$$

with the scalars $\hat{r}_i$, for $i \in \{1, \ldots, Z\}$, given by

$$\hat{r}_i \equiv \frac{\hat{w}\tau_{\mu,i}}{1+\hat{w}\tau_{\mu,i}}\left(\frac{\lambda\gamma}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\left(A-h'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A', \quad \textit{with} \quad \tau_{\mu,i} \equiv \frac{\sigma^2_{\mu,i}}{\sigma^2_{\eta,i}}.$$

Then, the equilibrium h satisfies

$$h' = A\left(\mathrm{I}_m + \hat{R}\right)\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1}.$$

Proof. From the definition of $\tilde{\Omega}_\mu$ and Δ in Lemma A.3, we have that

$$\tilde{\Omega}_\mu \equiv \left(\Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1} = \Omega_\mu + \frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}.$$

Thus, it follows that

$$\Delta \equiv \Gamma\Omega + \hat{w}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K} = \bar{\Delta} + \hat{w}\mathcal{K}'\left(\frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\mathcal{K} = \bar{\Delta} + s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K},$$

with $s \equiv \lambda\gamma/(1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A')$. Hence, it follows from the result in Lemma A.3 that

$$h' = A\left(\bar{\Delta} + s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'\left[B\left(\bar{\Delta} + s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'\right]^{-1},$$

and, therefore,

$$h'\left[B\left(\bar{\Delta} + s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'\right] = A\left(\bar{\Delta} + s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'.$$

Rearranging, we get

$$h'B\bar{\Delta}B' + s\hat{w}h'B\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}B' = A\bar{\Delta}B' + s\hat{w}A\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}B',$$

and right-multiplying both sides by $\left(B\bar{\Delta}B'\right)^{-1}$ yields

$$\begin{aligned} h' &= A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + s\hat{w}\left(A-h'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1} \\ &= A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{z}\hat{w}A\mathcal{K}'\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1}. \end{aligned}$$

Next, notice that

$$\mathcal{K}'\Omega_\mu\mathcal{K} = \mathcal{K}'\Omega_\mu\Omega_\eta^{-1}\mathcal{K}\Omega,$$

and

$$\bar{\Delta} = \left(\mathcal{K}'(\mathrm{I}_u + \hat{w}\Omega_\mu\Omega_\eta^{-1})\mathcal{K} + (1-\alpha)^{-1}\left(\mathrm{I}_m - \mathcal{K}'\mathcal{K}\right)\right)\Omega,$$

so we have

$$\mathcal{K}'\Omega_\mu\mathcal{K} = \mathcal{K}'\left(\Omega_\mu(\Omega_\eta + \hat{w}\Omega_\mu)^{-1}\right)\mathcal{K}\bar{\Delta}.$$

Thus, it follows that

$$h' = A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{z}\hat{w}A\mathcal{K}'\left(\Omega_\mu(\Omega_\eta + \hat{w}\Omega_\mu)^{-1}\right)\mathcal{K}\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1},$$

with the scalar $\hat{z}$ given by

$$\hat{z} \equiv \left(\frac{\lambda\gamma}{1 - \lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)(A - h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A'.$$

Further, we can write

$$h' = A\left(\mathrm{I}_m + \hat{R}\right)\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1},$$

with

$$\begin{aligned}
\hat{R} &= \mathcal{K}'\left(\hat{z}\hat{w}\Omega_\mu(\Omega_\eta + \hat{w}\Omega_\mu)^{-1}\right)\mathcal{K} \\
&= \mathcal{K}'\left(\hat{z}\hat{w}(\mathrm{I}_{T+1}\otimes\Sigma_\mu)((\mathrm{I}_{T+1}\otimes\Sigma_\eta) + \hat{w}(\mathrm{I}_{T+1}\otimes\Sigma_\mu))^{-1}\right)\mathcal{K} \\
&= \mathcal{K}'\left(\mathrm{I}_{T+1}\otimes(\hat{z}\hat{w}\Sigma_\mu(\Sigma_\eta + \hat{w}\Sigma_\mu)^{-1})\right)\mathcal{K} \\
&= \mathcal{K}'\left(\mathrm{I}_{T+1}\otimes\mathrm{diag}\left(\hat{z}\hat{w}\sigma_{\mu,1}^2(\sigma_{\eta,1}^2 + \hat{w}\sigma_{\mu,1}^2)^{-1}, \ldots, \hat{w}\sigma_{\mu,Z}^2(\sigma_{\eta,Z}^2 + \hat{w}\sigma_{\mu,Z}^2)^{-1}\right)\right)\mathcal{K} \\
&= \mathcal{K}'\left(\mathrm{I}_{T+1}\otimes\mathrm{diag}\left(\hat{r}_1, \ldots, \hat{r}_Z\right)\right)\mathcal{K},
\end{aligned}$$

which concludes the proof. □

Parts 1 and 2 of Proposition B.4. Given the result in Lemma B.9, we are left with taking the limit as $T \to \infty$ of the truncated problem. In particular, we have that

$$\lim_{T\to\infty} A\left(\mathrm{I}_m + \hat{R}\right)\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} = p\left(L; w, \alpha, \{r_i\}_{i=1}^Z\right), \quad \lim_{T\to\infty} A\mathcal{K}'\Omega_\mu\mathcal{K}A' = \mathbb{V}(\xi_t^\mu),$$

$$\lim_{T\to\infty}\,(A - h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}(A' - Bh') = \mathbb{V}(\xi_t^\mu - K_t^\mu), \quad \lim_{T\to\infty}(A - h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A' = \mathbb{COV}\left(\xi_t^\mu - K_t^\mu, \xi_t^\mu\right),$$

$$\lim_{T\to\infty}\frac{(A - h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A'}{A\mathcal{K}'\Omega_\mu\mathcal{K}A'} = 1 - \mathcal{S}^\mu.$$

Let $w \equiv \lim_{T\to\infty}\hat{w}$, and $r_i \equiv \lim_{T\to\infty}\hat{r}_i$, for $i \in \{1, \ldots, Z\}$. Then, we can show that

$$\begin{aligned}
r_i &= \lim_{T\to\infty}\frac{\hat{w}\tau_{\mu,i}}{1 + \hat{w}\tau_{\mu,i}}\frac{\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}{1 - \lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\frac{(A - h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A'}{A\mathcal{K}'\Omega_\mu\mathcal{K}A'} \\
&= \frac{w\tau_{\mu,i}}{1 + w\tau_{\mu,i}}\frac{\lambda\gamma\mathbb{V}\left(\xi_t^\mu\right)}{1 - \lambda\gamma\mathbb{V}\left(\xi_t^\mu\right)}\left(1 - \mathcal{S}^\mu\right),
\end{aligned}$$

and

$$
\begin{aligned}
w &= \lim_{T\to\infty} \frac{1}{1-\lambda\left(1-\alpha\right)\left(A-h'B\right)\mathcal{K}'\left(\Omega_\mu + \frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\mathcal{K}\left(A'-B'h\right)} \\
&= \lim_{T\to\infty} \frac{1}{1-\lambda\left(1-\alpha\right)\left(\left(A-h'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}\left(A'-B'h\right) + \frac{\lambda\gamma\left(\left(A-h'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A'\right)\left(A\mathcal{K}'\Omega_\mu\mathcal{K}\left(A'-B'h\right)\right)}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)} \\
&= \frac{1}{1-\lambda(1-\alpha)\left(\mathbb{V}\left(\xi_t^\mu - K_t^\mu\right) + \frac{\lambda\gamma\mathbb{V}(\xi_t^\mu)^2(1-\mathcal{S}^\mu)^2}{1-\lambda\gamma\mathbb{V}(\xi_t^\mu)}\right)}.
\end{aligned}
$$

Part 3 of Proposition B.4. All the steps used in the proof of part 3 of Proposition 4 hold without change except for the last step. From those derivations we have that

$$
h_0 = \chi\lambda\left(A-h'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A'\left(\mathrm{I_u} + \frac{\lambda\gamma\mathrm{A}\mathcal{K}'\Omega_\mu\mathcal{K}\mathrm{A}'}{1-\lambda\gamma\mathrm{A}\mathcal{K}'\Omega_\mu\mathcal{K}\mathrm{A}'}\right)
$$

Taking the limit we get

$$
\mathcal{B} = \lim_{T\to\infty} h_0 = \chi\lambda\tau_\mu\mathbb{C}\mathbb{O}\mathbb{V}(\xi_t^\mu - K_t^\mu, \xi_t^\mu)\left(1 + \frac{\lambda\gamma\mathbb{V}(\xi_t^\mu)}{1-\lambda\gamma\mathbb{V}(\xi_t^\mu)}\right) = \frac{\chi\lambda\mathbb{V}\left(\xi_t^\mu\right)}{1-\lambda\gamma\mathbb{V}\left(\xi_t^\mu\right)}\left(1-\mathcal{S}^\mu\right),
$$

which completes the proof of part 3 of the proposition.

□

C Proofs of Other Results

Proof of Proposition 1. Following the same arguments used in the proof of Proposition 3, the optimal linear strategy, $g(x_i) \equiv s^* x_i + \mathcal{B}$, solves the following fixed point problem

$$s^* x_i + \mathcal{B} = \int_\mu \mathbb{E}^\mu [\xi | x_i] \, \hat{p}(\mu | x_i) \, \mathrm{d}\mu = \frac{\sigma_\xi^2}{\sigma_\xi^2 + \sigma_\epsilon^2} x_i + \frac{\sigma_\epsilon^2}{\sigma_\xi^2 + \sigma_\epsilon^2} \int_\mu \mu \, \hat{p}(\mu | x_i) \, \mathrm{d}\mu,$$

with

$$\begin{aligned} \hat{p}(\mu | x_i) &\propto \exp\left(\lambda \mathbb{E}^\mu \left[(s^* x_i + \mathcal{B} - \xi)^2 - \chi \xi\right]\right) p(x_i | \mu) \, p(\mu) \\ &\propto \exp\left(\lambda (1 - s^*)^2 \mu^2 + 2\lambda (s^* - 1) \mathcal{B} \mu - \chi \mu - \frac{(x_i - \mu)^2}{2\left(\sigma_\xi^2 + \sigma_\epsilon^2\right)} - \frac{1}{2\sigma_\mu^2} \mu^2\right). \end{aligned}$$

Mapping it into the kernel of a normal distribution yields

$$\mu \sim \mathcal{N}\left(\frac{\frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} x_i + 2\lambda (s^* - 1) \mathcal{B} - \lambda \chi}{\frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} - 2\lambda (1 - s^*)^2}, \frac{1}{\frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} - 2\lambda (1 - s^*)^2}\right),$$

which implies that

$$\int_\mu \mu \, \hat{p}(\mu | x_i) \, \mathrm{d}\mu = \frac{\frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} x_i + 2\lambda (s^* - 1) \mathcal{B} - \lambda \chi}{\frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} - 2\lambda (1 - s^*)^2}.$$

Matching coefficients leads to the following conditions

$$s^* = \frac{\sigma_\xi^2}{\sigma_\xi^2 + \sigma_\epsilon^2} + \frac{\sigma_\epsilon^2}{\sigma_\xi^2 + \sigma_\epsilon^2} \frac{\frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2}}{\frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} - 2\lambda (1 - s^*)^2},$$

and

$$\mathcal{B} = \frac{\sigma_\epsilon^2}{\sigma_\xi^2 + \sigma_\epsilon^2} \frac{2\lambda (s^* - 1) \mathcal{B} - \lambda \chi}{\frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} - 2\lambda (1 - s^*)^2}.$$

Solving for s^* and $\mathcal{B}$ leads to the expressions stated in the proposition. □

Proof of Corollary 1. Aggregating the individual best response in equation (14) leads to

$$K_t = (1 - \alpha) \overline{\mathcal{F}}_t^1 [\xi_t] + \alpha \overline{\mathcal{F}}_t^1 [K_t].$$

Iterating forward using the definitions of subjective higher-order expectations, it follows that

$$\begin{aligned}
K_t &= (1-\alpha)\,\overline{\mathcal{F}}_t^1\left[\xi_t\right] + \alpha\,(1-\alpha)\,\overline{\mathcal{F}}_t^2\left[\xi_t\right] + \alpha^2\overline{\mathcal{F}}_t^2\left[K_t\right] \\
&= \cdots \\
&= (1-\alpha)\sum_{j=0}^{N}\alpha^j\overline{\mathcal{F}}_t^{j+1}\left[\xi_t\right] + \alpha^{N+1}\overline{\mathcal{F}}_t^{N+1}\left[K_t\right] \\
&= \cdots \\
&= (1-\alpha)\sum_{j=0}^{\infty}\alpha^j\overline{\mathcal{F}}_t^{j+1}\left[\xi_t\right],
\end{aligned}$$

which completes the proof. □

Proof of Corollary 2. This result follows directly from the fact that $p\,(L;w,\alpha)$ permits a finite state representation. □

Proof of Proposition 5. Applying Proposition 4, we obtain

$$\mathcal{F}_i\left[\xi\right] = \varsigma x_i - (1-\varsigma)\,\lambda\chi\sigma_\mu^2, \quad \text{with} \quad \varsigma \equiv \frac{(1+w)\,\sigma_\xi^2}{(1+w)\,\sigma_\xi^2 + \sigma_\epsilon^2}.$$

Aggregating over i, it follows that

$$\overline{\mathcal{F}}\left[\xi\right] = \varsigma\xi - (1-\varsigma)\,\lambda\chi\sigma_\mu^2.$$

Applying the operator $\mathcal{F}_i$ to both sides and aggregating again yields,

$$\overline{\mathcal{F}}^2\left[\xi\right] = \varsigma^2\xi - (1-\varsigma)\,(1+\varsigma)\,\lambda\chi\sigma_\mu^2.$$

Iterating forward, it follows that

$$\overline{\mathcal{F}}^m\left[\xi\right] = \varsigma^m\xi - (1-\varsigma)\sum_{k=0}^{m-1}\varsigma^k\lambda\chi\sigma_\mu^2 = \kappa_m\xi + \beta_m,$$

with

$$\kappa_m \equiv \varsigma^m, \quad \text{and} \quad \beta_m \equiv -\,(1-\varsigma)\sum_{k=0}^{m-1}\varsigma^k\lambda\chi\sigma_\mu^2.$$

Therefore, we have that

$$\beta_m = \beta_{m-1} - (1-\varsigma)\,\kappa_{m-1}\lambda\chi\sigma_\mu^2 = \beta_{m-1} + (\kappa_m - \kappa_{m-1})\,\lambda\chi\sigma_\mu^2,$$

which completes the proof of Part 1. Moreover, combing equation (15) with the fact that $\overline{\mathcal{F}}^m\left[\xi\right] = \kappa_m\xi + \beta_m$ leads to

$$K = (1-\alpha)\sum_{m=0}^{\infty}\alpha^m\kappa_m\,\xi + (1-\alpha)\sum_{m=0}^{\infty}\alpha^m\beta_m,$$

which completes the proof of Part 3.

To establish Part 2 notice that, from Proposition 4, we have

$$w = \left[\frac{1}{\tau_\mu} - \lambda\left(1-\alpha\right)\mathbb{V}\left(\xi - K\right)\right]^{-1},$$

which, differentiating with respect to α, yields

$$\frac{\mathrm{d}w}{\mathrm{d}\alpha} = \lambda w^2 \left[\left(1-\alpha\right)\frac{\mathrm{d}\mathbb{V}\left(\xi - K\right)}{\mathrm{d}\alpha} - \mathbb{V}\left(\xi - K\right)\right].$$

Since

$$\mathbb{V}\left(\xi - K\right) = \left(\frac{\sigma_\epsilon^2}{\left(1+w\right)\left(1-\alpha\right)\sigma_\xi^2 + \sigma_\epsilon^2}\right)^2 \sigma_\xi^2,$$

it follows that

$$\frac{\mathrm{d}\mathbb{V}\left(\xi - K\right)}{\alpha} = 2\left(\frac{\sigma_\xi^2}{\left(1+w\right)\left(1-\alpha\right)\sigma_\xi^2 + \sigma_\epsilon^2}\right)\mathbb{V}\left(\xi - K\right)\left(w - \left(1-\alpha\right)\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right),$$

and, therefore,

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}\alpha} &= \lambda w^2 \mathbb{V}\left(\xi - K\right)\left[2\left(1-\alpha\right)\left(\frac{\sigma_\xi^2}{\left(1+w\right)\left(1-\alpha\right)\sigma_\xi^2 + \sigma_\epsilon^2}\right)\left(w - \left(1-\alpha\right)\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right) - 1\right] \\ &= \frac{\lambda w^2 \mathbb{V}\left(\xi - K\right)\left(\left(w-1\right)\left(1-\alpha\right)\sigma_\xi^2 - \sigma_\epsilon^2\right)}{\left[1 + w + 2\lambda\left(1-\alpha\right)w^2\mathbb{V}\left(\xi - K\right)\right]\left(1-\alpha\right)\sigma_\xi^2 + \sigma_\epsilon^2}.\end{aligned}$$

Then, since, in the limit as α increases to 1, we have that $w \to \tau_\mu$, and $\mathbb{V}(\xi - K) \to \mathbb{V}(\xi)$, it follows that

$$\lim_{\alpha \to 1^-} \frac{\mathrm{d}w}{\mathrm{d}\alpha} = -\lambda \tau_\mu^2 \mathbb{V}(\xi) < 0.$$

On the other hand, notice that

$$\operatorname{sgn}\left[\lim_{\alpha \to 0^+} \frac{\mathrm{d}w}{\mathrm{d}\alpha}\right] = \operatorname{sgn}\left[\left(w-1\right)\sigma_\xi^2 - \sigma_\epsilon^2\right],$$

so that, since $w \geq \tau_\mu$, we have that

$$\tau_\mu > \frac{\sigma_\xi^2 + \sigma_\epsilon^2}{\sigma_\xi^2} \quad \Rightarrow \quad \lim_{\alpha \to 0^+} \frac{\mathrm{d}w}{\mathrm{d}\alpha} > 0.$$

Hence, w is non-monotonic in α if τ_μ is large enough. □

Proof of Proposition 6. It follows from Proposition 4 that, when $\gamma = 0$,

$$\mathcal{B} = \chi\lambda\tau_\mu \mathbb{V}(\xi_t)(1 - \mathcal{S}).$$

Therefore, to prove that $|\mathcal{B}|$ is increasing in α, it is sufficient to prove that the sensitivity $\mathcal{S}$ is decreasing in α. Since, by definition

$$\mathcal{S} = \frac{\mathbb{COV}(K_t, \xi_t)}{\mathbb{V}(\xi_t)},$$

with $\mathbb{V}(\xi_t)$ independent of α, it is sufficient to show that

$$\frac{\mathrm{d}\mathbb{COV}(K_t, \xi_t)}{\mathrm{d}\alpha} < 0.$$

Following the notation of the truncated economy introduced in the proof of Proposition 4, we have that

$$\mathbb{COV}(K_t, \xi_t) = h' B \Lambda \Omega A',$$

with h denoting the optimal forecasting rule

$$h = A\bar{\Omega}B'\left(B\bar{\Omega}B'\right)^{-1}, \quad \text{with} \quad \bar{\Omega} = (1+w)\Lambda\Omega + (1-\alpha)^{-1}(\mathrm{I}_m - \Lambda)\Omega.$$

Since Ω is diagonal, we can rewrite h as

$$h = A\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}, \quad \text{with} \quad \hat{\Omega} = \Lambda\Omega + m_\alpha(\mathrm{I}_m - \Lambda)\Omega, \quad \textit{and} \quad m_\alpha \equiv [(1-\alpha)(1+w)]^{-1}.$$

It follows that

$$\begin{aligned}
\frac{\mathrm{d}\mathbb{COV}(K_t, \xi_t)}{\mathrm{d}\alpha} &= A\Omega\Lambda B' \frac{\mathrm{d}\left(B\hat{\Omega}B'\right)^{-1}}{\mathrm{d}\alpha} B\Lambda\Omega A' \\
&= -A\Omega\Lambda B'\left(B\hat{\Omega}B'\right)^{-1} B\frac{\mathrm{d}\hat{\Omega}}{\mathrm{d}\alpha}B'\left(B\hat{\Omega}B'\right)^{-1} B\Lambda\Omega A' \\
&= -\left(z'(\mathrm{I}_m - \Lambda)\Omega z\right) m_\alpha^2 \left[(1+w) - (1-\alpha)\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right],
\end{aligned}$$

where z is a column vector,

$$z \equiv B'\left(B\hat{\Omega}B'\right)^{-1} B\Lambda\Omega A'.$$

Since $(\mathrm{I}_m - \Lambda)\Omega$ is positive semi-definite, it follows that

$$\mathrm{sgn}\left[\frac{\mathrm{d}\mathbb{COV}(K_t, \xi_t)}{\mathrm{d}\alpha}\right] = -\mathrm{sgn}\left[(1+w) - (1-\alpha)\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right].$$

Further, notice that since $w \geq \tau_\mu$ and $\lim_{\alpha\to 1^-} w = \tau_\mu$, we have that the $\lim_{\alpha\to 1^-} \mathrm{d}w/\mathrm{d}\alpha$ is bounded and, therefore,

$$\lim_{\alpha\to 1^-}\frac{\mathrm{d}\mathbb{COV}\left(K_t,\xi_t\right)}{\mathrm{d}\alpha}<0.$$

Finally, for a contradiction, suppose there exists some $\alpha\in[0,1)$ such that $\mathrm{d}\mathbb{COV}\left(K_t,\xi_t\right)/\mathrm{d}\alpha>0$. It follows from the intermediate value theorem and the continuity of $\mathrm{d}\mathbb{COV}\left(K_t,\xi_t\right)/\mathrm{d}\alpha$ that there must exist some $\alpha_\dagger$ such that

$$\left.\frac{\mathrm{d}\mathbb{COV}\left(K_t,\xi_t\right)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger}=0\ \Rightarrow\ \left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger}=\frac{1+w_\dagger}{1-\alpha_\dagger}>0,$$

where $w_\dagger$ denotes w evaluated at $\alpha_\dagger$. With $\gamma=0$, Proposition 4 implies that

$$w=\left[\frac{1}{\tau_\mu}-\lambda\left(1-\alpha\right)\mathbb{V}\left(\xi_t-K_t\right)\right]^{-1},$$

and it follows that

$$\frac{\mathrm{d}w}{\mathrm{d}\alpha}=-\lambda w^2\left[\mathbb{V}\left(\xi_t-K_t\right)-\left(1-\alpha\right)\frac{\mathrm{d}\mathbb{V}\left(\xi_t-K_t\right)}{\mathrm{d}\alpha}\right].$$

Using the fact that, similarly to $\mathbb{COV}\left(\xi_t,K_t\right)$, $\mathbb{V}\left(\xi_t-K_t\right)$ depends on α only through m_α, we have that

$$\left.\frac{\mathrm{d}\mathbb{V}\left(\xi_t-K_t\right)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger}=\left.\frac{\mathrm{d}\mathbb{V}\left(\xi_t-K_t\right)}{\mathrm{d}m_\alpha}\frac{\mathrm{d}m_\alpha}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger}=\frac{\mathrm{d}\mathbb{V}\left(\xi_t-K_t\right)}{\mathrm{d}m_\alpha}m_\alpha^2\left.\left[\left(1+w\right)-\left(1-\alpha\right)\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right]\right|_{\alpha=\alpha_\dagger}=0,$$

and, therefore,

$$\left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger}=-\lambda w^2\mathbb{V}\left(\xi_t-K_t\right)<0,$$

which yields the desired contradiction. □

Proof of Lemma 2. We start by characterizing the zero-inflation steady state. From the budget constraint of household i, we have that

$$C_{i,g,t+1}=\frac{Y_g-C_{i,g,t}}{1+\pi_{t+1}}.$$

Substituting $C_{i,g,t+1}$ into the utility function $U\left(C_{i,g,t},C_{i,g,t+1}\right)$ yields

$$U\left(C_{i,g,t},\pi_{t+1}\right)=\frac{C_{i,g,t}^{1-\nu}}{1-\nu}+\beta\frac{\left(\frac{Y_g-C_{i,g,t}}{1+\pi_{t+1}}\right)^{1-\nu}}{1-\nu}.$$

The Euler equation in the zero-inflation steady state implies that

$$\bar{C}_g^{-\nu}-\beta\left(Y_g-\bar{C}_g\right)^{-\nu}=0. \tag{C.1}$$

Let $c_{i,g,t}$ be the log-deviation from the zero-inflation steady state, that is

$$c_{i,g,t} \equiv \log C_{i,g,t} - \log \bar{C}_g.$$

The quadratic approximation of $U(C_{i,g,t}, \pi_{t+1})$ around the zero-inflation steady state leads to

$$\begin{aligned}
U(C_{i,g,t}, \pi_{t+1}) \approx\ & \mathcal{Q}(\hat{c}_t, \pi_{t+1}) \\
\equiv\ & const - \bar{C}_g^{1-\nu}\left(\frac{Y - \bar{C}_g}{\bar{C}_g}\right)\pi_{t+1} + (1-\nu)\,\bar{C}_g^{1-\nu} c_{i,g,t}\pi_{t+1} \\
& + \frac{1}{2}(1-\nu)\,\bar{C}_g^{1-\nu}\left(\frac{Y_g - \bar{C}_g}{\bar{C}_g}\right)\pi_{t+1}^2 - \frac{1}{2}\nu\bar{C}_g^{1-\nu}\left[1 + \left(\frac{Y_g - \bar{C}_g}{\bar{C}_g}\right)^{-1}\right] c_{i,g,t}^2 \\
=\ & const - \bar{C}_g^{1-\nu}\left(\frac{Y_g - \bar{C}_g}{\bar{C}_g}\right)\pi_{t+1} + \frac{1}{2}(1-\nu)\,\bar{C}_g^{1-\nu}\left(\frac{Y_g - \bar{C}_g}{\bar{C}_g} + \frac{1-\nu}{\nu}\frac{Y_g - \bar{C}_g}{Y_g}\right)\pi_{t+1}^2 \\
& - \frac{1}{2}\nu\bar{C}_g^{1-\nu}\frac{Y_g}{Y_g - \bar{C}_g}\left(c_{i,g,t} - \frac{1-\nu}{\nu}\frac{Y_g - \bar{C}_g}{Y_g}\pi_{t+1}\right)^2.
\end{aligned}$$

Given subjective beliefs $\mathcal{F}_{i,g,t}[\cdot]$, the optimal consumption must be proportional to the households subjective expectation about inflation:

$$\begin{aligned}
c_{i,g,t} =& \frac{1-\nu}{\nu}\frac{Y_g - \bar{C}_g}{Y_g}\mathcal{F}_{i,g,t}[\pi_{t+1}] \\
=& \frac{\beta^{1/\nu}}{1+\beta^{1/\nu}}\mathcal{F}_{i,g,t}[\pi_{t+1}],
\end{aligned}$$

where the last equality directly follows equation (C.1).

In the smooth model of ambiguity, similarly to the proof of Proposition 3, it can be shown that

$$c_{i,g,t} = \frac{1-\nu}{\nu}\frac{Y_g - \bar{C}_g}{Y_g}\int_{\mu^t}\mathbb{E}^{\mu^t}[\pi_{t+1}|\mathcal{I}_{i,g,t}]\,\hat{p}\left(\mu^t|\mathcal{I}_{i,g,t}\right)\mathrm{d}\mu^t,$$

where the distorted posterior $\hat{p}(\mu^t|\mathcal{I}_{i,g,t})$ is such that

$$\hat{p}\left(\mu^t|\mathcal{I}_{i,g,t}\right) \propto \exp\left(-\lambda\mathbb{E}^{\mu^t}[\mathcal{Q}(\hat{c}_t, \pi_{t+1})]\right).$$

Let the subjective belief of the household be such that

$$\mathcal{F}_{i,g,t}[\cdot] \equiv \int_{\mu^t}\mathbb{E}^{\mu^t}[\cdot|\mathcal{I}_{i,g,t}]\,\hat{p}\left(\mu^t|\mathcal{I}_{i,g,t}\right)\mathrm{d}\mu^t,$$

then, it follows that

$$c_{i,g,t} = \frac{\beta^{1/\nu}}{1+\beta^{1/\nu}}\mathcal{F}_{i,g,t}[\pi_{t+1}],$$

which yields equation (28). Substituting $c_{i,g,t}$ into $\mathcal{Q}(\hat{c}_t, \pi_{t+1})$, leads to equation (29) with

$$\chi_g \equiv Y_g^{1-\nu} \frac{\beta^{1/\nu}}{\left(1+\beta^{1/\nu}\right)^{1-\nu}}$$

$$\gamma_g \equiv \frac{1}{2} Y_g^{1-\nu} \frac{(\nu-1)\,\beta^{1/\nu}}{\left(1+\beta^{1/\nu}\right)^{1-\nu}} \frac{1+\nu\beta^{1/\nu}}{\nu\left(1+\beta^{1/\nu}\right)}$$

$$\delta_g \equiv \frac{1}{2} Y_g^{1-\nu} \frac{(1-\nu)^2\,\beta^{1/\nu}}{\nu\left(1+\beta^{1/\nu}\right)^{2-\nu}}.$$

Notice that δ_g, χ_g, and γ_g are all proportional to $Y_g^{\nu-1}$. Moreover, when $\nu > 1$, they are all positive and decreasing in Y_g. □

The following lemma is used in the proof of the next propositions.

Lemma C.1 (Kalman filter for AR(1))**.** *Given a state equation*

$$\xi_t = \rho\xi_{t-1} + \nu_t, \quad \textit{with} \quad \nu_t \sim \mathcal{N}\left(0, \sigma_\nu^2\right),$$

and an observation equation

$$x_t = \xi_t + u_t, \quad \textit{with} \quad u_t \sim \mathcal{N}\left(0, \sigma_u^2\right),$$

the steady-state Kalman gain is given by

$$\kappa = \frac{1}{2\rho}\left(\rho - \frac{\sigma_u^2 + \sigma_\nu^2}{\rho\sigma_u^2} - \sqrt{\left(\rho - \frac{\sigma_u^2 + \sigma_\nu^2}{\rho\sigma_u^2}\right)^2 + 4\frac{\sigma_\nu^2}{\sigma_u^2}}\right),$$

and the updating rule for the Bayesian forecast follows

$$\mathbb{E}_t\left[\xi_{t+1}\right] = \rho\,(1-\kappa)\,\mathbb{E}_{t-1}\left[\xi_t\right] + \rho\kappa x_t.$$

Proof of Proposition 8. Cosider Lemma C.1 with $\xi_t = \pi_t$, $\sigma_\nu^2 = \sigma_\eta^2$, and $\sigma_u^2 = \sigma_\varepsilon^2$, and define $\omega \equiv \rho\,(1-\kappa)$. Since every agent i in every group g has the same information structure with signals given by

$$x_{i,g,t} = \pi_t + \varepsilon_{i,g,t}, \quad \text{with} \quad \varepsilon_{i,g,t} \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right),$$

it immediately follows from Lemma C.1 that

$$\mathbb{E}_{i,g,t}\left[\pi_{t+1}\right] = \omega\mathbb{E}_{i,g,t-1}\left[\pi_t\right] + (\rho-\omega)\,x_{i,g,t},$$

and

$$\omega = \frac{1}{2}\left(\rho + \frac{\sigma_\varepsilon^2 + \sigma_\eta^2}{\rho\sigma_\varepsilon^2} - \sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + \sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4}\right).$$

Integrating the updating rule for the forecast, we have that

$$\int \mathbb{E}_{i,g,t}\left[\pi_{t+1}\right] = \omega \int \mathbb{E}_{i,g,t-1}\left[\pi_t\right] + (\rho-\omega)\int x_{i,g,t}$$

and, therefore,

$$\overline{\mathbb{E}}_{g,t}\left[\pi_{t+1}\right] = \omega\overline{\mathbb{E}}_{g,t-1}\left[\pi_t\right] + (\rho-\omega)\,\pi_t,$$

which can be rewritten as

$$\overline{\mathbb{E}}_{g,t}\left[\pi_{t+1}\right] = \frac{\rho-\omega}{1-\omega L}\pi_t.$$

The average forecast error is, then, given by

$$\begin{aligned} \pi_{t+1} - \overline{\mathbb{E}}_{g,t}\left[\pi_{t+1}\right] &= \pi_{t+1} - \frac{\rho-\omega}{1-\omega L}\pi_t \\ &= \frac{\eta_{t+1}}{1-\rho L} - \frac{\rho-\omega}{1-\omega L}\frac{L\eta_{t+1}}{1-\rho L} \\ &= \frac{\eta_{t+1}}{1-\omega L}, \end{aligned}$$

which concludes the proof. □

Proof of Proposition 9. It follows from Proposition 4 that

$$\mathcal{F}_{i,g,t}\left[\pi_{t+1}\right] = (1+r_g)\,\mathbb{E}_{i,g,t}\left[\pi_{t+1}\right] + \mathcal{B}_g$$

where $\mathbb{E}_{i,g,t}\left[\pi_{t+1}\right]$ denotes the periot-t Bayesian forecast of π_{t+1} of agent i in group g given the $(w_g, 0)$-modified information structure (notice that here $\alpha = 0$). Thus, setting $\xi_t = \pi_t$, $\sigma_\nu^2 = (1+w)\,\sigma_\eta^2$, and $\sigma_u^2 = \sigma_\varepsilon^2$, it follows from Lemma C.1 that

$$\mathbb{E}_{i,g,t}\left[\pi_{t+1}\right] = \rho\,(1-\kappa_g)\,\mathbb{E}_{i,g,t-1}\left[\pi_t\right] + \rho\kappa_g x_{i,g,t},$$

with

$$\kappa_g = \frac{1}{2\rho}\left(\left(\rho - \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right) - \sqrt{\left(\rho - \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 + 4\frac{(1+w_g)\,\sigma_\eta^2}{\sigma_\varepsilon^2}}\right).$$

It follows that

$$(1+r_g)\,\mathbb{E}_{i,g,t}\left[\pi_{t+1}\right] + \mathcal{B}_g = \rho\,(1-\kappa_g)\left((1+r_g)\,\mathbb{E}_{i,g,t-1}\left[\pi_t\right] + \mathcal{B}_g\right) + (1+r_g)\,\rho\kappa_g x_{i,g,t} - \rho\,(1-\kappa_g)\,\mathcal{B}_g + \mathcal{B}_g$$

and, therefore,

$$\mathcal{F}_{i,g,t}\left[\pi_{t+1}\right] = \rho\,(1-\kappa_g)\,\mathcal{F}_{i,g,t-1}\left[\pi_t\right] + (1+r_g)\,\rho\kappa_g x_{i,g,t} + (1-\rho\,(1-\kappa_g))\,\mathcal{B}_g.$$

Defining $\vartheta_g \equiv \rho\,(1-\kappa_g)$, we obtain

$$\mathcal{F}_{i,g,t}\left[\pi_{t+1}\right] = \vartheta_g\mathcal{F}_{i,g,t-1}\left[\pi_t\right] + (1+r_g)\,(\rho-\vartheta_g)\,x_{i,g,t} + (1-\vartheta_g)\,\mathcal{B}_g,$$

with

$$\vartheta_g = \frac{1}{2}\left(\rho + \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2} - \sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4}\right).$$

Integrating the updating rule for the forecast, we have that

$$\int \mathcal{F}_{i,g,t}\left[\pi_{t+1}\right] = \vartheta_g \int \mathcal{F}_{i,g,t-1}\left[\pi_t\right] + (1+r_g)\,(\rho - \vartheta_g)\int x_{i,g,t} + (1-\vartheta_g)\,\mathcal{B}_g$$

and, therefore,

$$\overline{\mathcal{F}}_{g,t}\left[\pi_{t+1}\right] = \vartheta_g \overline{\mathcal{F}}_{g,t-1}\left[\pi_t\right] + (1+r_g)\,(\rho - \vartheta_g)\,\pi_t + (1-\vartheta_g)\,\mathcal{B}_g,$$

which can be rewritten as

$$\overline{\mathcal{F}}_{g,t}\left[\pi_{t+1}\right] = \frac{(1+r_g)\,(\rho-\vartheta_g)\,\pi_t}{1-\vartheta_g L} + \mathcal{B}_g.$$

The average forecast error is, then, given by

$$\begin{aligned}
\pi_{t+1} - \overline{\mathcal{F}}_{g,t}\left[\pi_{t+1}\right] &= \pi_{t+1} - \frac{(1+r_g)\,(\rho-\vartheta_g)\,\pi_t}{1-\vartheta_g L} - \mathcal{B}_g \\
&= \frac{(1+r_g)\,\eta_{t+1}}{1-\vartheta_g L} - \frac{r_g}{1-\rho L}\eta_{t+1} - \mathcal{B}_g.
\end{aligned}$$

The fact that $r_g > 0$, $w_g > 0$, and $\mathcal{B}_g > 0$ follows immediately from Proposition 4 together with the fact that $\delta_g > 0$, $\chi_g > 0$, and $\gamma_g > 0$ established in Lemma 2 and that, by assumption, $\lambda > 0$ and $\sigma_\mu^2 > 0$. Finally, to see that $\vartheta_g < \omega$ notice that, from the triangle inequality, we have that

$$\sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + \sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4} + \sqrt{\left(\frac{w_g\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2} < \sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4},$$

so that

$$\frac{w_g\sigma_\eta^2}{\rho\sigma_\varepsilon^2} - \sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4} < -\sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + \sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4}.$$

Adding $\rho + \frac{\sigma_\varepsilon^2 + \sigma_\eta^2}{\rho\sigma_\varepsilon^2}$ and dividing by 2 yields the result. □

Proof of Proposition 10. Under rational expectations, the optimal inflation forecast is such that

$$\mathcal{F}_i\left[\pi\right] = \mathbb{E}_i\left[(1-\alpha)\,\pi^* + \alpha\overline{\mathcal{F}}\left[\pi\right]\right].$$

It follows from the the equivalence result in Huo and Pedroni (2020), that the optimal forecast is given by

$$\mathcal{F}_i\left[\pi\right] = \frac{\sigma_\pi^2}{\sigma_\pi^2 + (1-\alpha)^{-1}\,\sigma_\epsilon^2}\,x_i.$$

Aggregating, we obtain

$$\overline{\mathcal{F}}[\pi] = \frac{\sigma_\pi^2}{\sigma_\pi^2 + (1-\alpha)^{-1}\sigma_\epsilon^2}\pi^*.$$

Plugging this into the time-invariant inflation policy rule (34) completes the proof. □

Proof of Proposition 11. To ease notation, let

$$k_i \equiv \mathcal{F}_i[\pi], \quad \text{and} \quad K = \overline{\mathcal{F}}[\pi].$$

Plugging (34) into the utility function of the agent results in

$$\begin{aligned} u(k_i, K, \pi^*) &= -(k_i - (1-\alpha)\pi^* - \alpha K)^2 - \chi((1-\alpha)\pi^* + \alpha K) \\ &= -\left[(1-\alpha)(k_i - \pi^*)^2 + \alpha(k_i - K)^2\right] - (1-\alpha)\chi\pi^* + \alpha(1-\alpha)(K-\pi^*)^2 - \alpha\chi K. \end{aligned}$$

This is an inefficient economy, so we use Proposition B.3 to characterize the optimal forecasts. Let

$$\begin{aligned} &\lambda_{\text{ineff.}} \equiv 2\lambda, \quad \alpha_{\text{ineff.}} \equiv \alpha, \quad \gamma_{\text{ineff.}} \equiv 0, \quad \chi_{\text{ineff.}} \equiv \frac{1}{2}(1-\alpha)\chi, \\ &\psi_{\text{ineff.}} \equiv -\alpha(1-\alpha), \quad \phi_{\text{ineff.}} \equiv 0, \quad \text{and} \quad \varphi_{\text{ineff.}} \equiv \frac{1}{2}\alpha\chi, \end{aligned}$$

where parameters with a subscript "ineff." correspond to the ones in the setup of Proposition B.3. It follows that

$$w = \frac{\tau_\mu}{1 - 2\lambda(1-\alpha)^2\tau_\mu \mathbb{V}(\pi - K)}, \quad \text{and} \quad r = 0,$$

where $\tau_\mu \equiv \sigma_\mu^2/\sigma_\pi^2$ is the normalized amount of ambiguity. Moreover, the bias is given by

$$\mathcal{B} = \lambda(1-\alpha)\chi\tau_\mu\mathbb{V}(\pi)(1-\mathcal{S}) + \lambda\alpha\chi\tau_\mu\left[\mathbb{V}(\pi)(1-\mathcal{S}) - \mathbb{V}(\pi - K)\right].$$

Using $\mathbb{V}(\pi) = \sigma_\pi^2$ and $\mathbb{V}(\pi - K) = (1-\mathcal{S})^2\sigma_\pi^2$, we obtain the desired expressions for sensitivity $\mathcal{S}$ and bias $\mathcal{B}$. Finally, the implied inflation policy directly follows from equation (34), which completes the proof. □

Proof of Proposition 12. Since the loss function is continuous in σ_μ^2, it is sufficient to show that

$$\left.\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\sigma_\mu^2}\right|_{\sigma_\mu^2=0} < 0.$$

Fist notice that

$$\mathcal{L} = \frac{\omega}{\alpha}\left[(1-\mathcal{R})^2\sigma_\pi^2 + \mathcal{C}^2\right] \;\Rightarrow\; \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\sigma_\mu^2} = \frac{2\omega}{\alpha}\left[-(1-\mathcal{R})\sigma_\pi^2\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\sigma_\mu^2} + \mathcal{C}\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\sigma_\mu^2}\right].$$

If $\sigma_\mu^2 = 0$, it is optimal to set $\mathcal{R} < 1$ and $\mathcal{C} = 0$, so that it is sufficient to show that

$$\left.\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\sigma_\mu^2}\right|_{\sigma_\mu^2=0} > 0,$$

or, equivalently,

$$\left.\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\sigma_\mu^2}\right|_{\sigma_\mu^2=0} > 0,$$

since $\mathcal{R} = 1 - \alpha + \alpha\mathcal{S}$. Further, notice that sensitivity $\mathcal{S}$ depends on σ_μ^2 only through w and is monotonically increasing in w, it is then sufficient to show that

$$\left.\frac{\mathrm{d}w}{\mathrm{d}\sigma_\mu^2}\right|_{\sigma_\mu^2=0} > 0.$$

This, in turn, follows from the fact that $w = 0$ if $\sigma_\mu^2 = 0$, and $w > 0$ for any $\sigma_\mu^2 > 0$. □

Proof of Proposition 13. The optimal inflation forecast must satisfy

$$\mathcal{F}_i[\pi] = (1-\alpha)\,\mathcal{F}_i[\pi^*] + \alpha\mathcal{F}_i\left[\overline{\mathcal{F}}[\pi]\right].$$

With heterogeneous priors, the belief system of agent i is such that

$$\mathcal{F}_i[\pi^*] = \mathbb{E}_i[\pi^*] = \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right) x_i, \quad \text{and}$$
$$\mathcal{F}_i[\mathcal{F}_j[\pi^*]] = \mathcal{F}_i\left[\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\,x_j + \frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\,\mathcal{B}\right] = \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^2 x_i + \left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B}.$$

It follows that

$$\mathcal{F}_i\left[\overline{\mathcal{F}}[\pi^*]\right] = \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^2 x_i + \left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B}, \quad \text{and}$$
$$\mathcal{F}_i\left[\mathcal{F}_j\left[\overline{\mathcal{F}}[\pi^*]\right]\right] = \mathcal{F}_i\left[\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^2 x_j + \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B} + \left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B}\right],$$

and, therefore,

$$\mathcal{F}_i\left[\overline{\mathcal{F}}^2[\pi^*]\right] = \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^3 x_i + \left(\sum_{s=0}^{1}\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^s\right)\left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B}.$$

Continuing to iterate forwards, we obtain that, for all $k \geq 1$,

$$\begin{aligned}\mathcal{F}_i\left[\overline{\mathcal{F}}^k\left[\pi^*\right]\right] &= \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^k\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)x_i + \left(\sum_{s=0}^{k-1}\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^s\right)\left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B} \\ &= \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^k\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)x_i + \left(1-\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^k\right)\mathcal{B}.\end{aligned}$$

Notice that the optimal forecast of agent i can be expressed as a weighted sum of higher-order beliefs,

$$\begin{aligned}\mathcal{F}_i\left[\pi\right] &= (1-\alpha)\,\mathcal{F}_i\left[\pi^*\right] + (1-\alpha)\sum_{k=1}^{\infty}\alpha^k\mathcal{F}_i\left[\overline{\mathcal{F}}^k\left[\pi^*\right]\right] \\ &= (1-\alpha)\left(\sum_{k=0}^{\infty}\alpha^k\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^k\right)\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)x_i + (1-\alpha)\sum_{k=1}^{\infty}\alpha^k\left(1-\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^k\right)\mathcal{B} \\ &= \mathcal{S}^{\mathrm{RE}}\,x_i + \alpha\left(1-\mathcal{S}^{\mathrm{RE}}\right)\mathcal{B},\end{aligned}$$

where $\mathcal{S}^{\mathrm{RE}} \equiv \frac{\sigma_\pi^2}{\sigma_\pi^2+(1-\alpha)^{-1}\sigma_\epsilon^2}$ denotes the sensitivity under rational expectations.

From the inflation policy in equation (34), it follows that

$$\mathcal{R} = 1-\alpha+\alpha\mathcal{S}^{\mathrm{RE}} = \mathcal{R}^{\mathrm{RE}}, \quad \text{and} \quad \mathcal{C} = \alpha\left(\alpha-\alpha\mathcal{S}^{\mathrm{RE}}\right)\mathcal{B} = \alpha\left(1-\mathcal{R}^{\mathrm{RE}}\right)\mathcal{B}.$$

Finally, the social loss function is given by

$$\mathcal{L} = \frac{\omega}{\alpha}\left[\left(1-\mathcal{R}\right)^2\sigma_\pi^2+\mathcal{C}^2\right],$$

which is increasing in $\mathcal{B}$ since $\mathcal{C} = \alpha\left(1-\mathcal{R}^{\mathrm{RE}}\right)\mathcal{B}$. □

D Uniqueness and Linearity of Optimal Strategies without Strategic Interactions

In this Appendix, we prove that in the absence of strategic interactions, the optimal strategy is unique and linear in signals. It is worth noting that the uniqueness of the optimal strategy only requires concavity of the utility function $u(\cdot)$ and the $\phi(\cdot)$ function (Lemma D.1). Linearity, on the other hand, requires $u(\cdot)$ to be quadratic, $\phi(\cdot)$ to be of CAAA form, and the information structure to be Gaussian (Lemma D.2).

We base our analysis on the truncated economy outlined in the proof of Proposition 4, while shutting down strategic interactions by suppressing the dependence of the utility function on the aggregate action K:

$$\max_{\{k(x_i)\}} \int_\mu \phi\left(\mathbb{E}^\mu\left[u\left(k(x_i),\theta\right)\right]\right) p(\mu)\mathrm{d}\mu.$$

Agent i must choose an ex-ante strategy $k(x_i)$, a function of their entire history of private information, x_i.

Lemma D.1. *Without strategic interactions, there is a unique optimal strategy $k_i = g\left(x_i\right)$.*

Proof. To simplify notation, denote

$$\mathcal{W}\left(f\right) = \int_\mu \phi\left(\mathbb{E}^\mu\left[u\left(f,\theta\right)\right]\right) p\left(\mu\right)\mathrm{d}\mu, \quad \text{and} \quad \bar{\mathcal{W}} = \max_f \mathcal{W}\left(f\right).$$

Suppose there are at least two strategies $g_1\left(x_i\right)$ and $g_2\left(x_i\right)$ with $g_1 \neq g_2$ both achieving the optimum, that is, $\mathcal{W}\left(g_1\right) = \mathcal{W}\left(g_2\right) = \bar{\mathcal{W}}$. Consider an alternative strategy $h = \frac{g_1+g_2}{2}$. It follows that

$$\begin{aligned}
\mathcal{W}\left(h\right) &> \int_\mu \phi\left(\mathbb{E}^\mu\left[\frac{1}{2}u\left(g_1,\theta\right) + \frac{1}{2}u\left(g_2,\theta\right)\right]\right) p\left(\mu\right)\mathrm{d}\mu \\
&= \int_\mu \phi\left(\frac{1}{2}\mathbb{E}^\mu\left[u\left(g_1,\theta\right)\right] + \frac{1}{2}\mathbb{E}^\mu\left[u\left(g_2,\theta\right)\right]\right) p\left(\mu\right)\mathrm{d}\mu \\
&> \int_\mu \left(\frac{1}{2}\phi\left(\mathbb{E}^\mu\left[u\left(g_1,\theta\right)\right]\right) + \frac{1}{2}\phi\left(\mathbb{E}^\mu\left[u\left(g_1,\theta\right)\right]\right)\right) p\left(\mu\right)\mathrm{d}\mu \\
&= \frac{1}{2}\mathcal{W}\left(g_1\right) + \frac{1}{2}\mathcal{W}\left(g_2\right) = \bar{\mathcal{W}},
\end{aligned}$$

where the first and second inequalities use the concavity of u and ϕ, respectively. The condition $\mathcal{W}\left(h\right) > \bar{\mathcal{W}}$ contradicts the assumption that g_1 and g_2 are both optimal strategies. As a result, it must be the case that there exists a unique optimal strategy g. □

Lemma D.2. *If $u\left(\cdot\right)$ is quadratic, $\phi\left(\cdot\right)$ takes the CAAA form, and the information structure is Gaussian, the optimal strategy is unique and linear in signals, i.e., there exist unique h' and h_0 such that*

$$k_i = g\left(x_i\right) = h'x_i + h_0.$$

Proof. Notice that the economy under consideration is a special case of our model in Section II in which there are no strategic interactions, i.e., $\alpha = 0$. Then, invoking Proposition 2, we know that a linear optimal strategy exists. Combining this with the uniqueness result of Lemma D.1 completes the proof. □

E Robust Preferences: Derivations and Proofs

Lemma E.1. *Taking the law of motion of K_t as given, individual i's best response satisfies*

$$k_{it} = (1-\alpha)\,\mathcal{F}_{it}[\xi_t] + \alpha \mathcal{F}_{it}[K_t],$$

where $\mathcal{F}_{it}[\cdot]$ denotes agent i's subjective expectation, such that for any random variable X,

$$\mathcal{F}_{it}[X] \equiv \int X\, \tilde{\mathrm{p}}_{it}(X)\mathrm{d}X, \quad \textit{with} \quad \tilde{\mathrm{p}}_{it}(X) \propto \exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right) \mathrm{p}(X \mid x_i^t).$$

Proof of Lemma E.1. The first-order-condition for the minimization with respect to m_{it} is given by

$$u(k_{it}, K_t, \xi_t) + \frac{1}{\varpi}\log m_{it} + \frac{1}{\varpi} = 0.$$

Together with the fact that $\mathbb{E}_{it}[m_{it}] = 1$, it follows that

$$m_{it} = \frac{\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)}{\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\right]}.$$

Thus, problem (35) can be rewritten as the following problem with risk sensitivity:

$$\max_{k_{it}} -\frac{1}{\varpi}\log\left(\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\right]\right).$$

The first-order-condition for this problem with respect to k_{it} is given by

$$\frac{\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\frac{\partial u(k_{it}, K_t, \xi_t)}{\partial k_{it}}\right]}{\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\right]} = 0.$$

Since

$$\frac{\partial u(k_{it}, K_t, \xi_t)}{\partial k_{it}} = k_{it} - (1-\alpha)\,\xi_t - \alpha K_t,$$

it follows that

$$k_{it} = (1-\alpha)\,\mathbb{E}_{it}\left[\xi_t \frac{\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)}{\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\right]}\right] + \alpha \mathbb{E}_{it}\left[K_t \frac{\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)}{\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\right]}\right].$$

Letting $\frac{\exp(-\varpi u(k_{it}, K_t, \xi_t))}{\mathbb{E}_{it}[\exp(-\varpi u(k_{it}, K_t, \xi_t))]}$ be the Radon-Nikodym derivative completes the proof. □

Proof of Proposition 14. Consider the same truncated version of the model used in the proof of Proposition 4. From Lemma E.1 we have that the optimal strategy then satisfies that

$$k_i = (1-\alpha)\,\mathcal{F}[\theta \mid x_i] + \alpha \mathcal{F}[K \mid x_i], \tag{E.1}$$

with the distorted posterior given by

$$\tilde{p}(\eta|x_i) \propto \exp\left(-\varpi u\left(k_i, K, \theta\right)\right) p\left(\eta \mid x_i\right).$$

We proceed with a guess-and-verify strategy. First we guess that

$$k_i = h' B\nu_i + h_0.$$

Substituting this into equation (E.1), it follows that

$$k_i = \left(\left(1-\alpha\right) A\mathcal{K}' + \alpha h' B\mathcal{K}'\right) \mathcal{F}\left[\eta \mid B\nu_i\right] + \alpha h_0.$$

Thus, we need to determine the subjective conditional expectation $\mathcal{F}\left[\eta \mid B\nu_i\right]$. We proceed to characterize the distorted posterior $\tilde{p}\left(\eta \mid B\nu_i\right)$ by the following three steps:

1. First, the Bayesian posterior $p\left(\eta \mid B\nu_i\right)$ is such that

$$p\left(\eta \mid B\nu_i\right) \propto \exp\left(-\frac{1}{2}\left(\eta - \mu_{\eta|B\nu_i}\right)' \Sigma^{-1}_{\eta|B\nu_i} \left(\eta - \mu_{\eta|B\nu_i}\right)\right),$$

   with the conditional mean and variance of given by

$$\mu_{\eta|B\nu_i} = \mathcal{K}\Omega B' \left(B\Omega B'\right)^{-1} B\nu_i, \quad \text{and} \quad \Sigma_{\eta|B\nu_i} = \mathcal{K}\Omega\mathcal{K}' - \mathcal{K}\Omega B' \left(B\Omega B'\right)^{-1} B\Omega\mathcal{K}'.$$

2. Second, notice that

$$\begin{aligned} u\left(k, K, \theta\right) = & -\frac{1}{2}\left[\left(1-\alpha\right)\left(h' B\nu_i + h_0 - A\mathcal{K}'\eta\right)^2 + \alpha\left(h' B\nu_i - h' B\mathcal{K}'\eta\right)^2\right] - \chi A\mathcal{K}'\eta - \frac{1}{2}\gamma\eta'\mathcal{K}A'A\mathcal{K}'\eta \\ & = \text{constant} - \frac{1}{2}\gamma\eta'\mathcal{K}A'A\mathcal{K}'\eta - \frac{1}{2}\left[\left(1-\alpha\right)\eta'\mathcal{K}A'A\mathcal{K}'\eta + \alpha\eta'\mathcal{K}B'hh'B\mathcal{K}'\eta\right] \\ & + \frac{1}{2}\left[\left(1-\alpha\right)\left(h_0 + \nu_i' B' h\right) A + \alpha\nu_i' B' hh' B - \chi A\right]\mathcal{K}'\eta \\ & + \eta'\mathcal{K}\frac{1}{2}\left[\left(1-\alpha\right) A'\left(h_0 + h' B\nu_i\right) + \alpha B' hh' B\nu_i - \chi A'\right], \end{aligned}$$

   with the constant independent of η.

3. Finally, putting these results together, the distorted posterior must be such that

$$\tilde{p}\left(\eta \mid B\nu_i\right) \propto \exp\left(-\frac{1}{2}\eta'\tilde{\Sigma}^{-1}_{\eta|B\nu_i}\eta + \frac{1}{2}\tilde{\mu}'_{\eta|B\nu_i}\tilde{\Sigma}^{-1}_{\eta|B\nu_i}\eta + \frac{1}{2}\eta'\tilde{\Sigma}^{-1}_{\eta|B\nu_i}\tilde{\mu}_{\eta|B\nu_i}\right)$$

   where the distorted posterior variance and mean are given by

$$\tilde{\Sigma}^{-1}_{\eta|B\nu_i} \equiv \Sigma^{-1}_{\eta|B\nu_i} + \mathrm{Q} \quad \text{and} \quad \tilde{\mu}_{\eta|B\nu_i} \equiv \tilde{\Sigma}_{\eta|B\nu_i}\left(\Sigma^{-1}_{\eta|B\nu_i}\mu_{\eta|B\nu_i} + \mathrm{R}B\nu_i\right) + \pi_\mu,$$

with the matrices Q and R and the vector π_μ given by

$$\mathrm{Q} \equiv -\varpi\gamma\mathcal{K}A'A\mathcal{K}' - \varpi\left[(1-\alpha)\mathcal{K}A'A\mathcal{K}' + \alpha\mathcal{K}B'hh'B\mathcal{K}'\right], \tag{E.2}$$

$$\mathrm{R} \equiv -\varpi\mathcal{K}\left[(1-\alpha)A' + \alpha B'h\right]h', \tag{E.3}$$

$$\pi_\mu \equiv -\varpi\left(\Sigma^{-1}_{\eta|B\nu_i} + \mathrm{Q}\right)^{-1}\mathcal{K}\left[(1-\alpha)A'h_0 - \chi A'\right]. \tag{E.4}$$

The distorted expectation under robust preferences can, then, be written as

$$\tilde{\mathbb{E}}\left[\eta \mid B\nu_i\right] = \tilde{\mu}_{\eta|B\nu_i} = \mathrm{M}B\nu_i + \pi_\mu,$$

with the matrix M given by

$$\mathrm{M} \equiv \left(\Sigma^{-1}_{\eta|B\nu_i} + \mathrm{Q}\right)^{-1}\left(\Sigma^{-1}_{\eta|B\nu_i}\mathcal{K}\Omega B'\left(B\Omega B'\right)^{-1} + \mathrm{R}\right). \tag{E.5}$$

Thus, we that that

$$k_i = \left((1-\alpha)A\mathcal{K}' + \alpha h'B\mathcal{K}'\right)\left(\mathrm{M}B\nu_i + \pi_\mu\right) + \alpha h_0.$$

and for the initial guess to be correct the following fixed-point conditions must be satisfied:

$$h' = \left[(1-\alpha)A + \alpha h'B\right]\mathcal{K}'\mathrm{M}, \tag{E.6}$$

$$h_0 = \left[(1-\alpha)A + \alpha h'B\right]\mathcal{K}'\pi_\mu + \alpha h_0. \tag{E.7}$$

In what follows, we first characterize the responsiveness to signals h that solves equation (E.6) and then characterize the bias h_0 that solves equation (E.7).

Characterization of the responsiveness, h. We start by rewriting the equation for the matrix M. Substituting h' from equation (E.6) into equation (E.3), we obtain

$$\mathrm{R} = -\varpi\mathcal{K}\left((1-\alpha)A' + \alpha B'h\right)\left((1-\alpha)A + \alpha h'B\right)\mathcal{K}'\mathrm{M}$$

Plugging this expression for R into the definition of M, equation (E.5), it follows that

$$\left(\Sigma^{-1}_{\eta|B\nu_i} + \mathrm{Q}\right)\mathrm{M} = \left(\Sigma^{-1}_{\eta|B\nu_i}\mathcal{K}\Omega B'\left(B\Omega B'\right)^{-1} - \varpi\mathcal{K}\left((1-\alpha)A' + \alpha B'h\right)\left((1-\alpha)A + \alpha h'B\right)\mathcal{K}'\mathrm{M}\right).$$

Solving for M we get

$$\mathrm{M} = \left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\mathcal{K}\Omega B'\left(B\Omega B'\right)^{-1},$$

where the I_u is the identity matrix of dimension u and the matrix $\tilde{\mathrm{Q}}$ is given by

$$\begin{aligned}\tilde{\mathrm{Q}} &\equiv \mathrm{Q} + \varpi\mathcal{K}\left((1-\alpha)A' + \alpha B'h\right)\left((1-\alpha)A + \alpha h'B\right)\mathcal{K}' \\ &= -\varpi\gamma\mathcal{K}A'A\mathcal{K}' - \varpi\alpha(1-\alpha)\mathcal{K}\left(B'h - A'\right)\left(h'B - A\right)\mathcal{K}'.\end{aligned} \tag{E.8}$$

To ease notation, we define matrices

$$Z_1 \equiv -\varpi\gamma\mathcal{K}A' - \varpi\alpha\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right), \quad \text{and} \quad Z_2 \equiv -\varpi\alpha\left(1-\alpha\right)\mathcal{K}\left(B'h - A'\right),$$

so that

$$\tilde{Q} = \mathrm{Z}_1 A\mathcal{K}' + \mathrm{Z}_2 h' B\mathcal{K}'.$$

The fixed-point condition (E.6) can, then, be rewritten as

$$h' = \left[\left(1-\alpha\right)A + \alpha h' B\right]\mathcal{K}'\left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\Omega_\eta\mathcal{K}B'\left(B\Omega B'\right)^{-1},$$

where we used the fact that $\mathcal{K}\Omega = \Omega_\eta\mathcal{K}$. Using the Woodbury matrix identity, we obtain

$$\left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\Omega_\eta = \Omega_\eta - \left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\Omega_\eta,$$

so, we can further rewrite the fixed-point condition as

$$\begin{aligned} h' = & \left[\left(1-\alpha\right)A + \alpha h' B\right]\mathcal{K}'\left(\Omega_\eta - \left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\Omega_\eta\right)\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ &= \left[\left(1-\alpha\right)A + \alpha h' B\right]\mathcal{K}'\left(\Omega_\eta - \left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\Sigma_{\eta|B\nu_i}\left(\mathrm{Z}_1 A\mathcal{K}' + \mathrm{Z}_2 h' B\mathcal{K}'\right)\Omega_\eta\right)\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ &= \left(1-\alpha+\varkappa_1\right)A\Lambda\Omega B' + \left(\alpha - \varkappa_2\right)h' B\Lambda\Omega B', \end{aligned}$$

where $\Lambda = \mathcal{K}'\mathcal{K}$ and the endogenous scalars $\varkappa_1$ and $\varkappa_2$ are given by

$$\begin{aligned} \varkappa_1 \equiv & \; -\left[\left(1-\alpha\right)A + \alpha h' B\right]\mathcal{K}'\left(I_u + \Sigma_{\eta|B\nu_i}\tilde{Q}\right)^{-1}\Sigma_{\eta|B\nu_i}Z_1, \\ \varkappa_2 \equiv & \; \left[\left(1-\alpha\right)A + \alpha h' B\right]\mathcal{K}'\left(I_u + \Sigma_{\eta|B\nu_i}\tilde{Q}\right)^{-1}\Sigma_{\eta|B\nu_i}Z_2. \end{aligned}$$

Solving for h' we obtain

$$h' = \frac{1-\alpha+\varkappa_1}{1-\alpha+\varkappa_2}A\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}, \tag{E.9}$$

where the transformed variance-covariance matrix $\hat{\Omega}$ is given by

$$\hat{\Omega} \equiv \frac{1-\alpha+\varkappa_2}{1-\alpha}\Lambda\Omega + \frac{1}{1-\alpha}\left(\mathrm{I}_m - \Lambda\right)\Omega, \tag{E.10}$$

with I_m denoting the identity matrix of dimension m.

In what follows, we provide expressions for the two endogenous scalars $(\varkappa_1, \varkappa_2)$ such that we can take the limit as $T \to \infty$ and obtain the formulas in Proposition 14. For this purpose, it is useful to define

$$\mathrm{X} \equiv \left[\left(1-\alpha\right)A + \alpha h' B\right]\mathcal{K}'\left(\Sigma^{-1}_{\eta|B\nu_i} + \tilde{\mathrm{Q}}\right)^{-1}.$$

Notice that $(\varkappa_1, \varkappa_2)$ can then be written as

$$\varkappa_1 = -\mathrm{X}Z_1 = \varpi\gamma\mathrm{X}\mathcal{K}A' + \varkappa_2, \quad \text{and} \quad \varkappa_2 = -\mathrm{X}Z_2 = \varpi\alpha\left(1-\alpha\right)\mathrm{X}\mathcal{K}\left(A' - B'h\right).$$

Therefore, it follows that

$$\begin{aligned}
\mathrm{X} =& \left[\left(1-\alpha\right)A + \alpha h'B\right]\mathcal{K}'\left(\Sigma_{\eta|B\nu_i} - \left(\Sigma_{\eta|B\nu_i}^{-1} + \tilde{\mathrm{Q}}\right)^{-1}\tilde{\mathrm{Q}}\Sigma_{\eta|B\nu_i}\right) \\
=& \left[\left(1-\alpha\right)A + \alpha h'B\right]\mathcal{K}'\Sigma_{\eta|B\nu_i} - \mathrm{X}\tilde{\mathrm{Q}}\Sigma_{\eta|B\nu_i} \\
=& \left[\left(1-\alpha\right)A + \alpha h'B\right]\mathcal{K}'\Sigma_{\eta|B\nu_i} - \mathrm{X}\left(\mathrm{Z}_1 A\mathcal{K}' + \mathrm{Z}_2 h'B\mathcal{K}'\right)\Sigma_{\eta|B\nu_i} \\
=& \left[\left(1-\alpha\right)A + \alpha h'B\right]\mathcal{K}'\Sigma_{\eta|B\nu_i} + \left(\varkappa_1 A\mathcal{K}' - \varkappa_2 h'B\mathcal{K}'\right)\Sigma_{\eta|B\nu_i} \\
=& \left(1-\alpha+\varkappa_1\right)A\mathcal{K}'\Sigma_{\eta|B\nu_i} + \left(\alpha - \varkappa_2\right)h'B\mathcal{K}'\Sigma_{\eta|B\nu_i}.
\end{aligned}$$

Thus, since $\varkappa_1 - \varkappa_2 = \varpi\gamma\mathrm{X}\mathcal{K}A'$, we have that,

$$\varkappa_1 - \varkappa_2 = \varpi\gamma\left(1-\alpha+\varkappa_1\right)A\mathcal{K}'\Sigma_{\eta|B\nu_i}\mathcal{K}A' + \varpi\gamma\left(\alpha-\varkappa_2\right)h'B\mathcal{K}'\Sigma_{\eta|B\nu_i}\mathcal{K}A'. \tag{E.11}$$

Next, notice that

$$\mathrm{X} = \mathrm{X}\Sigma_{\eta|B\nu_i}^{-1}\Sigma_{\eta|B\nu_i} = \mathrm{X}\Sigma_{\eta|B\nu_i}^{-1}\mathcal{K}\Omega\mathcal{K}' - \mathrm{X}\Sigma_{\eta|B\nu_i}^{-1}\mathcal{K}\Omega B'\left(B\Omega B'\right)^{-1}B\Omega\mathcal{K}' = \mathrm{X}\Sigma_{\eta|B\nu_i}^{-1}\mathcal{K}\Omega\mathcal{K}' - h'B\Omega\mathcal{K}',$$

where the second equality uses the definition of $\Sigma_{\eta|B\nu_i}$ and the last equality uses the fact that

$$h' = \mathrm{X}\Sigma_{\eta|B\nu_i}^{-1}\mathcal{K}\Omega B'\left(B\Omega B'\right)^{-1}.$$

Rearranging terms and right-multiplying $\left(\mathcal{K}\Omega\mathcal{K}'\right)^{-1}\mathcal{K}\Omega B'$ to both sides of the equation, we obtain

$$\mathrm{X}\Sigma_{\eta|B\nu_i}^{-1}\mathcal{K}\Omega B' = \mathrm{X}\left(\mathcal{K}\Omega\mathcal{K}'\right)^{-1}\mathcal{K}\Omega B' + h'B\Omega\mathcal{K}'\left(\mathcal{K}\Omega\mathcal{K}'\right)^{-1}\mathcal{K}\Omega B' = \mathrm{X}\mathcal{K}B' + h'B\Lambda\Omega B'.$$

Further, since $\mathrm{X}\Sigma_{\eta|B\nu_i}^{-1}\mathcal{K}\Omega B' = h'B\Omega B'$, it follows that

$$\mathrm{X}\mathcal{K}B'h = h'B\left(\mathrm{I}_m - \Lambda\right)\Omega B'h.$$

Hence, we have that

$$\begin{aligned}
\varkappa_2 &= \varpi\alpha\left(1-\alpha\right)\mathrm{X}\mathcal{K}A' - \varpi\alpha\left(1-\alpha\right)\mathrm{X}\mathcal{K}B'h && \text{(E.12)} \\
&= \frac{\alpha\left(1-\alpha\right)}{\gamma}\left(\varkappa_1 - \varkappa_2\right) - \varpi\alpha\left(1-\alpha\right)h'B\left(\mathrm{I}_m - \Lambda\right)\Omega B'h, && \text{(E.13)}
\end{aligned}$$

where we use the fact that $\varpi\gamma\mathrm{X}\mathcal{K}A' = \varkappa_1 - \varkappa_2$.

Given the above results, we are left with taking the limit as $T \to \infty$ of the truncated problem. In particular, we have that

$$\lim_{T\to\infty} A\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} = p\left(L;w,\alpha\right) \qquad \lim_{T\to\infty} A\mathcal{K}'\Sigma_{\eta|B\nu_i}\mathcal{K}A' = \mathbb{V}_{it}\left(\xi_t\right)$$

$$\lim_{T\to\infty} h'B\mathcal{K}'\Sigma_{\eta|B\nu_i}\mathcal{K}A' = \mathbb{COV}_{it}\left(K_t,\xi_t\right) \qquad \lim_{T\to\infty} h'B\left(\mathrm{I}_m-\Lambda\right)\Omega B'h = \mathbb{DISP}\left(k_{it}\right)$$

which, together with equations (E.9), (E.10), (E.11), and (E.13), completes the characterization of the responsiveness to signals.

Characterization of the bias, h_0. From the fixed-point condition (E.7) and the definition of π_μ in equation (E.4), it follows that

$$(1-\alpha)\,h_0 = \varpi\left[(1-\alpha)\,A+\alpha h'B\right]\mathcal{K}'\left(\Sigma^{-1}_{\eta|B\nu_i}+\mathrm{Q}\right)^{-1}\mathcal{K}\left[\chi A'-(1-\alpha)\,A'h_0\right],$$

which can be solved for h_0 implying

$$h_0 = \frac{\chi\varpi\mathrm{Y}}{(1-\alpha)\left(1+\varpi\mathrm{Y}\right)},$$

with Y given by

$$\mathrm{Y} \equiv \left[(1-\alpha)\,A+\alpha h'B\right]\mathcal{K}'\left(\Sigma^{-1}_{\eta|B\nu_i}+\mathrm{Q}\right)^{-1}\mathcal{K}A'.$$

Using the definition of $\tilde{\mathrm{Q}}$ in equation (E.8) and the Woodbury matrix identity, it follows that

$$\begin{aligned}
\left(\Sigma^{-1}_{\eta|B\nu_i}+\mathrm{Q}\right)^{-1} &= \left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}-\varpi\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)\left((1-\alpha)\,A+\alpha h'B\right)\mathcal{K}'\right)^{-1} \\
&= \left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}\right)^{-1}+ \\
&\quad \frac{\varpi\left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}\right)^{-1}\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)\left((1-\alpha)\,A+\alpha h'B\right)\mathcal{K}'\left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}\right)^{-1}}{1-\varpi\left((1-\alpha)\,A+\alpha h'B\right)\mathcal{K}'\left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}\right)^{-1}\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)} \\
&= \left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}\right)^{-1}+\frac{\varpi\mathrm{X}'\mathrm{X}}{1-\varpi\mathrm{X}\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Y} &= \mathrm{X}\mathcal{K}A'+\frac{\varpi\left[(1-\alpha)\,A+\alpha h'B\right]\mathcal{K}'\mathrm{X}'\mathrm{X}\mathcal{K}A'}{1-\varpi\mathrm{X}\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)} \\
&= \frac{\mathrm{X}\mathcal{K}A'}{1-\varpi\mathrm{X}\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)} \\
&= \frac{\frac{\varkappa_1-\varkappa_2}{\varpi\gamma}}{1-\left(\frac{1-\alpha}{\gamma}\right)\left(\varkappa_1-\varkappa_2\right)-\varpi\alpha h'B\left(\mathrm{I}_m-\Lambda\right)\Omega B'h},
\end{aligned}$$

where the last equality uses the fact that

$$\varkappa_1-\varkappa_2 = \varpi\gamma\mathrm{X}\mathcal{K}A', \quad \text{and} \quad \mathrm{X}\mathcal{K}B'h = h'B\left(\mathrm{I}_m-\Lambda\right)\Omega B'h.$$

Therefore, we have that

$$h_0 = \frac{\chi(\varkappa_1 - \varkappa_2)}{(1-\alpha)\left(\gamma + \alpha\left(\varkappa_1 - \varkappa_2\right) - \gamma\varpi\alpha h' B\left(\mathrm{I}_m - \Lambda\right)\Omega B' h\right)}.$$

Finally, taking the limit as $T \to \infty$ leads to

$$\mathcal{B} = \lim_{T\to\infty} h_0 = \frac{\chi(\varkappa_1 - \varkappa_2)}{(1-\alpha)\left(\gamma + \alpha\left(\varkappa_1 - \varkappa_2\right) - \gamma\varpi\alpha\mathbb{DISP}\left(k_{it}\right)\right)}.$$

□

Proof of Corollary 3. Observe that, by using (24), the expression of w under smooth model (23) can be transformed into

$$w = \left[\frac{1}{\tau_\mu} - \lambda\left(1-\alpha\right)\left(\mathbb{V}\left(\xi_t - K_t\right) + r\frac{1+w}{w}\left(1-\mathcal{S}\right)\mathbb{V}\left(\xi_t\right)\right)\right]^{-1} \tag{E.14}$$

Take any pair (w, r) and the associated sensitivity $\mathcal{S}$ that would arise from robust preferences. We may solve $\left(\lambda, \sigma_\mu^2\right)$ from (24) and (E.14). Note that the first condition $w \geq 0, r \geq 0, \mathcal{S} \leq 1$ ensures that Assumption 2 can be satisfied and the second condition$(1-\mathcal{S})\left(\frac{\gamma w}{(1+w)r} - \frac{(1-\alpha)(1+w)r}{w}\right) + \gamma > (1-\alpha)\frac{\mathbb{V}(\xi_t - K_t)}{\mathbb{V}(\xi_t)}$ ensures that the resulted $\tau_\mu > 0$.

□

F Value of Information

In this appendix, we demonstrate that the value of information increases with the amount of ambiguity. To start with, as a simplification, we restrict our attention to the situation where the idiosyncratic noises share a common variance σ_ϵ^2. Specifically, we investigate the sign of the following cross-derivative for agent i:

$$\mathrm{D} \equiv -\frac{\mathrm{d}^2 \mathrm{V}\left(\sigma_\epsilon^2; \bar{g}\left(x_{-i}^t\right)\right)}{\mathrm{d}\sigma_\epsilon^2 \mathrm{d}\tau_\mu},$$

where

$$\mathrm{V}\left(\sigma_\epsilon^2; \bar{g}\left(x_{-i}^t\right)\right) \equiv \phi^{-1}\left(\int_{\mu^t} \phi\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}, K_t, \xi_t\right)\right]\right) p\left(\mu^t\right) \mathrm{d}\mu^t\right),$$

and $\bar{g}\left(x_{-i}^t\right)$ denotes the strategies taken by all other agents. The derivative $-\mathrm{dV}\left(\sigma_\epsilon^2; \bar{g}\left(x_{-i}^t\right)\right)/\mathrm{d}\sigma_\epsilon^2$ captures the effect on the agent's objective function of an increase in signal precision, thereby quantifying the value of extra information. As a result, a positive sign of the cross-derivative D reflects that a higher level of ambiguity increases the value of information.

We allow D to depend on the strategies of the other agents $\bar{g}\left(x_{-i}^t\right)$. This approach focuses our analysis on the value of information from the perspective of agent i, without imposing a symmetric equilibrium a priori. As a result, this notion of the value of information is ready to be incorporated into a rational inattention framework with some information acquisition cost function. This way of measuring the value of information is also consistent with our framework of persistent learning, where all private information shares the same precision so that a marginal change in σ_ϵ^2 changes the precision of all private information. In a generic environment where the precision of different sources of private information can differ substantially, our notion of the value of information can be equivalently understood as increasing the precision of all private information by the same amount.

In what follows, through the lens of a set of lemmas, we demonstrate that $\mathrm{D} > 0$, i.e., the value of information increases with the amount of ambiguity. We begin with Lemma F.1, which analytically characterizes the value of information.

Lemma F.1. *If $\phi(\cdot)$ takes the CAAA form, i.e., $\phi(x) = -\frac{1}{\lambda}\exp(-\lambda x)$, the value of information equals the equilibrium cross-sectional dispersion of actions:*

$$-\frac{\mathrm{dV}(\sigma_\epsilon^2; \bar{g}(x_{-i}^t))}{\mathrm{d}\sigma_\epsilon^2} = \frac{1}{2\sigma_\epsilon^2}\,\mathbb{E}[(k_{it} - K_t)^2].$$

Proof. We start the proof with the truncated economy as in the proof of Proposition 4. As a result, the strategies of individual agent i and of the other agents are respectively given by

$$k_i = h' B \nu_i + h_0, \quad \text{and} \quad K = \bar{h}' B \Lambda \nu_i + \bar{h}_0.$$

When $\phi(\cdot)$ takes the CAAA form, the ex-ante value of agent i is such that

$$\mathrm{V}(\sigma_\epsilon^2;\bar{h},h_0) \equiv -\frac{1}{\lambda}\ln\left(\int_\mu \exp\left(-\lambda\mathbb{E}^\mu[u(k_i,K,\theta)]\right)p(\mu)\mathrm{d}\mu\right)$$
$$\text{s.t.} \quad k_i = h'B\nu_i + h_0 \quad \text{and} \quad K = \bar{h}'B\Lambda\nu_i + h_0.$$

Taking derivative with respect to σ_ϵ^2 leads to

$$\begin{aligned}\frac{\mathrm{dV}(\sigma_\epsilon^2;\bar{h},h_0)}{\mathrm{d}\sigma_\epsilon^2} =& \frac{\int_\mu \exp\left(-\lambda\mathbb{E}^\mu[u(k_i,K,\theta)]\right)\left(\frac{\partial\mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}\sigma_\epsilon^2} + \frac{\partial\mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial h_0}\frac{\mathrm{d}h_0}{\mathrm{d}\sigma_\epsilon^2} + \frac{\partial\mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial\sigma_\epsilon^2}\right)p(\mu)\mathrm{d}\mu}{\int_\mu \exp\left(-\lambda\mathbb{E}^\mu[u(k_i,K,\theta)]\right)p(\mu)\mathrm{d}\mu}\\ =& \int_\mu \frac{\partial\mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial\sigma_\epsilon^2}\hat{p}(\mu)\mathrm{d}\mu + \frac{\mathrm{d}h}{\mathrm{d}\sigma_\epsilon^2}\int_\mu \frac{\partial\mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial h}\hat{p}(\mu)\mathrm{d}\mu + \frac{\mathrm{d}h_0}{\mathrm{d}\sigma_\epsilon^2}\int_\mu \frac{\partial\mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial h_0}\hat{p}(\mu)\mathrm{d}\mu,\end{aligned}$$

where $\hat{p}(\mu)$ is the (ex-ante) distorted subjective belief given by

$$\hat{p}(\mu) \propto \exp\left(-\lambda\mathbb{E}^\mu[u(k_i,K,\theta)]\right)p(\mu).$$

Note that the first-order conditions that pin down the optimal sensitivity h and bias h_0 are such that

$$\int_\mu \frac{\partial\mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial h}\hat{p}(\mu)\mathrm{d}\mu = \int_\mu \frac{\partial\mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial h_0}\hat{p}(\mu)\mathrm{d}\mu = 0.$$

Denote $\mathcal{K}$ and $\mathcal{G}$ by

$$\mathcal{K} \equiv \left[\mathrm{I}_u, 0_{u,m-u}\right], \quad \text{and} \quad \mathcal{G} \equiv \left[0_{m-u,u}, \mathrm{I}_{m-u}\right].$$

It can then be shown that

$$\begin{aligned}\mathbb{E}^\mu[u(k_i,K,\theta)] =& -\frac{1}{2}(1-\alpha)\mathbb{E}^\mu[(h'B(\mathcal{K}'\eta+\mathcal{G}'\epsilon_i)+h_0-a'\eta)^2]\\ &-\frac{1}{2}\alpha\mathbb{E}^\mu[(h'B(\mathcal{K}'\eta+\mathcal{G}'\epsilon_i)+h_0-\bar{h}'B\mathcal{K}'\eta-\bar{h}_0)^2] - \mathbb{E}^\mu[\chi a'\eta + \frac{1}{2}\gamma a'\eta\eta' a]\\ =& -\frac{1}{2}h'B(I-\Lambda)B'h\,\sigma_\epsilon^2 + \mathbb{Z}(\mu,\sigma_\eta^2,h,h_0,\bar{h},\bar{h}_0),\end{aligned}$$

where $\Lambda = \mathcal{K}'\mathcal{K}$, and $\mathbb{Z}(\mu,\sigma_\eta^2,h,h_0,\bar{h},\bar{h}_0)$ are independent of σ_ϵ. Therefore, we have

$$-\frac{\mathrm{dV}(\sigma_\epsilon^2;\bar{h},h_0)}{\mathrm{d}\sigma_\epsilon^2} = -\int_\mu \frac{\partial\mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial\sigma_\epsilon^2}\hat{p}(\mu)\mathrm{d}\mu = \frac{1}{2}h'B(I-\Lambda)B'h = \frac{1}{2\sigma_\epsilon^2}h'B(I-\Lambda)\Omega B'h.$$

Taking the limit as $T\to\infty$ of the truncated problem yields

$$\lim_{T\to+\infty} -\frac{\mathrm{dV}(\sigma_\epsilon^2;\bar{h},h_0)}{\mathrm{d}\sigma_\epsilon^2} = -\frac{\mathrm{dV}(\sigma_\epsilon^2;\bar{g}(x_{-i}^t))}{\mathrm{d}\sigma_\epsilon^2}, \quad \text{and} \quad \lim_{T\to+\infty} h'B(I-\Lambda)\Omega B'h = \mathbb{E}[(k_{it}-K_t)^2].$$

Thus, the value of information equals the equilibrium cross-sectional dispersion of actions. □

Does higher ambiguity increase the value of information? Providing an answer to this question is equivalent to analyzing whether the cross-sectional dispersion of actions increases with the amount of ambiguity τ_μ. Our equivalence result suggests that τ_μ shapes cross-sectional dispersion by affecting the two endogenous scalars w and r. In what follows, we first characterize how w and r affect the cross-sectional dispersion of actions (Lemma F.2). Intuitively, increases in either w or r should increase the cross-sectional dispersion, given that both higher w and r contribute to more overreactions. Lemma F.2 confirms this intuition.

Lemma F.2. *The cross-sectional dispersion of actions is increasing in both w and r:*

$$\frac{\partial \mathbb{E}\left[(k_{it}-K_t)^2\right]}{\partial w} > 0, \quad \textit{and} \quad \frac{\partial \mathbb{E}\left[(k_{it}-K_t)^2\right]}{\partial r} > 0.$$

Proof. Again, we start the proof with the truncated economy, in which $h(w,r)'B(I-\Lambda)\Omega B'h(w,r)$ denotes cross-sectional dispersion. Further, denote $\hat{h}'(w)$ as the truncated version of $p(L;w,\alpha)$, namely the forecasting rule of the (w,α)-modified signal process in Section 3.3. Then, we have that

$$h'(w,r) = (1+r)\hat{h}'(w),$$

which implies that

$$h(w,r)'B(I-\Lambda)\Omega B'h(w,r) = (1+r)^2\hat{h}(w)'B(I-\Lambda)\Omega B'\hat{h}(w).$$

It is then straightforward to see that

$$\frac{\partial h(w,r)'B(I-\Lambda)\Omega B'h(w,r)}{\partial r} > 0.$$

In what follows, we proceed to prove that $\hat{h}(w)'B(I-\Lambda)\Omega B'\hat{h}(w)$ is increasing in w. Utilizing our equivalence results, it can be shown that

$$\begin{aligned}
\hat{h}'(w) =& A\left((1+w)\Lambda\Omega + (1-\alpha)^{-1}(I-\Lambda)\Omega\right)B'\left(B\left((1+w)\Lambda\Omega + (1-\alpha)^{-1}(I-\Lambda)\Omega\right)B'\right)^{-1} \\
=& A\left(\Lambda\Omega + (1+w)^{-1}(1-\alpha)^{-1}(I-\Lambda)\Omega\right)B'\left(B\left(\Lambda\Omega + (1+w)^{-1}(1-\alpha)^{-1}(I-\Lambda)\Omega\right)B'\right)^{-1} \\
=& A\Omega B'\left(B\left(\Lambda\Omega + (1+w)^{-1}(I-\Lambda)\Omega\right)B'\right)^{-1} \\
=& A\Omega B'\left(B\left(\Lambda\Omega_\alpha + (1+w)^{-1}(I-\Lambda)\Omega_\alpha\right)B'\right)^{-1},
\end{aligned}$$

where $\Omega_\alpha \equiv \Lambda\Omega + (1-\alpha)^{-1}(I-\Lambda)\Omega$. As a result, taking the derivative with respect to w leads to

$$\frac{\mathrm{d}\hat{h}'(w)}{\mathrm{d}w} = (1+w)^{-2}\,\hat{h}'(w)B(I-\Lambda)\Omega_\alpha B'\left(B(\Lambda\Omega_\alpha + (1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}.$$

Therefore, we have that

$$
\begin{aligned}
&(1+w)^2 \frac{\mathrm{d}\hat{h}(w)'B(I-\Lambda)\Omega B'\hat{h}(w)}{\mathrm{d}w} \\
&= \hat{h}'(w)B(I-\Lambda)\Omega_\alpha B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}B(I-\Lambda)\Omega B'\hat{h}(w) \\
&\quad + \hat{h}(w)'B(I-\Lambda)\Omega B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}B(I-\Lambda)\Omega_\alpha B'\hat{h}(w) \\
&= (1-\alpha)^{-1}\hat{h}'(w)B(I-\Lambda)\Omega B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}B(I-\Lambda)\Omega B'\hat{h}(w) \\
&\quad + (1-\alpha)^{-1}\hat{h}(w)'B(I-\Lambda)\Omega B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}B(I-\Lambda)\Omega B'\hat{h}(w) \\
&= 2(1-\alpha)^{-1}\varpi\Pi^{-1}\varpi',
\end{aligned}
$$

where $\varpi \equiv \hat{h}'(w)B(I-\Lambda)\Omega B'$ and $\Pi \equiv B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'$. Notice that the matrix Π^{-1} is symmetric and positive semi-definite, hence so is Π. We then conclude that

$$
\frac{\mathrm{d}\hat{h}(w)'B(I-\Lambda)\Omega B'\hat{h}(w)}{\mathrm{d}w} > 0 \quad \Leftrightarrow \quad \frac{\partial h(w,r)'B(I-\Lambda)\Omega B'h(w,r)}{\partial w} > 0.
$$

Finally, taking the limit as $T \to \infty$ of the truncated problem results in

$$
\frac{\mathrm{d}\mathbb{E}[(k_{it}-K_t)^2]}{\mathrm{d}r} = \lim_{T\to\infty} \frac{\partial h(w,r)'B(I-\Lambda)\Omega B'h(w,r)}{\partial r} > 0,
$$

and

$$
\frac{\mathrm{d}\mathbb{E}[(k_{it}-K_t)^2]}{\mathrm{d}w} = \lim_{T\to\infty} \frac{\partial h(w,r)'B(I-\Lambda)\Omega B'h(w,r)}{\partial w} > 0.
$$

□

In the last step, we analyze how changes in τ_μ affect w and r directly. To enjoy an analytical result, we abstract out r by setting $\gamma = 0$.

Lemma F.3. *The endogenous scaler w is increasing in τ_μ if $\gamma = 0$.*

Proof. When $\gamma = 0$, it can be shown that

$$
w = \frac{1}{\frac{1}{\tau_\mu} - \lambda(1-\alpha)(A-h'B)\Lambda\Omega(A-h'B)'}. \tag{F.1}
$$

Similar to the proof of Lemma F.2 and using the same notation, it can be shown that

$$
h = A\Omega B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1},
$$

which implies that

$$
\frac{\mathrm{d}h}{\mathrm{d}w} = (1+w)^{-2}h'B(I-\Lambda)\Omega_\alpha B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}.
$$

Therefore, we can show that

$$\begin{aligned}
&\frac{\mathrm{d}(A-h'B)\Lambda\Omega(A-h'B)'}{\mathrm{d}w} \\
=& -2(1+w)^{-1}(A-h'B)\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B'h \\
=& \, 2(1+w)^{-1}(h'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B'h-h'B(I-\Lambda)\hat{\Omega}B'h) \\
=& \, 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B'-B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B\Lambda\hat{\Omega}B'-B(I-\Lambda)\hat{\Omega}B')h \\
=& \, 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B'-B(I-\Lambda)\hat{\Omega}B')h \\
=& \, 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}-I)B(I-\Lambda)\hat{\Omega}B'h \\
=& -2(1+w)^{-1}(h'B(I-\Lambda)\hat{\Omega}B')\left(B\hat{\Omega}B'\right)^{-1}(B(I-\Lambda)\hat{\Omega}B'h)<0,
\end{aligned}$$

where we denote $\hat{\Omega}=\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha$. It can be further shown that

$$\begin{aligned}
\frac{\mathrm{d}(A-h'B)\Lambda\Omega(A-h'B)'}{\mathrm{d}w} &= 2(1+w)^{-1}(h'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B'h \\
&= 2(1+w)^{-1}(h'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B'h-h'B(I-\Lambda)\hat{\Omega}B'h) \\
&= 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B'-B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B\Lambda\hat{\Omega}B'-B(I-\Lambda)\hat{\Omega}B')h \\
&= 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B'-B(I-\Lambda)\hat{\Omega}B')h \\
&= 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}-I)B(I-\Lambda)\hat{\Omega}B'h \\
&= -2(1+w)^{-1}(h'B(I-\Lambda)\hat{\Omega}B')\left(B\hat{\Omega}B'\right)^{-1}(B(I-\Lambda)\hat{\Omega}B'h)<0.
\end{aligned}$$

Denote the right-hand side of equation (F.1) by $\mathrm{RHS}(\tau_\mu,w)$ and the left-hand side by $\mathrm{LHS}(w)$. It is then straightforward to demonstrate that

$$\frac{\mathrm{dLHS}(w)}{\mathrm{d}w}>0, \quad \frac{\partial\mathrm{RHS}(\tau_\mu,w)}{\partial w}<0, \quad \text{and} \quad \frac{\partial\mathrm{RHS}(\tau_\mu,w)}{\partial\tau_\mu}<0,$$

which jointly proves that

$$\frac{\mathrm{d}w}{\mathrm{d}\tau_\mu}>0.$$

□

Lemma F.1, Lemma F.3, and Lemma F.2 combined establish the desired result, that the value of information increases with the amount of ambiguity if $\gamma=0$:

$$\mathrm{D}>0 \quad \text{if} \quad \gamma=0.$$

In the general case where $\gamma > 0$, proving that $\mathrm{D} > 0$ turns out to be challenging. However, extensive numerical exercises suggest that the value of information continues to increase with the level of ambiguity in this more complex scenario. Intuitively, with $\gamma > 0$, there is an additional channel of overreaction, namely, the scalar $r > 0$, which leads to a higher utilization of information. It is the intricate interaction between w and r, however, that complicates the analytical analysis.

G Ambiguity about Variance

In this section, we explore the cases in which there is ambiguity about the variance of the fundamental and about the variance of the noise, respectively.

G.1 Ambiguity about the variance of the fundamental

We start with the case that agents perceive ambiguity about the variance of the fundamental. Specifically, we assume that the fundamental ξ follows a normal distribution with mean 0 and variance $\sigma^2_{\xi,*}$: $\xi \sim \mathcal{N}\left(0, \sigma^2_{\xi,*}\right)$. Agents exhibit ambiguity regarding the true variance of the fundamental, $\sigma^2_{\xi,*}$. We let Γ_ξ be the the range of possible values for the variance of the fundamental, σ^2_ξ. Analysts believe that $\sigma^2_\xi \in \Gamma_\xi$ and have some prior belief about Γ_ξ with density distribution given by $p\left(\sigma^2_\xi\right)$. To ensure that strategies based on Bayesian inference and ambiguity neutrality coincide, we impose the following assumption on the agents' prior belief:

Assumption 1. *The prior belief of the agent is such that*

$$\int_{\Gamma_\xi} \sigma^2_\xi \, p\left(\sigma^2_\xi\right) \mathrm{d}\sigma^2_\xi = \sigma^2_{\xi,*} \, .$$

Similar to the setup of ambiguity about the mean of the fundamental, each agent receives a private signal

$$x_i = \xi + \varepsilon_i, \quad \text{with} \quad \varepsilon_i \sim \mathcal{N}\left(0, \sigma^2_\epsilon\right).$$

Agents are ambiguity-averse and select a strategy $g(x_i)$ to minimize the following objective:

$$\mathcal{L}(g) = \phi^{-1}\left(\int_{\Gamma_\xi} \phi\left(\mathbb{E}^{\sigma^2_\xi}\left[(g(x_i) - \xi)^2 - \chi\xi\right]\right) p(\sigma^2_\xi) \mathrm{d}\sigma^2_\xi\right),$$

where $\phi\left(x\right) = \frac{1}{\lambda}\exp\left(\lambda x\right)$ takes the CAAA form with λ representing the degree of ambiguity aversion. Finally, we restrict our analysis to linear strategies such that

$$g\left(x_i\right) = s x_i + b, \tag{G.1}$$

which facilitates a direct comparison with our baseline setup, where ambiguity pertains to the mean of the fundamental.

The following proposition suggests that ambiguity has a more limited effect, leading to an optimal linear strategy that exhibits higher sensitivity compared to the rational RE benchmark, but no bias.

Proposition G.1. *When agents are ambiguity-averse, $\lambda > 0$, the optimal linear strategy exhibits higher sensitivity than the RE benchmark and features no bias:*

$$s^* > s^{RE} \equiv \frac{\sigma^2_{\xi,*}}{\sigma^2_{\xi,*} + \sigma^2_\epsilon}, \quad \textit{and} \quad b^* = 0.$$

Proof. Given the restriction to linear strategies, the objective function of the agents can be written as a function of the sensitivity, s, and bias, b, as follows

$$\mathcal{L}(s,b)=\frac{1}{\lambda}\ln\left(\int_{\Gamma_\xi}\exp\left(\lambda\left((s-1)^2\sigma_\xi^2+s^2\sigma_\epsilon^2\right)\right)p\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2\right)+\frac{1}{2}b^2.$$

The zero-bias result is straight-forward: the FOC with respect to bias b is such that

$$\frac{\partial\mathcal{L}(s,b)}{\partial b}=b=0.$$

To characterize the optimal of sensitivity, s, we consider the corresponding FOC,

$$\frac{\partial\mathcal{L}(s,b)}{\partial s}=\frac{\int_{\Gamma_\xi}\exp\left(\lambda\left((s-1)^2\sigma_\xi^2+s^2\sigma_\epsilon^2\right)\right)\left[(s-1)\sigma_\xi^2+s\sigma_\varepsilon^2\right]p\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2}{\int_{\Gamma_\xi}\exp\left(\lambda\left((s-1)^2\sigma_\xi^2+s^2\sigma_\epsilon^2\right)\right)p\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2}=0,$$

which is equivalent to

$$s\,\sigma_\varepsilon^2=(1-s)\int_{\Gamma_\xi}\sigma_\xi^2\,\hat{p}\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2,$$

where the distorted belief $\hat{p}(\sigma_\xi^2)$ is such that

$$\hat{p}(\tau_\xi)\propto\exp\left(\lambda(s-1)^2\sigma_\xi^2\right)p\left(\sigma_\xi^2\right).$$

Notice that, relative to the agents' prior $p(\sigma_\xi^2)$, the distorted belief $\hat{p}(\sigma_\xi^2)$ puts higher weights on the larger σ_ξ^2 in Γ_ξ: $\hat{p}(\sigma_\xi^2)$ first-order stochastically dominates $p(\sigma_\xi^2)$. It follows that

$$\int_{\Gamma_\xi}\sigma_\xi^2\,\hat{p}\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2\geq\int_{\Gamma_\xi}\sigma_\xi^2\,p\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2=\sigma_{\xi,*}^2\,,$$

and, therefore,

$$s^*=\frac{\int_{\Gamma_\xi}\sigma_\xi^2\,\hat{p}\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2}{\int_{\Gamma_\xi}\sigma_\xi^2\,\hat{p}\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2+\sigma_\epsilon^2}>\frac{\sigma_{\xi,*}^2}{\sigma_{\xi,*}^2+\sigma_\epsilon^2}=s^{\mathrm{RE}}.$$

□

G.2 Ambiguity about the variance of signal noise

We proceed to analyze the effect of ambiguity about the variance of the noise instead. Similar to the setup of Section G.1, we assume that the fundamental ξ follows a normal distribution with mean 0 and variance σ_ξ^2, $\xi\sim\mathcal{N}\left(0,\sigma_\xi^2\right)$. Moreover, each agent receives a private signal

$$x_i=\xi+\varepsilon_i,\quad\text{with}\quad\varepsilon_i\sim\mathcal{N}\left(0,\sigma_{\epsilon,*}^2\right).$$

Agents face ambiguity regarding the true variance of the noise, denoted as $\sigma^2_{\epsilon,*}$. We let Γ_ϵ represent the range of possible values for this variance. Agents maintain a belief that σ^2_ϵ lies within Γ_ϵ and hold a prior distribution over this range, represented by $p(\sigma^2_\epsilon)$. To ensure that strategies based on Bayesian inference and ambiguity neutrality coincide, we impose the following assumption on the agents' prior belief:

Assumption 2. *The prior belief of the agent is such that*

$$\int_{\Gamma_\epsilon} \sigma^2_\epsilon \, p(\sigma^2_\epsilon) \mathrm{d}\sigma^2_\epsilon = \sigma^2_{\epsilon,*} \, .$$

Agents are ambiguity averse and select a strategy $g(x_i)$ to minimize the following objective:

$$\mathcal{L}(g) = \phi^{-1}\left(\int_{\Gamma_\epsilon} \phi\Big(\mathbb{E}^{\sigma^2_\epsilon}[(g(x_i)-\xi)^2 - \chi\xi]\Big) p(\sigma^2_\epsilon)\mathrm{d}\sigma^2_\epsilon\right),$$

where $\phi(x) = \frac{1}{\lambda}\exp(\lambda x)$ takes the CAAA form with λ representing the degree of ambiguity aversion. Finally, we restrict our analysis to linear strategies as in equation (G.1).

The following proposition states that ambiguity has not only a more limited effect but an opposite one on sensitivity when ambiguity is on the variance of noise: the optimal linear strategy exhibits lower sensitivity compared to the rational RE benchmark, while featuring no bias.

Proposition G.2. *When agents are ambiguity averse, $\lambda > 0$, the optimal linear strategy exhibits higher sensitivity than the RE benchmark and features no bias:*

$$s^* < s^{RE} \equiv \frac{\sigma^2_\xi}{\sigma^2_\xi + \sigma^2_{\epsilon *}}, \quad \textit{and} \quad b^* = 0.$$

Proof. Given the restriction to linear strategies, the objective function of the agents can be written as a function of the sensitivity, s, and bias, b, as follows:

$$\mathcal{L}(s,b) = \frac{1}{\lambda}\ln\left(\int_{\Gamma_\epsilon} \exp\left(\lambda\left((s-1)^2\sigma^2_\xi + s^2\sigma^2_\epsilon\right)\right) p(\sigma^2_\epsilon)\mathrm{d}\sigma^2_\epsilon\right) + \frac{1}{2}b^2.$$

The zero-bias result is straightforward: the first-order condition with respect to bias b is such that

$$\frac{\partial \mathcal{L}(s,b)}{\partial b} = b = 0.$$

To characterize the optimal sensitivity, s, we consider the corresponding first-order condition,

$$\frac{\partial \mathcal{L}(s,b)}{\partial s} = \frac{\int_{\Gamma_\epsilon} \exp\Big(\lambda\Big((s-1)^2\sigma^2_\xi + s^2\sigma^2_\epsilon\Big)\Big)\Big[(s-1)\sigma^2_\xi + s\sigma^2_\varepsilon\Big] p(\sigma^2_\epsilon)\mathrm{d}\sigma^2_\epsilon}{\int_{\Gamma_\epsilon} \exp\Big(\lambda\Big((s-1)^2\sigma^2_\xi + s^2\sigma^2_\epsilon\Big)\Big) p(\sigma^2_\epsilon)\mathrm{d}\sigma^2_\epsilon} = 0,$$

which is equivalent to

$$s\int_{\Gamma_\epsilon} \sigma^2_\varepsilon \, \hat{p}(\sigma^2_\epsilon)\mathrm{d}\sigma^2_\epsilon = (1-s)\sigma^2_\xi,$$

where the distorted belief $\hat{p}(\sigma_\epsilon^2)$ is such that

$$\hat{p}(\tau_\epsilon) \propto \exp\left(\lambda s^2 \sigma_\epsilon^2\right) p(\sigma_\epsilon^2).$$

Notice that, relative to the agents' prior $p(\sigma_\epsilon^2)$, the distorted belief $\hat{p}(\sigma_\epsilon^2)$ assigns higher weights to larger σ_ϵ^2 in Γ_ϵ: $\hat{p}(\sigma_\epsilon^2)$ first-order stochastically dominates $p(\sigma_\epsilon^2)$. It follows that

$$\int_{\Gamma_\epsilon} \sigma_\epsilon^2 \, \hat{p}(\sigma_\epsilon^2) \mathrm{d}\sigma_\epsilon^2 \geq \int_{\Gamma_\epsilon} \sigma_\epsilon^2 \, p(\sigma_\epsilon^2) \mathrm{d}\sigma_\epsilon^2 = \sigma_{\epsilon,*}^2 \ ,$$

and, therefore,

$$s^* = \frac{\sigma_\xi^2}{\sigma_\xi^2 + \int_{\Gamma_\epsilon} \sigma_\epsilon^2 \, \hat{p}(\sigma_\epsilon^2) \mathrm{d}\sigma_\epsilon^2} < \frac{\sigma_\xi^2}{\sigma_\xi^2 + \sigma_{\epsilon,*}^2} = s^{\mathrm{RE}}.$$

□

H Evidence on Inflation Expectations by Income Group

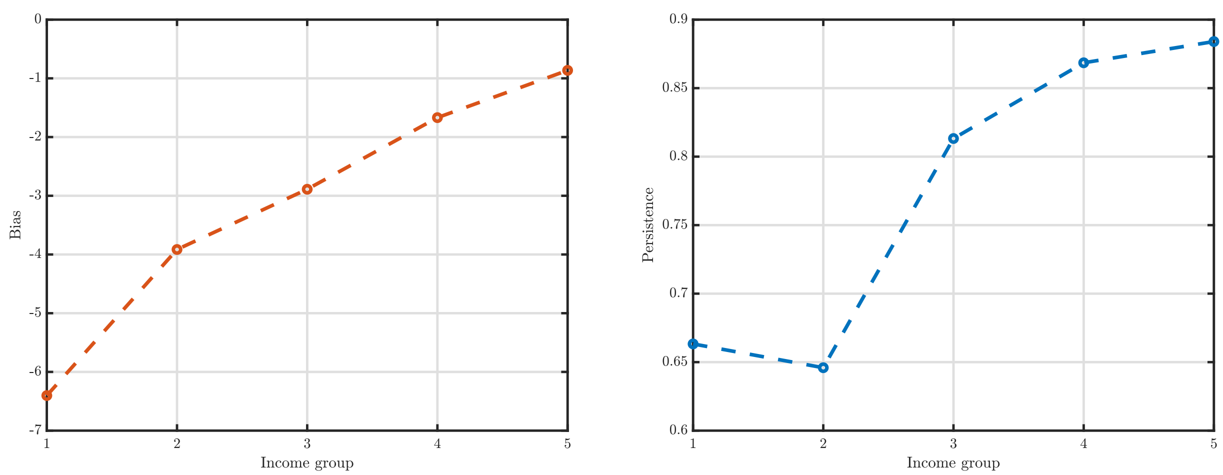
H.1 Forecast error bias and persistence

We investigate the joint behaviors of bias and persistence in forecast errors using both the Michigan Survey of Consumers (MSC) and the Survey of Consumer Expectations (SCE). We examine two regression equations:

$$\overline{\text{FE}}_{g,t} = \sum_{g=1}^{N} \beta_g \mathcal{I}_g + \omega_{g,t},$$
$$\overline{\text{FE}}_{g,t} = \sum_{g=1}^{N} \beta_g \mathcal{I}_g + \sum_{g=1}^{N} \alpha_g \overline{\text{FE}}_{g,t-1} + \omega_{g,t},$$

where $\overline{\text{FE}}_{g,t}$ represents the average forecast errors for group g at year-quarter t and $\mathcal{I}_g$ is the group dummy. For the MSC dataset, we divide individuals into $N = 7$ income groups, while for the SCE dataset, we divide individuals into $N = 5$ income groups. Table H.1 provides the results of our analysis. We use the poorest group (Group 1) as the reference group when reporting the results. The overall patterns of bias and persistence are similar in both the MSC and SCE datasets: as the income level increases, the amount of bias decreases, while the persistence of forecast errors increases. Similar to Figure 3 that displays the empirical patterns in MSC, Figure H.1 plot the point estimates of the biases and the persistence across different income groups in SCE.

FIGURE H.1: Bias and Persistence of Forecast Error in the Survey Data (NYSCE)



(A) Bias in forecast (B) Persistence of forecast error

Note: This figure reports bias (Panel A) and persistence (Panel B) of households' inflation forecasts in the cross-section of the income distribution. Bias and persistence of each income percentile are calculated by the mean and serial correlation of forecast errors of households' inflation expectations for the next 12 months. Data are obtained from FRBNY Survey of Consumer Expectations (2013:II-2022:I) and U.S. Consumer Price Index (2013:I-2022:IV).

Table H.1: Bias and Persistence of Forecast Errors: MSC and SCE

	MSC		SCE	
	Bias	Persistence	Bias	Persistence
Constant	-2.297^{***}	-1.055^{***}	-6.403^{***}	-2.187^{***}
	(0.072)	(0.081)	(0.164)	(0.204)
Group 2	0.235^{***}	0.366^{**}	2.488^{***}	0.760^{**}
	(0.060)	(0.085)	(0.110)	(0.155)
Group 3	0.766^{***}	0.564^{***}	3.514^{***}	1.650^{***}
	(0.053)	(0.053)	(0.142)	(0.150)
Group 4	1.103^{***}	0.713^{***}	4.734^{***}	1.980^{***}
	(0.057)	(0.032)	(0.142)	(0.092)
Group 5	1.258^{***}	0.810^{***}	5.539^{***}	2.105^{***}
	(0.054)	(0.025)	(0.163)	(0.143)
Group 6	1.535^{***}	0.876^{***}		
	(0.051)	(0.030)		
Group 7	1.924^{***}	0.959^{***}		
	(0.044)	(0.055)		
FE_{t-1}		0.537^{***}		0.663^{***}
		(0.044)		(0.031)
$FE_{t-1}\times$ Group 2		0.125^{**}		-0.017
		(0.034)		(0.069)
$FE_{t-1}\times$ Group 3		0.142^{**}		0.150
		(0.038)		(0.084)
$FE_{t-1}\times$ Group 4		0.171^{***}		0.205^{**}
		(0.023)		(0.049)
$FE_{t-1}\times$ Group 5		0.217^{***}		0.221^{**}
		(0.040)		(0.063)
$FE_{t-1}\times$ Group 6		0.218^{***}		
		(0.041)		
$FE_{t-1}\times$ Group 7		0.192^{***}		
		(0.042)		
Obs.	952	945	180	175

* p<0.1, ** p<0.05, *** p<0.01.

To address the concern that bias may be influenced by other observed individual characteristics, such as age and resident state, we introduce the following empirical specification at the individual level for both the MSC and the SCE:

$$\mathrm{FE}_{i,t} = \sum_{g=1}^{N} \beta_g \mathcal{I}_{i,g,t} + \gamma' \mathrm{X}_{i,t} + \delta_t + \omega_{i,t},$$

where $\mathcal{I}_{i,g,t}$ is a dummy variable that equals to 1 if individual i belongs to income group g at year-month t, and $\mathrm{X}_{i,t}$ is a vector of observed individual characteristics. For the MSC dataset, we control for age, gender, education, birth cohort, marital status, region, and the number of kids and adults in the household. It is worth noting that controlling for the birth cohort helps address concerns regarding the impact of inflation experiences on households' inflation expectations (Malmendier and Nagel, 2016). For the SCE dataset, we control for age group, numeracy, education, and region. Table H.2 reports the results. Again, we use the poorest group (Group 1) as the base group for both the MSC and SCE datasets. Even after controlling for additional individual characteristics, the biases in forecasts persist and exhibit a negative correlation with households' income levels.

Table H.2: Bias of Forecast Errors Controlling Individual Characteristics: MSC and SCE

	MSC	SCE
Constant	-2.370^{***}	-5.255^{***}
	(0.288)	(0.246)
Group 2	0.162^{***}	1.834^{***}
	(0.036)	(0.108)
Group 3	0.573^{***}	2.446^{***}
	(0.030)	(0.126)
Group 4	0.856^{***}	3.212^{***}
	(0.032)	(0.183)
Group 5	0.989^{***}	3.754^{***}
	(0.034)	(0.245)
Group 6	1.223^{***}	
	(0.024)	
Group 7	1.510^{***}	
	(0.031)	
Demographics	Yes	Yes
Birth Cohort	Yes	No
Age	Yes	Yes
Region	Yes	Yes
Time fixed effects	Yes	Yes
Obs.	146,622	135,434

* p<0.1, ** p<0.05, *** p<0.01.

H.2 CG and BGMS regressions

As a comparison to the group-specific CG and BGMS coefficients derived from our model, we run the corresponding CG and BGMS regressions using data from the Michigan Survey of Consumers and the Survey of Consumer Expectations. The term structure of the forecasts is not available in these datasets, preventing us from constructing exact forecast revisions. As a compromise, we consider the following closely related regressions instead:

$$\text{CG:} \qquad \pi_{t+1} - \bar{\mathbb{E}}_t\left[\pi_{t+1}\right] = \alpha + \beta_{\text{CG}}\left(\bar{\mathbb{E}}_t\left[\pi_{t+1}\right] - \bar{\mathbb{E}}_{t-1}\left[\pi_t\right]\right) + \epsilon_{t+1}, \tag{H.1}$$

$$\text{BGMS:} \qquad \pi_{t+1} - \mathbb{E}_{it}\left[\pi_{t+1}\right] = \alpha + \beta_{\text{BGMS}}\left(\mathbb{E}_{it}\left[\pi_{t+1}\right] - \mathbb{E}_{it-1}\left[\pi_t\right]\right) + \epsilon_{it+1}. \tag{H.2}$$

Columns (1)-(2) in Table H.3 display the results for the MSC, and columns (5)-(6) display the results for the SCE. At the individual level, the BGMS regression coefficients are more negative for poorer households, while the CG regression coefficients are larger for richer households. These results are broadly consistent with our model's predictions.

TABLE H.3: CG and BGMS Estimates: MSC and SCE

	MSC				SCE			
	(1) BGMS	(2) CG	(3) CG (IV)	(4) F-Stat	(5) BGMS	(6) CG	(7) CG (IV)	(8) F-Stat
Group 1	-0.546^{***}	-0.411^{***}	0.600^{*}	14.08	-0.510^{***}	-0.372^{***}	-0.455	6.50
	(0.048)	(0.101)	(0.349)		(0.184)	(0.127)	(0.760)	
Group 2	-0.435^{***}	-0.314^{**}	2.033^{**}	5.13	-0.440^{***}	-0.289^{**}	0.624	2.97
	(0.040)	(0.145)	(0.920)		(0.014)	(0.136)	(0.786)	
Group 3	-0.395^{***}	-0.207	1.080^{**}	9.04	-0.422^{***}	-0.295^{*}	2.777	0.62
	(0.025)	(0.266)	(0.489)		(0.015)	(0.175)	(3.588)	
Group 4	-0.393^{***}	-0.169	0.493^{*}	24.87	-0.408^{***}	0.202	2.590	4.90
	(0.031)	(0.230)	(0.285)		(0.019)	(0.333)	(2.334)	
Group 5	-0.375^{***}	-0.147	0.984^{**}	10.67	-0.384^{***}	0.281	2.888^{*}	15.24
	(0.028)	(0.260)	(0.383)		(0.035)	(0.331)	(1.628)	
Group 6	-0.394^{***}	0.054	0.797^{**}	17.56				
	(0.018)	(0.370)	(0.340)					
Group 7	-0.418^{***}	0.011	0.982^{**}	9.85				
	(0.018)	(0.301)	(0.483)					

* p<0.1, ** p<0.05, *** p<0.01.

However, due to the previously mentioned data limitations, the approximating regressions (H.1) and (H.2) may suffer from an endogeneity issue.[4] We follow Coibion and Gorodnichenko (2015) and use Spot Crude

[4]The error term ϵ_{t+1} in the CG specification above contains not only the rational expectations forecast errors $\hat{\epsilon}_{t+1}$ but also the expected change in inflation $\beta_{\text{CG}}\left(\bar{\mathbb{E}}_{t-1}\left[\pi_{t+1}\right] - \bar{\mathbb{E}}_{t-1}\left[\pi_t\right]\right)$. Under rational expectations, $\hat{\epsilon}_{t+1}$ is uncorrelated

Oil Price (1987-2022) as the instrumental variable. Unfortunately, while the instrumental variable is strong enough for the entire sample, it tends to be weak when segmenting the sample by different income groups (see the F-statistics in Columns (4) and (8)). Bearing in mind the weak IV issue, the CG coefficient generally increases with income, a trend that is more pronounced in the SCE data.

H.3 Balance-sheet effects

This section addresses the concern that balance-sheet effects may overturn the effects of inflation on labor income. We argue that this is unlikely to be the case.

First, notice that balance-sheet effects are primarily relevant for capital income, which constitutes a relatively small share of total income, especially for the income-poor. In Table H.4, we document the shares of different sources of income using the Survey of Consumer Finances, since this data is not available in the Michigan Survey.[5] For all households, capital and business income represent a relatively small share of total income, and this is especially true for the bottom four quintiles of income. The table also shows that the bottom quintiles of income have relatively low levels of net worth. With this in mind, one would expect that even the large proportional effects documented by Doepke and Schneider (2006) would be dominated by the effects of inflation on labor and transfer incomes.

TABLE H.4: Income Sources (%) by Quintiles of Income

	Quintiles of Income				
	1st	2nd	3rd	4th	5th
Labor	48.9	77.3	83.4	85.8	64.3
Capital	0.1	0.4	0.3	0.8	10.8
Business	6.2	5.4	5.9	5.6	18.7
Transfer	37.3	15.0	9.2	7.1	2.4
Other	7.5	1.8	1.2	0.7	3.7
Total Income	2.7	6.5	11.0	16.9	63.0
Net Worth	1.4	2.7	5.5	9.8	80.6

Notes: Calculated using data from the Survey of Consumer Finances (2016). We use the definitions from Kuhn and Ríos-Rull (2016) and limit the sample to heads of households aged 18 to 65, for comparability with the results in the paper. We also choose the 2016 wave of the survey as it is roughly in the middle of the time sample we use in the paper.

with the consensus forecast error $\pi_{t+1} - \bar{\mathbb{E}}_t[\pi_{t+1}]$. However, the covariance between the expected change in inflation $\beta_{\text{CG}}\left(\bar{\mathbb{E}}_{t-1}[\pi_{t+1}] - \bar{\mathbb{E}}_{t-1}[\pi_t]\right)$ and the consensus forecast error $\pi_{t+1} - \bar{\mathbb{E}}_t[\pi_{t+1}]$ is correlated as long as the inflation process is not a random walk. Therefore, the error term ϵ_{t+1} will be correlated with the forecast error on the left-hand side. Note that the reason for this endogeneity issue arises from the fact that neither the MSC nor the SCE provides the term structure of forecasts. As a result, forecasts are imperfectly overlapped.

[5] The seven groups from the MSC sample have average incomes, in thousands of 2016 dollars, of {12.9, 24.5, 40.5, 59.8, 74.5, 104.9, 216.8}, while the quintiles of income from the SCF show averages of {13.9, 33.0, 56.4, 90.3, 331.2}. Although the top income levels from the SCF are higher, reflecting its detailed approach to top-coding issues, the bottom four quintiles align relatively well with the MSC groups. The net worth levels for these income quintiles in the SCF, again in thousands of 2016 dollars, are {39.2, 75.0, 156.2, 287.0, 2322.3}.

Table H.5: Concern About Inflation (%) by Income Levels

	Income Levels (thousands of dollars)							
	<25	25–35	35–50	50–75	75–100	100–150	150–200	>200
Very concerned	66.5	62.2	60.9	57.3	53.9	46.1	38.7	24.0
Somewhat concerned	17.9	20.6	21.7	23.0	21.9	26.2	27.3	28.2
A little concerned	10.4	12.6	13.1	14.4	16.7	19.1	22.2	29.5
Not at all concerned	5.2	4.5	4.2	5.3	7.5	8.6	11.8	18.3

Notes: Calculated using data from Household Pulse Survey (2024). This survey started in 2020, so we selected the most recent wave to try to mitigate the impact of Covid-related concerns. Similar results are reported, using data from 2021, by Jayashankar and Murphy (2023).

One way to assess the overall effect of inflation on different households is to estimate a quantitative structural model incorporating the relevant mechanisms and heterogeneity, and then compute the conditional welfare effects for different groups. This approach is pursued by Cao, Meh, Ríos-Rull, and Terajima (2021). They find that poorer households are more negatively affected by inflation:

> *"An increase in inflation from 2% to 5% costs 13% of one-year consumption. [...] From the point of view of consumption class, the poor lose a lot more than the rich: 37.0% of 2010 consumption versus 5.6% for the poorest and richest quintiles."*

The same conclusions can be drawn from the Census Bureau's Household Pulse Survey data, which includes the question, "In the area where you live and shop, how concerned are you, if at all, that prices will increase in the next six months?" In Table H.5, we present the results categorized by income brackets. A clear pattern can be observed, with the inflation concern monotonically decreasing as income levels increase.

References

Angeletos, G.-M., and A. Pavan (2007): "Efficient Use of Information and Social Value of Information," *Econometrica*, 75(4), 1103–1142.

Cao, S., C. A. Meh, J.-V. Ríos-Rull, and Y. Terajima (2021): "The welfare cost of inflation revisited: The role of financial innovation and household heterogeneity," *Journal of Monetary Economics*, 118, 366–380.

Coibion, O., and Y. Gorodnichenko (2015): "Information Rigidity and the Expectations Formation Process: A Simple Framework and New Facts," *American Economic Review*, 105(8), 2644–78.

Doepke, M., and M. Schneider (2006): "Inflation and the redistribution of nominal wealth," *Journal of Political Economy*, 114(6), 1069–1097.

FRBNY Survey of Consumer Expectations (2013:II-2022:I): "Micro data of FRBNY Survey of Consumer Expectations," https://www.newyorkfed.org/microeconomics/sce#/ (accessed July 16, 2024).

Hanany, E., and P. Klibanoff (2009): "Updating Ambiguity Averse Preferences," *The B.E. Journal of Theoretical Economics*, 9.

Hanany, E., P. Klibanoff, and S. Mukerji (2020): "Incomplete Information Games with Ambiguity Averse Players," *American Economic Journal: Microeconomics*, 12(2), 135–87.

Household Pulse Survey (2024): "Phase 4.0 Cycle 1 PUF Releases of Household Pulse Survey 2024," https://www.census.gov/programs-surveys/household-pulse-survey/data/datasets.html#phase-4-1 (accessed July 16, 2024).

Huo, Z., and M. Z. Pedroni (2020): "A Single-Judge Solution to Beauty Contests," *American Economic Review*, 110(2), 526–68.

Jayashankar, A., and A. Murphy (2023): "High Inflation Disproportionately Hurts Low-Income Households," Federal Reserve Bank of Dallas.

Kuhn, M., and J.-V. Ríos-Rull (2016): "2013 Update on the US Earnings, Income, and Wealth Distributional Facts: A View from Macroeconomics," *Quarterly Review*, (April), 1–75.

Malmendier, U., and S. Nagel (2016): "Learning from inflation experiences," *The Quarterly Journal of Economics*, 131(1), 53–87.

Spot Crude Oil Price (1987-2022): "Federal Reserve Bank of St. Louis, Spot Crude Oil Price: West Texas Intermediate (WTI)," retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/WTISPLC (accessed July 11, 2024).

Survey of Consumer Finances (2016): "The 2016 Survey of Consumer Finances," https://www.federalreserve.gov/econres/scfindex.htm (accessed July 15, 2024).

U.S. Consumer Price Index (2013:I-2022:IV): "Consumer Price Indices (CPIs, HICPs), COICOP 1999: Consumer Price Index: Total for United States," retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/CPALTT01USM661S (accessed July 11, 2024).