# Job Matching under Constraints

**Fuhito Kojima, Ning Sun, and Ning Neil Yu**

## Online Appendix

### A. Proof of Lemma 1

We first present two conditions equivalent to the substitutes condition when there are constraints. It is useful to define, for each $A \subset D$, $\mathbf{e}^A \in \mathbb{R}^D$ to be the **indicator function for** $A$, i.e., the salary schedule with all outputs zero except that $\mathbf{e}^A(d) = 1$ for every $d \in A$. Write $\mathbf{e}^d := \mathbf{e}^{\{d\}}$ and $\mathbf{1} := \mathbf{e}^D$ for simplicity.

**Lemma 3.** *For a revenue function $R$ and a feasibility collection $\mathcal{F}$, the following statements are equivalent.*

1. *(Substitutes Condition) $R|_{\mathcal{F}}$ satisfies the substitutes condition.*
2. *(Single-Improvement Property) For any salary schedule $\mathbf{s}$ and $A \in \mathcal{F}$ such that $A \notin X(\mathbf{s}; \mathcal{F})$, there exists $A' \in \mathcal{F}$ such that $V(A; \mathbf{s}) < V(A'; \mathbf{s})$, $|A \setminus A'| \leq 1$, and $|A' \setminus A| \leq 1$.*
3. *(Monotone Substitutes Condition) For any two salary schedules $\mathbf{s}$ and $\mathbf{s}'$ with $\mathbf{s}' \geq \mathbf{s}$, and any $A \in X(\mathbf{s}; \mathcal{F})$, there exists $A' \in X(\mathbf{s}'; \mathcal{F})$ such that $I(A, \mathbf{s}, \mathbf{s}') \subset A'$ and $|A'| \leq |A|$.*

*Proof.* There is a short proof of the equivalence between the substitutes condition and the single-improvement property in our context; it is almost identical to the one in Sun and Yang (2009), so we skip it. The monotone substitutes condition obviously strengthens the substitutes condition.

We only need to show that the substitutes condition implies the monotone substitutes condition. Take an arbitrary salary schedule $\mathbf{s}$ and $A \in X(\mathbf{s}; \mathcal{F})$. For a fixed doctor $d$, define $\sigma(\delta) = \mathbf{s} + \delta \mathbf{e}^d$, where $\delta \geq 0$. Clearly, it suffices to prove the conclusion for the case of $\mathbf{s}' = \sigma(\bar{\delta})$ where $\bar{\delta}$ is an arbitrary positive number. When $d \notin A \in X(\mathbf{s}; \mathcal{F})$, $A \in X(\mathbf{s}'; \mathcal{F})$, so setting $A' = A$ establishes the lemma. As a result, we assume $d \in A$.

If all feasible sets contain $d$, $X(\mathbf{s}'; \mathcal{F}) = X(\mathbf{s}; \mathcal{F})$, and setting $A' = A$ establishes the lemma. Otherwise let $\delta^* := V(A; \mathbf{s}) - \phi^*$, where $\phi^* := \max\{V(B; \mathbf{s}) : d \notin B \in \mathcal{F}\}$. It is possible that $\delta^* = 0$ or $\delta^* > 0$. We can make some observations.

Whenever $0 \le \delta \le \delta^*$, $A$ is optimal and $\Pi(\sigma(\delta);\mathcal{F}) = \Pi(\mathbf{s};\mathcal{F}) - \delta$: when $0 \le \delta < \delta^*$, $X(\sigma(\delta);\mathcal{F}) = X(\mathbf{s};\mathcal{F})$; at $\delta^*$, $\Pi(\sigma(\delta^*);\mathcal{F}) = \phi^*$ and $X(\sigma(\delta^*);\mathcal{F}) = X(\mathbf{s};\mathcal{F}) \cup \mathcal{B}$, where $\mathcal{B} \coloneqq \{B \in \mathcal{F} : d \notin B \text{ and } V(B;\mathbf{s}) = \phi^*\}$. Whenever $\delta > \delta^*$, $\Pi(\sigma(\delta);\mathcal{F}) = \phi^*$ and $X(\sigma(\delta);\mathcal{F}) = \mathcal{B}$: those in $X(\mathbf{s};\mathcal{F})$ and containing $d$ are no longer optimal.

If $\bar{\delta} \le \delta^*$, $A \in X(\mathbf{s}';\mathcal{F})$, so setting $A' = A$ establishes this part of the proof. If $\bar{\delta} > \delta^*$, choose $\epsilon > 0$ such that $\Pi(\sigma(\delta^*);\mathcal{F}) - \epsilon > V(C;\sigma(\delta^*))$ for every $C \in \mathcal{F}$ not in $X(\sigma(\delta^*);\mathcal{F})$. Since $A \notin X(\sigma(\delta^* + \epsilon);\mathcal{F})$, by the single-improvement property, there exists $A' \subset \mathcal{F}$ such that $|A \setminus A'| \le 1$, $|A' \setminus A| \le 1$, and

$$V(A';\sigma(\delta^* + \epsilon)) > V(A;\sigma(\delta^* + \epsilon)) = V(A;\sigma(\delta^*)) - \epsilon = \Pi(\sigma(\delta^*);\mathcal{F}) - \epsilon,$$

which implies $d \notin A'$. Hence, $I(A,\mathbf{s},\mathbf{s}') \subset A'$ and $|A'| \le |A|$. We also have $V(A';\sigma(\delta^*)) = V(A';\sigma(\delta^* + \epsilon)) > \Pi(\sigma(\delta^*);\mathcal{F}) - \epsilon$. Since $\epsilon$ is smaller than the gap between the maximal profit and second best (if it exists) under $\sigma(\delta^*)$, $A' \in X(\sigma(\delta^*);\mathcal{F})$. Since $d \notin A'$, we also have $A' \in X(\mathbf{s}';\mathcal{F})$. $\square$

Given this lemma, we can use the substitutes condition interchangeably with the others. The single-improvement property is a natural extension of the concept in Gul and Stacchetti (1999) to our setting with constraints; it states that any feasible but suboptimal set of doctors can be improved upon by adding or dropping one doctor, or replacing one doctor with another. The monotone substitutes condition is seemingly stronger than the substitutes condition, but the above lemma demonstrates that they are in fact equivalent.[1,2]

For $A \subset D$ and $d \in D$, let $A + d \coloneqq A \cup \{d\}$ and $A - d \coloneqq A \setminus \{d\}$. A feasibility collection $\mathcal{F}$ is **exchangeable** if for any $A, B \in \mathcal{F}$ and $d \in A \setminus B$, either $A - d \in \mathcal{F}$ and $B + d \in \mathcal{F}$, or there exists $d' \in B \setminus A$ such that $A - d + d' \in \mathcal{F}$ and $B + d - d' \in \mathcal{F}$, or both.[3] It is well known that for $R|_{\mathcal{F}}$ to satisfy the substitutes condition, $\mathcal{F}$ must be exchangeable (Murota, 2016). As a result, preserving the substitutes condition for any unrestricted revenue function $R$ also requires $\mathcal{F}$ to be exchangeable.

Exchangeability requires that if a set of doctors and one of its proper subsets

[1] It resembles the monotone substitutability condition of Hatfield et al. (2019), but our result is not directly implied by theirs.

[2] Our equivalence result implies the **law of aggregate demand** (Hatfield and Milgrom, 2005) when the substitutes condition is satisfied and demands are single-valued, that is, whenever $\mathbf{s}' \ge \mathbf{s}$, $X(\mathbf{s};\mathcal{F}) = \{A\}$, and $X(\mathbf{s}';\mathcal{F}) = \{A'\}$, we have $|A'| \le |A|$.

[3] This property is called $\mathrm{B}^\natural$-EXC by Murota (2016).

are both feasible, all sets between those two are feasible.

**Lemma 4** (Discrete Continuity)**.** *For an exchangeable feasibility collection $\mathcal{F}$, $A, B \in \mathcal{F}$ and $A \subset C \subset B$ imply $C \in \mathcal{F}$.*

*Proof.* Let $C \setminus A = \{d_1, d_2, \ldots, d_l\}$. We can apply exchangeability to $B, A \in \mathcal{F}$ and $d_1 \in B \setminus A$, and find that $A \cup \{d_1\} \in \mathcal{F}$. We can then apply exchangeability to $B, A \cup \{d_1\} \in \mathcal{F}$, and $d_2 \in B \setminus (A \cup \{d_1\})$, and find that $A \cup \{d_1, d_2\} \in \mathcal{F}$. The result that $A \cup \{d_1, d_2, \ldots, d_l\} \in \mathcal{F}$ follows from induction. $\square$

For a feasibility collection $\mathcal{F}$, we have $\overline{\chi}(\mathcal{F}) \coloneqq \cap_{A \in \mathcal{F}} A$ as its always-hired set and $\underline{\chi}(\mathcal{F}) \coloneqq D \setminus (\cup_{A \in \mathcal{F}} A)$ as its never-hired set, so $\hat{\chi}(\mathcal{F}) \coloneqq D \setminus (\overline{\chi}(\mathcal{F}) \cup \underline{\chi}(\mathcal{F}))$ is its real-decision set. If $\mathcal{F}$ satisfies $\hat{\chi}(\mathcal{F}) = D$, we call it **proper**. For example, a nondegenerate interval constraint, whose floor is not $M$ and ceiling is not 0, is proper. A proper and exchangeable feasibility collection can "separate" any two distinct doctors.

**Lemma 5** (Separability)**.** *For a proper and exchangeable feasibility collection $\mathcal{F}$ and two distinct doctors $d$ and $d'$, there exists $A \in \mathcal{F}$ such that $d \in A$ and $d' \notin A$.*

*Proof.* Suppose, for contradiction, that for every $A \in \mathcal{F}$ such that $d \in A$, we have $d' \in A$. Because $\mathcal{F}$ is proper, there exist $B, C \in \mathcal{F}$ such that $d, d' \in B$ and $d' \notin C$. But then $d \notin C$. We can apply exchangeability to $B, C \in \mathcal{F}$ and $d \in B \setminus C$, and find that either $C + d$ is feasible, or there exists $d'' \in C$ such that $C + d - d''$ is feasible, or both. Because $d' \notin C$, the assumption at the beginning of the proof is contradicted. $\square$

A feasibility collection $\mathcal{F}$ is **anonymous within** $A \subset D$ if $B \in \mathcal{F}$, $d \in A \cap B$, and $d' \in A \setminus B$ imply $B - d + d' \in \mathcal{F}$. In other words, we can replace a member of $A$ in a feasible set with another member of $A$ not in it to create a new feasible set. The next lemma shows that if a proper feasibility collection preserves the substitutes condition for a revenue function that is binary disjunctive over two doctors, then it is anonymous between those two doctors.

**Lemma 6** (Binary Anonymity)**.** *Given a binary disjunctive revenue function $R$ over $d, d' \in D$ and a proper feasibility collection $\mathcal{F}$, if $R|_{\mathcal{F}}$ satisfies the substitutes condition, then $\mathcal{F}$ is anonymous within $\{d, d'\}$.*

*Proof.* Consider an arbitrary $A \in \mathcal{F}$ such that $d \in A$ and $d' \notin A$. Since $\mathcal{F}$ must be exchangeable, according to the last lemma, there exists $B \in \mathcal{F}$ such that $d' \in B$ and $d \notin B$. Consider a salary schedule that is 0 except that $s_d = 1$. The profit of $B$ is 1 and the profit of $A$ is 0. To achieve single improvement (Lemma 3) over $A$, adding or removing a doctor is not helpful, which leaves the option of replacing one doctor with another. Only replacing $d$ with $d'$ strictly increases profit, so $A - d + d' \in \mathcal{F}$.

The case of $d' \in A$ and $d \notin A$ is symmetric. □

A proper feasibility collection not defined by an interval constraint fails to preserve the substitutes condition for the class of binary disjunctive revenue functions. Before the proof, for a collection of finite sets $\mathcal{G}$, denote the **maximal cardinality** of its elements by $\overline{\omega}(\mathcal{G})$, and let $\overline{\Omega}(\mathcal{G}) := \{G \in \mathcal{G} : |G| = \overline{\omega}(\mathcal{G})\}$; denote the **minimal cardinality** by $\underline{\omega}(\mathcal{G})$, and let $\underline{\Omega}(\mathcal{G}) := \{G \in \mathcal{G} : |G| = \underline{\omega}(\mathcal{G})\}$.

**Lemma 7.** *If a proper feasibility collection preserves the substitutes condition for all binary disjunctive revenue functions, then it is defined by an interval constraint.*

*Proof.* By Lemma 6, $\mathcal{F}$ is anonymous within all pairs of doctors, and thus within $D$. By anonymity, all sets of doctors with the minimal cardinality $\underline{\omega}(\mathcal{F})$ and the maximal cardinality $\overline{\omega}(\mathcal{F})$ are feasible. We only need to show that for any $A \subset D$ such that $\underline{\omega}(\mathcal{F}) < |A| < \overline{\omega}(\mathcal{F})$, $A \in \mathcal{F}$. But $\mathcal{F}$ must contain a proper subset of $A$ with the minimal cardinality and a set with the maximal cardinality that includes $A$. We can apply Lemma 4 to these two sets and see $A \in \mathcal{F}$. □

*Proof of Lemma 1.* We prove the contrapositive. Let $\mathcal{F}$ be a feasibility collection not defined by a generalized interval constraint. Consider a proper feasibility collection $\mathcal{F}' := \{A \cap \hat{\chi}(\mathcal{F}) : A \in \mathcal{F}\}$ on $D' := \hat{\chi}(\mathcal{F})$; $\mathcal{F}'$ is proper but not defined by an interval constraint, so by Lemma 7, it fails to preserve the substitutes condition for all binary disjunctive revenue functions on $2^{D'}$. Hence, there is a binary disjunctive $R' : 2^{D'} \to \mathbb{R}$ such that $R'|_{\mathcal{F}'}$ fails the substitutes condition. Extend $R'$ to $2^D$ by making $R(A) = R'(A \cap \hat{\chi}(\mathcal{F}))$ for every $A \subset D$. It is routine to check that $R$ is binary disjunctive but $R|_{\mathcal{F}}$ fails the substitutes conditions. □

## B. Proof of Theorem 1

We establish several properties of demand correspondences that satisfy the substitutes condition. First, assuming the substitutes condition, the following lemma demonstrates a rich structure of the collection of demand sets given a fixed salary schedule. Whenever these demand sets feature multiple cardinalities, say, with $\overline{m}$ as the maximum and $\underline{m}$ as the minimum, the collection is a union of $(\overline{m} - \underline{m} + 1)$-element chains whose elements, with cardinalities covering all integers from $\underline{m}$ to $\overline{m}$, are totally ordered by strict set inclusion.

**Lemma 8.** *If a demand correspondence $X(\,\cdot\,;\mathcal{F})$ satisfies the substitutes condition, then for any salary schedule $\mathbf{s}$ and any $A \in X(\mathbf{s};\mathcal{F})$, there exists a sequence $(A_j)_{j\in J}$, where $J := \{\underline{\omega}(X(\mathbf{s};\mathcal{F})), \ldots, \overline{\omega}(X(\mathbf{s};\mathcal{F}))\}$, such that $A_{|A|} = A$ and for every $j$, $|A_j| = j$, $A_j \in X(\mathbf{s};\mathcal{F})$, and $A_j \subset A_{j+1}$.*

*Proof.* We first prove the existence of $A_j$ for $j \in \{|A|+1, \ldots, \overline{\omega}(X(\mathbf{s};\mathcal{F}))\}$, when $|A| < \overline{\omega}(X(\mathbf{s};\mathcal{F}))$. Consider $\epsilon > 0$ such that $M\epsilon < \Pi(\mathbf{s};\mathcal{F}) - \max\{V(C;\mathbf{s}) : C \in \mathcal{F} \setminus X(\mathbf{s};\mathcal{F})\}$. Let $\mathbf{s}' = \mathbf{s} - \epsilon\mathbf{1}$. Under $\mathbf{s}'$, a suboptimal set $C$ under $\mathbf{s}$ can never generate a profit of at least $\Pi(\mathbf{s};\mathcal{F})$, so $X(\mathbf{s}';\mathcal{F}) = \overline{\Omega}(X(\mathbf{s};\mathcal{F}))$. Thus, $A \notin X(\mathbf{s}';\mathcal{F})$. By the single-improvement property, there exists $B \in \mathcal{F}$ such that $|A \setminus B| \leq 1$, $|B \setminus A| \leq 1$, and

$$V(A;\mathbf{s}) + |A|\epsilon \;=\; V(A;\mathbf{s}') \;<\; V(B;\mathbf{s}') \;=\; V(B;\mathbf{s}) + |B|\epsilon.$$

But $A \in X(\mathbf{s};\mathcal{F})$, so $V(A;\mathbf{s}) = \Pi(\mathbf{s};\mathcal{F}) \geq V(B;\mathbf{s})$. Therefore, $|B| = |A| + 1$ and $A \subset B$. Moreover,

$$V(B;\mathbf{s}) \;>\; V(A;\mathbf{s}) - \epsilon \;=\; \Pi(\mathbf{s};\mathcal{F}) - \epsilon,$$

which implies $B \in X(\mathbf{s};\mathcal{F})$. Thus, $B$ can serve as $A_{|A|+1}$. Applying the same argument to $A_{|A|+1}$, we get $A_{|A|+2}$. Iteratively, other $A_j$ with $j > |A|$ can be obtained.

Now, we show the existence of $A_j$ for $j \in \{\underline{\omega}(X(\mathbf{s};\mathcal{F})), \ldots, |A|-1\}$, when $|A| > \underline{\omega}(X(\mathbf{s};\mathcal{F}))$. Let $\mathbf{s}'' = \mathbf{s} + \epsilon\mathbf{1}$. Similar arguments give us $X(\mathbf{s}'';\mathcal{F}) = \underline{\Omega}(X(\mathbf{s};\mathcal{F}))$. Thus, $A \notin X(\mathbf{s}'';\mathcal{F})$. By the single-improvement property, there exists $B' \in \mathcal{F}$ such that $|A \setminus B'| \leq 1$, $|B' \setminus A| \leq 1$, and

$$V(A;\mathbf{s}) - |A|\epsilon \;=\; V(A;\mathbf{s}'') \;<\; V(B';\mathbf{s}'') \;=\; V(B';\mathbf{s}) - |B'|\epsilon.$$

But $A \in X(\mathbf{s};\mathcal{F})$, so $V(A;\mathbf{s}) = \Pi(\mathbf{s};\mathcal{F}) \geq V(B';\mathbf{s})$. Therefore, $|B'| = |A|-1$ and $B' \subset A$. Moreover,

$$V(B';\mathbf{s}) \;>\; V(A;\mathbf{s}) - \epsilon \;=\; \Pi(\mathbf{s};\mathcal{F}) - \epsilon,$$

which implies $B' \in X(\mathbf{s};\mathcal{F})$. Thus, $B'$ can serve as $A_{|A|-1}$. Applying the same argument to $A_{|A|-1}$, we get $A_{|A|-2}$. Iteratively, other $A_j$ with $j < |A|$ can be obtained. $\square$

The next lemma shows how a hospital's demand sets change when we adjust salaries uniformly (uniform changes of salaries turn out to play an important role in subsequent analysis).

**Lemma 9.** *For a fixed salary schedule* $\mathbf{s}$*, let* $\sigma(\delta) \coloneqq \mathbf{s} + \delta\mathbf{1}$ *for every* $\delta \in \mathbb{R}$*. For any revenue function* $R$ *and feasibility collection* $\mathcal{F}$ *with* $\overline{\omega}(\mathcal{F}) > \underline{\omega}(\mathcal{F})$*, there exists a unique strictly increasing sequence* $(\delta_k)_{k\in\mathcal{K}}$*, where* $\mathcal{K} = \{1, 2, \ldots, K\}$*, such that the following is true:*

*(i) for any* $k \in \mathcal{K}$*,* $\overline{\omega}(X(\sigma(\delta_k);\mathcal{F})) > \underline{\omega}(X(\sigma(\delta_k);\mathcal{F}))$*;*

*(ii) for any* $\delta < \delta_1$*,* $X(\sigma(\delta);\mathcal{F}) \subset \overline{\Omega}(\mathcal{F})$*; for any* $\delta > \delta_K$*,* $X(\sigma(\delta);\mathcal{F}) \subset \underline{\Omega}(\mathcal{F})$*;*

*(iii) for any* $k \in \mathcal{K} \setminus \{K\}$ *and* $\delta \in (\delta_k, \delta_{k+1})$*,*

$$\underline{\Omega}(X(\sigma(\delta_k);\mathcal{F})) = X(\sigma(\delta);\mathcal{F}) = \overline{\Omega}(X(\sigma(\delta_{k+1});\mathcal{F})).$$

*Moreover, given any* $\underline{\delta} < \overline{\delta}$*, if the demand correspondence* $X(\,\cdot\,;\mathcal{F})$ *satisfies the substitutes condition, then*

*(i) for any* $\underline{A} \in X(\sigma(\underline{\delta});\mathcal{F})$ *and* $\overline{m} \in \{\underline{\omega}(X(\sigma(\overline{\delta});\mathcal{F})), \ldots, \overline{\omega}(X(\sigma(\overline{\delta});\mathcal{F}))\}$*, there exists* $\overline{A} \in X(\sigma(\overline{\delta});\mathcal{F})$ *such that* $|\overline{A}| = \overline{m}$ *and* $\overline{A} \subset \underline{A}$*;*

*(ii) for any* $\overline{B} \in X(\sigma(\overline{\delta});\mathcal{F})$ *and* $\underline{m} \in \{\underline{\omega}(X(\sigma(\underline{\delta});\mathcal{F})), \ldots, \overline{\omega}(X(\sigma(\underline{\delta});\mathcal{F}))\}$*, there exists* $\underline{B} \in X(\sigma(\underline{\delta});\mathcal{F})$ *such that* $|\underline{B}| = \underline{m}$ *and* $\overline{B} \subset \underline{B}$*.*

*Proof.* Starting from a sufficiently low $\hat{\delta} \in \mathbb{R}$ such that $X(\sigma(\hat{\delta});\mathcal{F}) \subset \overline{\Omega}(\mathcal{F})$, imagine that $\delta$ grows at a uniform rate of 1 per unit time to become $\hat{\delta}+t$ at time $t$, so for any $C \in \mathcal{F}$, $V(C;\sigma(\delta))$ decreases at a uniform rate of $|C|$ per unit time. We construct the sequence $(\delta_k)_{k\in\mathcal{K}}$ as follows.

Set $t_1 = \inf\{t > 0 : \overline{\omega}(X(\sigma(\hat{\delta}+t);\mathcal{F})) > \underline{\omega}(X(\sigma(\hat{\delta}+t);\mathcal{F}))\}$, and $\delta_1 = \hat{\delta} + t_1$. Clearly, $t_1 < \infty$, $\overline{\omega}(X(\sigma(\delta_1);\mathcal{F})) > \underline{\omega}(X(\sigma(\delta_1);\mathcal{F}))$, and $X(\sigma(\hat{\delta});\mathcal{F}) = \overline{\Omega}(X(\sigma(\delta_1);\mathcal{F}))$. Iteratively, given $t_k$ and $\delta_k$ such that $\underline{\omega}(X(\sigma(\delta_k);\mathcal{F})) > \underline{\omega}(\mathcal{F})$, set

$$t_{k+1} = \inf\{t > t_k : \overline{\omega}(X(\sigma(\hat{\delta}+t);\mathcal{F})) > \underline{\omega}(X(\sigma(\hat{\delta}+t);\mathcal{F}))\},$$

and $\delta_{k+1} = \hat{\delta} + t_{k+1}$. Clearly, $t_{k+1} < \infty$, $\overline{\omega}(X(\sigma(\delta_{k+1}); \mathcal{F})) > \underline{\omega}(X(\sigma(\delta_{k+1}); \mathcal{F}))$, and for any $\delta \in (\delta_k, \delta_{k+1})$, $\underline{\Omega}(X(\sigma(\delta_k); \mathcal{F})) = X(\sigma(\delta); \mathcal{F}) = \overline{\Omega}(X(\sigma(\delta_{k+1}); \mathcal{F}))$. The iteration has to stop at some $t_K$ and $\delta_K$ such that $\underline{\omega}(X(\sigma(\delta_K); \mathcal{F})) = \underline{\omega}(\mathcal{F})$.

The uniqueness of $(\delta_k)_{k \in \mathcal{K}}$ is obvious.

For the second half of the lemma, first consider the case that there is no $\delta_k$ in $[\underline{\delta}, \overline{\delta}]$. The first statement implies $X(\sigma(\overline{\delta}); \mathcal{F}) = X(\sigma(\underline{\delta}); \mathcal{F}))$ and $\overline{\omega}(X(\sigma(\overline{\delta}); \mathcal{F})) = \underline{\omega}(X(\sigma(\overline{\delta}); \mathcal{F}))$. We can set $\overline{A} = \underline{A}$ and $\underline{B} = \overline{B}$.

Second, consider the case where there exists some $\delta_k$ in $[\underline{\delta}, \overline{\delta}]$. Let $\underline{k} = \min\{k \in \mathcal{K} | \delta_k \in [\underline{\delta}, \overline{\delta}]\}$ and $\overline{k} = \max\{k \in \mathcal{K} | \delta_k \in [\underline{\delta}, \overline{\delta}]\}$. From the first statement of the lemma, we know $\underline{A} \in X(\sigma(\delta_{\underline{k}}); \mathcal{F})$ and $\overline{B} \in X(\sigma(\delta_{\overline{k}}); \mathcal{F})$. By Lemma 8, there exists $A_{\underline{k}} \in \underline{\Omega}(X(\sigma(\delta_{\underline{k}}); \mathcal{F})) = \overline{\Omega}(X(\sigma(\delta_{\underline{k}+1}); \mathcal{F}))$ such that $A_{\underline{k}} \subset \underline{A}$. Iteratively, given $A_{\underline{k}+i}$, we can find $A_{\underline{k}+i+1}$ such that $A_{\underline{k}+i+1} \in \underline{\Omega}(X(\sigma(\delta_{\underline{k}+i+1}); \mathcal{F})) = \overline{\Omega}(X(\sigma(\delta_{\underline{k}+i+2}); \mathcal{F}))$ and $A_{\underline{k}+i+1} \subset A_{\underline{k}+i} \subset \ldots \subset \underline{A}$, until we obtain $A_{\overline{k}-1}$. We can apply Lemma 8 at $\delta_{\overline{k}}$ again to obtain $\overline{A}$.

Similarly, we can find $B_{\overline{k}} \in \overline{\Omega}(X(\sigma(\delta_{\overline{k}}); \mathcal{F}))$ such that $B_{\overline{k}} \supset \overline{B}$, and iteratively, $B_{\overline{k}-1} \supset B_{\overline{k}}$ with $B_{\overline{k}-1} \in \overline{\Omega}(X(\sigma(\delta_{\overline{k}-1}); \mathcal{F}))$, $B_{\overline{k}-2} \supset B_{\overline{k}-1}$ with $B_{\overline{k}-2} \in \overline{\Omega}(X(\sigma(\delta_{\overline{k}-2}); \mathcal{F}))$, and so on, until we obtain $B_{\underline{k}+1}$. We can apply Lemma 8 at $\delta_{\underline{k}}$ again to obtain $\underline{B}$. $\square$

Figure 1 illustrates the first half of Lemma 9. Note that this part of the lemma does not require the substitutes condition. Lemma 8 implies that when the substitutes condition is assumed, for each $\delta_k$ in Figure 1, every integer between the maximal cardinality and the minimal one must have a dot in that figure. Lemma 9 demonstrates that if a hospital's demand correspondence satisfies the substitutes condition, uniform adjustments of its salary schedule produce closely-related demand sets.

Lemma 10 below shows that, under the substitutes condition, imposing a binding interval constraint is equivalent to making a certain uniform adjustment, in a particular sense: when a hospital's innate demand falls short of the floor, the compelled demand is equal to the innate demand at a uniformly subsidized salary schedule; when a hospital's innate demand exceeds the ceiling, the compelled demand is equivalent to the innate demand at a uniformly taxed salary schedule.

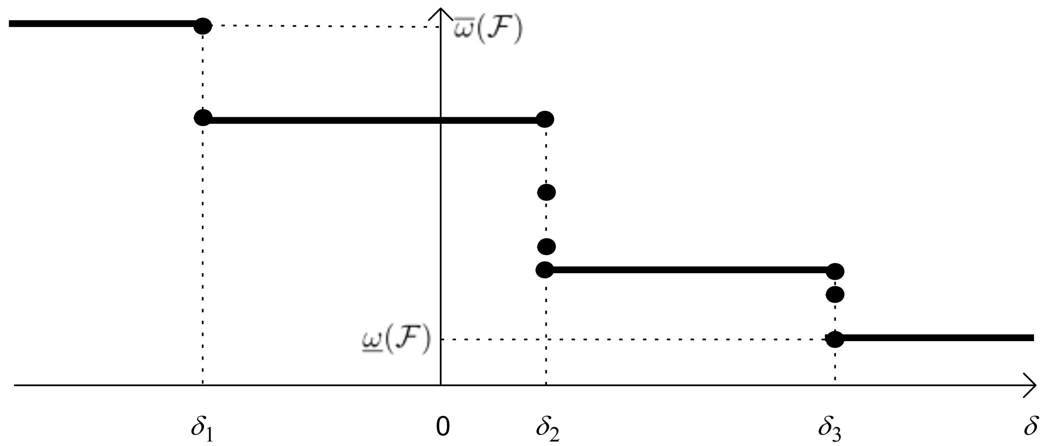


Figure 1: An example of cardinalities of demand sets for each $\sigma(\delta)$. The horizontal axis measures the parameter $\delta$ while the vertical axis measures the cardinalities of the demand sets. A thick line or dot means that there exists a demand set of the corresponding cardinality.

**Lemma 10.** *Suppose an innate demand correspondence $X(\,\cdot\,;\mathcal{F}^0)$ satisfies the substitutes condition, and the government-imposed feasibility collection is $\mathcal{D}_{[f,c]}$, with $\mathcal{F} \coloneqq \mathcal{F}^0 \cap \mathcal{D}_{[f,c]} \neq \emptyset$. For every salary schedule $\mathbf{s}$,*

- *(i) if $\overline{\omega}(X(\mathbf{s};\mathcal{F}^0)) < f$, there exists a unique $\delta^* < 0$ such that $X(\mathbf{s};\mathcal{F}) = X(\mathbf{s}+\delta^*\mathbf{1};\mathcal{F}^0) \cap \mathcal{D}_f$ and for any $\delta > \delta^*$, $\overline{\omega}(X(\mathbf{s}+\delta\mathbf{1};\mathcal{F}^0)) < f$;*
- *(ii) if $\underline{\omega}(X(\mathbf{s};\mathcal{F}^0)) > c$, there exists a unique $\delta^\diamond > 0$ such that $X(\mathbf{s};\mathcal{F}) = X(\mathbf{s}+\delta^\diamond\mathbf{1};\mathcal{F}^0) \cap \mathcal{D}_c$ and for any $\delta < \delta^\diamond$, $\underline{\omega}(X(\mathbf{s}+\delta\mathbf{1};\mathcal{F}^0)) > c$.*

*Proof.* Define $\sigma(\delta) = \mathbf{s}+\delta\mathbf{1}$ for $\delta \in \mathbb{R}$, and by Lemma 9, obtain the unique increasing sequence $(\delta_k)_{k\in\mathcal{K}}$, where $\mathcal{K} = \{1, 2, \ldots, K\}$, as described.

In case (i), let $\delta^*$ be the greatest $\delta_k$ such that $\overline{\omega}(X(\sigma(\delta_k);\mathcal{F}^0)) \geq f$: we know $\delta^*$ exists because $\overline{\omega}(\mathcal{F}^0) \geq f$. Since $\overline{\omega}(X(\mathbf{s};\mathcal{F}^0)) < f$, we know $\delta^* < 0$. We need to show $X(\mathbf{s};\mathcal{F}) = X(\sigma(\delta^*);\mathcal{F}^0) \cap \mathcal{D}_f$. By Lemma 8, $X(\sigma(\delta^*);\mathcal{F}^0) \cap \mathcal{D}_f \neq \emptyset$. Choose any $A' \in X(\mathbf{s};\mathcal{F})$, and $A'' \in X(\sigma(\delta^*);\mathcal{F}^0) \cap \mathcal{D}_f$. We have $V(A';\sigma(\delta^*)) \leq V(A'';\sigma(\delta^*))$

and $V(A'';\mathbf{s}) \le V(A';\mathbf{s})$. Therefore,

$$\begin{aligned} V(A';\mathbf{s}) - |A'|\,\delta^* &= V(A';\sigma(\delta^*)) \\ &\le V(A'';\sigma(\delta^*)) \\ &= V(A'';\mathbf{s}) - |A''|\,\delta^* \\ &\le V(A';\mathbf{s}) - f\,\delta^*, \end{aligned}$$

which implies $|A'| \le f$. Since $|A'| \ge f$, these are all equalities, so $|A'| = f$, $A' \in X(\sigma(\delta^*);\mathcal{F}^0)$, and $A'' \in X(\mathbf{s};\mathcal{F})$.

In case (ii), let $\delta^\diamond$ be the smallest $\delta_k$ such that $\underline{\omega}(X(\sigma(\delta_k);\mathcal{F}^0)) \le c$: we know $\delta^\diamond$ exists because $\underline{\omega}(\mathcal{F}^0) \le c$. Since $\underline{\omega}(X(\mathbf{s};\mathcal{F}^0)) > c$, we know $\delta^\diamond > 0$. We need to show $X(\mathbf{s};\mathcal{F}) = X(\sigma(\delta^\diamond);\mathcal{F}^0) \cap \mathcal{D}_c$. By Lemma 8, $X(\sigma(\delta^*);\mathcal{F}^0) \cap \mathcal{D}_f \neq \emptyset$. Choose any $A' \in X(\mathbf{s};\mathcal{F})$, and $A'' \in X(\sigma(\delta^\diamond);\mathcal{F}^0) \cap \mathcal{D}_c$. We have $V(A';\sigma(\delta^\diamond)) \le V(A'';\sigma(\delta^\diamond))$ and $V(A'';\mathbf{s}) \le V(A';\mathbf{s})$. Therefore,

$$\begin{aligned} V(A';\mathbf{s}) - |A'|\,\delta^\diamond &= V(A';\sigma(\delta^\diamond)) \\ &\le V(A'';\sigma(\delta^\diamond)) \\ &= V(A'';\mathbf{s}) - |A''|\,\delta^\diamond \\ &\le V(A';\mathbf{s}) - c\,\delta^\diamond, \end{aligned}$$

which implies $|A'| \ge c$. Since $|A'| \le c$, these are all equalities, so $|A'| = c$, $A' \in X(\sigma(\delta^\diamond);\mathcal{F}^0)$, and $A'' \in X(\mathbf{s};\mathcal{F})$. □

We know from this lemma that, under the substitutes condition and an interval constraint, when a hospital's innate demand falls short of the floor, the compelled demand meets the floor exactly. A similar statement can be made about the ceiling.

The next lemma uncovers an additional relationship between an innate demand correspondence and the associated compelled demand correspondence under an interval constraint. Assuming the substitutes condition for a hospital's innate demand correspondence, when an innate demand set falls short of the floor, we can find a compelled demand set that includes it; when it exceeds the ceiling, we can find a compelled demand set that is included by it.

**Lemma 11.** *Suppose an innate demand correspondence $X(\,\cdot\,;\mathcal{F}^0)$ satisfies the substitutes condition and the government-imposed feasibility collection is $\mathcal{D}_{[f,c]}$, with $\mathcal{F} := \mathcal{F}^0 \cap \mathcal{D}_{[f,c]} \neq \emptyset$. For any salary schedules $\mathbf{s}$ and any $A \in X(\mathbf{s};\mathcal{F}^0)$,* *(i) if $|A| < f$, there exists $B \in X(\mathbf{s};\mathcal{F})$ such that $A \subset B$ and $|B| = f$;*

*(ii) if $|A| > c$, there exists $B \in X(\mathbf{s};\mathcal{F})$ such that $B \subset A$ and $|B| = c$.*

*Proof.* When $\overline{\omega}(X(\mathbf{s};\mathcal{F}^0)) < f$, $|A| < f$. By Lemma 10, there exists $\delta^* < 0$ such that $X(\mathbf{s};\mathcal{F}) = X(\mathbf{s} + \delta^*\mathbf{1};\mathcal{F}^0) \cap \mathcal{D}_f$. Applying Lemma 9 at $\underline{\delta} = \delta^*$ and $\overline{\delta} = 0$ for $\overline{B} = A \in X(\sigma(0);\mathcal{F}^0)$, we can obtain $B \supset A$, with $B \in X(\mathbf{s} + \delta^*\mathbf{1};\mathcal{F}^0) \cap \mathcal{D}_f = X(\mathbf{s};\mathcal{F})$.

When $\underline{\omega}(X(\mathbf{s};\mathcal{F}^0)) > c$, $|A| > c$. By Lemma 10, there exists $\delta^\diamond > 0$ such that $X(\mathbf{s};\mathcal{F}) = X(\mathbf{s} + \delta^\diamond\mathbf{1};\mathcal{F}^0) \cap \mathcal{D}_c$. Applying Lemma 9 at $\underline{\delta} = 0$ and $\overline{\delta} = \delta^\diamond$ for $\underline{A} = A \in X(\sigma(0);\mathcal{F}^0)$, we can obtain $B \subset A$, with $B \in X(\mathbf{s} + \delta^\diamond\mathbf{1};\mathcal{F}^0) \cap \mathcal{D}_c = X(\mathbf{s};\mathcal{F})$.

When $\overline{\omega}(X(\mathbf{s};\mathcal{F}^0)) \geq f$ and $\underline{\omega}(X(\mathbf{s};\mathcal{F}^0)) \leq c$, $[\underline{\omega}(X(\mathbf{s};\mathcal{F}^0)), \overline{\omega}(X(\mathbf{s};\mathcal{F}^0))]$ and $[f, c]$ overlap. Consequently, $X(\mathbf{s};\mathcal{F}) = X(\mathbf{s};\mathcal{F}^0) \cap \mathcal{F}$. By Lemma 8, there exists a sequence $(A_j)_{j\in J}$ of sets of doctors, where $J = \{\underline{\omega}(X(\mathbf{s};\mathcal{F}^0)), \ldots, \overline{\omega}(X(\mathbf{s};\mathcal{F}^0))\}$, such that $A_{|A|} = A$ and for every $j$, $|A_j| = j$, $A_j \in X(\mathbf{s};\mathcal{F}^0)$, and $A_j \subset A_{j+1}$. If $|A| < f$, we can set $B = A_f$; if $|A| > c$, we can set $B = A_c$. $\square$

Lemma 12 is close to the sufficiency part of Theorem 1. The gist of the proof is that Lemmas 10 and 11 allow us to translate hospitals' compelled optimization problems under government-imposed interval constraints into those without government-imposed constraints, the structure of which is well-explored in earlier lemmas.

**Lemma 12.** *A feasibility collection defined by an interval constraint preserves the substitutes condition.*

*Proof.* Let a demand correspondence $X(\cdot;\mathcal{F}^0)$ satisfy the substitutes condition, and a government-imposed feasibility collection be $\mathcal{D}_{[f,c]}$, with $\mathcal{F} := \mathcal{F}^0 \cap \mathcal{D}_{[f,c]} \neq \emptyset$. We need to show that $X(\,\cdot\,;\mathcal{F})$ satisfies the substitutes condition. In other words, pick any two salary schedules $\mathbf{s}$ and $\mathbf{s}'$ with $\mathbf{s}' \geq \mathbf{s}$, and any $A \in X(\mathbf{s};\mathcal{F})$. We claim that there exists $A' \in X(\mathbf{s}';\mathcal{F})$ such that $I(A, \mathbf{s}, \mathbf{s}') \subset A'$.

When $A \in X(\mathbf{s};\mathcal{F}^0)$, it follows from the monotone substitutes condition that there exists $B \in X(\mathbf{s}';\mathcal{F}^0)$ such that $I(A, \mathbf{s}, \mathbf{s}') \subset B$ and $|B| \leq |A| \leq c$. If $|B| \geq f$, $B \in X(\mathbf{s}';\mathcal{F})$, and we can set $A' = B$. Otherwise, $|B| < f$, and by Lemma 11, there exists $A' \in X(\mathbf{s}';\mathcal{F})$ such that $I(A, \mathbf{s}, \mathbf{s}') \subset B \subset A'$. The claim is true.

When $A \notin X(\mathbf{s};\mathcal{F}^0)$, we have $X(\mathbf{s};\mathcal{F}^0) \cap \mathcal{D}_{[f,c]} = \emptyset$. By Lemma 8, either $\overline{\omega}(X(\mathbf{s};\mathcal{F}^0)) < f$ or $\underline{\omega}(X(\mathbf{s};\mathcal{F}^0)) > c$.

**Case 1.** $\overline{\omega}(X(\mathbf{s};\mathcal{F}^0)) < f$

By Lemma 10, there exists $\delta^* < 0$ such that $A \in X(\mathbf{s};\mathcal{F}) = X(\mathbf{s}+\delta^*\mathbf{1};\mathcal{F}^0)\cap\mathcal{D}_f$. Let $\mathbf{s}^* = \mathbf{s} + \delta^*\mathbf{1}$ and $\mathbf{s}'^* = \mathbf{s}' + \delta^*\mathbf{1}$. Since $\mathbf{s}'^* \geq \mathbf{s}^*$ and $A \in X(\mathbf{s}^*;\mathcal{F}^0)$, by the monotone substitutes condition, there exists $A'^* \in X(\mathbf{s}'^*;\mathcal{F}^0)$ such that $I(A,\mathbf{s},\mathbf{s}') = I(A,\mathbf{s}^*,\mathbf{s}'^*) \subset A'^*$ and $|A'^*| \leq |A| = f$.

We have $\overline{\omega}(X(\mathbf{s}';\mathcal{F}^0)) \leq \underline{\omega}(X(\mathbf{s}'^*;\mathcal{F}^0)) \leq |A'^*| \leq f$, where the first inequality follows from Lemma 9 and $\mathbf{s}'^* = \mathbf{s}' + \delta^*\mathbf{1}$ with $\delta^* < 0$, and the second and third inequalities come from the definition of $A'^*$. If $\overline{\omega}(X(\mathbf{s}';\mathcal{F}^0)) = f$, Lemma 9 says $A'^* \in X(\mathbf{s}';\mathcal{F}^0) \cap \mathcal{D}_f \subset X(\mathbf{s}';\mathcal{F})$. We can set $A' = A'^* \supset I(A,\mathbf{s},\mathbf{s}')$ to finish the proof. Now, we assume $\overline{\omega}(X(\mathbf{s}';\mathcal{F}^0)) < f$.

Define $\sigma(\delta) = \mathbf{s}' + \delta\mathbf{1}$ for $\delta \in \mathbb{R}$. By Lemma 10, there exists a unique $\delta' < 0$ such that $X(\mathbf{s}';\mathcal{F}) = X(\sigma(\delta');\mathcal{F}^0) \cap \mathcal{D}_f$ and for any $\delta > \delta'$, $\overline{\omega}(X(\sigma(\delta)) < f$.

When $\delta' < \delta^*$, Lemma 9 implies that there exists $A' \supset A'^* \supset I(A,\mathbf{s},\mathbf{s}')$ such that $A' \in X(\sigma(\delta');\mathcal{F}^0)\cap\mathcal{D}_f = X(\mathbf{s}';\mathcal{F})$. When $\delta' = \delta^*$, Lemma 8, instead of Lemma 9, implies the existence of $A'$ with the desirable properties above. When $\delta^* < \delta' < 0$, Lemma 9 implies $f \leq \overline{\omega}(X(\sigma(\delta');\mathcal{F}^0)) \leq \underline{\omega}(X(\mathbf{s}'^*;\mathcal{F}^0)) \leq f$. So $|A'^*| = f$. We can set $A' = A'^* \in X(\sigma(\delta');\mathcal{F}^0)\cap\mathcal{D}_f = X(\mathbf{s}';\mathcal{F})$, where the membership relation follows from Lemma 9.

**Case 2.** $\underline{\omega}(X(\mathbf{s};\mathcal{F}^0)) > c$

By Lemma 10, there exists $\delta^\diamond > 0$ such that $A \in X(\mathbf{s};\mathcal{F}) = X(\mathbf{s}+\delta^\diamond\mathbf{1};\mathcal{F}^0)\cap\mathcal{D}_c$. Let $\mathbf{s}^\diamond = \mathbf{s} + \delta^\diamond\mathbf{1}$ and $\mathbf{s}'^\diamond = \mathbf{s}' + \delta^\diamond\mathbf{1}$. Since $\mathbf{s}'^\diamond \geq \mathbf{s}^\diamond$ and $A \in X(\mathbf{s}^\diamond;\mathcal{F}^0)$, by the monotone substitutes condition, there exists $A'^\diamond \in X(\mathbf{s}'^\diamond;\mathcal{F}^0)$ such that $I(A,\mathbf{s},\mathbf{s}') = I(A,\mathbf{s}^\diamond,\mathbf{s}'^\diamond) \subset A'^\diamond$ and $|A'^\diamond| \leq |A| = c$.

Suppose $[f,c]$ and $[\underline{\omega}(X(\mathbf{s}';\mathcal{F}^0)),\overline{\omega}(X(\mathbf{s}';\mathcal{F}^0))]$ overlap, say, at integer $m$. Since $\mathbf{s}'^\diamond = \mathbf{s}' + \delta^\diamond\mathbf{1}$, Lemma 9 implies that there exists $A' \in X(\mathbf{s}';\mathcal{F}^0) \cap \mathcal{D}_m \subset X(\mathbf{s}';\mathcal{F})$, such that $A' \supset A'^\diamond \supset I(A,\mathbf{s},\mathbf{s}')$.

When $\overline{\omega}(X(\mathbf{s}';\mathcal{F}^0)) < f$, we can first obtain $\bar{A}^\diamond \supset A'^\diamond$ such that $\bar{A}^\diamond \in X(\mathbf{s}';\mathcal{F}^0)$ by Lemma 9. Lemma 11 further implies that there exists $A' \supset \bar{A}^\diamond \supset A'^\diamond \supset I(A,\mathbf{s},\mathbf{s}')$ such that $A' \in X(\mathbf{s}';\mathcal{F})$.

When $\underline{\omega}(X(\mathbf{s}';\mathcal{F}^0)) > c$, by Lemma 10, there exists a unique $\delta'' > 0$ such that $X(\mathbf{s}';\mathcal{F}) = X(\sigma(\delta'');\mathcal{F}^0) \cap \mathcal{D}_c$ and for any $\delta < \delta''$, $\underline{\omega}(X(\sigma(\delta);\mathcal{F}^0)) > c$. Since $\underline{\omega}(X(\mathbf{s}';\mathcal{F}^0)) > c$ and $|A'^\diamond| \leq c$, we know $\delta'' \in (0,\delta^\diamond]$. If $\delta'' < \delta^\diamond$, we can apply

Lemma 9 to obtain $A' \supset A'^{\diamond} \supset I(A, \mathbf{s}, \mathbf{s}')$ such that $A' \in X(\sigma(\delta''); \mathcal{F}^0) \cap \mathcal{D}_c = X(\mathbf{s}'; \mathcal{F})$; if $\delta'' = \delta^{\diamond}$, by Lemma 8, the same is true. $\square$

*Proof of Theorem 1.* The necessity part is apparently a corollary of Lemma 1. For sufficiency, let $\mathcal{F}^g$ be defined by a generalized interval constraint. Suppose that $\mathcal{F}^g$ does not preserve the substitutes condition. There is a revenue function $R : 2^D \to \mathbb{R}$ and a self-imposed feasibility collection $\mathcal{F}^0$ such that $R|_{\mathcal{F}^0}$ satisfies the substitutes condition but $R|_{\mathcal{F}}$, where $\mathcal{F} := \mathcal{F}^0 \cap \mathcal{F}^g$, does not.

Consider a feasibility constraint $\mathcal{F}'^g := \{A \cap \hat{\chi}(\mathcal{F}) : A \in \mathcal{F}\}$ on $D' := \hat{\chi}(\mathcal{F}^g)$; it is defined by an interval constraint. Construct a revenue function $R'$ on $2^{D'}$, where $R'(A) = R(A \cup \overline{\chi}(\mathcal{F}^g))$ for every $A \subset D'$; let

$$\mathcal{F}'^0 := \{A \cap D' : A \in \mathcal{F}^0,\ \overline{\chi}(\mathcal{F}^g) \subset A, \text{ and } A \cap \underline{\chi}(\mathcal{F}^g) = \emptyset\}.$$

It is routine to check that $R'|_{\mathcal{F}'^0}$ satisfies the substitutes condition but $R'|_{\mathcal{F}'^0 \cap \mathcal{F}'^g}$ does not. In other words, $\mathcal{F}'^g$ does not preserve the substitutes condition as an interval constraint on $D'$, a contradiction to Lemma 12. $\square$

## C. Proof of Lemma 2

Define a **vectorization function** $\tau : 2^D \to \mathbb{Z}^{\mathcal{P}}$ such that for each $A \subset D$ and $P \in \mathcal{P}$, $\tau(A)(P) = |A \cap P|$. In other words, $\tau$ maps $A \subset D$ to a $|\mathcal{P}|$-dimensional vector that specifies how many doctors in each $P \in \mathcal{P}$ belong to $A$. We list two simple results regarding $\tau$.

**Lemma 13.** *If a feasibility collection $\mathcal{F}$ satisfies $\mathcal{F} = \tau^{-1}(\mathbf{Z})$ for a subset $\mathbf{Z}$ of $\mathbb{Z}^{\mathcal{P}}$, then it is anonymous within each group.*

*Proof.* Consider an arbitrary group $P \in \mathcal{P}$. For any $A \in \mathcal{F}$, $d \in P \cap A$, and $d' \in P \setminus A$, we have $\tau(A - d + d') = \tau(A)$. Hence, based on the assumption that $\mathcal{F} = \tau^{-1}(\mathbf{Z})$, we get $A - d + d' \in \mathcal{F}$. $\square$

**Lemma 14.** *A feasibility collection $\mathcal{F}$ is anonymous within each group if and only if $\mathcal{F} = \tau^{-1}(\tau(\mathcal{F}))$.*

*Proof.* The sufficiency part is a corollary of Lemma 13. For necessity, we know "$\subset$" is trivial. To show "$\supset$," consider $A \in \tau^{-1}(\tau(\mathcal{F}))$, that is, there exists $B \in \mathcal{F}$ such that $\tau(A) = \tau(B)$. By anonymity, we can convert $B$ to $A$ by replacement within groups and maintain feasibility. $\square$

In the space $\mathbb{Z}^{\mathcal{P}}$, denote the **indicator function for** $P \in \mathcal{P}$ by $\mathbf{i}^P$. A function $W : \mathbf{Z} \to \mathbb{R}$, where $\mathbf{Z} \subset \mathbb{Z}^{\mathcal{P}}$, is **M$^\natural$-concave** if for any $\mathbf{z}, \mathbf{z}' \in \mathbf{Z}$ and $P \in \mathcal{P}$ such that $\mathbf{z}(P) > \mathbf{z}'(P)$, either $W(\mathbf{z})+W(\mathbf{z}') \leq W(\mathbf{z}-\mathbf{i}^P)+W(\mathbf{z}'+\mathbf{i}^P)$,[4] or there exists $P' \in \mathcal{P}$ such that $\mathbf{z}'(P') > \mathbf{z}(P')$ and $W(\mathbf{z}) + W(\mathbf{z}') \leq W(\mathbf{z} - \mathbf{i}^P + \mathbf{i}^{P'}) + W(\mathbf{z}' + \mathbf{i}^P - \mathbf{i}^{P'})$, or both.[5]

A subset $\mathbf{Z}$ of $\mathbb{Z}^{\mathcal{P}}$ is **vector exchangeable** if for any $\mathbf{z}, \mathbf{z}' \in \mathbf{Z}$ and $P \in \mathcal{P}$ such that $\mathbf{z}(P) > \mathbf{z}'(P)$, either $\mathbf{z} - \mathbf{i}^P \in \mathbf{Z}$ and $\mathbf{z}' + \mathbf{i}^P \in \mathbf{Z}$, or there exists $P' \in \mathcal{P}$ such that $\mathbf{z}'(P') > \mathbf{z}(P')$, $\mathbf{z} - \mathbf{i}^P + \mathbf{i}^{P'} \in \mathbf{Z}$, and $\mathbf{z}' + \mathbf{i}^P - \mathbf{i}^{P'} \in \mathbf{Z}$, or both. Apparently, if $W : \mathbf{Z} \to \mathbb{R}$ is M$^\natural$-concave, then $\mathbf{Z}$ must be vector exchangeable. We list two simple results that connect exchangeability and vector exchangeability with $\tau$.

**Lemma 15.** *If a feasibility collection $\mathcal{F}$ satisfies $\mathcal{F} = \tau^{-1}(\mathbf{Z})$ for a vector exchangeable subset $\mathbf{Z}$ of $\mathbb{Z}^{\mathcal{P}}$, then it is exchangeable.*

*Proof.* Consider any $A, B \in \mathcal{F}$ and $d \in A \setminus B$. There exists $P \in \mathcal{P}$ such that $d \in P$. There are two cases. If $|A \cap P| \leq |B \cap P|$, there exists $d' \in (B \cap P) \setminus A$. By Lemma 13, $A - d + d'$, $B + d - d' \in \mathcal{F}$.

In the second case of $|A \cap P| > |B \cap P|$, we have $\tau(A)(P) > \tau(B)(P)$, so we can apply vector exchangeability to $\tau(A)$, $\tau(B)$, and $P \in \mathcal{P}$, and learn that either $\tau(A) - \mathbf{i}^P \in \mathbf{Z}$ and $\tau(B) + \mathbf{i}^P \in \mathbf{Z}$, or there exists $P' \in \mathcal{P}$ such that $\tau(B)(P') > \tau(A)(P')$, $\tau(A) - \mathbf{i}^P + \mathbf{i}^{P'} \in \mathbf{Z}$, and $\tau(B) + \mathbf{i}^P - \mathbf{i}^{P'} \in \mathbf{Z}$, or both. The first subcase translates into $A - d$, $B + d \in \mathcal{F}$; the second subcase means there exists $d' \in B \setminus (A \cup P)$ such that $A - d + d'$, $B + d - d' \in \mathcal{F}$. □

**Lemma 16.** *If a feasibility collection $\mathcal{F}$ is exchangeable and anonymous within each group, then $\tau(\mathcal{F})$ is vector exchangeable.*

*Proof.* Consider any $\mathbf{z}, \mathbf{z}' \in \tau(\mathcal{F})$ and $\hat{P} \in \mathcal{P}$ such that $\mathbf{z}(\hat{P}) > \mathbf{z}'(\hat{P})$. There exist $A, B \in \mathcal{F}$ such that $\tau(A) = \mathbf{z}$ and $\tau(B) = \mathbf{z}'$. Since $\mathcal{F}$ is anonymous within each $P \in \mathcal{P}$, we can convert $A$ to $A' \in \mathcal{F}$ by replacement within groups such that for each $P \in \mathcal{P}$, $|A' \cap P| = |A \cap P|$, $\mathbf{z}(P) < \mathbf{z}'(P)$ implies $A' \cap P \subsetneq B \cap P$, $\mathbf{z}(P) = \mathbf{z}'(P)$ implies $A' \cap P = B \cap P$, and $\mathbf{z}(P) > \mathbf{z}'(P)$ implies $B \cap P \subsetneq A' \cap P$. In particular, $B \cap \hat{P} \subsetneq A' \cap \hat{P}$, and there exists $d \in (\hat{P} \cap A') \setminus B$.

[4] In this case, it is implicitly assumed that $\mathbf{z} - \mathbf{i}^P \in \mathbf{Z}$ and $\mathbf{z}' + \mathbf{i}^P \in \mathbf{Z}$. In the other case, $\mathbf{z} - \mathbf{i}^P + \mathbf{i}^{P'} \in \mathbf{Z}$ and $\mathbf{z}' + \mathbf{i}^P - \mathbf{i}^{P'} \in \mathbf{Z}$.

[5] This property is M$^\natural$-EXC[$\mathbb{Z}$] in Murota (2016); "vector exchangeability" below is B$^\natural$-EXC[$\mathbb{Z}$].

Exchangeability of $\mathcal{F}$ can be applied to $A', B \in \mathcal{F}$ and $d \in A' \setminus B$: either $A' - d \in \mathcal{F}$ and $B + d \in \mathcal{F}$, or there exists $d' \in B \setminus A'$ such that $A' - d + d' \in \mathcal{F}$ and $B + d - d' \in \mathcal{F}$, or both. In the first case, $\mathbf{z} - \mathbf{i}^{\hat{P}} = \tau(A' - d) \in \tau(\mathcal{F})$ and $\mathbf{z}' + \mathbf{i}^{\hat{P}} = \tau(B + d) \in \tau(\mathcal{F})$. In the second case, $d'$ must be from another group $\hat{P}'$, so $\mathbf{z} - \mathbf{i}^{\hat{P}} + \mathbf{i}^{\hat{P}'} = \tau(A' - d + d') \in \tau(\mathcal{F})$ and $\mathbf{z}' + \mathbf{i}^{\hat{P}} - \mathbf{i}^{\hat{P}'} = \tau(B + d - d') \in \tau(\mathcal{F})$. $\square$

Lemma 17 is from Fujishige (1991, Theorem 3.58) and Murota (2003, Page 117); it characterizes all vector exchangeable sets in $\mathbb{Z}^{\mathcal{P}}$. We say $\mathbf{Z} \subset \mathbb{Z}^{\mathcal{P}}$ is a **generalized polymatroid** if there exist functions $\mu, \rho : 2^{\mathcal{P}} \to \mathbb{Z}$ such that $\mu$ is supermodular, $\rho$ is submodular, $\mu(\mathcal{Q}) - \mu(\mathcal{Q} \setminus \mathcal{Q}') \leq \rho(\mathcal{Q}') - \rho(\mathcal{Q}' \setminus \mathcal{Q})$ for any $\mathcal{Q}, \mathcal{Q}' \subset \mathcal{P}$, and

$$\mathbf{Z} = \Big\{ \mathbf{z} \in \mathbb{Z}^{\mathcal{P}} : \mu(\mathcal{Q}) \leq \sum_{P \in \mathcal{Q}} \mathbf{z}(P) \leq \rho(\mathcal{Q}) \text{ for every } \mathcal{Q} \subset \mathcal{P} \Big\}. \tag{A1}$$

**Lemma 17.** *A subset $\mathbf{Z}$ of $\mathbb{Z}^{\mathcal{P}}$ is vector exchangeable if and only if it is a generalized polymatroid.*

A feasibility collection $\mathcal{F}$ is defined by a **polyhedral constraint** if there exist functions $\mu, \rho : 2^{\mathcal{P}} \to \mathbb{Z}$ such that $\mu$ is supermodular, $\rho$ is submodular, $\mu(\mathcal{Q}) - \mu(\mathcal{Q} \setminus \mathcal{Q}') \leq \rho(\mathcal{Q}') - \rho(\mathcal{Q}' \setminus \mathcal{Q})$ for any $\mathcal{Q}, \mathcal{Q}' \subset \mathcal{P}$, and

$$\mathcal{F} = \{ A \subset D : \mu(\mathcal{Q}) \leq |A \cap (\cup \mathcal{Q})| \leq \rho(\mathcal{Q}) \text{ for every } \mathcal{Q} \subset \mathcal{P} \}. \tag{A2}$$

Here, $\mu(\mathcal{Q})$ and $\rho(\mathcal{Q})$ respectively dictate the floor and ceiling on $\cup \mathcal{Q}$. Equation (A2) simply states that $\mathcal{F}$ is defined by a family of interval constraints on unions of groups. Clearly, for such $\mathcal{F}$, $\tau(\mathcal{F})$ is a generalized polymatroid and thus, according to Lemma 17, vector exchangeable. Also, $\mathcal{F} = \tau^{-1}(\tau(\mathcal{F}))$; by Lemma 15, $\mathcal{F}$ is exchangeable.

**Lemma 18.** *If a proper feasibility collection preserves the substitutes condition for all binary disjunctive revenue functions over pairs of doctors in the same groups, then it is defined by a polyhedral constraint.*

*Proof.* Consider a proper feasibility collection $\mathcal{F}$ that preserves the substitutes condition for all binary disjunctive revenue functions over pairs of doctors in the same groups. By Lemma 6, $\mathcal{F}$ is then anonymous within each pair of doctors who belong to the same group, and thus within each group.

Because $\mathcal{F}$ has to be exchangeable too (remember the comments before Lemma 4), by Lemma 16, $\tau(\mathcal{F})$ is vector exchangeable. By Lemma 17, $\tau(\mathcal{F})$ must be a generalized polymatroid in $\mathbb{Z}^{\mathcal{P}}$. Since $\mathcal{F}$ is anonymous within each group, by Lemma 14, $\mathcal{F} = \tau^{-1}(\tau(\mathcal{F}))$. Plugging in the right-hand side of Equation (A1), we obtain Equation (A2). $\square$

The proof of Lemma 2 from Lemma 18 is almost identical to the proof of Lemma 1 from Lemma 7, so we skip it.

## D. Proof of Theorem 2

The following two results about exact constraints follow from Lemmas 8-11. Lemma 19[6] states that given an innate demand correspondence that satisfies the substitutes condition and a demand set under an exact constraint, when the constraint is raised by 1, we can find a doctor to add to the original demand set and obtain a new one; when the constraint is lowered by 1, we can find a doctor to remove and obtain a new demand set.

**Lemma 19.** *Suppose an innate demand correspondence $X(\,\cdot\,;\mathcal{F}^0)$ satisfies the substitutes condition and an integer $f$ satisfies $\underline{\omega}(\mathcal{F}^0) \le f \le \overline{\omega}(\mathcal{F}^0)$. For any salary schedules $\mathbf{s}$ and any $A \in X(\mathbf{s};\mathcal{F}^0 \cap \mathcal{D}_f)$,*

*(i) if $f < \overline{\omega}(\mathcal{F}^0)$, there exists $B \in X(\mathbf{s};\mathcal{F}^0 \cap \mathcal{D}_{f+1})$ such that $A \subset B$;*

*(ii) if $f > \underline{\omega}(\mathcal{F}^0)$, there exists $C \in X(\mathbf{s};\mathcal{F}^0 \cap \mathcal{D}_{f-1})$ such that $C \subset A$.*

*Proof.* When $\underline{\omega}(X(\mathbf{s};\mathcal{F}^0)) \le f \le \overline{\omega}(X(\mathbf{s};\mathcal{F}^0))$, by Lemma 8, $A \in X(\mathbf{s};\mathcal{F}^0)$, so both (i) and (ii) follow from Lemma 11.

When $f > \overline{\omega}(X(\mathbf{s};\mathcal{F}^0))$, by Lemma 10, there exists a unique $\delta^* < 0$ such that $X(\mathbf{s};\mathcal{F}^0 \cap \mathcal{D}_f) = X(\mathbf{s} + \delta^*\mathbf{1};\mathcal{F}^0) \cap \mathcal{D}_f$ and for any $\delta > \delta^*$, $\overline{\omega}(X(\mathbf{s} + \delta\mathbf{1};\mathcal{F}^0)) < f$.

For (i), by Lemma 10, there exists a unique $\delta'^* < 0$ such that $X(\mathbf{s};\mathcal{F}^0 \cap \mathcal{D}_{f+1}) = X(\mathbf{s} + \delta'^*\mathbf{1};\mathcal{F}^0) \cap \mathcal{D}_{f+1}$. Then $\delta'^* \le \delta^*$. By Lemma 8 (when $\delta'^* = \delta^*$) or Lemma 9 (when $\delta'^* < \delta^*$), $B$ exists.

For (ii), first assume $f - 1 = \overline{\omega}(X(\mathbf{s};\mathcal{F}^0))$, which implies $X(\mathbf{s};\mathcal{F}^0 \cap \mathcal{D}_{f-1}) = X(\mathbf{s};\mathcal{F}^0) \cap \mathcal{D}_{f-1}$, and then notice that the existence of $C$ follows from Lemma 9. We are left with the case of $f - 1 > \overline{\omega}(X(\mathbf{s};\mathcal{F}^0))$. By Lemma 10, there exists a unique $\delta''^* < 0$ such that $X(\mathbf{s};\mathcal{F}^0 \cap \mathcal{D}_{f-1}) = X(\mathbf{s} + \delta''^*\mathbf{1};\mathcal{F}^0) \cap \mathcal{D}_{f-1}$ and for any $\delta > \delta''^*$,

[6] The unconstrained version of this lemma is essentially known (Segal-Halevi, Hassidim, and Aumann, 2016; Paes Leme, 2017).

$\overline{\omega}(X(\mathbf{s}+\delta\mathbf{1};\mathcal{F}^0)) < f-1$. Then $\delta^* \leq \delta''^*$. By Lemma 8 (when $\delta^* = \delta''^*$) or Lemma 9 (when $\delta^* < \delta''^*$), $C$ exists.

The case of $f < \underline{\omega}(X(\mathbf{s};\mathcal{F}^0))$ is symmetric. $\square$

Lemma 20 implies that maximal profits as a function of $f$, where $f$ specifies the exact constraint, is extensible to a concave function on $\mathbb{R}$.

**Lemma 20.** *If an innate demand correspondence $X(\,\cdot\,;\mathcal{F}^0)$ satisfies the substitutes condition and an integer $f$ satisfies $\underline{\omega}(\mathcal{F}^0) < f < \overline{\omega}(\mathcal{F}^0)$, then for any salary schedules* $\mathbf{s}$*,*

$$\Pi(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_{f+1}) - \Pi(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_f) \;\leq\; \Pi(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_f) - \Pi(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_{f-1}).$$

*Proof.* Pick an arbitrary $A \in X(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_f)$. By Lemma 19, there exist $d, d' \in D$ such that $A + d \in X(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_{f+1})$ and $A - d' \in X(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_{f-1})$. So

$$\begin{aligned}
\Pi(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_{f+1}) - \Pi(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_f) &= V(A+d;\mathbf{s}) - V(A;\mathbf{s})\\
&\leq V(A+d-d';\mathbf{s}) - V(A-d';\mathbf{s})\\
&\leq V(A;\mathbf{s}) - V(A-d';\mathbf{s})\\
&= \Pi(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_f) - \Pi(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_{f-1}),
\end{aligned}$$

where the first inequality follows from the fact that $V(\,\cdot\,;\mathbf{s})$ satisfies the substitutes condition, which implies submodularity (Murota, 2016); the second is from $A \in X(\mathbf{s};\mathcal{F}^0\cap\mathcal{D}_f)$. $\square$

The following lemma is the last piece needed.

**Lemma 21.** *A feasibility collection defined by a polyhedral constraint preserves the substitutes condition for all group separable revenue functions.*

*Proof.* Consider a group separable revenue function $R$ on $2^D$ such that $R(A) = \sum_{P\in\mathcal{P}} R_P(A\cap P)$ for every $A \subset D$ and each $R_P : 2^P \to \mathbb{R}$ satisfies the substitutes condition on $2^P$. For a feasibility collection defined by a polyhedral constraint $\mathcal{F}$, we show that $R|_{\mathcal{F}}$ satisfies the single-improvement property and thus the substitutes condition.

Given a price schedule $\mathbf{s}$ and a suboptimal set of doctors $A \notin X(\mathbf{s};\mathcal{F})$, we need to find a single-improvement opportunity. For each revenue function $R_P$ defined on $2^P$, denote the profit function and maximal profit function by $V_P$ and

$\Pi_P$. If there is $P \in \mathcal{P}$ such that for some $d, d' \in P$, $d \in A$, $d' \notin A$, and $V_P((A \cap P) - d + d'; \mathbf{s}|_P) > V_P(A \cap P; \mathbf{s}|_P)$, then we know $V(A - d + d'; \mathbf{s}) > V(A; \mathbf{s})$.[7] Because a feasibility collection defined by a polyhedral constraint is anonymous within each group, we have found a single-improvement opportunity in $A - d + d' \in \mathcal{F}$. Assume nonexistence of such $P \in \mathcal{P}$.

For each $R_P$, define $\Phi_P : \{0, 1, \ldots, |P|\} \to \mathbb{R}$ such that $\Phi_P(f) = \Pi_P(\mathbf{s}|_P; \mathcal{G}_f^P)$, where $\mathcal{G}_f^P := \{B \subset P : |B| = f\}$, is the maximal profit under the exact constraint of $f$. By the assumption in the second paragraph, for each $P \in \mathcal{P}$, $V_P(A \cap P; \mathbf{s}|_P) = \Phi_P(|A \cap P|)$. Define $W : \tau(\mathcal{F}) \to \mathbb{R}$ such that $W(\mathbf{z}) = \sum_{P \in \mathcal{P}} \Phi_P(\mathbf{z}(P))$.

We will show that $W$ is M$^\natural$-concave. To do so, remember that $\tau(\mathcal{F})$ is vector exchangeable. As a result, for any $\mathbf{z}, \mathbf{z}' \in \tau(\mathcal{F})$ and $\hat{P} \in \mathcal{P}$ such that $\mathbf{z}(\hat{P}) > \mathbf{z}'(\hat{P})$, either $\mathbf{z} - \mathbf{i}^{\hat{P}} \in \tau(\mathcal{F})$ and $\mathbf{z}' + \mathbf{i}^{\hat{P}} \in \tau(\mathcal{F})$, or there exists $\hat{P}' \in \mathcal{P}$ such that $\mathbf{z}'(\hat{P}') > \mathbf{z}(\hat{P}')$, $\mathbf{z} - \mathbf{i}^{\hat{P}} + \mathbf{i}^{\hat{P}'} \in \tau(\mathcal{F})$, and $\mathbf{z}' + \mathbf{i}^{\hat{P}} - \mathbf{i}^{\hat{P}'} \in \tau(\mathcal{F})$, or both. In the first case,

$$\begin{aligned}
&W(\mathbf{z} - \mathbf{i}^{\hat{P}}) + W(\mathbf{z}' + \mathbf{i}^{\hat{P}}) - W(\mathbf{z}) - W(\mathbf{z}') \\
=& \sum_{P \in \mathcal{P}} \Phi_P((\mathbf{z} - \mathbf{i}^{\hat{P}})(P)) + \sum_{P \in \mathcal{P}} \Phi_P((\mathbf{z}' + \mathbf{i}^{\hat{P}})(P)) - \sum_{P \in \mathcal{P}} \Phi_P(\mathbf{z}(P)) - \sum_{P \in \mathcal{P}} \Phi_P(\mathbf{z}'(P)) \\
=& \Phi_{\hat{P}}(\mathbf{z}(\hat{P}) - 1) + \Phi_{\hat{P}}(\mathbf{z}'(\hat{P}) + 1) - \Phi_{\hat{P}}(\mathbf{z}(\hat{P})) - \Phi_{\hat{P}}(\mathbf{z}'(\hat{P})) \ \geq \ 0,
\end{aligned}$$

where the inequality follows from $\mathbf{z}(\hat{P}) > \mathbf{z}'(\hat{P})$ and Lemma 20. In the second case, we can similarly show $W(\mathbf{z} - \mathbf{i}^{\hat{P}} + \mathbf{i}^{\hat{P}'}) + W(\mathbf{z}' + \mathbf{i}^{\hat{P}} - \mathbf{i}^{\hat{P}'}) \geq W(\mathbf{z}) + W(\mathbf{z}')$.

Consider $A' \in X(\mathbf{s}; \mathcal{F})$. By the suboptimality of $A$ and the assumption in the second paragraph, $W(\tau(A')) = V(A'; \mathbf{s}) = \Pi(\mathbf{s}; \mathcal{F}) > V(A; \mathbf{s}) = W(\tau(A))$. Since an $M^\natural$-concave function satisfies a multi-unit single-improvement property (Murota, 2003; Milgrom and Strulovici, 2009), at least one of the following 3 statements must be true: there exists $\tilde{P} \in \mathcal{P}$ such that $W(\tau(A) + \mathbf{i}^{\tilde{P}}) > W(\tau(A))$; there exists $\tilde{P} \in \mathcal{P}$ such that $W(\tau(A) - \mathbf{i}^{\tilde{P}}) > W(\tau(A))$; there exist $\tilde{P}, \tilde{P}' \in \mathcal{P}$ such that $W(\tau(A) - \mathbf{i}^{\tilde{P}} + \mathbf{i}^{\tilde{P}'}) > W(\tau(A))$.

In the first case, we infer that $\Phi_{\tilde{P}}(|A \cap \tilde{P}| + 1) > \Phi_{\tilde{P}}(|A \cap \tilde{P}|) = V_{\tilde{P}}(A \cap \tilde{P}; \mathbf{s}|_{\tilde{P}})$. According to Lemma 19, there exists $d \in \tilde{P} \setminus A$ such that $V_{\tilde{P}}((A \cap \tilde{P}) + d; \mathbf{s}|_{\tilde{P}}) = \Phi_{\tilde{P}}(|A \cap \tilde{P}| + 1)$. So $A + d \in \mathcal{F}$ is a single-improvement opportunity. Similarly, in the second case, we can find $A - d \in \mathcal{F}$ with $d \in \tilde{P} \cap A$ as a single-improvement

[7] By convention, $\mathbf{s}|_P$ is the restriction of salary schedule $\mathbf{s}$ to $P$.

opportunity; in the third case, we can find $A - d + d' \in \mathcal{F}$ with $d \in \tilde{P} \cap A$ and $d' \in \tilde{P}' \setminus A$ as a single-improvement opportunity. $\square$

The necessity part of Theorem 2 is apparently a corollary of Lemma 2. The proof of the sufficiency part from Lemma 21 is almost identical to the proof of the sufficiency part of Theorem 1 from Lemma 12, so we skip it.

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