Dual Moral Hazard and the Tyranny of Success
(Online Appendix - Not for Publication)
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1 Discussion and Extensions

In this Appendix, I discuss some extensions and describe some ways in which the model can be enriched, pointing to how the analysis would change.

1.1 The Role of Uncertainty

Some interesting implications can be obtained when analyzing the role of uncertainty in the principal's commitment problem:

Corollary 1.

(i) $v_0^{bonus} \to w_0^{fb}$ *and* $v_0^* \to w_0^{fb}$ *as* $\bar{q} \to 1$.

(ii) For any $\bar{q} \in (0,1)$, $v_0^{bonus} \to w_0^{fb}$ *and* $v_0^* \to w_0^{fb}$ *as* $\lambda \to \infty$ *and* $\Pi \to 0$ *with* $g \equiv \lambda\Pi$ *constant.*

Proof. I will show that $v_0^{bonus} \to w_0^{fb}$, since this immediately implies that $v_0^* \to w_0^{fb}$ as a time-contingent contract can always be made time-independent.

For (i), note that $w(\bar{q}; s) \to g$ as $\bar{q} \to 1$, so $s^{fb} \to g - rf$ when I take this limit. Thus, $w_0^{fb} \to \max\{g - rf, s\}$ as $\bar{q} \to 1$. Now, $v(\bar{q}; B, s) \to g - \lambda\mu B/(1+\mu)$ and $u(\bar{q}; B, s) \to \lambda\mu B/(1+\mu)$ as $\bar{q} \to 1$, so in the limit the principal solves:

$$\max_{B \geq 0}\{s + \mathbb{1}_{\lambda\mu B/(1+\mu) \geq rf}\left[g - \lambda\mu B/(1+\mu) - s\right]\}$$

where $\mathbb{1}_X$ is the indicator function of the event X. The solution to this problem is $\lambda\mu B^{bonus}/(1+\mu) = rf$ if $g - rf > s$ and $B^{bonus} = 0$ otherwise, so $v_0^{bonus} \to \max\{g - rf, s\}$ as $\bar{q} \to 1$ also.

For (ii), note that $w(\bar{q}; s) \to g\bar{q} + s(1-\bar{q})$ as $\lambda \to \infty$ and $\Pi \to 0$, keeping $g \equiv \lambda\Pi$ constant, so $w_0^{fb} \to \max\{g\bar{q} + s(1-\bar{q}) - rf, s\}$. Now, $v(\bar{q}; B, s) \to \bar{q}(g - \lambda\mu B/(1+\mu)) + s(1-\bar{q})$ and $u(\bar{q}; B, s) \to \bar{q}\lambda\mu B/(1+\mu)$ as $\lambda \to \infty$ and $\Pi \to 0$ keeping $g \equiv \lambda\Pi$ constant, so in the limit the principal solves:

$$\max_{B \geq 0}\{s + \mathbb{1}_{\bar{q}\lambda\mu B/(1+\mu) \geq rf}\left[\bar{q}(g - \lambda\mu B/(1+\mu)) + s(1-\bar{q}) - s\right]\}$$

The solution to this problem is $\bar{q}\lambda\mu B^{bonus}/(1+\mu) = rf$ if $\bar{q}(g - s) > rf$ and $B^{bonus} = 0$ otherwise. Thus, $v_0^{bonus} \to \max\{g\bar{q} + s(1-\bar{q}) - rf, s\}$ also as $\lambda \to \infty$ keeping g constant. □

The first part of Corollary 1 shows the necessity of ex-ante uncertainty for the principal's commitment problem to arise, stating that the dual moral hazard problem disappears—even if the principal is restricted to offer a fixed bonus—as players become ex-ante certain that R is superior to S. This is because when the principal becomes ex-ante more optimistic about R's prospects, there is a decrease in the ex-ante risk of the principal ex-post inefficiently switching to S early.

Interestingly, this result could explain some anecdotal evidence suggesting that, even though they usually struggle to innovate first, incumbent firms are relatively successful second-mover adopters, see, e.g., Markides and Geroski (2005) and Gans (2016, pp. 123-124). This is because—according to the model presented here—once a new entrant develops an innovation, the uncertainty surrounding the innovation decreases, ameliorating the organizational rigidities that were deterring successful incumbents from adopting it.

The second part of Corollary 1 shows, however, that ex-ante uncertainty is necessary but not sufficient for the commitment problem to arise. In particular, it states that when the expected instantaneous profit of R conditional on $\theta = 1$ (equal to $\lambda\Pi$) is held fixed at g, the incentive problem also disappears—again, even with a time-independent bonus—as the arrival rate of R's payments, λ, approaches infinity. That is, there is no dual moral hazard problem if the organization can learn infinitely quickly whether the risky arm is profitable.

Intuitively, the principal's incentives to experiment with R increase with the rate at which she can discern its profitability. Hence, a higher λ induces the principal to wait for more evidence of R's unprofitability before switching back to S. As a result, there is less scope for a disagreement between the principal and the agent over the correct allocation of resources once R is installed.

This last result is important because it shows that not all innovations will be equally difficult for successful firms to adopt. In particular, a more profitable status quo will not lead to organizational rigidities in the case of easily scalable innovations where preliminary experiments, prototypes, and minimum viable products provide a good sense of the value of the overall opportunity. Conversely, it reinforces the difficulty of developing innovations requiring extensive time and testing within successful firms.

1.2 Timing of Adoption

As noted in Section 2 of the main text, the organization's problem in the baseline model is to decide whether to install R, not when to install it. This is without loss since no useful information is gained when the organization does not install the risky arm.

Consider instead an alternative model in which the principal also has access to information that does not depend on R's installation. This adds an option value problem to the analysis. Both in equilibrium and in the first-best case, the organization installs R whenever q_t is greater than an "installation" cutoff. This cutoff, however, will be higher under the optimal time-contingent bonus (and in the fixed bonus case) than in the first-best. The reason is that the principal can alleviate her commitment problem by decreasing the ex-ante uncertainty surrounding R's profitability, as mentioned above. Hence, her incentives to wait for more evidence on R's profitability before installing the arm are stronger in equilibrium than in the first-best.

Thus, the main difference between the baseline model and this richer setting would be where the inefficiency shows up. In the baseline model, the incentive friction appears *either* in the agent's increased compensation or in the organization's inability to install R. In contrast, in the model involving the timing of adoption, the friction appears in the agent's increased compensation *and* in the greater delay relative to the efficient time of adoption.[1]

1.3 Inconclusive Evidence

The baseline model considers only the case of conclusive evidence. However, it is unlikely that changing this assumption will dramatically affect the results.

For instance, in a previous version of this paper, I showed that the first-best outcome and the equilibrium with the fixed-bonus are qualitatively similar if players also observe the following "news" process that imperfectly reveals R's type once the arm is installed:[2]

$$dX_t = \theta k_t dt + k_t^{1/2} \sigma dZ_t, \text{ where } \{Z_t\}_{t \geq 0} \text{ is a standard Brownian motion}$$

Regarding time-contingent schedules, characterizing the optimal contract with inconclusive news is left for future research. That said, one would expect that the value to the principal from offering a time-contingent contract relative to the time-independent bonus will likely be lower in the case of inconclusive news compared to the case with fully conclusive evidence, as time without a breakthrough becomes a worse proxy for players' pessimism on R's profitability. Thus, the equilibrium with a time-contingent bonus will probably be closer to the equilibrium with a fixed bonus in the inconclusive news case than in the conclusive evidence case.

1.4 Observability of the Installation Decision

The assumption in the main text that the agent's installation decision is observable is not crucial, but it simplifies the analysis. In particular, it is easy to see that the equilibrium with the time-contingent bonus described in Section 4 of the main text is an equilibrium of the equivalent game where the agent's installation is not observable to the principal (the same applies to the fixed-bonus

[1] The organization always eventually installs R in equilibrium since the incentive friction vanishes when the belief that $\theta = 1$ at the time of installation is sufficiently close to one (see Corollary 1).

[2] The benefit of using the diffusion process to create partial evidence—rather than the typical Poisson Bandit model with inconclusive news—is that players' value functions and the principal's experimentation cutoff can be obtained in closed form when characterizing the first-best and the fixed bonus case.

equilibrium of Section 3).

However, when the agent's installation decision is unobservable, there are additional Pareto-dominated equilibria (dominated, in particular, by the equilibrium described in Section 4). This is because the agent's installation decision and the principal's allocation of resources are complements, so there is scope for miscoordination. For instance, there is an equilibrium where, irrespective of the contract offered by the principal at $t = 0^-$, the agent never installs R and, therefore, the principal never allocates resources to that arm.

Hence, the results of the main text can be equivalently understood as the equilibrium of a game where the installation decision is not observable and Pareto-dominated equilibria are discarded.

1.5 Contract Renegotiation

Finally, it is easy to see that allowing contract renegotiation improves efficiency but does not completely solve the dual moral hazard problem.

More precisely, renegotiation allows the organization to avoid the inefficiency on the "intensive" margin. This is because as long as experimenting with R is efficient, both the principal and the agent have incentives to change the contract's terms—lowering the agent's compensation—whenever the principal is about to switch to S.

However, renegotiation does not help with the inefficiency on the "extensive margin," as even if the parties can renegotiate the contract, the agent is still being expropriated ex-post (which is anticipated by the agent ex-ante). This implies that the principal's ex-ante payoff is still discontinuous at some $s \in (0, s^{fb})$ when approaching from the left. Moreover, it also implies that a well-designed renegotiation procedure (as in e.g., Aghion, Dewatripont, and Rey, 1994) does not solve the organization's dual moral hazard problem.

2 Proof of Proposition 1

To prove this proposition, I begin with the following lemma.

Lemma 2.1. *If the organization installs R, then there is a cutoff belief q^{fb} given by:*

$$q^{fb} = \frac{\mu s}{\mu s + (1+\mu)(g-s)}$$

such that below the cutoff it is optimal to allocate all the resource to S and above it is optimal to allocate all the resource to R. The value function $w(q; s)$ for this problem is given by:

$$w(q;s) = \begin{cases} gq + (s - gq^{fb}) \left(\dfrac{1-q}{1-q^{fb}}\right) \left(\dfrac{\Omega(q)}{\Omega(q^{fb})}\right)^{\mu} & \text{if } q \geq q^{fb} \\ s & \text{otherwise} \end{cases} \tag{1}$$

where $\Omega(q) \equiv (1-q)/q$ *denotes the odds ratio at the belief* q*, and* $\mu = r/\lambda$*.*

Proof. If the organization installs R, it faces a single-player experimentation problem with exponential bandits and conclusive news. The solution to this problem is characterized in Keller, Rady and Cripps (2005, Proposition 3.1). □

Lemma 2.1 implies that installing R gives the organization an average discounted present value of $w(\bar{q}; s) - rf$. Not installing R, on the other hand, gives the organization an average discounted present value of s, as the organization will always allocate all the resource to S in this case. Hence, it is (strictly) optimal to install R whenever $w(\bar{q}; s) - rf - s > 0$.

Using (1), note then that $w(\bar{q}; s) - s$ can be written as $w(\bar{q}; s) - s = \omega(T^{fb}(s); s)$, where:

$$\omega(T; s) = \bar{q}(g-s)\Big(1 - e^{-\lambda(1+\mu)T}\Big) - s(1-\bar{q})\Big(1 - e^{-\lambda\mu T}\Big)$$

and $T^{fb}(s) \equiv \big(\ln\Omega(q^{fb}(s)) - \ln\Omega(\bar{q})\big)/\lambda$ (note that I am making explicit that $q^{fb}(s)$ depends on s). Using the fact that the derivative of $\omega(T^{fb}(s); s)$ with respect to s taking into account that T^{fb} depends on s is the same as the derivative taking T^{fb} fixed (this is due to the Envelope Theorem, as $T^{fb}(s) = \arg\max_T \omega(T; s)$), it follows that:

$$\frac{\partial}{\partial s}\omega(T^{fb}(s); s) = -\bar{q}\Big(1 - e^{-\lambda(1+\mu)T^{fb}(s)}\Big) - (1-\bar{q})\Big(1 - e^{-\lambda\mu T^{fb}(s)}\Big) < 0$$

Hence, $w(\bar{q}; s) - rf - s = \omega(T^{fb}(s); s) - rf$ is strictly decreasing in s. Furthermore, as $s \to 0$, $w(\bar{q}; s) - s - rf$ converges to $\bar{q}g - rf > 0$, and as $s \to g$, to $-rf < 0$ (recall that $g \equiv \lambda\Pi$). Hence, it is optimal to install R whenever $s < s^{fb}$, where $s^{fb} \in (0, g)$ is the (unique) solution to $w(\bar{q}; s^{fb}) - rf = s^{fb}$. The latter also implies that the first-best expected payoff at $t = 0$ is $w_0^{fb} = \max\{w(\bar{q}; s) - rf, s\}$.

It is then not difficult to prove that s^{fb} is strictly decreasing in f and strictly increasing in $g = \lambda\Pi$ and $\bar{q}$. This is intuitive: A higher installation cost decreases the attractiveness of R *vis-a-vis* S, while the opposite occurs when the risky arm is more likely to be profitable or provides higher profits conditional on being profitable.

Finally, I show that w_0^{fb} is strictly increasing in s. When $s > s^{fb}$, the result follows immediately since $w_0^{fb} = s$. When $s \leq s^{fb}$, in turn, I differentiate $w_0^{fb} = w(\bar{q}; s) - rf$ with respect to s (noting that $s \leq s^{fb}$ implies $\bar{q} > q^{fb}$ and that q^{fb} depends on s) to obtain:

$$\frac{\partial w_0^{fb}}{\partial s} = \left(\frac{1-q}{1-q^{fb}}\right)\left(\frac{\Omega(q)}{\Omega(q^{fb})}\right)^{\mu} > 0$$

Hence, w_0^{fb} is also strictly increasing in s in this case.[3]

[3]Note that the derivative of w_0^{fb} with respect to s taking into account that q^{fb} depends on s is the same as the

3 Proof of Proposition 2

For easy of exposition, I split the proof into two steps. In the first step, I characterize the principal's equilibrium resource allocation policy at any t for a given bonus $B \geq 0$ conditional on the agent installing R. This allows me to obtain players' continuation payoffs immediately after R's installation. In the second step, I use these continuation payoffs to characterize the optimal bonus offered by the principal at $t = 0^-$ and the agent's equilibrium installation decision at $t = 0^+$.

3.1 Optimal Resource Allocation given $B \geq 0$ and $a_0 = 1$

The following lemma characterizes the equilibrium resource allocation policy at any t for a given bonus $B \geq 0$ conditional on the agent installing R.

Lemma 3.1. *If the agent installs R, then there is a cutoff belief q^{bonus} given by:*

$$q^{bonus} = \frac{\mu s}{\mu s + (1+\mu)(g-s) - \lambda\mu B}$$

such that below the cutoff the principal allocates all the resource to S and above the cutoff she allocates all the resource to R. The principal's value function $v(q; B, s)$ is then given by:

$$v(q;B,s) = \begin{cases} gq + (s - gq^{bonus})\left(\frac{1-q}{1-q^{bonus}}\right)\left(\frac{\Omega(q)}{\Omega(q^{bonus})}\right)^{\mu} - u(q;B,s) & \text{if } q \geq q^{bonus} \\ s & \text{otherwise} \end{cases}$$

where $u(q; B, s)$ is the agent's value function, which is given by:

$$u(q;B,s) = \begin{cases} \lambda B q\left(\frac{\mu}{1+\mu}\right)\left[1 - \left(\frac{\Omega(q)}{\Omega(q^{bonus})}\right)^{1+\mu}\right] & \text{if } q \geq q^{bonus} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If the agent installs R given a bonus $B \geq 0$, the principal's resource allocation problem is a dynamic programming problem with the current belief q as the state variable. The Hamilton-Jacobi-Bellman equation (HJB) associated with this problem is (throughout the proof, I will omit the dependency of $v(\cdot)$ on B and s to simplify notation):

$$rv(q) = \max_{k\in[0,1]} \left\{r\left[(1-k)s + qk(g - \lambda B)\right] + \lambda kq(g - v(q)) - \lambda kq(1-q)v'(q)\right\} \quad (2)$$

derivative taking q^{fb} fixed. This is due to the envelope theorem given that:

$$q^{fb} = \arg\max_{x} \left\{g\bar{q} + (s - gx)\left(\frac{1-\bar{q}}{1-x}\right)\left(\frac{\Omega(\bar{q})}{\Omega(x)}\right)^{\mu}\right\}$$

Hence, the right-hand side of the HJB equation implies that:

$$k^*(q) = \begin{cases} 1 & \text{if } r[q(g-\lambda B)-s]+\lambda q(g-v(q))-\lambda q(1-q)v'(q)>0 \\ [0,1] & \text{if } r[q(g-\lambda B)-s]+\lambda q(g-v(q))-\lambda q(1-q)v'(q)=0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

I then conjecture that the principal follows a cutoff strategy (as in the first-best): $k^*(q) = 1$ if $q \geq q^{bonus}$, and $k^*(q) = 0$, otherwise. If so, the differential equation (2), plus the value-matching and the smooth-pasting conditions $v(q^{bonus}) = s$ and $v'(q^{bonus}) = 0$, imply that $q^{bonus} = \mu s/(\mu s + (1+\mu)(g-s) - \lambda\mu B)$, and that $v(q)$ is given as in the statement of the lemma. Moreover, with this last expression is easy to prove that the conjecture was correct, i.e., that (3) holds when using the expression of $v(q)$ just found. The optimality of this cutoff strategy follows from standard verification arguments.

To find the agent's value function, note that since $k^*(q) = 0$ for $q < q^{bonus}$, then $u(q) = 0$ for $q < q^{bonus}$ as no breakthrough is possible in this region of the belief space (I am, again, omitting the dependency of $u(\cdot)$ on B and s to simplify notation). In contrast, if $q \geq q^{bonus}$, then $k^*(q) = 1$, so $u(q)$ must satisfy $ru(q) = \lambda q(rB - u(q)) - \lambda q(1-q)u'(q)$ with terminal condition $u(q^{bonus}) = 0$. Solving yields that $u(q)$ is as in the statement of the lemma. □

3.2 Equilibrium Bonus and Installation Decision

Consider first the agent's installation decision at $t = 0^+$ given $B \geq 0$. If the agent does not install R, then he obtains a continuation value of zero, and the principal obtains s. In contrast, if the agent installs R, he obtains $u(\bar{q}; B, s) - rf$ while the principal obtains $v(\bar{q}; B, s)$. Clearly, then:

$$a_0^*(B; f, s) = \begin{cases} 1 & \text{if } rf \leq u(\bar{q}; B, s) \\ 0 & \text{otherwise} \end{cases}$$

So at $t = 0^-$, the principal solves:

$$\max_{B\geq 0} \left\{ s + a_0^*(B; f, s)\left[v(\bar{q}; B; s) - s\right] \right\}$$

To solve this problem, define $\hat{B}(s) \equiv \arg\max_B u(\bar{q}; B, s)$. The bonus $\hat{B}(s)$ always exists and is unique given that $u(\bar{q}; B, s)$ is strictly quasiconcave in B. Using $\hat{B}(s)$, plus the fact that $u(\bar{q}; B, s)$ is strictly decreasing in s, it is possible to prove that there exists a unique $s^{bonus} \in (0, s^{fb})$ such that $u(\bar{q}; \hat{B}(s), s) \geq rf$ if $s \leq s^{bonus}$, and $u(\bar{q}; \hat{B}(s), s) < rf$, otherwise.

With these results at hand, Lemma 3.2 characterizes the solution to the principal's problem at $t = 0^+$, while Lemma 3.3 establishes the non-monotonicy of the principal's ex-ante equilibrium payoff with respect to s. The statement of Proposition 2 then follows directly from combining these two lemmas and Lemma 3.1 above.

Lemma 3.2. *The equilibrium bonus offered by the principal is:*

$$B^{bonus} = \begin{cases} \underline{B}(f,s) & \text{if } s \leq s^{bonus} \\ [0,\infty) & \text{otherwise} \end{cases}$$

where $\underline{B}(f,s) \equiv \inf\{B \geq 0 : u(\bar{q}; B, s) = rf\}$. *In equilibrium, the agent installs* R *whenever* $s \leq s^{bonus}$, *and the players' equilibrium expected payoffs at* $t = 0$ *are:*

$$u_0^{bonus} = 0 \qquad \text{and} \qquad v_0^{bonus} = \begin{cases} v(\bar{q}; \underline{B}(f,s), s) > s & \text{if } s \leq s^{bonus} \\ s & \text{otherwise} \end{cases}$$

Proof. Clearly, motivating the agent to install R is impossible when $s > s^{bonus}$, as $u(\bar{q}; B, s) \leq u(\bar{q}; \hat{B}(s), s) < rf$ in this case. Thus, any B is optimal when $s > s^{bonus}$.

Consider $s \leq s^{bonus}$ instead. Then, the principal always finds it optimal to incentivize the agent to install R. Indeed, for the agent to install R, $u(\bar{q}; B, s) \geq rf$ necessarily. However, since $f > 0$, this requires that B is such that $q^{bonus} < \bar{q}$, implying that $v(\bar{q}; B, s) > s$.

Now, $v(\bar{q}; B, s)$ is strictly decreasing in B:[4]

$$\frac{\partial}{\partial B} v(\bar{q}; B, s) = -\lambda \left(\frac{\mu}{1+\mu} \right) \left[1 - \left(\frac{\Omega(q)}{\Omega(q^{bonus})} \right)^{1+\mu} \right] < 0$$

Thus, the principal offers the agent the minimum B that satisfies $u(\bar{q}; B, s) = rf$, i.e., $B^{bonus} = \underline{B}(f,s)$ which always exists given that $s \leq s^{bonus}$. Based on this result, it is easy to see that the principal's expected payoff at $t = 0$, v_0^{bonus}, is as given in the statement. On the other hand, since the agent is always indifferent between installing and not installing R, his expected payoff at $t = 0$ is $u_0^{bonus} = 0$. $\square$

Lemma 3.3. *The principal's equilibrium expected payoffs at* $t = 0$ *is strictly decreasing in* s *for* $s \in (s^{bonus} - \epsilon, s^{bonus})$ *(where* $\epsilon > 0$*). Moreover, it exhibits a discontinuous downward jump at* $s = s^{bonus}$ *when approaching from the left, i.e.,*

$$\lim_{s \uparrow \bar{s}} v_0^{bonus} > \lim_{s \downarrow \bar{s}} v_0^{bonus} = s$$

[4] This follows due to the envelope theorem given that:

$$q^{bonus} = \arg\max_x \left\{ g\bar{q} + (s - gx) \left(\frac{1-\bar{q}}{1-x} \right) \left(\frac{\Omega(\bar{q})}{\Omega(x)} \right)^{\mu} - \lambda B \bar{q} \left(\frac{\mu}{1+\mu} \right) \left[1 - \left(\frac{\Omega(\bar{q})}{\Omega(x)} \right)^{1+\mu} \right] \right\}$$

Proof. That v_0^{bonus} exhibits a discontinuous downward jump at $s = s^{bonus}$ when approaching from the left comes directly from the fact that $v(\bar{q}; \underline{B}(f,s), s) > s$ for $s \leq s^{bonus}$ and that $v(\bar{q}; \underline{B}(f,s), s) = s$ for $s > s^{bonus}$.

To show, in turn, that v_0^{bonus} is strictly decreasing in s for $s \in (s^{bonus} - \epsilon, s^{bonus})$, differentiate v_0^{bonus} with respect to s to obtain:

$$\frac{\partial v_0^{bonus}}{\partial s} = \frac{\bar{q}}{q^{bonus}} \left(\frac{\Omega(\bar{q})}{\Omega(q^{bous})} \right)^{1+\mu} - \lambda \bar{q} \left(\frac{\mu}{1+\mu} \right) \left[1 - \left(\frac{\Omega(\bar{q})}{\Omega(q^{bous})} \right)^{1+\mu} \right] \frac{\partial}{\partial s} \underline{B}(f,s) \tag{4}$$

where $\underline{B}(f,s) \equiv \inf\{B \geq 0 : u(\bar{q}; B, s) = rf\}$. By the implicit function theorem, I then have that:

$$\frac{\partial}{\partial s} \underline{B}(f,s) = \frac{-\frac{\partial}{\partial s} u(\bar{q}; B, s)\big|_{B=\underline{B}(f,s)}}{\frac{\partial}{\partial B} u(\bar{q}; B, s)\big|_{B=\underline{B}(f,s)}}$$

As I argued above, $u(\bar{q}; B, s)$ is strictly decreasing in s and $u(\bar{q}; B, s)$ is strictly quasiconcave in B. This implies that $\partial \underline{B}(f,s)/\partial s > 0$ as $\underline{B}(f,s) \leq \arg\max_B = u(\bar{q}; B, s)$.

Consequently, the first term on the right-hand side is strictly positive (4), while the second term is strictly negative. I then claim that $\partial \underline{B}/\partial s \to \infty$ as s approaches s^{bonus} from the left, implying that there exists a $\epsilon > 0$ such that $\partial v_0^{bonus}/\partial s < 0$ for all $s \in (s^{bonus} - \epsilon, s^{bonus})$. Indeed, notice that s^{bonus} is given by the (unique) solution to $\max_B u(\bar{q}; B, s^{bonus}) = rf$, and that at such point $\underline{B}(f, s^{bonus})$ is unique and satisfies:

$$\frac{\partial}{\partial B} u(\bar{q}; B, s^{bonus})\big|_{B=\underline{B}(f,s^{bonus})} = 0$$

Hence, as s approaches s^{bonus} from the left, then $(\partial u(\bar{q}; B, s)/\partial B)|_{B=\underline{B}(f,s)} \to 0$ from above and, therefore, $\partial \underline{B}/\partial s \to \infty$. □

4 Efficiency when Contracting over the Resource Policy

In this appendix, I formally show that the principal can obtain her first-best payoff if parties can contract directly or indirectly upon the allocation of resources.

4.1 Directly Contracting over the Resource Policy

I begin by showing that if parties can contract upon the allocation of resources, the principal can obtain her first-best payoff by offering the agent a fixed bonus payable upon a breakthrough.

Formally, a strategy for the principal consists of a fixed bonus $B \geq 0$ and a resource allocation policy $k = \{k_t, 0 \leq t \leq \infty\}$ to be offered to the agent at $t = 0^-$. The resource allocation policy is such that k_t is measurable with respect to (i) the sequence of resources allocated to each arm up to t (not including t) and (ii) the sequence of profits delivered by each of the arms up to and including time t.[5]

A strategy for the agent, in turn, corresponds to an installation choice at $t = 0^+$, $a_0 \in \{0, 1\}$, as a function of the bonus and the resource allocation policy offered by the principal at $t = 0^-$.

As in the main text, let $\tau \geq 0$ be the time at which R's first success (a "breakthrough") occurs. Given the players' actions, the principal's and the agent's expected payoffs at time 0, expressed in per-period units, are equal to:

$$\begin{aligned} v_0 =& \mathbb{E}\left[\int_0^\infty re^{-rt}\big[(a_0 k_t \lambda\theta)\Pi + s(1-k_t)\big]dt - re^{-r\tau}B\right] \\ u_0 =& \mathbb{E}\left[re^{-r\tau}B\right] - ra_0 f \end{aligned}$$

The next lemma characterizes the equilibrium of this game:

Lemma 4.1. *In equilibrium, the principal achieves her first-best payoffs, $v_0^{ck} = w_0^{fb}$ (where the superscript "ck" stands for "contractible resources"), by offering the fixed bonus:*

$$B^{ck} = \frac{rf}{\bar{q}}\left(1 + \frac{1}{\mu}\right)\left[1 - \left(\frac{\Omega(\bar{q})}{\Omega(q^{fb})}\right)^{1+\mu}\right]^{-1}$$

and committing to the first-best resource allocation policy:

$$k_t^{fb}(N_t = 0) = \begin{cases} 1 & \textit{if } t \leq T^{fb} = \frac{1}{\lambda}\ln\left(\frac{\Omega(q^{fb})}{\Omega(\bar{q})}\right) \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and} \quad k_t^{fb}(N_t \geq 1) = 1$$

where N_t is the number of successes with R observed up to and including time t.

Proof. To prove this lemma, it suffices to show that under the proposed contract: (i) the agent installs R, and (ii) the principal obtains her first-best payoff if the agent installs R.

To show (i), note that under the first-best resource policy, the belief that $\theta = 1$ at $t \in [0, T^{fb}]$ in the absence of a breakthrough is $q_t = \bar{q}/(\bar{q} + (1-\bar{q})e^{\lambda t})$. This implies that if the agent installs

[5] This measurability condition aims to capture the idea that the resource allocation policy is verifiable and that it is actually being written in a formal contract. An alternative assumption would be to assume that the principal can credibly commit $k = \{k_t, 0 \leq t \leq \infty\}$ without having to actually include the policy in a formal contract, due to, for example, reputational concerns. In this last case, it would be more natural to assume that k_t is measurable with respect to all the information available at t. In the current setting, both assumptions yield equivalent outcomes in that both allow the principal to achieve her first-best payoff. The latter, however, is not true in general; for example, in the case where R's payments give only inconclusive evidence about the arm's desirability.

R, he obtains:

$$\begin{aligned}\mathbb{E}\left[re^{-r\tau}B^{ck}\right]-rf&=\int_0^{T^{fb}} re^{-rt}(\lambda q_t B^{ck})\left(\frac{1-\bar{q}}{1-q_t}\right)dt-rf\\&=\lambda B^{ck}\bar{q}\left(\frac{\mu}{1+\mu}\right)\left[1-\left(\frac{\Omega(\bar{q})}{\Omega(q^{fb})}\right)^{1+\mu}\right]-rf=0\end{aligned}$$

Hence, the agent installs R as he is indifferent between doing so or not.

To show (ii), I directly compute the principal's payoffs taking into account that the agent has incentives to install R:

$$\begin{aligned}v_0^{ck}&=\mathbb{E}\left[\int_0^{\infty} re^{-rt}\big[(k_t^{fb}\lambda\theta)\Pi+s(1-k_t^{fb})\big]dt-re^{-r\tau}B^{ck}\right]\\&=\int_0^{T^{fb}} re^{-rt}q_t g\left(1+\frac{\lambda}{r}\right)\left(\frac{1-\bar{q}}{1-q_t}\right)dt+se^{-rT^{fb}}\left(\frac{1-\bar{q}}{1-q_{T^{fb}}}\right)-rf\\&=g\bar{q}+(s-gq^{fb})\left(\frac{1-\bar{q}}{1-q^{fb}}\right)\left(\frac{\Omega(\bar{q})}{\Omega(q^{fb})}\right)^{\mu}-rf\\&=w_0^{fb}\end{aligned}$$

where I am again using that in the absence of a breakthrough, $q_t=\bar{q}/(\bar{q}+(1-\bar{q})e^{\lambda t})$ for $t\in[0,T^{fb}]$ under the first-best resource policy. □

4.2 Indirectly Contracting over the Resource Policy

I now show that even if directly contracting upon $k=\{k_t,0\leq t\leq\infty\}$ is unfeasible, the principal can still achieve efficiency by indirectly contracting upon k. The latter is done by signing a contract that promises the agent damages of $D^*>s$ every time S delivers profits before the efficient stopping time T^{fb}.

As a first step, I define the idea of a *compensation scheme* $c=\{c_t,0\leq t\leq\infty\}$. This is a nonnegative, nondecreasing process, where c_t is interpreted as the cumulative transfers the principal has made to the agent up to and including time t. Since directly contracting upon $k=\{k_t,0\leq t\leq\infty\}$ is unfeasible, c_t is only measurable with respect to the sequence of profits delivered by each of the arms up to and including time t.

A strategy for the principal then consists of committing to a compensation scheme $c=\{c_t,0\leq t\leq\infty\}$ at $t=0^-$, and making a resource allocation decision, $k_t\in[0,1]$, at every $t>0$. The resource allocation decision is contingent on all the information available at t, which can be summarized in (i) the compensation scheme that the principal promised to the agent at $t=0^-$, (ii) whether the agent installed R at $t=0^+$ or not, and (iii) the belief at time t that $\theta=1$.

A strategy for the agent, in turn, corresponds to an installation choice at $t=0^+$, $a_0\in\{0,1\}$, as a function of the compensation scheme offered by the principal at $t = 0^-$.

Given the players' actions, the principal's and the agent's expected payoffs at time 0, expressed in per-period units, are equal to:

$$v_0 = \mathbb{E}_0 \left[\int_0^\infty re^{-rt} \big[a_0 k_t \lambda \theta \Pi + s(1 - k_t) \big] dt - \int_0^\infty re^{-rt} dc_t \right]$$
$$u_0 = \mathbb{E}_0 \left[\int_0^\infty re^{-rt} dc_t \right] - ra_0 f$$

Lemma 4.2. *Even if directly contracting upon $k = \{k_t, 0 \leq t \leq \infty\}$ is unfeasible, the principal can achieve her first-best payoff, $v_0^{ick} = w_0^{fb}$ (where the superscript "ick" stands for "indirectly contracting upon resources"), by offering:*

$$dc_t^{ick} = \begin{cases} \mathbb{1}_{N_{t^-}=0} B^{ck} dN_t + \mathbb{1}_{\{S \text{ delivers profits at } t\}} D & \text{if } t \leq T^{fb} \\ \mathbb{1}_{N_{t^-}=0} B^{ck} dN_t & \text{otherwise} \end{cases} \tag{5}$$

where $D > s$, and B^{ck} is defined as in Lemma 4.1. This compensation scheme induces the principal to follow a stopping-time allocation policy with a fixed stopping time T^{fb} irrespective of whether the agent installs R and leaves the agent indifferent between installing R or not.

Proof. To prove this proposition, it suffices to show that under the stated contract, (i) the principal follows the first-best resource policy if the agent installs R, and (ii) the agent installs R and breaks even.

To show (i), suppose that the agent installs R. If $t < T^{fb}$, then it is easy to see that $k_t = 1$ is optimal for the principal. Indeed, if $N_t = 1$, then $q_t = 1$ and the bonus B^{ck} has already been disbursed. Hence $k_t = 1$ is optimal as $g > s$. If $N_t = 0$, on the other hand, then playing $k_t = 1$ is optimal since $D > s$ and the principal's continuation payoff in the event of a breakthrough is $r(\Pi - B^{ck}) + g$, which is always strictly greater than zero given the value of B^{ck} and the fact that $s < s^{fb}$.

Now consider $t \geq T^{fb}$. If $N_t = 1$, then $k_t = 1$ is optimal, as $q_t = 1$, the bonus B^{ck} has already been disbursed, and $g > s$. If $N_t = 0$, then $k_t = 0$ is optimal given that the principal no longer has to pay damages for S's profits, and $q_t \leq q^{fb}$ (given that $k_t = 1$ is optimal for all $t \leq T^{fb}$, as I showed above).

To show (ii), note that because the principal follows the efficient resource allocation policy upon installation, then given B^{ck}, the agent obtains an expected payoff of zero if he installs R. Notice, further, that since $D > s$, the principal has incentives to allocate all the resources to R during $t \in (0, T^{fb})$ even if the agent fails to install R. Hence, not installing R also gives the agent zero. Thus, the agent installs R and breaks even. $\square$

An important feature of the contract described in Lemma 4.2 is that the damages must be sufficiently high so that the principal always has the incentives to avoid paying them. Note, in particular, that this implies that the principal still has incentives to allocate all the resources to R during $t \in (0, T^{fb}]$ even if the agent fails to install R. This feature of the contract is important since, otherwise, the agent would deviate and not install R to obtain the damages payments promised in the contract.

5 Proof of Lemma 2

I first show the existence and uniqueness of $\bar{B}_t^C$. Then, I show the existence and uniqueness of $\bar{B}_t$ and that this bonus coincides with $\bar{B}_t^C$ when $t \in \mathcal{T}$. Finally, I show that $\bar{B}_t$ is pointwise higher than any other bonus that satisfies the local incentive-compatibility constraint for all $t \in [0, \bar{T}]$, which necessarily implies that $\bar{B}_t^C$ is pointwise higher than any other local incentive compatible bonus defined over $\mathcal{T}$.

5.1 Construction of $\bar{B}_t^C$

To show the existence and uniqueness of $\bar{B}_t^C$, I will construct such a bonus starting from the last time interval in $[0, \bar{T}]$ that entails $k_t > 0$. For this purpose, it is notationally more convenient to work with $M_t \equiv g(1 + \lambda/r) - \lambda B_t$ rather than directly with B_t.

Let $[T_1 - \Delta_1, T_1)$, where $T_1 \leq \bar{T}$, be the last time interval at which the recommendation prescribes allocating resources to R. For any $t \in [T_1 - \Delta_1, T_1)$, then $\lambda \bar{B}_t^C = g(1 + \lambda/r) - \bar{M}_t^{(1)}$, where $\bar{M}_t^{(1)}$ satisfies the incentive-compatibility constraint (4) with equality:

$$q_t \bar{M}_t^{(1)} - s = \lambda q_t \left[\int_t^{T_1} e^{-r(z-t)} \left[s(1-k_z) + k_z \bar{M}_t^{(1)} \right] e^{-\int_t^z \lambda k_u du} dz \right. \\ \left. + \underbrace{\int_{T_1}^{\bar{T}} e^{-r(z-t)} s e^{-\int_t^{T_1} \lambda k_u du} dz + \frac{s}{r} e^{-r(\bar{T}-t)} e^{-\int_t^{\bar{T}} \lambda k_u du}}_{= \frac{s}{r} e^{-r(T_1 - t)} e^{-\int_t^{T_1} \lambda k_u du}} \right] \quad (6)$$

Note that evaluating at $t = T_1$, the above condition immediately implies that:

$$q_{T_1} \bar{M}_{T_1}^{(1)} - s = \frac{\lambda q_{T_1} s}{r} \Longrightarrow \bar{M}_{T_1}^{(1)} = s \left[1 + \frac{\lambda}{r} + \left(\frac{1 - q_{T_1}}{q_{T_1}} \right) \right] \quad (7)$$

Now, to solve (6), differentiate both sides of the equation with respect to t (taking into account that $q_t = \bar{q}/(\bar{q} + (1-\bar{q}) e^{\int_0^t \lambda k_u du})$ depends on t) to obtain:

$$r \bar{M}_t^{(1)} - \frac{d}{dt} \bar{M}_t^{(1)} = rsh(t), \text{ where } h(t) \equiv 1 + \frac{\lambda}{r} + \left(\frac{1-\bar{q}}{\bar{q}} \right) e^{\int_0^t \lambda k_u du}$$

$\bar{M}_t^{(1)}$ then comes from solving this differential equation using (7) as border condition:

$$\bar{M}_t^{(1)} = \int_t^{T_1} rse^{-r(z-t)}h(z)dz + se^{-r(T_1-t)}h(T_1) \tag{8}$$

With this at hand, I then move to the second-to-last interval at which $k_t = 1$. Let $[T_2 - \Delta_2, T_2]$, where $T_2 \leq T_1 - \Delta_1$ be such interval. Then for any $t \in [T_2 - \Delta_2, T_2)$, $\lambda \bar{B}_t^C = g(1 + \lambda/r) - \bar{M}_t^{(2)}$, where $\bar{M}_t^{(2)}$ satisfies the incentive-compatibility constraint (4) with equality. Such constraint, in this case, can be written as:

$$q_t \bar{M}_t^{(2)} - s = \lambda q_t \left[\int_t^{T_2} e^{-r(z-t)} \left[s(1-k_z) + k_z \bar{M}_t^{(2)} \right] e^{-\int_t^z \lambda k_u du} dz + e^{-r(T_2-t)} e^{-\int_t^{T_2} \lambda k_u du} I^{(1)} \right] \tag{9}$$

where:

$$I^{(1)} = \frac{s}{r}(1 - e^{-r(T_1-\Delta_1-T_2)}) + \frac{e^{-r(T_1-\Delta_1-T_2)}}{\lambda} \left[\bar{M}_{T_1-\Delta_1}^{(1)} - \frac{s}{q_{T_1-\Delta_1}} \right]$$

To solve (9), differentiate both sides of the equation with respect to t to obtain:

$$r\bar{M}_t^{(2)} - \frac{d}{dt}\bar{M}_t^{(2)} = rsh(t) \tag{10}$$

The border condition, in turn, comes evaluating (9) at $t = T_2$:

$$\bar{M}_{T_2}^{(2)} = \lambda I^{(1)} + \frac{s}{q_{T_2}}$$

Solving (10) using this border conditions yields:

$$\begin{aligned} \bar{M}_t^{(2)} &= \int_t^{T_2} rse^{-r(z-t)}h(z)dz + e^{-r(T_2-t)} \left(\lambda I^{(1)} + \frac{s}{q_{T_2}} \right) \\ &= \int_t^{T_2} rse^{-r(z-t)}h(z)dz + se^{-r(T_2-t)}(1 - e^{-r(T_1-\Delta_1-T_2)})h(T_2) + e^{-r(T_1-\Delta_1-T_2)} \bar{M}_{T_1-\Delta_1}^{(1)} \end{aligned}$$

Where the last equality follows from the expression for $I^{(1)}$ and the fact that $q_{T_2} = q_{T_1-\Delta_1}$ since $k_t = 0$ for $t \in [T_2, T_1 - \Delta_1]$.

For $n \geq 2$, let $[T_n - \Delta_n, T_n]$ be the nth-to-last interval at which $k_t = 1$. By induction, $\bar{M}_t^{(n)}$ satisfies the following differential equation:

$$r\bar{M}_t^{(n)} - \frac{d}{dt}\bar{M}_t^{(n)} = rsh(t) \text{ with border condition } \bar{M}_{T_n}^{(n)} = \lambda I^{(n-1)} + \frac{s}{q_{T_n}}$$

where:

$$I^{(n)} = \frac{s}{r}(1 - e^{-r(T_n-\Delta_n-T_{n+1})}) + \frac{e^{-r(T_n-\Delta_n-T_{n+1})}}{\lambda} \left[\bar{M}_{T_n-\Delta_n}^{(n)} - \frac{s}{q_{T_n-\Delta_n}} \right]$$

The solution to this differential equation is:

$$\bar{M}_t^{(n)} = \int_t^{T_n} rse^{-r(z-t)}h(z)dz + se^{-r(T_n-t)}(1 - e^{-r(T_{n-1}-\Delta_{n-1}-T_n)})h(T_n) \\ + e^{-r(T_{n-1}-\Delta_{n-1}-T_n)}\bar{M}^{(n-1)}_{T_{n-1}-\Delta_{n-1}} \quad (11)$$

The function $\bar{B}_t^C$ in this interval is then given by $\lambda\bar{B}_t^C = g(1+\lambda/r) - \bar{M}_t^{(n)}$. □

5.2 Construction of $\bar{B}_t$

To show the existence and uniqueness of $\bar{B}_t$, I will explicitly solve the differential equation that arises from making the local incentive constraint (4) bind at every $t \in [0, \bar{T}]$. The incentive constraint can be written as:

$$q_t\bar{M}_t - s = \lambda q_t \left[\int_t^{\bar{T}} e^{-r(z-t)} \left[s(1-k_z) + k_z\bar{M}_z\right] e^{-\int_t^z \lambda k_u du} dz + \frac{s}{r} e^{-r(\bar{T}-t)} e^{-\int_t^{\bar{T}} \lambda k_u du}\right] \quad (12)$$

where $\bar{M}_t \equiv g(1+\lambda/r) - \lambda\bar{B}_t$ and $q_t = \bar{q}/(\bar{q} + (1-\bar{q})e^{\int_0^t \lambda k_u du})$. Note that evaluating at $t = \bar{T}$, the above condition immediately implies that:

$$q_{\bar{T}}\bar{M}_{\bar{T}} - s = \frac{\lambda q_{\bar{T}} s}{r} \Longrightarrow \bar{M}_{\bar{T}} = s\left[1 + \frac{\lambda}{r} + \left(\frac{1-q_{\bar{T}}}{q_{\bar{T}}}\right)\right] \quad (13)$$

where $q_{\bar{T}} = \bar{q}/(\bar{q} + (1-\bar{q})e^{\int_0^{\bar{T}} \lambda k_u du})$. Now, to solve (12), note that rearranging terms one obtains:

$$\int_t^{\bar{T}} e^{-r(z-t)} \left[s(1-k_z) + k_z\bar{M}_z\right] e^{-\int_t^z \lambda k_u du} dz = \frac{\bar{M}_t}{\lambda} - \frac{s}{\lambda q_t} - \frac{s}{r} e^{-r(\bar{T}-t)} e^{-\int_0^{\bar{T}} \lambda k_u du}$$

Differentiating both sides of this last expression with respect to t (taking into account that q_t depends on t) yields:

$$r\bar{M}_t - \frac{d}{dt}\bar{M}_t = rsh(t), \text{ where } h(t) \equiv 1 + \frac{\lambda}{r} + \left(\frac{1-\bar{q}}{\bar{q}}\right) e^{\int_0^t \lambda k_u du}$$

Solving this differential equation for $\bar{M}_t$ using (13) as border condition one obtains that:

$$\bar{M}_t = \int_t^{\bar{T}} rse^{-r(z-t)}h(z)dz + se^{-r(\bar{T}-t)}h(\bar{T}) \quad (14)$$

Because $\bar{M}_t = g(1+\lambda/r) - \lambda\bar{B}_t$, I can then solve for $\bar{B}_t$ to obtain (recall that $\mu \equiv r/\lambda$):

$$\lambda\bar{B}_t = g\left(1 + \frac{1}{\mu}\right) - \int_t^{\bar{T}} rse^{-r(z-t)}h(z)dz - se^{-r(\bar{T}-t)}h(\bar{T})$$

That $\bar{B}_t$ coincides with $\bar{B}_t^C$ when $t \in \mathcal{T}$ then follows directly from the fact that the right-hand side of (12) only depends on the future bonuses that can be paid with strictly positive probability, i.e., only at those times $t \in \mathcal{T}$. Alternatively (and significantly more cumbersomely), it can be shown that (14) coincides with (11) when $t \in \mathcal{T}$. □

5.3 $\bar{B}_t$ is Pointwise Higher Among the Locally Incentive Compatible Bonuses

The following proof is adapted from Ely, Georgiadis and Rayo (2023, Proposition 2). Consider an arbitrary bonus B_t that satisfies local incentive compatibility for all $t \in [0, \bar{T}]$, and let $M_t \equiv g(1 + \lambda/r) - \lambda B_t$. Define also the function β_t as:

$$\beta_t \equiv s\left[\frac{1}{q_t} + \frac{\lambda}{r}e^{-r(\bar{T}-t)}e^{-\int_t^{\bar{T}}\lambda k_u du} + \lambda\int_t^{\bar{T}} e^{-r(z-t)}s(1-k_z)e^{-\int_t^z \lambda k_u du}dz\right] > 0$$

Because B_t is locally incentive compatible, then it must satisfy the incentive constraint (4) of the main text. This constraint can be written as:

$$M_t \geq \beta_t + \int_t^{\bar{T}} \lambda k_z M_z e^{-r(z-t)}e^{-\int_t^{\bar{T}}\lambda k_u du}dz \tag{15}$$

Note that because $\beta_t > 0$ for all $t \in [0, \bar{T}]$, then the incentive constraint (15) implies that $M_t > 0$ for all $t \in [0, \bar{T}]$. Indeed, the incentive constraint at $t = \bar{T}$ requires that $M_{\bar{T}} \geq \beta_{\bar{T}} > 0$, which then implies that for $\epsilon > 0$ but low enough:

$$M_{\bar{T}-\epsilon} \geq \beta_{\bar{T}-\epsilon} + \int_{\bar{T}-\epsilon}^{\bar{T}} \lambda k_z M_z e^{-r(z-t)}e^{-\int_t^{\bar{T}}\lambda k_u du}dz > 0 \qquad \text{(given that } M_{\bar{T}} > 0)$$

The exact same argument can then be used to show that $M_{\bar{T}-\epsilon} > 0$ implies that $M_{\bar{T}-2\epsilon} > 0$, and so on, until eventually obtaining that $M_t > 0$ for all $t \in [0, \bar{T}]$.

With this in mind, define $\mathrm{Z}_t^{(1)}$ as the right-hand side of (15), i.e.,

$$\mathrm{Z}_t^{(1)} \equiv \beta_t + \int_t^{\bar{T}} \lambda k_z M_z e^{-r(z-t)}e^{-\int_t^{\bar{T}}\lambda k_u du}dz$$

Notice that, by construction, the function $\mathrm{Z}_t^{(1)} \leq M_t$ for all t. Moreover, because $M_z > 0$ and $\lambda k_z e^{-r(z-t)}e^{-\int_t^{\bar{T}}\lambda k_u du} > 0$, it also follows that $\mathrm{Z}_t^{(1)} \geq \beta_t$ for all t. For all $k \geq 2$, define then the function $\mathrm{Z}_t^{(k)}$ by:

$$\mathrm{Z}_t^{(k)} = \beta_t + \int_t^{\bar{T}} \lambda k_z \mathrm{Z}_t^{(k-1)} e^{-r(z-t)}e^{-\int_t^{\bar{T}}\lambda k_u du}dz$$

Given that $\lambda k_z e^{-r(z-t)}e^{-\int_t^{\bar{T}}\lambda k_u du} > 0$ and $0 < \mathrm{Z}_t^{(1)} \leq M_t$, it follows that $\beta_t \leq \mathrm{Z}_t^{(2)} \leq \mathrm{Z}_t^{(1)}$. By induction, I then have that $\beta_t \leq \mathrm{Z}_t^{(k)} \leq \mathrm{Z}_t^{(k-1)}$ for all t. I have, therefore, constructed a pointwise decreasing sequence of functions bounded below by the function β_t. Let Z_t be the pointwise limit. By the dominated convergence theorem, I have that:

$$\mathrm{Z}_t = \beta_t + \int_t^{\bar{T}} \lambda k_z \mathrm{Z}_z e^{-r(z-t)} e^{-\int_t^{\bar{T}} \lambda k_u du} dz \tag{16}$$

Define then $\bar{M}_t = Z_t$. Note that $\bar{M}_t$ is weakly pointwise lower than the original M_t and that it satisfies (16) with equality, so $\bar{M}_t$ is unique and given by (14). Thus, $\bar{M}_t$ is pointwise lower than any other locally incentive compatible M_t, as M_t was arbitrary. Using the fact that $M_t = g(1+\lambda/r) - \lambda B_t$ and $\bar{M}_t = g(1+\lambda/r) - \lambda \bar{B}_t$, this immediately implies that $\bar{B}_t$ is pointwise higher than any bonus that satisfies the local incentive compatibility constraint at all $t \in [0, \bar{T}]$. □

6 Proof of Lemma 3

6.1 Preliminaries

To solve this problem, let us omit the agent's limited liability constraint for a moment (I later verify that it is indeed satisfied). Note then that the agent's payoffs—the objective in this case—is pointwise increasing in B_t. Hence, it is clear that the agent's preferred bonus is $B_t^A = \bar{B}_t$. If so, the problem at hand becomes choosing a recommended resource policy $\{k_t, 0 \leq t \leq \bar{T}\}$ to maximize:

$$\int_0^{\bar{T}} re^{-rt}(\lambda q_t k_t \bar{B}_t) \left(\frac{1-\bar{q}}{1-q_t}\right) dt \tag{17}$$

where $\bar{B}_t$ is given as in Lemma 2 and $q_t = \bar{q}/(\bar{q} + (1-\bar{q})e^{\int_0^t \lambda k_u du})$.

To solve this problem, let me conjecture that the optimal policy is such that $k_0 > 0$ (I will later verify that this is indeed the case). If so, the objective function can be rewritten in a slightly different (but more useful) way:

Lemma 6.1. *Recall that $\mu \equiv r/\lambda$. If in the optimum $k_0 > 0$, then the objective (17) can be equivalently written as:*

$$\bar{q} \int_0^{\bar{T}} re^{-rt} \left[\left(s(1-k_t) + k_t g \left(1 + \frac{1}{\mu}\right) \right) e^{-x_t} - s(1+\mu) - \mu s \left(\frac{1-\bar{q}}{\bar{q}}\right) e^{x_t} \right] dt$$
$$+ \bar{q}\mu s \left[(1-e^{-r\bar{T}}) - \frac{e^{-r\bar{T}}}{\mu}(1-e^{-x_{\bar{T}}}) + \left(\frac{1-\bar{q}}{\bar{q}}\right)(1-e^{-r\bar{T}}e^{x_{\bar{T}}}) \right]$$

where $x_t \equiv \int_0^t \lambda k_z dz$.

Proof. Observe that $q_t(1-\bar{q})/(1-q_t) = \bar{q}\exp(-\int_0^t \lambda k_u du) = \bar{q}\exp(-x_t)$. This implies that:

$$\int_0^{\bar{T}} re^{-rt}(\lambda q_t k_t \bar{B}_t)\left(\frac{1-\bar{q}}{1-q_t}\right) dt = \bar{q}\int_0^{\bar{T}} re^{-rt}\lambda k_t \bar{B}_t e^{-x_t} dt$$

Now, the conjecture that $k_0 > 0$ then implies that the local incentive constraint (4) characterized in Lemma 1 of the main text binds at $t = 0$:

$$\bar{q}\left(g\left(1+\frac{1}{\mu}\right) - \lambda\bar{B}_0\right) - s = \bar{q}\left[\frac{s}{\mu}e^{-r\bar{T}}e^{-x_{\bar{T}}} + \lambda\int_0^{\bar{T}} e^{-rt}\left[s(1-k_t) + k_t\left(g\left(1+\frac{1}{\mu}\right) - \lambda\bar{B}_t\right)\right]e^{-x_t}dt\right]$$

Rearranging terms and noting that, according to Lemma 2 of the main text, $\lambda\bar{B}_0 = g\left(1+\frac{1}{\mu}\right) - \int_0^{\bar{T}} rse^{-rt}h(t)dt - se^{-r\bar{T}}h(\bar{T})$, where $h(t) \equiv 1 + 1/\mu + e^{x_t}(1-\bar{q})/\bar{q}$, one obtains that:

$$\begin{aligned}\bar{q}\int_0^{\bar{T}} re^{-rt}\lambda k_t\bar{B}_t e^{-x_t}dt \\ = \bar{q}\int_0^{\bar{T}} re^{-rt}\left[\left(s(1-k_t) + k_t g\left(1+\frac{1}{\mu}\right)\right)e^{-x_t} - s(1+\mu) - \mu s\left(\frac{1-\bar{q}}{\bar{q}}\right)e^{x_t}\right]dt \\ + \bar{q}\mu s\left[(1-e^{-r\bar{T}}) - \frac{e^{-r\bar{T}}}{\mu}(1-e^{-x_{\bar{T}}}) + \left(\frac{1-\bar{q}}{\bar{q}}\right)(1-e^{-r\bar{T}}e^{x_{\bar{T}}})\right]\end{aligned}$$

□

Lemma 6.1 implies that the problem at hand is equivalent maximizing $r\bar{q}[\int_0^{\bar{T}} e^{-rt}j(k_t, x_t)dt + \psi(\bar{T}, x_{\bar{T}})]$ where:

$$\begin{aligned}j(k_t, x_t) &\equiv \left[s(1-k_t) + k_t g\left(1+\frac{1}{\mu}\right)\right]e^{-x_t} - s(1+\mu) - \mu s\left(\frac{1-\bar{q}}{\bar{q}}\right)e^{x_t} \\ \psi(\bar{T}, x_{\bar{T}}) &\equiv \frac{s}{\lambda}\left[(1-e^{-r\bar{T}}) - \frac{e^{-r\bar{T}}}{\mu}(1-e^{-x_{\bar{T}}}) + \left(\frac{1-\bar{q}}{\bar{q}}\right)(1-e^{-r\bar{T}}e^{x_{\bar{T}}})\right]\end{aligned}$$

choosing the control $k_t \in [0,1]$ for $t \in [0, \bar{T}]$, subject to $\dot{x}_t = \lambda k_t$, $x_0 = 0$, $x_{\bar{T}}$ free, and $\bar{T}$ given. However, since r and $\bar{q}$ are constants, maximizing $r\bar{q}[\int_0^{\bar{T}} e^{-rt}j(k_t, x_t)dt + \psi(\bar{T}, x_{\bar{T}})]$ is equivalent as maximizing $\int_0^{\bar{T}} e^{-rt}j(k_t, x_t)dt + \psi(\bar{T}, x_{\bar{T}})$. The latter is the problem I will solve next.

6.1.1 Necessary Conditions for Optimality

Let $\mathcal{H}(k, x, z) \equiv j(k, x) + z\lambda k$ be the current value Hamiltonian associated with the control problem defined above, where z_t is the Lagrange multiplier associated with the state equation $\dot{x}_t = \lambda k_t$.

Given that this is a fixed-time problem with terminal date $\bar{T}$, free terminal state x_T, and a salvage value of $\psi(\bar{T}, x_{\bar{T}})$, the necessary conditions for optimality are (see Kamien and Schwartz, 1991, p. 160):

$$\text{(U.1)} \quad \dot{x}_t = \lambda k_t \text{ with } x_0 = 0 \qquad\qquad \text{(U.2)} \quad k \in \arg\max_{\tilde{k}\in[0,1]} \mathcal{H}(\tilde{k}, x, z)$$

$$\text{(U.3)} \quad \dot{z}_t = r z_t - \frac{\partial j}{\partial x} \qquad\qquad \text{(U.4)} \quad z_{\bar{T}} = e^{r\bar{T}} \frac{\partial \psi}{\partial x_{\bar{T}}}$$

I then conjecture that for $\bar{T}$ sufficiently high, the optimal policy is a stopping time policy with fixed time T^A, i.e., $k_t = 1$ for $t \leq T^A$ and $k_t = 0$ for $t \in (T^A, \bar{T}]$. Note if the conjectured policy is indeed a candidate for an optimum, this verifies the conjecture that the optimal policy must be such that $k_0 > 0$. Moreover, it also verifies the claim (stated at the end of Section 4.2) that is without loss to restrict attention to allocations with the property that $k_t = 0$ for all $t \geq \bar{T}$, at least when solving this particular problem.

Now, given the conjectured optimal policy, (U.1) can be written as:

$$x_t = \begin{cases} \lambda t & \text{if } t \in [0, T^A] \\ \lambda T^A & \text{if } t \in (T^A, \bar{T}] \end{cases}$$

while (U.3) as:

$$\dot{z}_t = r z_t + \begin{cases} g\left(1 + \dfrac{1}{\mu}\right) e^{-\lambda t} + \mu s \left(\dfrac{1-\bar{q}}{\bar{q}}\right) e^{\lambda t} & \text{if } t \in [0, T^A] \\ s e^{-\lambda T^A} + \mu s \left(\dfrac{1-\bar{q}}{\bar{q}}\right) e^{\lambda T^A} & \text{if } t \in (T^A, \bar{T}] \end{cases}$$

Solving for z_t using (U.4) as border condition, yields:

$$z_t = \begin{cases} -\dfrac{1}{\lambda}\left[\dfrac{e^{-\lambda t}}{\mu}\left(g - (g-s)e^{-\lambda(1+\mu)(T^A - t)}\right) \right. \\ \qquad \left. + s e^{\lambda t}\left(\dfrac{1-\bar{q}}{\bar{q}}\right)\left(\dfrac{\mu}{\mu-1}\right)\left(1 - \dfrac{e^{-\lambda(\mu-1)(T^A-t)}}{\mu}\right)\right] & \text{if } t \in [0, T^A] \\ -\dfrac{s}{\lambda}\left[\dfrac{e^{-\lambda T_A}}{\mu} + \left(\dfrac{1-\bar{q}}{\bar{q}}\right) e^{\lambda T^A}\right] & \text{if } t \in (T^A, \bar{T}] \end{cases}$$

The final optimality condition is (U.2). Because $\mathcal{H}(k, x, z)$ is linear in k, this condition can be written as:

$$\frac{\partial \mathcal{H}}{\partial k_t} = \left[g\left(1 + \frac{1}{\mu}\right) - s\right] e^{-x_t} + \lambda z_t \begin{cases} \geq 0 & \text{if } t \in [0, T^A] \\ \leq 0 & \text{if } t \in (T^A, \bar{T}] \end{cases}$$

For this condition to hold, it must then be that:

$$\left[g\left(1+\frac{1}{\mu}\right)-s\right]e^{-\lambda T^A}+\lambda z_{T^A}=0$$

Solving for T^A one obtains that $T^A = T^{fb}/2$. Moreover, because x_t and z_t are constant for $t \in (T^A, \bar{T}]$, the latter immediately implies that $\partial\mathcal{H}/\partial k_t = 0$ for $t \in (T^A, \bar{T}]$. Thus, the only remaining loose end is checking that $\partial\mathcal{H}/\partial k_t \geq 0$ for $t \in [0, T^A)$.

To do this, note that when $t \in [0, T^A)$, then:

$$\frac{\partial\mathcal{H}}{\partial k_t} = \underbrace{(g-s)e^{-\lambda t}\left[1+\frac{e^{-\lambda(1+\mu)(T^A-t)}}{\mu}\right]-se^{\lambda t}\left(\frac{1-\bar{q}}{\bar{q}}\right)\left(\frac{\mu}{\mu-1}\right)\left(1-\frac{e^{-\lambda(\mu-1)(T^A-t)}}{\mu}\right)}_{\equiv\gamma(t)}$$

Note then that $\gamma(T^A) = 0$ and that:

$$\gamma'(t) = -\lambda(g-s)e^{-\lambda t}\left[1-e^{-\lambda(1+\mu)(T^A-t)}\right] \\ -\lambda se^{\lambda t}\left(\frac{1-\bar{q}}{\bar{q}}\right)\left(\frac{\mu}{\mu-1}\right)\left(1-e^{-\lambda(\mu-1)(T^A-t)}\right)<0$$

Thus, for $t \in [0, T^A)$, $\partial\mathcal{H}/\partial k_t = \gamma(t) > \gamma(T^A) = 0$, so the conjectured solution satisfies all the necessary conditions for optimality. □

6.1.2 Sufficiency

In the previous subsection, I characterized a candidate solution to the problem at hand based on the necessary conditions for optimality. Thus, we are left with proving that such a candidate is indeed optimal.

While the optimization program described above is not necessarily concave in x_t, observe that, defining $y_t \equiv e^{-x_t}$, the problem can be rewritten as maximizing $\int_0^{\bar{T}} e^{-rt}\tilde{j}(k_t, y_t)dt + \tilde{\psi}(\bar{T}, y_{\bar{T}})$ where $\dot{y}_t = -\lambda y_t k_t$, $y_0 = 1$, $y_{\bar{T}}$ free, and:

$$\tilde{j}(k_t, y_t) = \left[s(1-k_t)+k_t g\left(1+\frac{1}{\mu}\right)\right]y_t - s(1+\mu) - \frac{\mu s}{y_t}\left(\frac{1-\bar{q}}{\bar{q}}\right)$$
$$\tilde{\psi}(\bar{T}, y_{\bar{T}}) = \frac{s}{\lambda}\left[(1-e^{-r\bar{T}}) - \frac{e^{-r\bar{T}}}{\mu}(1-y_{\bar{T}}) + \left(\frac{1-\bar{q}}{\bar{q}}\right)\left(1-\frac{e^{-r\bar{T}}}{y_t}\right)\right]$$

Let $\mathcal{H}^\circ(y,z)$ be the maximized Hamiltonian, i.e., $\mathcal{H}^\circ(y,z) \equiv \max_{k\in[0,1]}\{\tilde{j}(k,y) - z\lambda yk\}$. Note then that $\partial^2\mathcal{H}^\circ/\partial y^2 = -2\mu s(1-\bar{q})/y^3 < 0$, so $\mathcal{H}^\circ(y,m)$ is strictly concave in y. Hence, sufficiency follows from Arrow's Sufficiency Theorem (Seierstad and Sydsaeter, 1987, Thm. 3.17). □

6.1.3 Limited Liability

The final step is verifying that the optimum described above satisfies the agent's limited liability. The latter, however, is easy. Since the optimal allocation is a stopping-time policy with fixed stopping time $T^{fb}/2$, this implies that the offered bonus is $B_t^A = \bar{B}_t^{stop}(T^{fb}/2)$. Since $\bar{B}_t^{stop}(T^{fb}/2)$ is strictly decreasing in t, and given that $\bar{B}_t^{stop}(T^{fb}/2) > 0$ when $t = T^{fb}/2$, it follows that $\bar{B}_t^{stop}(T^{fb}/2) > 0$ for all $t \in [0, T^{fb}/2]$. □

7 Proof of Lemma 4

Recall that $\mu \equiv r/\lambda$. I will focus on the case where $\mu \neq 1$ since the proof for when $\mu = 1$ is basically the same. Now, recall that $\bar{s}$ is defined as the unique solution to $\bar{u}(T^{fb}(\bar{s})/2; \bar{s}) = rf$, where:

$$\bar{u}(T; s) \equiv \bar{q}(g - s)\Big(1 - e^{-\lambda(1+\mu)T}\Big) - s(1 - \bar{q})\left(\frac{\mu}{\mu - 1}\right)\Big(1 - e^{-\lambda(\mu-1)T}\Big)$$

On the other hand, s^{fb} is defined as the unique solution to $w(\bar{q}; s^{fb}) - rf = s^{fb}$, where $w(q; s)$ is defined in the proof of Proposition 1 found in Section 2 of this online Appendix. However, to show that $\bar{s} < s^{fb}$, it is convenient to rewrite the condition $w(\bar{q}; s^{fb}) - rf = s^{fb}$ in a slightly difference way. In particular, using the expressions found in the proof of Proposition 1, it is straightforward to prove that $w(\bar{q}; s^{fb}) - rf = s^{fb}$ is equivalent to $\omega(T^{fb}(s^{fb}); s^{fb}) = rf$, where:

$$\omega(T; s) = \bar{q}(g - s)\Big(1 - e^{-\lambda(1+\mu)T}\Big) - s(1 - \bar{q})\Big(1 - e^{-\lambda\mu T}\Big)$$

Note then that:[6]

$$\frac{\partial}{\partial s}\bar{u}(T^{fb}(s)/2; s) = -\bar{q}\Big(1 - e^{-\lambda(1+\mu)T^{fb}(s)/2}\Big) - (1 - \bar{q})\left(\frac{\mu}{\mu - 1}\right)\Big(1 - e^{-\lambda(\mu-1)T^{fb}(s)/2}\Big) < 0$$
$$\frac{\partial}{\partial s}\omega(T^{fb}(s); s) = -\bar{q}\Big(1 - e^{-\lambda(1+\mu)T^{fb}(s)}\Big) - (1 - \bar{q})\Big(1 - e^{-\lambda\mu T^{fb}(s)}\Big) < 0$$

Hence, to prove that $\bar{s} < s^{fb}$ it is sufficient to show that $\bar{u}(T^{fb}(s)/2; s) < \omega(T^{fb}(s); s)$ for all $s \leq s^{fb}$. To do this, notice that:

$$\omega(T^{fb}(s); s) - \bar{u}(T^{fb}(s)/2; s) = \underbrace{\frac{(1 - \bar{q})s}{\mu^2 - 1}\Big[1 + \mu + (\mu - 1)M^{-\mu} - 2\mu M^{(1-\mu)/2}\Big]}_{\equiv \rho(M)}$$

[6]Note that the derivatives of $\bar{u}(T^{fb}(s)/2; s)$ and $\omega(T^{fb}(s); s)$ with respect to s taking into account that T^{fb} depends on s is the same as the derivative taking T^{fb} fixed. This follows by the envelope theorem since $T^{fb}/2 = \arg\max_T \bar{u}(T; s)$ and $T^{fb} = \arg\max_T \omega(T; s)$.

where:

$$M \equiv \frac{(1+\mu)(g-s)\bar{q}}{\mu s(1-\bar{q})} > 1 \quad (\text{since } q^{fb} < \bar{q} \text{ as } s \leq s^{fb})$$

Note then that $\rho(1) = 0$ and that:

$$\rho'(M) = \frac{(1-\bar{q})\mu s M^{-\mu}}{(1+\mu)M}\left[M^{(1+\mu)/2} - 1\right] > 0$$

Hence, $\rho(M) > \rho(1)$ for all $M > 1$, so $\bar{u}(T^{fb}(s)/2; s) < \omega(T^{fb}(s); s)$ for all $s \leq s^{fb}$. □

8 Proof of Lemma 6

Notice that $q_t(1-\bar{q})/(1-q_t) = \bar{q}\exp(-\int_0^t \lambda k_u du) = \bar{q}\exp(-x_t)$, where $x_t \equiv \int_0^t \lambda k_u du$. Hence,

$$\int_0^{\bar{T}} re^{-rt}\left[s(1-k_t) + q_t k_t g\left(1+\frac{\lambda}{r}\right)\right]\left(\frac{1-\bar{q}}{1-q_t}\right)dt = r\bar{q}\int_0^{\bar{T}} e^{-rt} f(k_t, x_t)dt$$

$$se^{-r\bar{T}}\left(\frac{1-\bar{q}}{1-q_{\bar{T}}}\right) - rf = r\bar{q}\phi(\bar{T}, x_{\bar{T}})$$

Consequently, given that r and $\bar{q}$ are constant, maximizing the principal's expected payoff is equivalent maximizing:

$$\int_0^{\bar{T}} e^{-rt} f(k_t, x_t)dt + \phi(\bar{T}, x_{\bar{T}})$$

Rewriting the constraint requires more work. Note first that:

$$\int_0^{\bar{T}} re^{-rt}(\lambda q_t k_t \bar{B}_t)\left(\frac{1-\bar{q}}{1-q_t}\right)dt = \bar{q}\int_0^{\bar{T}} re^{-rt}\lambda k_t \bar{B}_t e^{-x_t} dt$$

Now, the conjecture that $k_0 > 0$ implies that the local incentive constraint (4) characterized in Lemma 1 of the main text binds at $t = 0$:

$$\bar{q}\left(g\left(1+\frac{1}{\mu}\right) - \lambda\bar{B}_0\right) - s = \bar{q}\left[\frac{s}{\mu}e^{-r\bar{T}}e^{-x_{\bar{T}}} + \lambda\int_0^{\bar{T}} e^{-rt}\left[s(1-k_t) + k_t\left(g\left(1+\frac{1}{\mu}\right) - \lambda\bar{B}_t\right)\right]e^{-x_t}dt\right]$$

where I am already using the fact that $x_t \equiv \int_0^t \lambda k_u du$ and that $\mu \equiv r/\lambda$. Rearranging terms and noting that, according to Lemma 2 of the main text, $\lambda\bar{B}_0 = g\left(1+\frac{1}{\mu}\right) - \int_0^{\bar{T}} rse^{-rt}h(t)dt -$

$se^{-r\bar{T}}h(\bar{T})$, where $h(t) \equiv 1 + 1/\mu + e^{x_t}(1-\bar{q})/\bar{q}$, one obtains that:

$$\begin{aligned} &\bar{q}\int_0^{\bar{T}} re^{-rt}\lambda k_t \bar{B}_t e^{-x_t} dt \\ &\quad = \bar{q}\int_0^{\bar{T}} re^{-rt}\left[\left(s(1-k_t) + k_t g\left(1+\frac{1}{\mu}\right)\right)e^{-x_t} - s(1+\mu) - \mu s\left(\frac{1-\bar{q}}{\bar{q}}\right)e^{x_t}\right]dt \\ &\qquad + \bar{q}\mu s\left[(1-e^{-r\bar{T}}) - \frac{e^{-r\bar{T}}}{\mu}(1-e^{-x_{\bar{T}}}) + \left(\frac{1-\bar{q}}{\bar{q}}\right)(1-e^{-r\bar{T}}e^{x_{\bar{T}}})\right] \end{aligned}$$

Hence,

$$\int_0^{\bar{T}} re^{-rt}(\lambda q_t k_t \bar{B}_t)\left(\frac{1-\bar{q}}{1-q_t}\right)dt = r\bar{q}\left[\int_0^{\bar{T}} e^{-rt}G(k_t, x_t)dt + \varphi(x_{\bar{T}}, \bar{T})\right]$$

where I am using the fact that $\mu s/r = s/\lambda$. Consequently,

$$\int_0^{\bar{T}} re^{-rt}(\lambda q_t k_t \bar{B}_t)\left(\frac{1-\bar{q}}{1-q_t}\right)dt = rf \iff \int_0^{\bar{T}} e^{-rt}G(k_t, x_t)dt + \varphi(x_{\bar{T}}, \bar{T}) = \frac{f}{\bar{q}}$$

□

9 Proof of Lemma 7

Recall that the necessary conditions for optimality were:

(O.1) $\dot{x}_t = \lambda k_t$ with $x_0 = 0$

(O.2) $k \in \arg\max_{\tilde{k}\in[0,1]} \mathcal{H}(\tilde{k}, x, m, \xi)$

(O.3) $\dot{m}_t = rm_t - \dfrac{\partial f}{\partial x} - \xi\dfrac{\partial G}{\partial x}$

(O.4) $m_{\bar{T}} = e^{r\bar{T}}\left(\dfrac{\partial \phi}{\partial x_{\bar{T}}} + \xi\dfrac{\partial \varphi}{\partial x_{\bar{T}}}\right)$

(O.5) $\displaystyle\int_0^{\bar{T}} e^{-rt}G(k_t, x_t)dt + \varphi(\bar{T}, x_{\bar{T}}) = \frac{f}{\bar{q}}$

(O.6) $\xi \geq 0$

where $\mathcal{H}(k, x, m, \xi) \equiv f(k, x) + m\lambda k + \xi G(k, x)$ is current value Hamiltonian of the problem.

I then conjecture that for $\bar{T}$ sufficiently high, the optimal policy is a stopping time policy with fixed time T^*, i.e., $k_t^* = 1$ for $t \leq T^*$ and $k_t^* = 0$ for $t \in (T^*, \bar{T}]$. Under this policy, condition (O.5) can be written as $\bar{u}(T^*; s) = rf$, where $\bar{u}(T; s)$ is given by equation (7) of the main text. Note that because $s \leq \bar{s}$, such a T^* always exists. Moreover, because $\bar{u}(T; s)$ is strictly quasiconcave in T with a maximum at $T = T^{fb}/2$, then there are two such T^*: One to the left of $T^{fb}/2$, denoted by T^*_-, and one to the right of $T^{fb}/2$, denoted by T^*_+.

The conjecture that the optimal policy is a stopping time policy with fixed time T^* also implies that (O.1) can be written as:

$$x_t = \begin{cases} \lambda t & \text{if } t \in [0, T^*] \\ \lambda T^* & \text{if } t \in (T^*, \bar{T}] \end{cases}$$

while (O.3) as:

$$\dot{m}_t = r m_t + \begin{cases} (1+\xi) g \left(1 + \dfrac{1}{\mu}\right) e^{-\lambda t} + \xi \mu s \left(\dfrac{1-\bar{q}}{\bar{q}}\right) e^{\lambda t} & \text{if } t \in [0, T^*] \\ (1+\xi) s e^{-\lambda T^*} + \xi \mu s \left(\dfrac{1-\bar{q}}{\bar{q}}\right) e^{\lambda T^*} & \text{if } t \in (T^*, \bar{T}] \end{cases}$$

Solving for m_t using (O.4) as border condition, yields:

$$m_t = \begin{cases} -\dfrac{(1+\xi)}{\lambda} \left[\dfrac{e^{-\lambda t}}{\mu} \left(g - (g-s) e^{-\lambda(1+\mu)(T^*-t)} \right) \right. \\ \qquad \left. + s e^{\lambda t} \left(\dfrac{\xi}{1+\xi}\right) \left(\dfrac{1-\bar{q}}{\bar{q}}\right) \left(\dfrac{\mu}{\mu-1}\right) \left(1 - \dfrac{e^{-\lambda(\mu-1)(T^*-t)}}{\mu}\right) \right] & \text{if } t \in [0, T^*] \\ -\dfrac{s(1+\xi) e^{-\lambda T^*}}{r} \left[1 + \mu \left(\dfrac{\xi}{1+\xi}\right) \left(\dfrac{1-\bar{q}}{\bar{q}}\right) e^{\lambda T^*} \right] & \text{if } t \in (T^*, \bar{T}] \end{cases}$$

The final optimality condition is (O.2). Because $\mathcal{H}(k, x, m, \xi)$ is linear in k, this condition can be written as:

$$\frac{\partial \mathcal{H}}{\partial k_t} = (1+\xi) \left[g \left(1 + \frac{1}{\mu}\right) - s \right] e^{-x_t} - s \left(\frac{1-\bar{q}}{\bar{q}}\right) + \lambda m_t \quad \begin{cases} \geq 0 & \text{if } t \in [0, T^*] \\ \leq 0 & \text{if } t \in (T^*, \bar{T}] \end{cases}$$

For this condition to hold, it must then be that:

$$(1+\xi) \left[g \left(1 + \frac{1}{\mu}\right) - s \right] e^{-\lambda T^*} - s \left(\frac{1-\bar{q}}{\bar{q}}\right) + \lambda m_{T^*} = 0$$

so the Lagrange multiplier ξ is given by:

$$\xi = e^{\lambda T^*} (e^{\lambda T^*} - 1) \left[e^{2\lambda T^*} - \frac{\bar{q}(1+\mu)(g-s)}{\mu s (1-\bar{q})} \right]^{-1} - 1$$

Because $\xi \geq 0$, it must be:

$$e^{2\lambda T^*} - \frac{\bar{q}(1+\mu)(g-s)}{\mu s (1-\bar{q})} \geq 0$$

which holds if and only if $T^* \geq T^{fb}/2$. Thus, $T^* = T^*_+$, or equivalently, $T^* = \max\{T > 0 : \bar{u}(T; s) = rf\}$. Moreover, because x_t and m_t are constant for $t \in (T^*, \bar{T}]$, the latter immediately implies that $\partial \mathcal{H}/\partial k_t = 0$ for $t \in (T^*, \bar{T}]$. Thus, the only remaining loose end is checking that $\partial \mathcal{H}/\partial k_t \geq 0$ for $t \in [0, T^*)$.

To do this, note that when $t \in [0, T^*)$, then:

$$\frac{\partial \mathcal{H}}{\partial k_t} = (1+\xi)(g-s)e^{-\lambda t}\left[1 + \frac{e^{-\lambda(1+\mu)(T^*-t)}}{\mu}\right] - s\left(\frac{1-\bar{q}}{\bar{q}}\right)\left[1 + e^{\lambda t}\xi\left(\frac{\mu}{\mu-1}\right)\left(1 - \frac{e^{-\lambda(\mu-1)(T^*-t)}}{\mu}\right)\right] \equiv \Sigma(t)$$

Moreover, notice that $\Sigma(T^*) = 0$ and that:

$$\Sigma'(t) = -\lambda(1+\xi)(g-s)e^{-\lambda t}\left[1 - e^{-\lambda(1+\mu)(T^*-t)}\right] - \lambda s e^{\lambda t}\xi\left(\frac{1-\bar{q}}{\bar{q}}\right)\left(\frac{\mu}{\mu-1}\right)\left(1 - e^{-\lambda(\mu-1)(T^*-t)}\right) < 0$$

where the inequality follows because $\xi \geq 0$. Thus, for $t \in [0, T^*)$, $\partial\mathcal{H}/\partial k_t = \Sigma(t) > \Sigma(T^*) = 0$, so the candidate solution satisfies all the necessary conditions for optimality. □

10 Efficiency with Belief-Contingent Contracts

In this appendix, I show that if the principal could offer the agent a single belief-contingent bonus payable upon a breakthrough, the principal would obtain her first-best payoff in equilibrium (as noted in footnote 24 of the main text). Of course, as discussed in the main text (see footnote **??**), it is hard to envision a contract being contingent upon players' beliefs. Hence, rather than being taken seriously, the results that follow should be understood only as an interesting benchmark to keep in mind.

As a first step, I begin by formally defining players' strategies and payoffs. A strategy for the principal consists of choosing a belief-contingent bonus $B(q)$ at $t = 0^-$, and making a resource allocation decision, $k_t \in [0, 1]$, at every $t > 0$.

The bonus $B(q)$ is interpreted as follows: If a breakthrough occurs at time t, the agent receives a single payment by the amount $B(q_{t^-})$ where q_{t^-} is the belief that $\theta = 1$ immediately before the breakthrough occurred. Note, moreover, that due to the agent's limited liability and lack of wealth, the bonus must be such that $B(q) \geq 0$ for all q. The resource allocation decision, in turn, is contingent on all the information available at t, which can be summarized in (i) the belief-contingent bonus that the principal promised to the agent at $t = 0^-$, (ii) whether the agent installed R at $t = 0^+$ or not, and (iii) the belief at time t that $\theta = 1$.

A strategy for the agent, in turn, corresponds to an installation choice at $t = 0^+$, $a_0 \in \{0, 1\}$, as a function of the belief-contingent bonus offered by the principal at $t = 0^-$.

Let $\tau \geq 0$ be the time at which a breakthrough occurs. Given the players' actions, the principal's and the agent's expected payoffs at time 0, expressed in per-period units, are equal to:

$$\begin{aligned} v_0 =& \mathbb{E}\left[\int_0^\infty re^{-rt}\big[(a_0k_t\lambda\theta)\Pi + s(1-k_t)\big]dt - re^{-r\tau}B(q_{\tau^-})\right] \\ u_0 =& \mathbb{E}\left[re^{-r\tau}B(q_{\tau^-})\right] - ra_0f \end{aligned}$$

where q_t evolves according to $dq_t = -\lambda a_0 k_t q_t(1-q_t)dt$ with initial condition $q_0 = \bar{q}$.

The next lemma characterizes the equilibrium of this game:

Lemma 10.1. *In equilibrium, the principal achieves her first-best payoffs,* $v_0^{belief} = w_0^{fb} = w(\bar{q}; s) - rf$ *(where the superscript "belief" stands for "belief-contingent bonus"), by offering the following belief-contingent bonus:*

$$B(q) = \begin{cases} \dfrac{rf}{\lambda(w_0^{fb} + rf - s)}\left[\left(1+\dfrac{1}{\mu}\right)(g-s) - s\left(\dfrac{1-q}{q}\right)\right] & \text{if } q \geq q^{fb} \\ 0 & \text{otherwise} \end{cases}$$

In particular, this bonus induces the principal to follow the first-best resource allocation policy of Proposition 1, and leaves the agent indifferent between installing R or not.

To prove this lemma, it suffices to show that under the proposed contract: (i) the principal follows the first-best resource allocation policy if the agent installs R, and (ii) that the agent is indifferent between installing R or not. Claims 10.2 and 10.3 prove points (i) and (ii), respectively.

Claim 10.2. *If the principal commits to* $B(q)$ *as in Lemma 10.1, then she follows the first-best resource allocation policy if the agent installs R.*

Proof. If the agent installs R, the principal's resource allocation problem is a dynamic programming problem with the belief q as the state variable. The Hamilton-Jacobi-Bellman equation (HJB) associated with this problem is:

$$rv(q) = \max_{k\in[0,1]}\left\{r\left[(1-k)s + qk(g-\lambda B(q))\right] + \lambda kq(g - v(q)) - \lambda kq(1-q)v'(q)\right\}$$

where $B(q)$ is given as in Lemma 10.1.

I then conjecture that the principal follows a cutoff strategy (as in the first-best): $k^*(q) = 1$ if $q \geq q^\dagger$, and $k^*(q) = 0$, otherwise. If so, the HJB equation plus the value-matching and the smooth-pasting conditions $v(q^\dagger) = s$ and $v'(q^\dagger) = 0$, imply that $q^\dagger = q^{fb}$, and that $v(q)$ is given as follows:

$$v(q) = \begin{cases} (1-\gamma)gq + \gamma s + (1-\gamma)(s - gq^{fb})\left(\dfrac{1-q}{1-q^{fb}}\right)\left(\dfrac{\Omega(q)}{\Omega(q^{fb})}\right)^\mu & \text{if } q \geq q^{fb} \\ s & \text{otherwise} \end{cases} \tag{18}$$

where $\gamma \equiv rf/(w_0^{fb} + rf - s) < 1$ (since $s < s^{fb}$). Note that $v(q)$ is continuously differentiable in q, so the standard verification/sufficiency arguments hold (e.g., Yong and Zhou, 1999, Theorem 3.7, p. 241). Thus, to show that the principal follows the first-best resource allocation policy, it is sufficient to show that:

$$k^{fb}(q) = \arg\max_{k\in[0,1]} \left\{ r\left[(1-k)s + qk(g - \lambda B(q))\right] + \lambda kq(g - v(q)) - \lambda kq(1-q)v'(q) \right\}$$

where:

$$k^{fb}(q) = \begin{cases} 1 & \text{if } q \geq q^{fb} \\ 0 & \text{if } q < q^{fb} \end{cases}$$

This is equivalent as showing that $\phi(q) \geq 0$ if $q \geq q^{fb}$ and that $\phi(q) < 0$, otherwise, where:

$$\phi(q) \equiv r\left[-s + q(g - \lambda B(q))\right] + \lambda q(g - v(q)) - \lambda q(1-q)v'(q) \tag{19}$$

To show this last point, consider first $q \geq q^{fb}$. Using the relevant expressions for $B(q)$ and $v(q)$ for this case yields that:

$$\phi(q) = \lambda s(1-\gamma)\left(\frac{\mu}{1+\mu}\right)\left[\mu(q - q^{fb}) - q^{fb}(1-q)\left(1 - \left(\frac{\Omega(q)}{\Omega(q^{fb})}\right)^{\mu}\right)\right] \geq 0$$

where the last inequality follows because $\gamma < 1$ and the fact that the term in square brackets is strictly increasing in q and is equal to zero when $q = q^{fb}$.

Consider now $q < q^{fb}$. Then using the relevant expression for $v(q)$ for this case, yields:

$$\phi(q) = -\lambda s\mu(1-\gamma)\left(1 - \frac{q}{q^{fb}}\right) < 0 \quad \text{as } \gamma < 1 \text{ and } q < q^{fb}$$

□

Claim 10.3. *If the principal commits to $B(q)$ as in Lemma 10.1, then the agent is indifferent between installing R or not.*

Proof. If the agent does not install R, then he gets zero. To obtain the agent's expected payoff if he installs R, on the other hand, let $u(q)$ be the agent's value function when R is installed, a breakthrough has not occurred, and the belief that $\theta = 1$ is q.

Since the principal plays $k^{fb}(q) = 0$ for $q < q^{fb}$, then $u(q) = 0$ for $q < q^{fb}$ as no breakthrough is possible in this region of the belief space. In contrast, if $q \geq q^{fb}$, then $k^{fb}(q) = 1$, so $u(q)$ must satisfy $ru(q) = \lambda q(rB(q) - u(q)) - \lambda q(1-q)u'(q)$ with terminal condition $u(q^{fb}) = 0$ where $B(q)$

as in Lemma 10.1. Solving this differential equation I obtain that:

$$u(q) = \begin{cases} \gamma \left[gq + (s - gq^{fb}) \left(\frac{1-q}{1-q^{fb}} \right) \left(\frac{\Omega(q)}{\Omega(q^{fb})} \right)^{\mu} - s \right] & \text{if } q \geq q^{fb} \\ 0 & \text{otherwise} \end{cases}$$

where $\gamma \equiv rf/(w_0^{fb} + rf - s) < 1$. Hence, the agent's expected payoff if he installs R is:

$$\begin{aligned} u(\bar{q}) - rf &= \gamma \left[g\bar{q} + (s - gq^{fb}) \left(\frac{1-\bar{q}}{1-q^{fb}} \right) \left(\frac{\Omega(\bar{q})}{\Omega(q^{fb})} \right)^{\mu} - s \right] - rf \\ &= \gamma[w_0^{fb} + rf - s] - rf \\ &= \left(\frac{rf}{w_0^{fb} + rf - s} \right) [w_0^{fb} + rf - s] - rf = 0 \end{aligned}$$

Thus, the agent's expected payoff if he installs R is also zero, so the agent is indifferent between installing R or not. □

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