THE DETERMINANTS OF THE SEPARATION HAZARD IN A MODEL WITH LEARNING AND TIME-VARYING MATCH QUALITY

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ABSTRACT. People and organizations enter relationships, learn about them, adapt to them, and sometimes decide to leave them. This paper develops a learning model of relationship dissolution. The model also allows relationship quality to vary over time, following an AR(1) process. The model nests two possible assumptions about the causes of relationship dissolution. One is that relationships dissolve when agents find out that they are in fact bad (pure learning model). The other is that relationships dissolve because they change over time (pure shocks model). This paper analyzes the effect of parameters on agents' separation decision and the resulting separation hazard. It also examines how one can empirically distinguish between the pure learning model, the pure shocks model, and a mixed model. Observing an increasing and then decreasing separation hazard is not sufficient evidence for the pure learning model. The shape of the variance of the transition function is enough to distinguish between the three models. One can also use the impact of a bad observation on the separation hazard to make such a distinction.

1. Introduction

When people enter a relationship, be it professional or personal, they usually do not know with certainty how good this relationship is for them. Moreover, a relationship that is good today may become undesirable tomorrow. Given this uncertainty, how do people and organizations decide whether to continue or separate from a relationship? In non-experimental empirical settings, we never observe the full information available to agents, but we are typically able to observe their separation decisions and determine how long the relationship was at the time of separation. These observations allow us to empirically estimate the separation hazard, i.e. the probability that a relationship is terminated given that it has survived so far. This paper develops a model which can ultimately allow the researcher to infer the type of the hidden information agents base their separation decision on. The model assumes that agents have an unbiased prior belief about the distribution of match qualities among potential partners, i.e. they know how likely they are to find a given match quality when sampling from the population of potential partners. Then, agents observe signals of relationship quality over time, and decide whether to separate or not based on their updated belief about quality, the costs of separation, and their discount factor. Match quality is assumed to be normally distributed and to evolve over time according to an AR(1) process. Such a model nests two different theories of relationship dissolution. One theory suggests that, when partners form a relationship, they are uncertain about its true quality, and they learn about it over time. In this case, the relationship is an experience good (e.g. Jovanovic, 1979). Crucially, such a model in its pure form assumes that relationship quality does not change at all over time, the only thing that changes being the partners' *information about* their relationship. Such a learning model has been proposed by Farber(1994) to explain the shape of job separation hazard and by Svarer (2004) to explain the shape of the divorce hazard. The other reason why relationships may end is because they have changed: for example, in the job match context, a worker may develop an alcohol problem that he did not have when the employer first hired him. In this case, the relationship is an inspection good but changes over time (e.g. Mortensen-Pissarides, 1994). Using a model that nests both these theories, I analyze the effect of belief-shaping parameters, the cost of separation and the discount factor on the threshold for separation (i.e. the match quality such that the agent is indifferent between continuing and separating) and on the hazard of separation. Separation costs and a lower discount factor (or higher discount rate) decrease the separation threshold, and thus the separation hazard in all cases: this is intuitive since both parameters diminish the returns to separating and looking for a better option. The effects of parameters entering the belief are too complex to discuss in this introduction. I also determine how one can empirically distinguish between the pure learning model (a la Jovanovic(1979)), the pure shocks model (no uncertainty about relationship quality but relationship quality changes over time) and the mixed model (uncertainty about relationship quality and relationship quality changes over time).

The contribution of this paper to the literature is three-fold. First, this paper uses a general model of relationship separation that nests two types of pre-existing models. While some general results about the separation hazard have been derived by Jovanovic(1979) in a model where match quality was assumed to be normal and constant over time, this paper directly computes the quantitative effect of various parameters on the hazard of separation in a more general model. Second, I introduce time-varying separation costs and analyze the impact of such variation on the separation hazard. Third, and most importantly, I derive empirically testable implications which allows the researcher to distinguish between the pure learning model, the pure shocks model and the mixed model. The pure learning model has often been proposed as an explanation of observed empirical separation hazards, even though such a model is not entirely consistent with the observed pattern. Indeed, estimated job separation hazards (Farber(1994),Marinescu(2006a)) or divorce hazards (Weiss-Willis(1997), Svarer(2004)) are prima facie inconsistent with a constant match quality since they do not decline to 0 with relationship length. Moreover, this paper shows that observing an increasing and then decreasing separation hazard is not sufficient evidence in favor of the pure learning model, since both the mixed model and the pure shocks model can yield the same pattern.

This paper is organized as follows. In section 2, I describe the proposed model of the optimal separation decision and analyze the impact of parameters on the separation threshold and hazard in this context. Since it is not possible to derive quantitative estimates of the impact of parameters in the general case, I move on to more specific assumptions about the distribution of match quality in section 3. Section 4 discusses how one can empirically distinguish between the pure learning model, the pure shocks model and the mixed model. Section 5 discusses the limitations and implications of the results. Finally, section 6 concludes.

2. The optimal separation decision as a Partially Observable Markov Decision Process (POMDP)

The goal of this paper is to model the decision of an agent[1] to continue or separate from a relationship. The relationship links the agent with a partner. It is assumed that the agent entering a new relationship does not know the exact value of such a relationship. The quality of the relationship, or match quality, is what makes the relationship valuable to the agent. The agent holds a prior belief about the distribution of quality in the population of partners that it encounters. Then, at each period, the agent observes a signal of the quality of the relationship. Based on these signals, the agent updates its belief using Bayes' rule, and decides whether to continue with the current relationship, or end it and start a new one[2]. If the agent decides to end the relationship, it has to pay a cost $f(k)$ which is a function of the length of the relationship k. Indeed, it is not uncommon that separation costs go up with the length of the relationship: for example, in the case of the employment relationship, employees go through probation, which means that dismissing them is cheaper for the employer at low tenures.

Such a decision process can be modeled using the framework of Partially Observable Markov Decision Processes (see Hauskrecht(2002) for a full description of such models). This model allows solving for the optimal policy of the agent. The model is fully specified by states, actions, transition and observation functions, reward function, discount and planning horizon.

2.1. **Definitions.**

2.1.1. *States,actions.* The *state of the world* is defined by a vector of two variables: the length of the current relationship k, and the quality of the agent-partner match, q. The length of the relationship is perfectly observed by the agent. Moreover, to be realistic and

[1]The agent may be a person or an organization.

[2]This means that there is no explicit account for search in this model, and agents cannot choose to stay unmatched. For a discussion of the implications of such assumptions, see the discussion in section 5.

simplify calculations, we assume that the length of the relationship is limited to some length k_{max}. As to match quality, it is assumed to take a finite number of values; if match quality is a continuous distribution, this distribution is approximated through discretization. All these hypotheses imply together that the state space is finite.

Using information coming from previous experience or some other source, the agent forms an idea of how likely it is that, when forming a relationship, that relationship will turn out to be of a given quality. The agent thus has a prior belief about match quality, which is defined by a prior distribution of qualities $P(q_0)$. Assume that this prior distribution is normal with variance σ_0^2. I will denote by q_k the value of match quality at length k, thus allowing it to be time-varying.

At every time step, the agent has two possible actions a. it can continue the current relationship ($a = C$) or separate from the current partner and begin a new relationship with another partner ($a = S$).

2.1.2. *Transition and observation functions.* The transition function attributes a probability to each new possible state as a function of the current state and the agent's action. Before I define this function, a few remarks are in order about the notation. There is a perfect correspondence between the length of the relationship and the action taken, and so, to simplify notation, I will dispense with the specification of the action when the latter is evident given the length of the relationship. More precisely, a relationship length of 1 indicates a separation during the previous time period, and any $k > 1$ indicates a decision to continue the relationship at the previous period.

We are now ready to specify the match quality transition function, i.e. the probability of a given match quality next period given current match quality. The evolution of match quality is governed by the following equations:

$$P(q_1|q_k) = P(q_0) \tag{2.1}$$

$$P(q_{k+1}|q_k) = N(\rho q_k + c, \sigma_p^2) \tag{2.2}$$

where $P(q_0)$ is the prior distribution of match qualities. Thus, match quality is assumed to evolve over time according an AR(1) process, with $0 <= \rho <= 1$. If $\rho = 1$, $c = 0$ and $\sigma_p = 0$, then match quality is constant over time. The transition function is such that the state of the world at time $t + 1$ only depends on the action of the agent and the state of the world at time t, and not on the whole history of actions and states, i.e. the state process is Markovian.

The observation function gives, for each action and actual match quality, the probability of observing a given signal z, i.e. $P(z|a, q)$. Note that the observation function is assumed to be independent of the length of the relationship. I will denote by z_k the value of the observation at length k. The observation is defined as::

$$P(z_k|q_k) = N(q_k, \sigma_{obs}^2) \tag{2.3}$$

2.1.3. *Belief and belief transition function.* A belief state is a distribution of probability over the states of the world. While the length of the relationship is known with certainty, the belief about match quality needs to be specified as a probability distribution. Given the prior distribution of match qualities, the transition function and the observation function, it is possible to use Bayes' rule and compute the belief as a function of $z_{1:k}$, the history of observations from length 1 to the current length k of the relationship. The belief distribution at length k, $P(q_k|z_{1:k})$, can be summarized by $(\hat{q}_k, k)$, i.e. the expected value of the belief distribution and the length of the relationship. To simplify notation, one can use $\hat{q}_k$ to summarize the belief distribution. Since beliefs fully summarize what the agent knows about the system, it is convenient to express the agent's decision problem in the belief space.

Accordingly, from now on, whenever I refer to the state, I will mean belief state, and by transition function, I will mean the belief transition function. It is important to note here that the decision process expressed in terms of beliefs (rather than actual states of the world) is Markovian. That is, it can be shown that in a POMDP, the belief about the state of the world at time $t+1$ only depends on the action of the agent and the belief of the agent about the state of the world at time t, and not on the whole history of actions and beliefs (see for example Cassandra 1998).

Specifically, if the agent separates, its belief is defined by:

$$P(q_1|z_{1:k-1}) = P(q_1) = \sum_{q_0} P(q_1|q_0)P(q_0) \tag{2.4}$$

Thus, if the agent separates from the current relationship, the next belief state does not depend on the current belief state, but on the prior belief. If the agent continues, the best estimate $\hat{q}_k$ of q_k given (2.2) and (2.3) is given by the Kalman filter solutions (see Arulampalam et al. (2001)).

$$P(q_k|z_{1:k}) = N(\hat{q}_k, \sigma_k^2) \tag{2.5}$$

$$P(q_{k+1}|z_{1:k}) = N(\hat{q}_{k+1|k}, \sigma_{k+1|k}^2) \tag{2.6}$$

where

$$\hat{q}_{k+1|k} = \rho\hat{q}_k + c \tag{2.7}$$

$$\sigma_{k+1|k}^2 = \sigma_p^2 + \rho^2\sigma_k^2 \tag{2.8}$$

$$\hat{q}_{k+1} = \hat{q}_{k+1|k} + K_{k+1}(z_{k+1} - \hat{q}_{k+1|k}) \tag{2.9}$$

$$\sigma_{k+1}^2 = (1 - K_{k+1})\sigma_{k+1|k}^2 \tag{2.10}$$

In equations (2.9) and (2.10), K_{k+1} is the Kalman gain and is defined as:

$$K_{k+1} = \frac{\sigma^2_{k+1|k}}{\sigma^2_{k+1|k} + \sigma^2_{obs}} = \frac{\sigma^2_p + \rho^2\sigma^2_k}{\sigma^2_p + \rho^2\sigma^2_k + \sigma^2_{obs}} \tag{2.11}$$

The belief transition function, i.e. the probability of going from belief state $\hat{q}_k$ to state $\hat{q}_{k+1}$ can be determined by first expressing the probability of a given observation at the next period, z_{k+1}. Indeed, z_{k+1} observation and the current belief $\hat{q}_k$ fully determine the belief at the next period $\hat{q}_{k+1}$. Specifically, the probability of observing z_{k+1} given $\hat{q}_k$ is:

$$P(z_{k+1}|\hat{q}_k) = N(\rho\hat{q}_k + c, \rho^2\sigma^2_k + \sigma^2_p + \sigma^2_{obs}3) \tag{2.12}$$

Since $\hat{q}_{k+1} = \rho\hat{q}_k + c + K_{k+1}(z_{k+1} - \rho\hat{q}_k - c)$ is a linear function of z_{k+1}, the transition function in the belief space can be readily specified as:

$$P(\hat{q}_{k+1}|\hat{q}_k) = N(\rho\hat{q}_k + c, K^2_{k+1}(\rho^2\sigma^2_k + \sigma^2_p + \sigma^2_{obs})) = N\left(\rho\hat{q}_k + c, \frac{(\rho^2\sigma^2_k + \sigma^2_p)^2}{\rho^2\sigma^2_k + \sigma^2_p + \sigma^2_{obs}}\right) \tag{2.13}$$

One interesting question is whether beliefs become more precise over time, i.e. whether the variance of the belief σ^2_k decreases with k. If match quality is fixed over time, i.e. if $\sigma^2_p = 0$ and $\rho = 1$, then we have $\sigma^2_{k+1} = (1 - K_{k+1})\sigma^2_k$, and since $0 \leq K_{k+1} < 1$, this implies that indeed σ^2_k decreases with k. In other terms, with fixed match quality, beliefs do get more precise over time, and in the limit σ^2_k converges to 0, so that beliefs become perfectly accurate. However, if $\sigma^2_p > 0$, then σ^2_k no longer necessarily decreases with k. This is because while the agent accumulates observations, match quality changes, and therefore whether the belief gets more precise as relationship length increases depends on whether observations are sufficiently informative given the parameters of the match quality process. More precisely, we have:

$$\sigma^2_{k+1} < \sigma^2_k \Leftrightarrow (\sigma^2_p + \rho^2\sigma^2_k)(\sigma^2_{obs} - \sigma^2_k) - \sigma^2_{obs}\sigma^2_k < 0 \tag{2.14}$$

Since the variance of the transition function in equation (2.12) is an increasing function of σ_k, and σ_p^2 and σ_{obs}^2 do not depend on k, condition (2.14) also indicates when the variance of the transition function decreases over time.

2.1.4. *Reward, discount, horizon.* The reward function R associates a reward to each possible combination of belief and action continue (C) or separate (S):

$$R_C(\hat{q}_k) = \hat{q}_k \tag{2.15}$$

$$R_S(\hat{q}_k) = \bar{q} - f(k) \tag{2.16}$$

where $\bar{q}$ is the mean of the prior, $f(k)$ is a separation cost which depends on the length of the relationship k, and it is assumed that the agent discounts the future at rate $\delta \in [0, 1]$.

The reward function can be derived from two possible hypotheses about the observability of the per period benefit of the relationship to the agent. Either the benefit is not directly observed but is known to be equal to the relationship quality, and to be realized after the observation: in this case, the benefit is trivially equal to the agent's belief. Or the benefit is equal to the observation next period: in this case, if we define the observation to have the same expected value as the actual quality, then $\hat{q}_k$ is indeed the agent's best estimate of the expected value of the observation, and hence the reward, at the next period.

The separation cost $f(k)$ covers the direct cost of ending the current relationship, such as a firing cost in the case of the employment relationship. It also covers the costs of beginning a new relationship, such as hiring costs. If the two partners have diverging interests over separation, i.e. if for example it is harder for the worker to find a new job than for the firm to find a new worker, then the model is not complete because it does not explicitly account for both partners' rewards. However, if these diverging interests are known ex ante and do not depend on match quality, then the party that is relatively more advantaged by the separation can agree ex ante to make a fixed payment to the other party. This case is covered by the model since the cost $f(k)$ can also include any such payments.

The definition of the reward function is compatible with a Nash bargaining solution where the two partners split the surplus, so that, while the relationship continues, each partner gets a fixed share. Suppose that α is the share received by the agent. The reward of the agent would then be $\alpha\hat{q}_k$ if continuing and $\alpha\bar{q}$ if separating; but this change is not substantial since it simply amounts to rescaling the distribution of match quality.

The planning horizon of the agent is assumed to be infinite. This means that the agent is infinitely lived; or alternatively, the agent's retirement from the relationships market is at some date so far away in the future that given the discount factor, it does not play any role in the agent's current decisions. The model is thus not quite adequate for explaining the behavior of "old" agents. That is typically not a problem if the agent considered is an organization, but may be relevant if the agent is a person.

2.1.5. *Value function.* We now need to define what it means for the agent to follow an optimal strategy. To do so, I will first define the notion of a strategy or policy, and the value function for a policy. Define a policy π, which gives for each belief and relationship length the action to be taken, i.e. either continue or separate. Define the Q function $Q^\pi(\hat{q}_k, a)$ as the expected return of taking action a today and then following the policy π in the future. The value function $V^\pi(\hat{q}_k)$ gives the current and future rewards of the agent as a function of current belief, assuming that the agent follows policy π from now on. The optimal policy maximizes $V^\pi(\hat{q}_k)$, and gives rise to the optimal value function $V^*(\hat{q}_k)$. The optimal action value function Q^* is defined as a function of the optimal value $V^*(\hat{q}_k)$:

$$Q^*(\hat{q}_k, C) = \hat{q}_k + \delta \sum_{\hat{q}_{k+1}} P(\hat{q}_{k+1}|\hat{q}_k) V^*(\hat{q}_{k+1}) \tag{2.17}$$

$$\begin{aligned} Q^*(\hat{q}_k, S) &= \bar{q} - f(k) + \delta \sum_{\hat{q}_1} P(\hat{q}_1|\bar{q}) V^*(\hat{q}_1) \\ &= V_{new} - f(k), \text{ where } V_{new} = \bar{q} + \delta \sum_{\hat{q}_1} P(\hat{q}_1|\bar{q}) V^*(\hat{q}_1) \end{aligned} \tag{2.18}$$

The optimal value is given by the Bellman equation:

$$V^*(\hat{q}_k) = \max_{a\in[C,S]} Q^*(\hat{q}_k, a) \tag{2.19}$$

In this framework, the optimal policy followed by the agent is uniquely defined by $\tau(k)$, the belief such that the agent is indifferent between continuing or separating from its partner at relationship length k. In other terms, the threshold for separation $\tau(k)$ is defined by the equalization[3] of Q functions for the actions "continue" (equation (2.17)) and "separate" (equation (2.18)), i.e.:

$$\tau(k) + \delta \sum_{\hat{q}_{k+1}} P(\hat{q}_{k+1}|\tau(k))V^*(\hat{q}_{k+1}) = \bar{q} - f(k) + \delta \sum_{\hat{q}_1} P(\hat{q}_1|\bar{q})V^*(\hat{q}_1) \tag{2.20}$$

$$\Leftrightarrow \tau(k) - \bar{q} + f(k) + \delta[\sum_{\hat{q}_{k+1}} P(\hat{q}_{k+1}|\tau(k))V^*(\hat{q}_{k+1}) - \sum_{\hat{q}_1} P(\hat{q}_1|\bar{q})V^*(\hat{q}_1)] = 0 \tag{2.21}$$

2.2. **Computing the value function and the optimal policy.** To compute the optimal policy, one starts at the highest possible relationship length, i.e. k_{max}. At that point, because relationships come to a final ending, the value of a relationship is exactly equal to the value of a new relationship, minus final separation costs, i.e. $V_{new} - f(k_{max})$.

The algorithm starts with giving V_{new} some arbitrary value. Then, at length $k_{max} - 1$, $Q(\hat{q}_k, S)$ and $Q(\hat{q}_k, C)$ are computed using equations 2.18 and 2.17. The optimal policy at $k_{max} - 1$ is then given by equation 2.19. These calculations are repeated for $k_{max} - 2, k_{max} - 3, ...$

It is thus possible to recursively compute the value up to length 0. V_{new} is then defined as the value of a relationship with length 0 and quality $\bar{q}$ (the expected value of the prior distribution). We start the loop over again until V_{new} is numerically identical to its value in

[3]When performing computations, we only consider a finite number of match qualities. Therefore there will typically be no belief that makes the agent *indifferent* between continuing and separating: rather, the optimal action will be "separate" for some belief and "continue" for the next higher belief. In practice, I defined as the threshold the minimum expected belief such that it is optimal for the agent to continue the relationship.

the previous iteration[4]. One can thus determine the value function and optimal actions for all beliefs and relationship lengths.

2.3. **The impact of parameters on the optimal policy.** In order to assess the impact of parameters on the optimal policy, one first needs to show how the condition defining the threshold in equation (2.21) is affected when parameters change. Define J as the left-hand side of equation (2.21), i.e.:

$$J = \tau(k) - \bar{q} + f(k) + \delta[\sum_{\hat{q}_{k+1}} P(\hat{q}_{k+1}|\tau(k))V^*(\hat{q}_{k+1}) - \sum_{\hat{q}_1} P(\hat{q}_1|\bar{q})V^*(\hat{q}_1)]$$

Consider some parameter x: we are interested in the sign of $\frac{\partial\tau(k)}{\partial x}$. Given that the threshold is defined by $J = 0$, we can use properties of implicit functions to determine the sign of $\frac{\partial\tau(k)}{\partial x}$. It is a known result that, if $J(\tau(k), x) = 0$, then $\frac{\partial\tau(k)}{\partial x} = -\frac{\partial J/\partial x}{\partial J/\partial\tau(k)}$. It is easy to show that J increases with $\tau(k)$, which implies that $\partial J/\partial\tau(k) > 0$. Hence, we have:

$$sign(\partial\tau(k)/\partial x) = -sign(\partial J/\partial x) \tag{2.22}$$

In the general case, it is not possible to determine $sign(\partial J/\partial x)$. It becomes feasable, however, if one uses a few approximations[5]. First, I show that, if k is small, then $\hat{q}_k = \hat{q}_{k+1}$ implies that $V^*(\hat{q}_k) \approx V^*(\hat{q}_{k+1})$. This allows me to drop k from $\hat{q}_k$. I then proceed to determine the sign of $\partial J/\partial x$, which will involve some more approximations.

Early in the relationship, if $\hat{q}_k = \hat{q}_{k+1}$, there is only a negligible difference between $V^*(\hat{q}_k)$ and $V^*(\hat{q}_{k+1})$. This is because, given the existence of a discount factor, the maximum possible length k_{max} is too far away in the future to influence the current value. Thus, at

[4]This is a special case of the "value iteration" algorithm, which has been shown to converge to the solution of the Partially Observed Markov Decision Problem (see Hauskrecht(2002)). Note however that this algorithm is not the fastest possible to establish the optimal policy, because we compute the values for all possible beliefs, whereas it is clear that if for some belief it is optimal to separate, then for all beliefs with lower expected value, it is also optimal to separate. If computation time were a concern, one could therefore use a faster algorithm.

[5]For each of these approximations, I will explain why it may be correct. Moreover, all the approximations used are indeed good approximations in the cases for which I explicitly compute the threshold in section 3.

short relationship lengths, the value of a given belief does not change with relationship length k.

Using the approximation $V^*(\hat{q}_k) = V^*(\hat{q}_{k+1})$ for $\hat{q}_k = \hat{q}_{k+1}$ allows us to drop the subscript of $\hat{q}$, and the condition defining the threshold can then be rewritten as:

$$\tau(k) - \bar{q} + f(k) + \delta \sum_{\hat{q}} [P(\hat{q}|\tau(k)) - P(\hat{q}|\bar{q})] V^*(\hat{q}) = 0$$

The function J is consequently redefined as:

$$J = \tau(k) - \bar{q} + f(k) + \delta \sum_{\hat{q}} [P(\hat{q}|\tau(k)) - P(\hat{q}|\bar{q})] V^*(\hat{q}) \tag{2.23}$$

I now show that typically:

$$\sum_{\hat{q}} [P(\hat{q}|\tau(k)) - P(\hat{q}|\bar{q})] V^*(\hat{q}) < 0 \tag{2.24}$$

This inequality will be important in determining the sign of the derivative of J with respect to some parameter x. Let $P_\tau(\hat{q})$ be a shorthand for $P(\hat{q}|\tau(k))$ and $P_{\bar{q}}(\hat{q})$ a shorthand for $P(\hat{q}|\bar{q})$. Note first that $V^*(\hat{q})$ trivially increases in $\hat{q}$. The expression $\sum_{\hat{q}} [P_{\tau(k)}(\hat{q}) - P_{\bar{q}}(\hat{q})] V^*(\hat{q})$ is the difference of two Gaussian-weighted means of the increasing series $V(\hat{q})$. These means are respectively taken in the neighborhood of $E(P_\tau)$ and $E(P_{\bar{q}})$. If the distributions P_τ and $P_{\bar{q}}$ have the same variance, and $E(P_\tau) < E(P_{\bar{q}})$, then inequality 2.24 is satisfied. Now I show that $E(P_\tau) < E(P_{\bar{q}})$. If $\tau(k) = \bar{q}$, then equation (2.3) implies that $f(k) = 0$. If $f(k) > 0$, then $\tau(k) < \bar{q}$, i.e. in the presence of positive separation costs, the threshold for separation is lower than the expected quality of a new match. We have $E(P_\tau) = \rho\tau(k) + c$ and $E(P_{\bar{q}}) = \rho\bar{q} + c$, and therefore, $E(P_\tau) < E(P_{\bar{q}})$.

In a more general case, the variances of P_τ and $P_{\bar{q}}$ may differ and so the means $\sum_{\hat{q}} P(\hat{q}|\tau(k)) V^*(\hat{q})$ and $\sum_{\hat{q}} P(\hat{q}|\bar{q}) V^*(\hat{q})$ use different weights. Denote by σ_1 the standard deviation of $P_\tau(\hat{q})$ and σ_2 the standard deviation of $P_{\bar{q}}(\hat{q})$. As long as σ_1 is not much greater than σ_2, we still have $\sum_{\hat{q}}[P(\hat{q}|\tau(k)) - P(\hat{q}|\bar{q})]V^*(\hat{q}) < 0$. In almost all cases that we will examine, $\sigma_1 < \sigma_2$[6] , and so we typically have $\sum_{\hat{q}}[P(\hat{q}|\tau(k)) - P(\hat{q}|\bar{q})]V^*(\hat{q}) < 0$.

2.3.1. *Impact of the separation cost on the optimal policy.* Taking the derivative of J with respect to $f(k)$, we get:

$$\frac{\partial J}{\partial f(k)} = 1 + \delta \sum_{\hat{q}} (P_\tau(\hat{q}) - P_{\bar{q}}(\hat{q})) \frac{\partial V^*(\hat{q})}{\partial f(k)} \tag{2.25}$$

$\partial V^*(\hat{q})/\partial f(k)$ decreases with $\hat{q}$ because higher quality matches have a lower probability of being eventually dissolved and thus the agent is less likely to bear the firing cost for higher quality matches. Since $\partial V^*(\hat{q})/\partial f(k)$ decreases with $\hat{q}$, the second term of equation (2.25) is positive[7]. Thus, $\partial J/\partial f(k) > 0$, which implies that $\partial \tau(k)/\partial f(k) < 0$. As is intuitive, this means that higher separation costs make the agent more willing to pursue relationships of lower value. Hence, we have:

Proposition 1. *Higher separation costs $f(k)$ lower the threshold for separation $\tau(k)$.*[8]

This implies in particular that if the separation cost increases over time, then the threshold decreases over time. For example, if there is some sort of probation period, with a low constant separation cost followed by a higher constant separation cost, then at the period when the separation cost increases, the threshold will decrease. Moreover, in such a case, the threshold will slightly increase at the end of the probation period (i.e. the initial period with low separation cost). This is explained by the following consideration. As the end of the probation period approaches, the value of separating from the relationship stays the same but the value of continuing the relationship decreases because of the increased probability

[6]This is because it is typically the case that the agent's belief gets more precise over time, and so since $P(\hat{q}|\bar{q})$ is taken at length 0, its variance is greater than the variance of $P(\hat{q}|\tau(k))$, which is taken at some length at least equal to 1.

[7]This is for the same reasons why the fact that $V^*(\hat{q})$ increases in $\hat{q}$ implies that $\sum_{\hat{q}}[P(\hat{q}|\tau(k)) - P(\hat{q}|\bar{q})]V^*(\hat{q}) < 0$.

[8]Most propositions in this paper (including this one) are dependent on the approximations used. However, they are still useful to understand the logic of the model, and they relate to each other in such a way that it is useful to number them. When a proposition depends on approximations, I will signal it in a footnote.

that a bad quality relationship will have to be terminated under the higher post-probation separation cost. Since the value of continuation decreases relative to the value of separation as the agent approaches the end of the probation period, the threshold increases.

2.3.2. *The impact of the discount factor on the threshold of separation.* Taking the derivative of J with respect to δ, we get:

$$\frac{\partial J}{\partial \delta} = \sum_{\hat{q}} [P_\tau(\hat{q}) - P_{\bar{q}}(\hat{q})][\delta \frac{\partial V^*(\hat{q})}{\partial \delta} + V^*(\hat{q})] \tag{2.26}$$

It seems plausible that $\partial V^*(\hat{q})/\partial \delta$ is roughly constant over $\hat{q}$, as a lower discount factor roughly proportionally reduces the value of all levels of match quality[9]. If this assumption is valid, then $\sum_{\hat{q}}[P_\tau(\hat{q}) - P_{\bar{q}}(\hat{q})]\delta \frac{\partial V^*(\hat{q})}{\partial \delta} = 0$: this is because both $P_\tau(\hat{q})$ and $P_{\bar{q}}(\hat{q})$ add up to 1 over $\hat{q}$. Then, equation (2.26) simplifies to:

$$\frac{\partial J}{\partial \delta} = \sum_{\hat{q}} [P_\tau(\hat{q}) - P_{\bar{q}}(\hat{q})]V^*(\hat{q}) \tag{2.27}$$

We have already established that the above expression is negative. Thus, from equation (2.22), we infer that:

Proposition 2. *The threshold of separation $\tau(k)$ increases with a higher discount rate δ*[10].

The intuition for this result is that a higher discount rate has an impact that is similar to that of a lower separation cost. Indeed, a higher discount rate implies that a good quality match yields a higher discounted value, and therefore it is more advantageous to end a mediocre match today in the hope of getting a better match in the future.

2.3.3. *The impact a change in the transition function on the threshold of separation.* We are now interested in the effect of some parameter x that enters both P_τ and $P_{\bar{q}}$. The derivative

[9]This is verified in all specific cases I will analyze in section 3.
[10]This proposition depends on the approximations used.

of J with respect to such a parameter is:

$$\frac{\partial J}{\partial x} = \sum_{\hat{q}} \left\{ \left(\frac{\partial P_\tau(\hat{q})}{\partial x} - \frac{\partial P_{\bar{q}}(\hat{q})}{\partial x} \right) V^*(\hat{q}) + (P_\tau(\hat{q}) - P_{\bar{q}}(\hat{q})) \frac{\partial V^*(\hat{q})}{\partial x} \right\} \tag{2.28}$$

Signing the expression above does not seem to be feasible unless one uses a substantial number of approximations. Therefore, the impact of parameters that enter both P_τ and $P_{\bar{q}}$ on the threshold is ambiguous and will need to be determined for specific parameter values.

2.4. **Hazard of separation.** Deriving the impact of parameters on the hazard of separation is an important task because the separation hazard can be computed from empirical data, while the threshold for separation is typically not observed. The theoretical hazard of separation is the result of infinitely many agents confronted with the same separation problem; it summarizes the average separation behavior of agents over relationship lengths.

One can compute the theoretical separation hazard once the threshold for separation is known. Note that at length 0, when no observation has been made yet, $\hat{q}_0 = \bar{q}$ for all matches, i.e. for all agents the belief is the same as the prior. Let $p_k(\hat{q}_k)$ be the density of agents who hold a belief with mean $\hat{q}_k$ at length k, given that they follow the optimal policy embodied in $\tau(k)$. The initial values for the distribution of agents' expected beliefs about match quality are:

$$p_0(\hat{q}_0) = \begin{cases} 1 \text{ if } \hat{q}_0 = \bar{q} \\ 0 \text{ otherwise} \end{cases} \tag{2.29}$$

$$p_1(\hat{q}_1) = \sum_{\hat{q}_0} p_0(\hat{q}_0) P(\hat{q}_1|\hat{q}_0) = P(\hat{q}_1|\bar{q}) \tag{2.30}$$

The hazard of separation at length k, h_k, can then be computed recursively, starting at $k = 1$ and using the following steps:

(2.31) $$h_k = \sum_{\hat{q}_k=\hat{q}_{min}}^{\hat{q}_k=\tau(k)} p_k(\hat{q}_k)$$

(2.32) $$p_k(\hat{q}_k) = 0 \text{ if } \hat{q}_k \leq \tau(k)$$

(2.33) $$p_k(\hat{q}_k) = \frac{p_k(\hat{q}_k)}{\sum p_k(\hat{q}_k)}$$

(2.34) $$p_{k+1}(\hat{q}_{k+1}) = \sum_{\hat{q}_k} p_k(\hat{q}_k) P(\hat{q}_{k+1}|\hat{q}_k)$$

Equation (2.33) insures that the mass of agents is always normalized to 1.

2.5. **The impact of parameters on the hazard of separation.** Parameters affect the hazard of separation through their effect on the threshold and the transition function. Therefore, we will first discuss the impact on the hazard of changes in the threshold and the transition function, and then proceed to the full analysis of the impact of parameters.

2.5.1. *Impact of the threshold and the transition function on the hazard of separation.* For a given distribution $p_k(\hat{q}_k)$, the effects of a change in the threshold of separation $\tau(k)$ or the transition function $P(\hat{q}_k|\hat{q}_{k-1})$ on the hazard of separation h_k are straightforwardly described. The impact of the transition function on the hazard of separation for a given threshold is defined by equations (2.30) and (2.34).

Proposition 3. *For a given threshold $\tau(k)$ and a given distribution of continuing relationships $p_k(\hat{q}_k)$, a higher variance for the transition probability $P(\hat{q}_k|\hat{q}_{k-1})$ implies a higher separation hazard h_k.*

Indeed, a higher variance for the transition probability $P(\hat{q}_k|\hat{q}_{k-1})$ implies that more matches will cross the threshold from one period k to the next (see equation 2.34). Thus, any change in parameters that increases the standard error of the transition function $P(\hat{q}_k|\hat{q}_{k-1})$, will, for a given policy and distribution of continuing relationships, increase the hazard of separation at length k.

For a given transition function, the threshold for separation affects the hazard of separation as described by equation (2.31).

Proposition 4. *The higher the threshold* $\tau(k)$*, the higher the hazard of separation* h_k*.*

Indeed, a higher threshold implies that a higher expected value of the belief distribution is needed for the agent to continue the relationship.

2.5.2. *Impact of parameters entering the transition function on the separation hazard.* If a parameter affects the variance of the transition function but has little impact on the threshold, then proposition 3 determines the impact of such a parameter. If, however, the parameter has a significant impact on both the threshold and the transition function, the effect on the hazard function cannot be predicted but has to be calculated numerically. Let us now examine the impact of the first type of parameters: can we say anything about it since proposition 3 is conditional on the distribution of continuing relationships $p_k(\hat{q}_k)$?

First, note that no change in the variance of the transition function can affect the starting point for the calculation hazard: indeed, from equation (2.29), $p_0(\hat{q}_0)$ only depends on $\bar{q}$. Second, note that the hazard of separation at length k is determined by the successive application of the transition function to the initial distribution $p_0(\hat{q}_0)$, with a truncation of the distribution below the threshold at each step. Thus, if a parameter increases the variance of the transition function at all lengths and does not affect the threshold, then, from Proposition 3, it increases the hazard of separation at length 1 for sure. However, this effect may be reversed with increasing length, i.e. it can be that a parameter that increases the variance of the transition function at all lengths and does not affect the threshold actually *decreases* the hazard of separation at longer lengths. Section 3, will show how this effect occurs under some specific parameter values. Thus, we have that:

Proposition 5. *A parameter that increases the variance of the transition function at short lengths, and does not affect the threshold, increases the hazard of separation at short lengths.*

2.5.3. *Impact of separation costs on the separation hazard.* Higher separation costs decrease the threshold and do not influence the transition function, therefore, from proposition 4:

Proposition 6. *Higher separation costs $f(k)$ decrease the separation hazard h_k for all lengths k*[11]*.*

If separation costs do not depend on k, then higher separation costs decrease the hazard of separation at all lengths. If there is a probation period, then the hazard of separation will be higher during the probation period relative to the post-probation period. Moreover, because the threshold increases when approaching the end of the probation period, the hazard of separation also increases, creating a spike at the end of the probation period.

2.5.4. *Impact of the discount factor on the separation hazard.* A higher discount rate increases the separation threshold and does not affect the transition function, therefore, from proposition 4:

Proposition 7. *A higher discount factor δ increases the separation hazard h_k for all lengths k*[12]*.*

We have now completed the exploration of what can be said in general about the effects of parameters on the threshold of separation and the separation hazard. In the next section, I will compute and analyze these effects using specific parameters.

3. Impact of parameters on the hazard of separation

Since the impact of parameters on the separation hazard cannot be fully determined analytically, it is useful to analyze the model for some given parameters.

[11]This proposition depends on the approximations used in assessing the effect of separation costs on the threshold of separation.

[12]This proposition depends on the approximations used in assessing the effect of the discount rate on the threshold of separation.

3.1. **Choice of parameters.** The specific parameters used in this section can be found in Table 1, reference case 1. The important point here is that parameters were chosen to be such that the hazard of separation first increases and then decreases with relationship length, a pattern that is typically found in studies of the firing hazard (Farber(1994),Marinescu(2006a)) or the divorce hazard (Weiss-Willis(1997), Svarer(2004)). Note that the specific value of the mean of the prior, $\bar{q}$, is not substantial for the calculations since the normal distribution is symmetric and defined over $\mathbb{R}$. $\bar{q}$ only matters relative to the separation cost. To make the interpretation of $\hat{q}_k$ more intuitive, I chose the minimal value of $\hat{q}_k$ to be 0 and its maximal value to be $2\bar{q}$. Thus a positive separation cost is commensurate with the per period value of the relationship. The parameters ρ and c, i.e. the parameters of the AR(1) process were chosen so as to yield a random walk ($\rho = 1$ and $c = 0$); this is to keep the model as close as possible from the Jovanovic(1979) learning model, in which match quality is deterministic *and* constant over time. Finally, for computational purposes, the belief space is divided in discrete steps. Specifically, the discretization uses equally spaced values between some minimal and some maximal value of $\hat{q}_k$, as specified in Table 1. To perform the computations in this section, I choose for each parameter a few values in the neighborhood of the reference value, and I compute the variance of the transition function, the separation threshold, and the resulting hazard of separation.

3.2. **Results.**

3.2.1. *The impact of separation costs.* As already pointed out, separation costs have no impact on the transition function. The computations confirm that higher separation costs do indeed lower the separation threshold. Note, moreover, that the threshold is almost constant over relationship lengths; it increases very slightly as the relationship approaches the maximal length. Since a higher separation cost decreases the threshold, it lowers the separation hazard, as can be seen in Figure 1.

One can also examine the impact of a probation period, i.e. instead of having constant separation costs over the length of the relationship, separation costs are constant in the beginning of the relationship, and they then increase to a higher and constant level after some given length. In this case, I use a separation cost of 30 in the beginning, and 40 after the end of the probation period. I also show results for two different lengths of the probation period, that is 12 and 24 periods. The separation thresholds are plotted in Figure 2. As predicted, one observes an increase in the threshold before the end of the probation period, and lower thresholds afterward. The hazards are plotted in Figure 3. As a result of the variations in the thresholds, the hazards increase right before the end of the probation period, producing spikes in separations. The spike is higher with a shorter probation period because, at lower lengths, there are more low quality matches that are close to the threshold, and therefore a larger proportion of relationships is terminated due to the increase in the separation threshold right before the end of the probationary period.

3.2.2. *The impact of the discount factor.* A higher discount factor has the same qualitative effect as lower firing costs, and so I do not reproduce any graphs illustrating its impact here.

3.2.3. *The impact of the process variance.* As shown in Figure 4, an increase in the standard error of the process leads to a higher variance of the transition function at all lengths. Moreover, when the process variance is 1, the variance of the transition function decreases with length, whereas it increases with length for values greater or equal to 2. Variances converge to a constant value, which corresponds to the process variance[13]. As for the impact on the threshold, a higher standard deviation of the process decreases the threshold (not shown). Since a higher process variance increases the variance of the transition function but decreases the threshold, the impact on the separation hazard is ambiguous.

[13]It is easy to show that if $\rho = 1$, then $\sigma_k^2 = \sigma_{k+1}^2$ implies that the variance of the belief transition function in equation 2.13 is equal to the process variance σ_p^2. This implication does not hold however if $\rho < 1$.

Figure 5 shows that the impact of the process variance on the separation hazard is overall positive. The impact of an increase in process variance on the hazard is mainly explained by the fact that a higher process variance increases the variance of the transition function.

What happens as the standard error of the process goes to 0, i.e. as the model becomes closer and closer to a pure learning model a la Jovanovic(1979)? Figure 6 shows that, as the standard error of the process decreases, the variance of the transition function keeps decreasing at all relationship lengths. One also notices that when the standard deviation is 0, i.e. in the pure learning model, the variance of the transition function no longer converges to a strictly positive value beyond a certain relationship length, but tends to 0 instead. One can also note that as the process variance increases from 0, the variance of the transition function converges earlier and earlier: this is because, as match quality evolves in a more and more unpredictable way, the initial improvement in belief precision stemming from observing additional signals of match quality reaches its limits earlier. Figure 7 shows how the separation hazard changes as the process standard deviation increases above 0[14]. In the pure learning model just as in the reference model, the hazard goes up and then down with relationship length. As the standard deviation of the process increases a little, the hazard increases at all lengths, and especially so at higher lengths. With a process standard deviation of 0.4, the hazard does not eventually decrease with relationship length. However, with a process standard deviation of 0.8, we observe again that the hazard decreases at higher lengths.

3.2.4. *The impact of the observation variance.* Because a higher observation variance implies that the agent acquires information at a slower pace (each observation is less informative), the higher the observation variance the closer the agent's belief should stay to the prior at low relationship lengths. In other terms, one can expect that at low relationship lengths a higher observation variance will lead to a lower variance of the transition function. Figure

[14]The little bumps in the hazards are due to discretization artifacts.

8 shows that, indeed, an increase in the observation variance decreases the variance of the transition function at low relationship lengths, at least if we consider a standard deviation of 2 or more. Eventually, regardless of the observation variance, the variance of the transition function converges to the same value[15]. Since the variance of the transition function and the threshold of separation both decrease with the standard deviation of the observation, we expect the hazard of separation to decrease with the standard deviation of the observation at low lengths.

Plotting the separation hazard in Figure 9, we see that, as expected, the hazard decreases with the observation variance at low lengths. At higher lengths the hazard increases with the observation variance. The explanation for this subsequent increase is given by the evolution of the distribution of continuing relationships with length. Intuitively, a noisier observation does not allow the agent to detect the "lemons" as fast, which means that the hazard of separation at low lengths is smaller. On the other hand, since with a noisier observation the agent has not been able to sort out the lemons so well in the beginning of relationships, there are more lemons left among continuing relationships. This is what drives the higher hazard of separation at longer relationship lengths.

To understand this in the specific context of the model, let's look at Figure 10. At length 2, we observe that the distribution of continuing relationships (after separations took place but before the next observation was obtained) with an observation standard deviation of 6 is narrower than the distribution with an observation standard deviation of 0. This implies that fewer low quality relationships are terminated at length 2 when the observation standard deviation is 6 versus 0. At length 10, the distribution of continuing relationships corresponding to a standard deviation of 0 has a higher density in the neighborhood of the threshold. The higher proportion of low quality relationships at length 10 in the case where the observation standard deviation is 6 is explained by the fact that overall fewer

[15]This is only the case because $\rho = 1$.

low quality relationships have been terminated before length 10, and so there are more low quality relationships to terminate at length 10 and later.

One last important thing to note here is that even if the observation standard deviation is 0, i.e. there is no uncertainty about current match quality, we still observe that the hazard of separation increases and then decreases with relationship length. This is very important since it shows that a hazard of separation that increases and then decreases with relationship length is not necessarily a sign of "‘learning"’, if by learning we mean that the agent is uncertain about the true value of current match quality and learns about it over time. One may be curious about how a model without learning can generate such a pattern in the separation hazard. The explanation for that is fairly simple. If separation costs are high enough, it is not optimal to terminate all those matches that are below the mean of the prior. Indeed, some of these matches, through random drift, may end up being good enough to keep. In other terms, the threshold of separation lies to the left of the mean of the prior. Now as match quality evolves over time and some relationships receive negative shocks, more and more matches are pushed below the threshold. However, eventually most of the mass of relationships near the threshold will have drifted below the threshold, and continuing matches will be concentrated far away to the right of the threshold. In fact, assuming a random walk as in the reference case, many matches randomly become extremely valuable (there is no upward limit to the process) over time and the proportion of these matches among surviving matches tends to increase over time. If the quality process was not a random walk but converged to some relatively high value, we would also observe that the hazard eventually decreases, since the density of continuing matches would tend to concentrate to the right of the separation threshold.

3.2.5. *The impact of the prior variance.* As shown in Figure 11, a higher prior variance increases the variance of the transition function at low lengths. However, the variance of the transition function eventually converges to the same value, irrespective of the prior variance.

The convergence is explained by the fact that, if match quality is time-varying, the prior ceases to matter after a while. Note that, with a prior standard deviation smaller than 3, the variance of the transition function increases with length. Moreover, a higher standard deviation for the prior increases the threshold for separation. The effects on the variance of the transition function and the threshold together imply that the hazard of separation should increase with prior variance at short lengths.

Figure 12 shows that indeed with a higher standard deviation of the prior, the hazard of separation is higher at low lengths. However, at higher lengths, the hazard is higher for a lower standard deviation of the prior. This is explained by the fact — mentioned earlier when discussing the impact of the observation variance — that a lower variance of the transition function at short lengths leads to a permanently higher separation hazard all other things equal.

3.2.6. *The impact of the drift.* The drift in the match quality process does not affect the variance of the transition function. It does however change its mean in a straightforward additive way, as seen in equation (2.12): the smaller the drift, the less relationship quality improves over time. Thus, for a given threshold, a smaller drift means that, at each length, more relationships cross the threshold, which should lead to a higher separation hazard. On the other hand, it turns out that a larger drift has a positive effect on the separation threshold. Since these two effects go in opposite directions, the impact on the hazard of separation is ex ante ambiguous.

In fact, a larger drift has a negative impact on the hazard of separation, which means that the effect on the mean of the transition function dominates the effect on the threshold.

3.2.7. *The impact of the AR(1) parameter of the process.* The AR(1) parameter ρ of the process has a positive impact on the variance of the transition function, as illustrated by Figure 13. This positive impact is small at low lengths and increases thereafter. A higher

ρ increases the separation threshold. These elements together imply that a higher ρ should increase the hazard of separation at low lengths.

Figure 14 shows that a higher ρ has very small positive effect on the hazard at very low lengths, but decreases it for longer lengths. This is because the effect of ρ on mean match quality given previous match quality (see equation 2.2) dominates: a larger ρ implies that relationship quality does not deteriorate as fast, and thus a larger ρ decreases the separation hazard.

4. Differences between the pure learning and the pure shocks models

In many empirical applications, the observation of a hazard of separation that increases and then decreases with relationship duration has been taken as evidence in favor of a pure learning model a la Jovanovic(1979). However, we have seen that the learning model is not the only one that can yield such a result. It is possible to obtain such a pattern for the separation hazard with both a pure shocks model ($\sigma_{obs} = 0$ and $\sigma_p > 0$) and a mixed model ($\sigma_{obs} > 0$ and $\sigma_p > 0$). This section discusses how one can distinguish the pure learning model ($\sigma_p = 0$, $rho = 1$, $c = 0$) from a mixed model with $\sigma_p > 0$, and a pure shocks model with $\sigma_{obs} = 0$.

If one can observe the variance of the transition function at various relationship lengths, one can infer which model is most likely to be correct. If the variance of the transition function does not change with relationship length, then the pure shocks model is correct and the two other models are rejected. If the variance of the transition function increases with length then the mixed model is correct and the two other models are rejected. Finally, if the variance of the transition function decreases with length, either the pure learning or the mixed model can explain the data, while the pure shocks model is rejected. If one can observe the relationship for long enough, then even if the variance of the transition function decreases with length, one can distinguish the pure learning model from the mixed model.

Indeed, in the pure learning model the variance of the transition function will eventually converge to 0, which will not be the case in the mixed model.

If we do not observe the transition function variance, we can still make some distinctions between models based on the observation of the separation hazard. If separation costs are constant (as assumed here) or increasing with relationship duration, then the pure learning model implies that the hazard decreases to 0 as relationship duration tends toward infinity. This is because in the limit the agent knows with certainty what the quality of the relationship is, and so only good relationships persist (the variance of the transition function converges to 0). If, on the other hand, match quality evolves over time (pure changes model or mixed model), then the hazard of separation never drops to 0 (unless separation costs become exceedingly high). This is because if the quality of a relationship can change, even a very good relationship can eventually become bad and be dissolved after a sequence of unfavorable changes (the variance of the transition function converges to a positive value).

Another feature that distinguishes the pure learning model from the others is the impact of a negative observation on the hazard of separation at different relationship durations. A negative observation is some observation that signals low match quality; for example, it could be an absence from the job in the case of a job match. Specifically, assume that if the a negative shock occurred at relationship length k, then the agent observes $z_k < z^*$, where z^* is some low value of the observation. The hazard h_b given a bad observation $z_k < z^*$ is:

$$h_b(k) = \sum_{\hat{q}_k < \tau(k), \hat{q}_{k-1}} p_{k-1}(\hat{q}_{k-1}) \frac{P(\hat{q}_k | \hat{q}_{k-1}, z_k < z^*)}{P(z_k < z^* | \hat{q}_{k-1})} \tag{4.1}$$

$$= \sum_{\hat{q}_{k-1}} p_{k-1}(\hat{q}_{k-1}) \frac{\sum_{\hat{q}_k = \hat{q}_{min}}^{\hat{q}_k = min(g(\hat{q}_{k-1}, z^*), \tau(k))} P(\hat{q}_k | \hat{q}_{k-1})}{P(\hat{q}_k < g(\hat{q}_{k-1}, z^*))} \tag{4.2}$$

where p_k is defined as in equation 2.34, and the function g gives the value of $\hat{q}_k$, such that given $\hat{q}_{k-1}$, this value corresponds to the observation of z^* at period k. Indeed, $\hat{q}_{k-1}$ and z_k uniquely determine $\hat{q}_k$ given distributional assumptions.

Similarly, the hazard h_g given a relatively good observation $z_k > z^*$ is:

$$h_g(k) = \sum_{\hat{q}_k < \tau(k), \hat{q}_{k-1}} p_{k-1}(\hat{q}_{k-1}) \frac{P(\hat{q}_k | \hat{q}_{k-1}, z_k > z^*)}{1 - P(z_k < z^* | \hat{q}_{k-1})} \tag{4.3}$$

$$= \sum_{\hat{q}_{k-1}} I[g(\hat{q}_{k-1}, z^*) \leq \tau(k)] p_{k-1}(\hat{q}_{k-1}) \frac{\sum_{\hat{q}_k = g(\hat{q}_{k-1}, z^*)}^{\hat{q}_k = \tau(k)} P(\hat{q}_k | \hat{q}_{k-1})}{1 - P(\hat{q}_k < g(\hat{q}_{k-1}, z^*))} \tag{4.4}$$

where I is an indicator function.

It is important to note that for each relationship length k, these hazards are computed assuming either $z_k < z^*$ or $z_k > z^*$, *and, in both cases,* a history of observations up to z_{k-1} that is consistent with the distributional assumptions and the optimal strategy of the agent.

Figure 15 plots the $h_b(k)$ and $h_g(k)$ under the pure learning model, i.e. assuming that $\sigma_p = 0$. The parameters used in this case and for all the figures in this section are in Table 1, reference case 2; these slightly different parameters were chosen to make the effects of interest more easily visible on graphs. To facilitate the reading of Figure 15, h_g, the hazard with a relatively good observation (i.e. $z_k > z^*$), is multiplied by 8.6, which is equal to $h_b(5)/h_g(5)$. We can see that the h_g begins to increase later than h_b, eventually catches up, and remains roughly proportional to h_b. The fact that h_g begins to increase later than h_b is explained by the fact that initially the prior belief has a strong influence on the agent's current belief. This means that the agent needs a really bad observation to separate. However, by definition h_g implies that she got a relatively good observation ($z > z*$), which is not sufficient to override the prior (and separation costs) and trigger separation. In Figure 18, we can see that the hazard ratio in the fixed match quality model decreases and then stabilizes as the relationship duration increases.

In Figure 16, I plot $h_b(k)$ and $h_g(k)$ under the mixed model. The hazards calculated in Figure 16 use the same parameters as those calculated in Figure 15, except that $\sigma_p = 5$ instead of $\sigma_p = 0$. In this case like in the pure learning case, h_g begins to increase later than h_b and eventually catches up. However, past 5 periods, the two hazards are not proportional:

instead, h_g decreases faster than h_b. To understand why this is the case, first note that h_g is driven by low quality relationships only, since only they are led to dissolve even in the absence of a bad observation. By contrast h_b is driven by both low quality relationships and mediocre relationships that end up dissolving due to a bad observation. Now, the proportion of low quality relationships decreases fairly steadily over time as more and more low quality relationships dissolve. However, the proportion of mediocre relationships decreases slower and slower over time: this is because while most relationships start out well above the separation threshold, some of them eventually randomly drift into the "'mediocre"' quality zone. When looking at the hazard ratio in Figure 18 for the mixed model with changing match quality, we see that it first decreases with relationship duration, and subsequently increases again. The initial decrease is due to learning ($\sigma_{obs} > 0$) while the subsequent increase is due to changes in relationship quality ($\sigma_p > 0$). Accordingly, by performing the same calculations with different values of σ_{obs}, one can see that the initial decrease in the hazard ratio is stronger with a higher σ_{obs}, i.e. when there is more scope for the agent to keep learning new things about the relationship as time goes by. Similarly, the subsequent increase in the hazard ratio is stronger as σ_p increases.

Finally, in Figure 17, I plot $h_b(k)$ and $h_g(k)$ under the pure shocks model. The model uses the same parameters as the mixed model, except that here we have $\sigma_{obs} = 0$. In this case, the hazard of separation in the presence of a bad observation is 1. This is because there is no uncertainty about the current value of the match, so a bad observation implies that the match is in fact just as bad as that observation. And since the definition of a bad observation is such that this bad observation is smaller than the threshold of separation, all relationships that get this bad observation separate. On the other hand, in this case, the hazard of separation in the absence of a bad observation declines with relationship length. This implies that the hazard ratio increases with relationship length, as can be seen in Figure 18. Note however that, as previously mentioned, it is possible to have a hazard of separation that increases and then decreases with length, even in the case of a pure shocks model.

Therefore, it is possible for the pure shocks model to yield a hazard ratio that decreases and then increases with length.

So far, we have examined the effect of observing a bad signal at length k on separation at length k. However, it is also interesting to ask how this effect evolves over time: relative to those relationships that did not get a bad signal at length k, how much more likely are relationships who did get a bad signal at length k to dissolve at lengths $k+1$, $k+2$, ..., $k+n$? It is fairly straightforward to compute the hazard of separation at lengths $k+1$,..., $k+n$ separately for those relationships that did get a bad signal at k and those that did not. Figure 19 plots the ratio of these two hazards for $k = 5$ (the two hazards are the hazard if a bad signal was observed at length 5, and the hazard if no bad signal was observed at length 5) for both the pure learning model ($\sigma_p = 0$) and the mixed model ($\sigma_p = 5$) [16]. The figure shows that in both models the hazard ratio is largest in the periods immediately following period 5. Thereafter, we can see that in the pure learning model, the hazard for those relationships that got a bad observation at 5 remains larger than the hazard for those relationships that did not get a bad observation at 5 (the ratio is no lower than 2.2). By contrast, in the mixed model, the difference between the two hazards disappears with time (the ratio converges to 1). These qualitative conclusions do not depend on the choice of $k = 5$. The intuition for these results is as follows. In the pure learning model, the true quality of the relationship is fixed. As a result, relationships that got a bad observation at 5 are on average worse than those that did not, and this difference persists over time. For both those relationships that got a bad observation at 5 and the others, learning continues until hazards converge to zero; however, until then, there will be more separations among relationships that got a bad observation at 5 because they are worse on average. In the mixed model, relationship quality changes continuously. As in the pure learning model, those relationships that got a bad observation at 5 are on average worse than those that did

[16] Performing the same calculation for the pure shocks model is not interesting, since in that case all those matches that got a bad observation are immediately terminated.

not. However, since relationships evolve according to a random walk after length 5, their quality at length 5 is less and less informative about their present quality as time goes by, and after a while the two groups no longer differ in their separation hazards[17].

4.1. **Summary of testable implications of the model.** We can now summarize the testable implications of the model. First, if we can observe the variance of the transition function, it is fairly easy to distinguish between the models at hand. If the variance of the transition function decreases with relationship length and converges to 0, then the pure learning model is correct. If the variance of the transition function changes over relationship lengths for low relationship lengths but converges to a strictly positive value, then the mixed model is correct. And finally, if the variance of the transition function is constant, then the pure shocks model is correct. Second, the theory predicts that, as long as separation costs are not too high and do not strongly decrease over the course of the relationship, the hazard of separation will go to 0 in the case of a pure learning model, but not in the case of the pure changes or the mixed model. Third, the ratio between the hazard with a bad observation and the hazard without such a bad observation decreases and eventually stabilizes with relationship duration in the case of the pure learning model. In the case of a mixed model or the pure shocks model under some parameters, the hazard ratio decreases and then increases with relationship duration. The pure shocks model, under a different parametrization, can yield a the hazard ratio that only increases with length. Fourth, under both the pure learning model and the mixed model, the impact of a bad observation on separation is positive and decreases with time elapsed since the shock. However, while under the pure learning model, the impact of labor market shocks remains positive, under the mixed model the impact of a labor market shock tends toward 0 as time passes after the shock.

[17]This also holds if the AR(1) process is stationary. Indeed, since all relationships converge to the long-run mean, past history becomes less and less relevant.

5. Discussion

5.1. **Effort and labor supply.** The models presented have not explicitly integrated agents' efforts. This is a very important issue as match quality could be in part determined by agents' efforts. For example, in the employment relationship, the employee can affect output by supplying more or less unobserved effort, as in Holmstrom(1999). In a formal framework similar to the one I use here, Holmstrom(1999) shows that labor supply will decline to 0 if an employment relationship continues indefinitely and the worker's ability is fixed. On the other hand, if ability evolves stochastically, labor supply will be positive and stable over time (after an initial period of adjustment). Homlstrom's results imply that models with fixed match quality are inconsistent in the presence of a serious moral hazard problem; these models assume indeed that the agent (wrongly) believes that the benefits from the relationship do not depend on the partner's effort. If match quality evolves over time, then the model is not necessarily inconsistent, even in the presence of moral hazard. Further exploration of this issue is left, however, to future work.

5.2. **General equilibrium.** The analysis developed in this paper is in partial equilibrium; it does not attempt to model the influence of the behavior of each agent on other agents. If relationship quality is entirely match specific, then this is not a problem as the prior distribution of match qualities faced by the agent is not influenced by the behavior of other agents. If, however, match quality is at least in part due to some general characteristics that make a partner desirable to all agents, then a change in behavior by other agents is likely to change the distribution of prior match qualities. For example, if firms face higher firing costs and, as a result, decrease their threshold for separation, then the distribution of prior match qualities should have a slightly lower mean since now workers who were terminated and are looking for a new job are a bit worse on average[18]. The feedback mechanism from

[18]This effect is smaller the more workers with no prior experience enter the labor market, and the more match-specific productivity is.

agents' optimal behavior to the distribution of prior match qualities could in principle be modeled within the framework used here, and it would be useful to do so in future work[19].

Another related issue is that this model does not allow agents not to be in a relationship at all. By assumption, the agent can only continue the current relationship or separate and start a new one immediately. This is an important limitation in contexts such as the labor market where vacancies and permanent layoffs do exist and are essential for understanding the dynamics of the labor market. The model, however, already contains the tools to analyze these issues, at least in a limited sense. Indeed, one can assume that at length 1 the separation cost is extremely low, and call period 1 the screening period: thus, in period 1 the agent meets a partner, gets a signal about match quality and decides to pursue the relationship or not. This application will be developed in future work and can allow to determine, for example, if firing costs reduce hiring (where hiring means not firing at length 1) more than firing (at lengths 2 and above) and under which conditions this is the case.

5.3. **Learning about match quality, learning on the job, and random shocks.** The model presented here can simultaneously account for learning about match quality, learning on the job, and random shocks to match quality. These three elements are typically included in separate models in the literature about match quality in the employment relationship. Learning about match quality is thus the main component in Jovanovic(1979). Teulings and van der Ende(2001) develop a model where match quality is subject to random shocks. Using a model that integrates all these empirically relevant effects at the same time[20] is more efficient for empirical analysis because parameters can be determined jointly from a single statistical model.

[19]One important challenge is that the feedback from behavior to the distribution of prior match qualities would likely make the latter distribution non normal. Computations are greatly eased if one assumes normality of the distribution of prior match qualities, and to preserve these desirable properties, one would have to devise a meaningful way to approximate the non normal distribution of prior match qualities by a normal one.

[20]Nagypal(2004) offers a somewhat different way of integrating these effects in her model.

6. Conclusion

This paper has developed a model of optimal matching and separation, allowing for partially observed and time-varying match quality. Despite the limitations discussed in section 5 — some of which could be overcome in future work — the model is already very general and sheds useful light on the mechanisms at play in relationship evolution and dissolution. Specifically, I have shown that, in all models considered, higher separation costs and a lower discount factor decrease the separation threshold and thus the separation hazard. The effect of parameters entering the belief on the separation hazard mainly depends on the effect of these parameters on the belief transition function[21], i.e. the probability of the agent's holding a certain belief next period given the agent's current belief. In all cases, an increase in the variance of the belief transition function leads to a higher hazard of separation at short lengths. A lower observation variance, a higher variance of the prior, and a higher variance of the error in the AR(1) process all increase the variance of the transition function at short lengths, and thus increase the hazard at short lengths. The effect of parameters entering the transition function at longer lengths is not so clear cut. If match quality is assumed to be constant over time, then the separation hazard converges to 0, and so there will be little effect at higher lengths. If however match quality follows and AR(1) process, then an increase in the variance of the transition function at all lengths typically *lowers* the separation hazard at higher lengths.

The class of models developed here lends itself to applications in various contexts. As already mentioned, domains of choice would be the employment relationship, marriage, and firm-suppliers relationships. Empirically, hazards of separation from an employment relationship and hazards of divorce both increase and decrease over the length of the relationship, but do not decline to 0. This implies that, very likely, the underlying match quality is time-varying and separation costs are positive. In general, it is possible to determine which

[21] This is only true if we assume that the expected value of the prior does not change.

parameters best fit[22] an empirically observed separation hazard and thus gain useful information about the matching process. The model is also useful in predicting the impact of a parameter change on the hazard of separation. For example, in Marinescu(2006), I examined the impact of a change in the probationary period on the hazard of an employment relationship being terminated by the employer. Finally, this paper develops some simple empirical tests that can allow researchers to determine whether the pure learning model, the pure shocks model or a mixed model is best able to explain the data at their disposal.

[22]We have to keep in mind, however, that the greater the number of unobserved parameters, the less precise the estimates. For example, I already pointed out that one cannot typically distinguish separation costs from the discount factor just by looking at the separation hazard.

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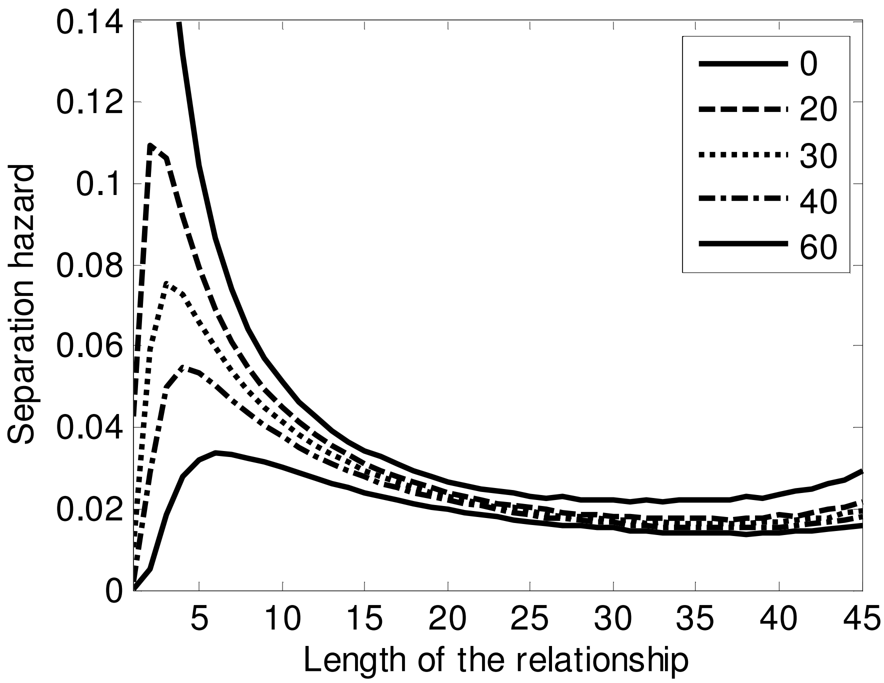
Table 1: Parameters for the reference case

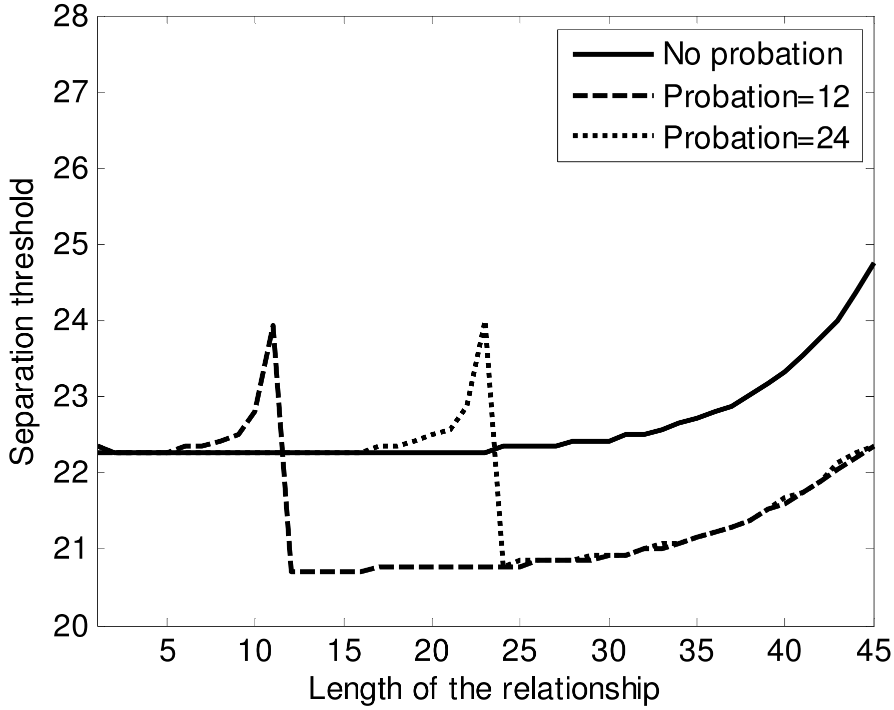
	Reference case 1	Reference case 2
	Parameters of interest	
Mean of prior	30	30
Standard deviation of prior	2	5
Standard deviation of process	4	5
Drift of process	0	0
Auto-correlation of process	1	1
Standard deviation of observation	4	10
Separation cost	30	30
Discount factor	0.85	0.85
	Technical parameters	
Range of match qualities	[0,60]	[0,60]
Number of match quality values	801	801
Maximal length	50	50

Figure 1: Separation hazard for different separation costs



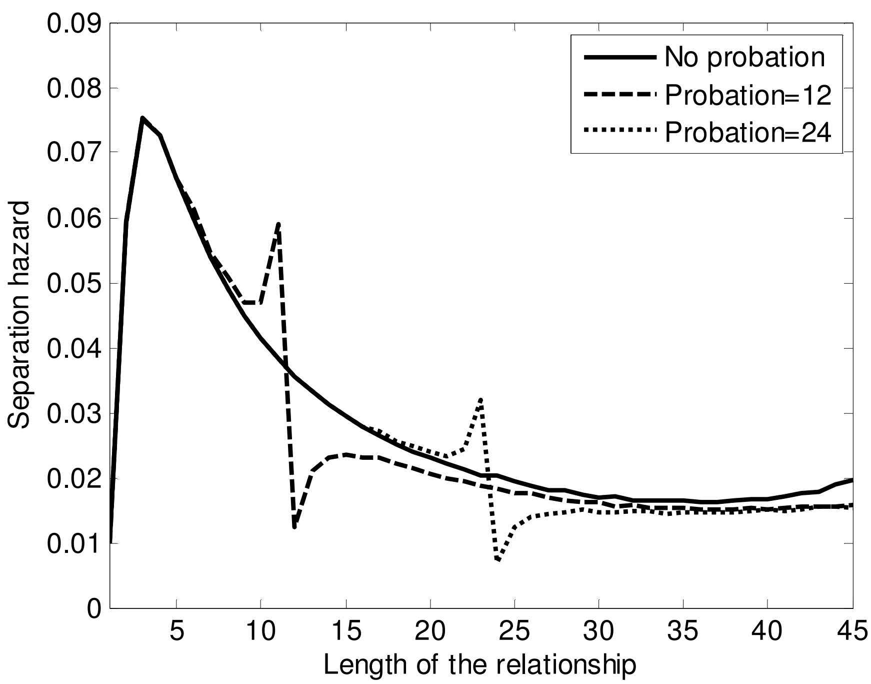
Note: Parameter values other than separation costs are in Table 1, reference case 1.

Figure 2: Separation threshold with a probation period



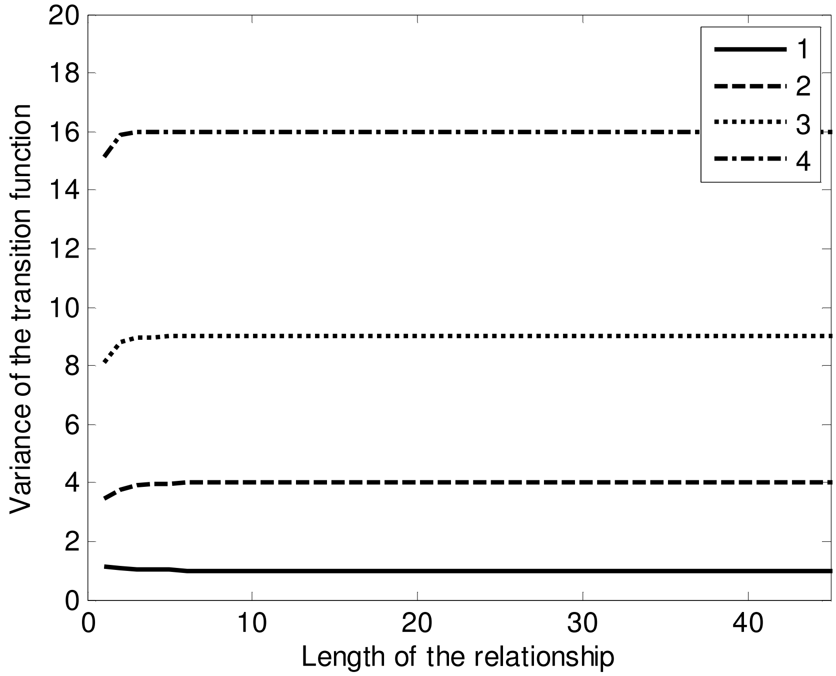
Note: Parameter values other than separation costs and probation are in Table 1, reference case 1.

Figure 3: Separation hazard with a probation period



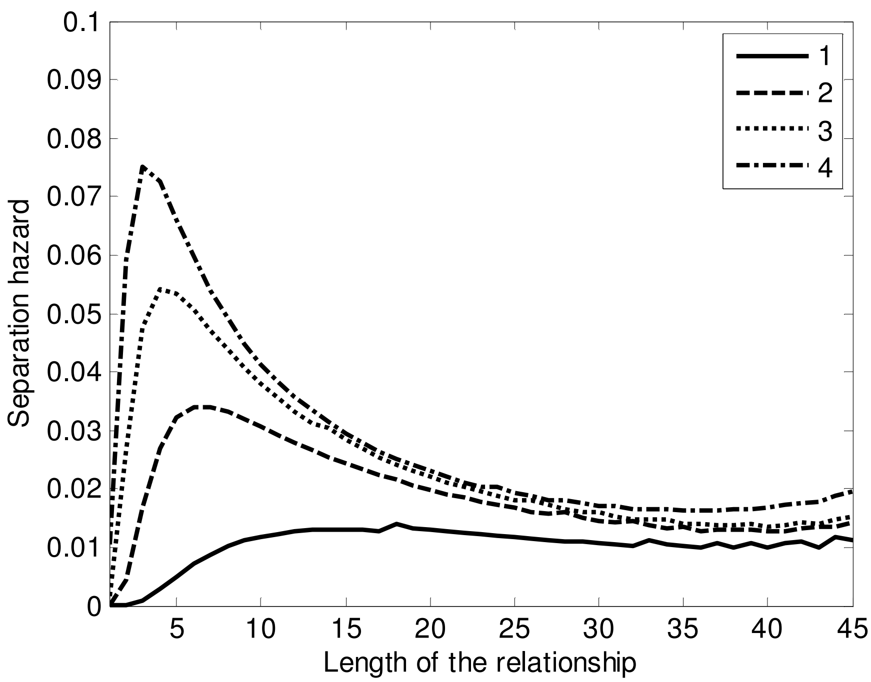
Note: Parameter values other than separation costs and probation are in Table 1, reference case 1.

Figure 4: Variance of the transition function for different process standard deviations



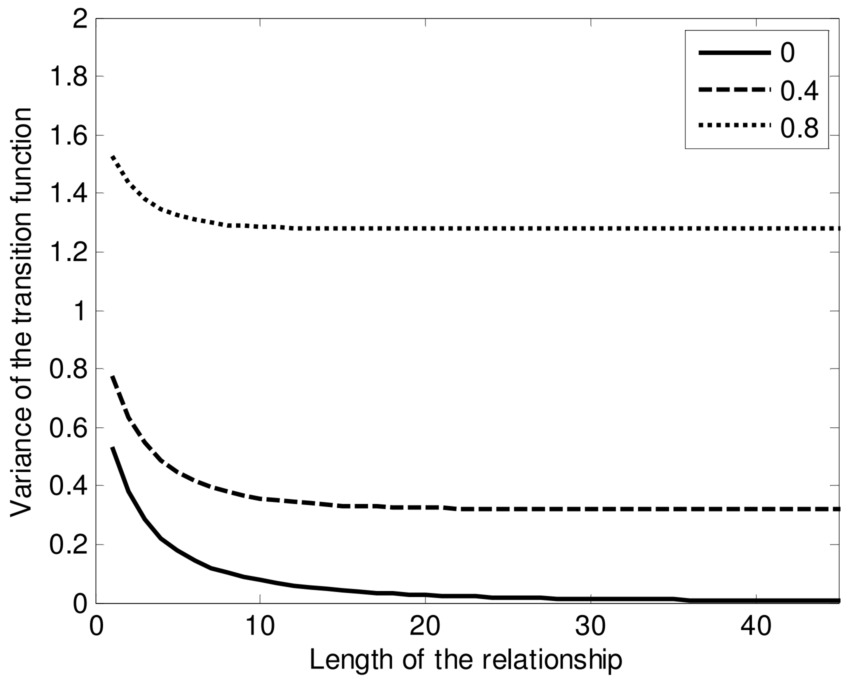
Note: Parameter values other than the process standard deviation are in Table 1, reference case 1.

Figure 5: Hazard for different process standard deviations



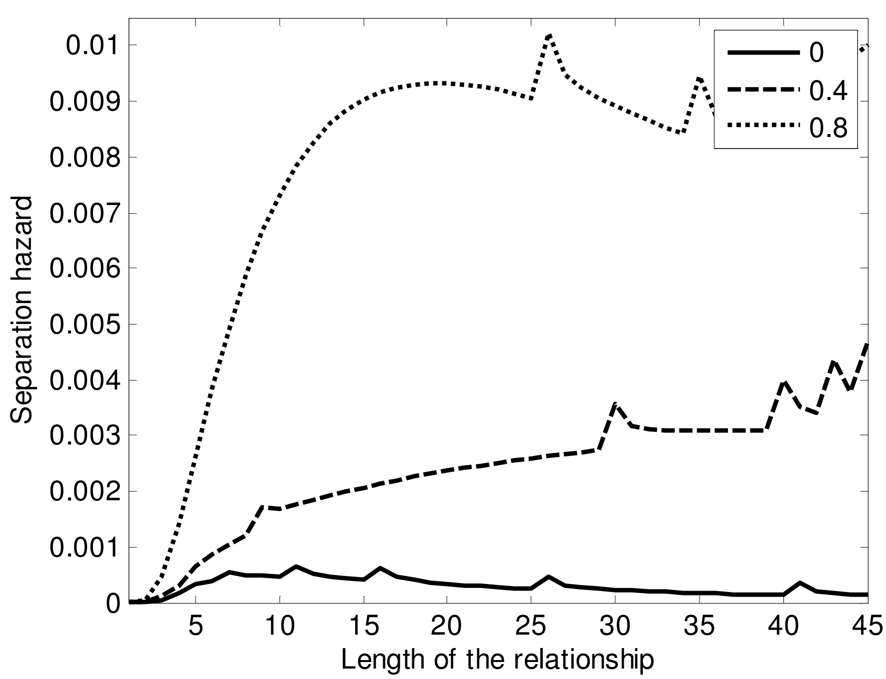
Note: Parameter values other than the process standard deviation are in Table 1, reference case 1.

Figure 6: Variance of the transition function for different process standard deviations (small values)



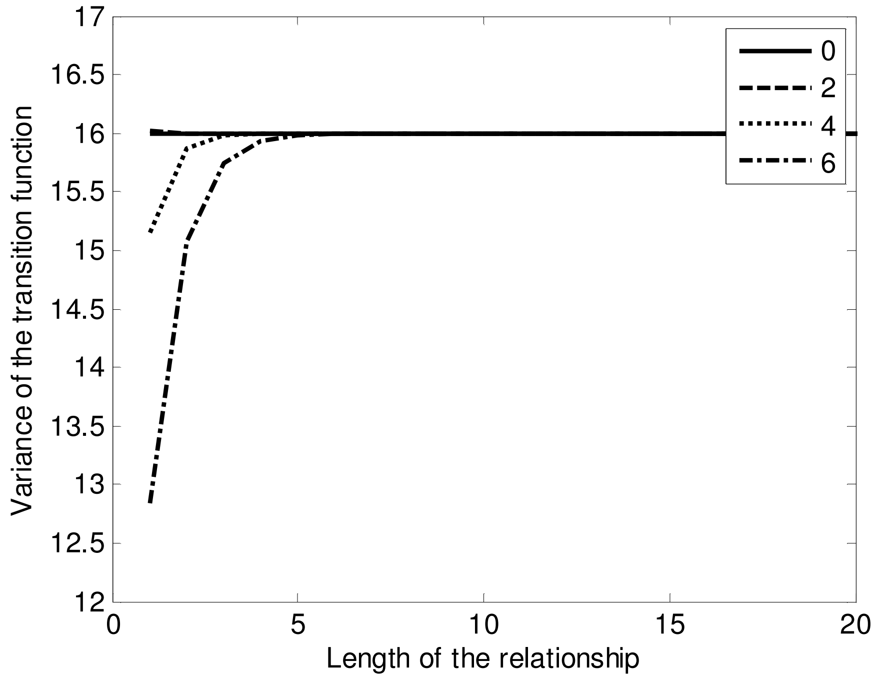
Note: Parameter values other than the process standard deviation are in Table 1, reference case 1.

Figure 7: Hazard for different process standard deviations, small values



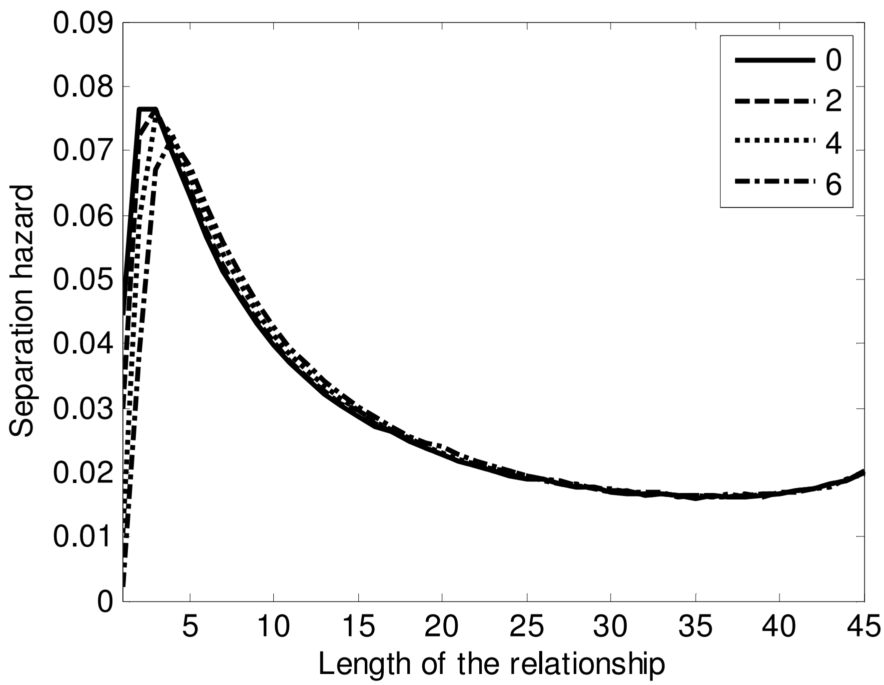
Note: Parameter values other than the process standard deviation are in Table 1, reference case 1.

Figure 8: Variance of the transition function for different observation standard deviations



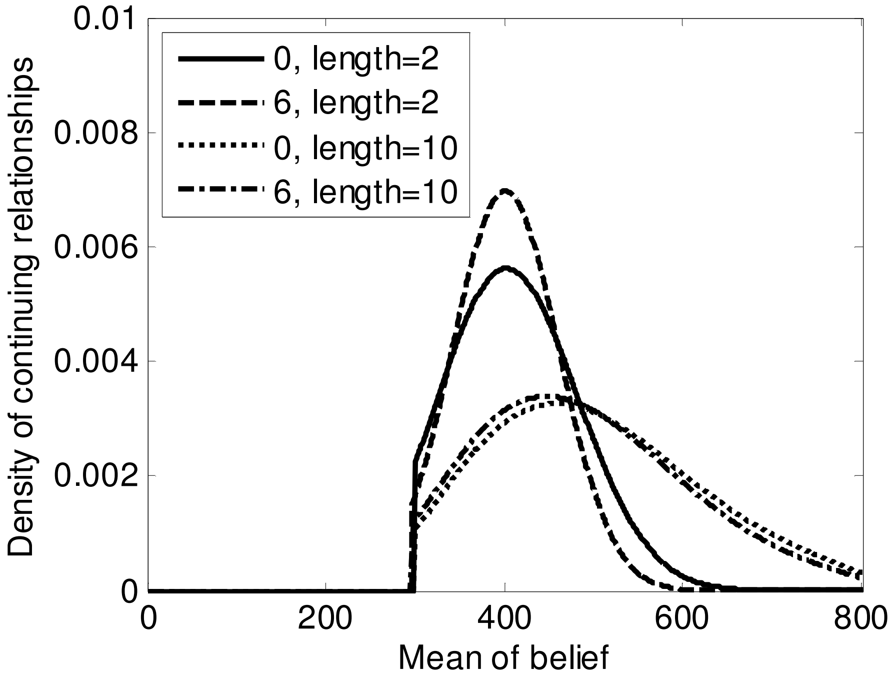
Note: Parameter values other than the observation standard deviation are in Table 1, reference case 1.

Figure 9: Hazard for different observation standard deviations



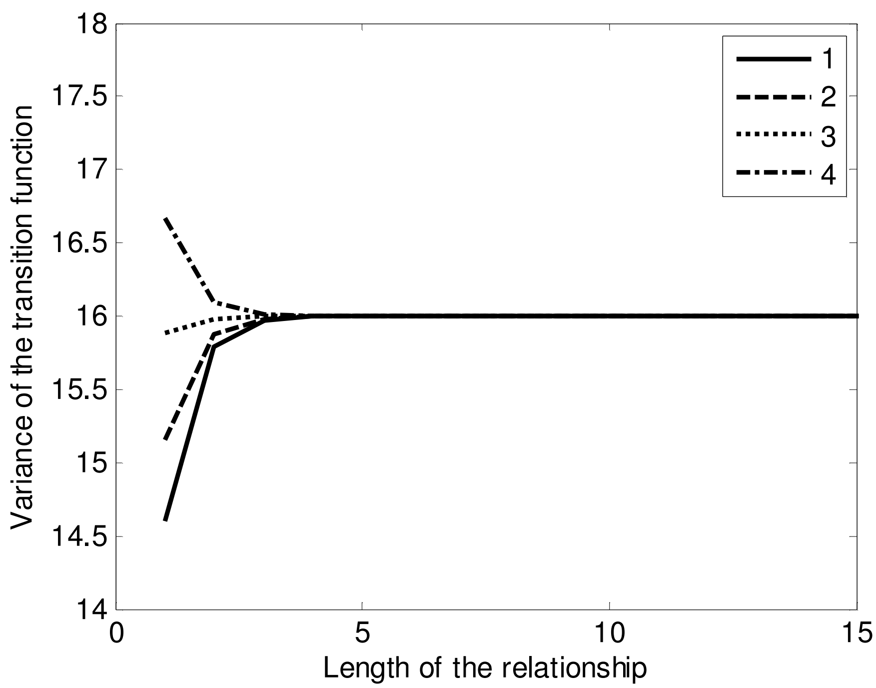
Note: Parameter values other than the observation standard deviation are in Table 1, reference case 1.

Figure 10: Distribution of continuing relationships for different observation standard deviations



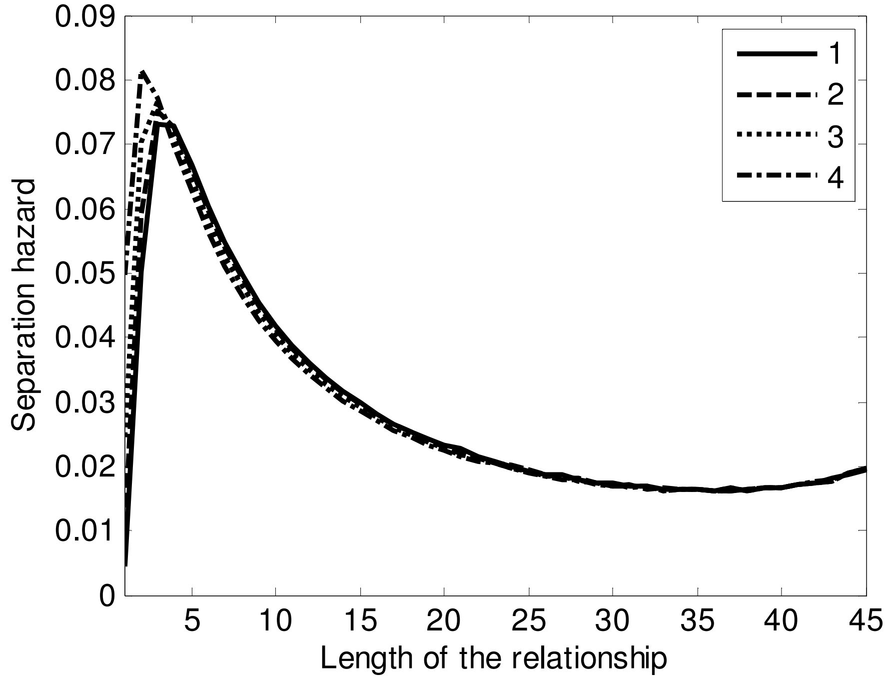
Note: Parameter values other than the observation standard deviation are in Table 1, reference case 1.

Figure 11: Variance of the transition function for different prior standard deviations



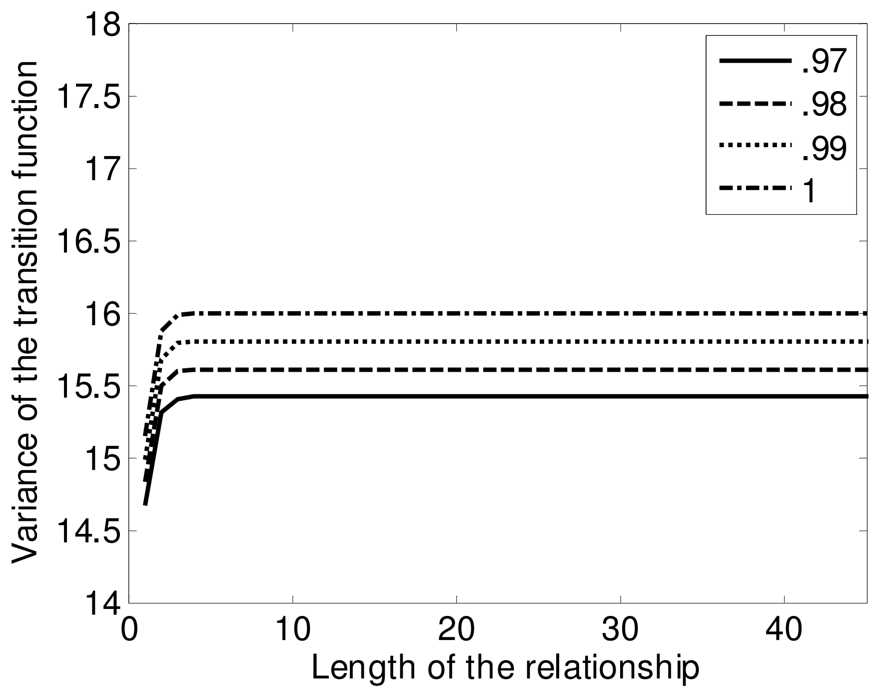
Note: Parameter values other than the prior standard deviation are in Table 1, reference case 1.

Figure 12: Separation hazard for different prior standard deviations



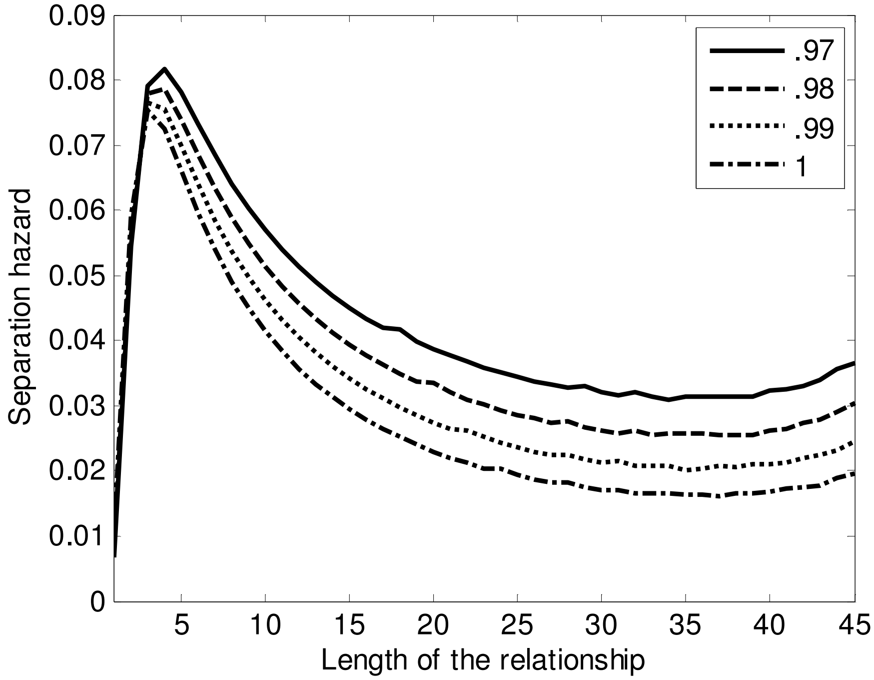
Note: Parameter values other than the prior standard deviation are in Table 1, reference case 1.

Figure 13: Variance of the transition function for different AR(1) parameters



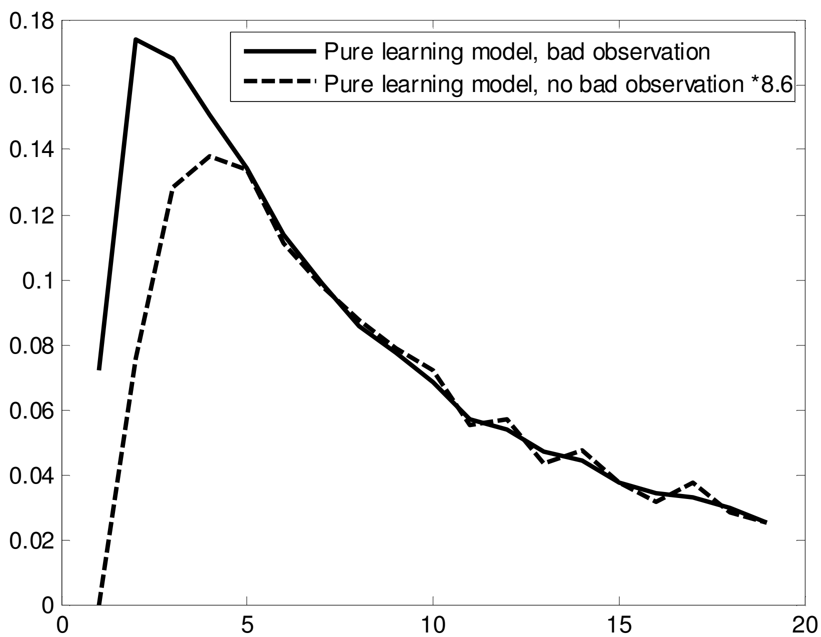
Note: Parameter values other than the AR(1) parameter are in Table 1, reference case 1.

Figure 14: Separation hazard for different AR(1) parameters



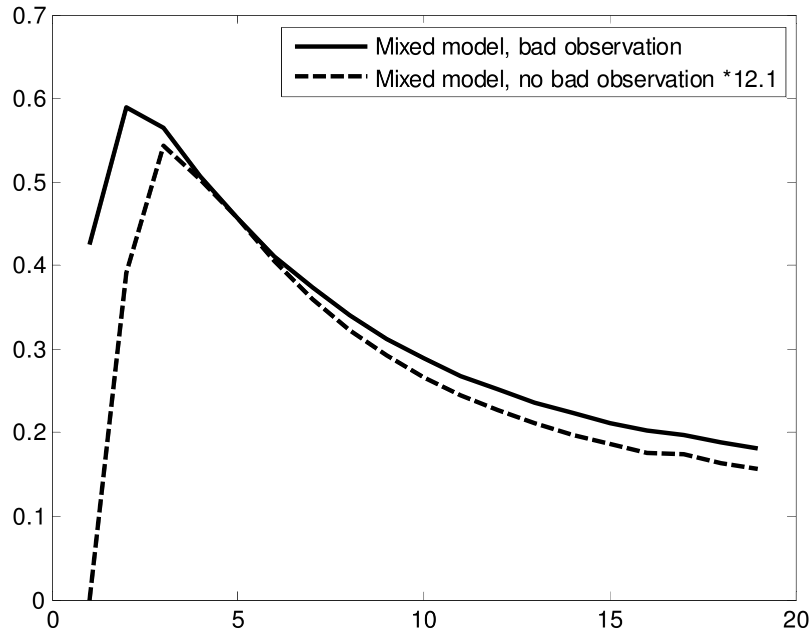
Note: Parameter values other than the AR(1) parameter are in Table 1, reference case 1.

Figure 15: Hazards of separation with and without a bad observation in the pure learning model



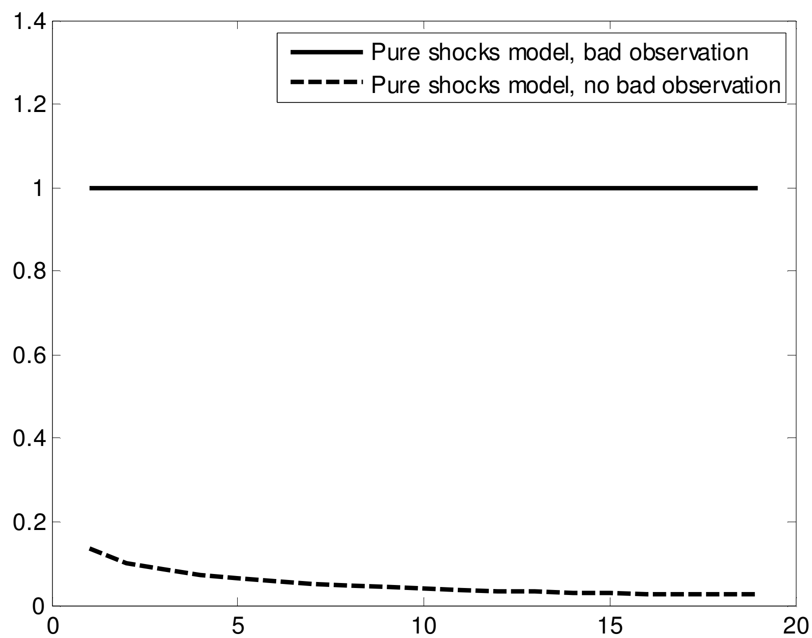
Note: Parameter values other than the process standard deviation (set to 0 here) are in Table 1, reference case 2.

Figure 16: Hazards of separation with and without a bad observation in the pure learning model



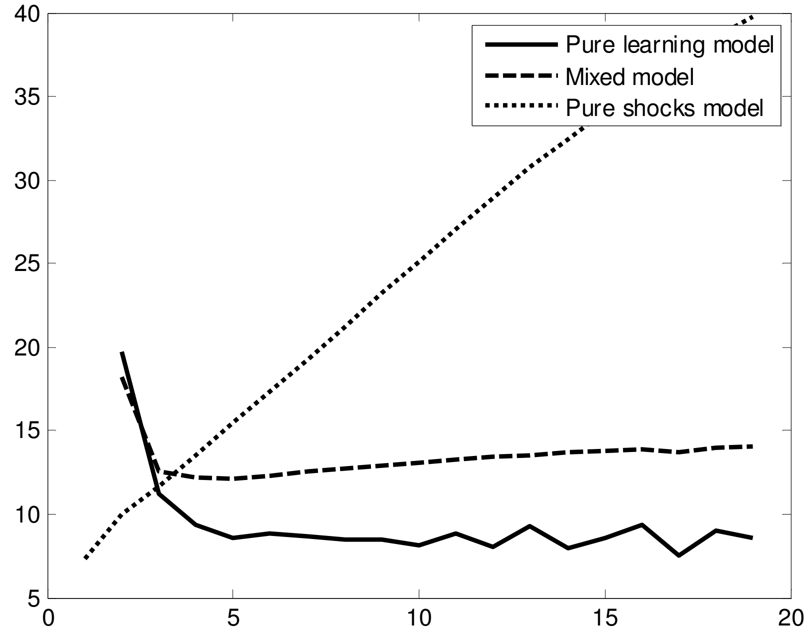
Note: Parameter values are in Table 1, reference case 2.

Figure 17: Hazards of separation with and without a bad observation in the pure learning model



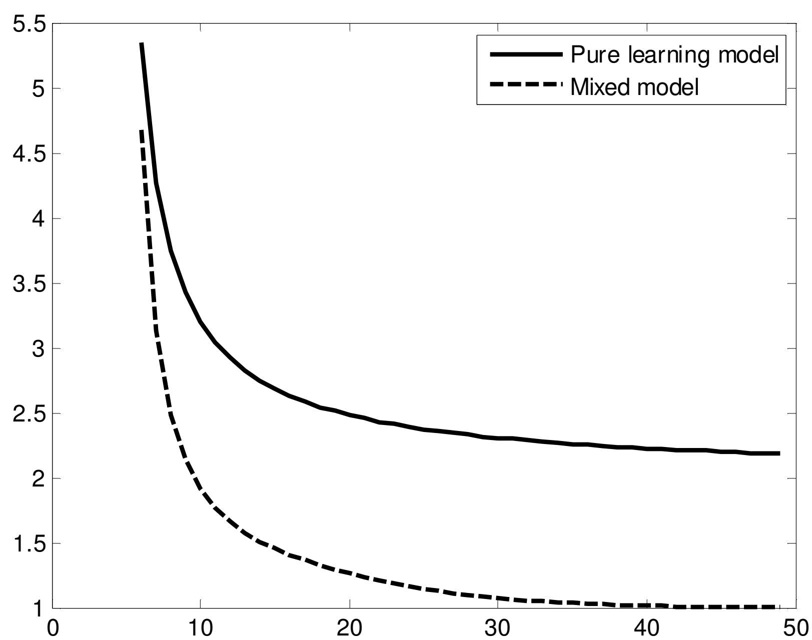
Note: Parameter values other than the observation standard deviation (set to 0 here) are in Table 1, reference case 2.

Figure 18: Ratio between the hazard with a bad observation and the hazard without a bad observation



Note: Parameter values other than the observation standard deviation and the process standard deviation are in Table 1, reference case 2.

Figure 19: Ratio between the hazard with a bad observation at period 5 and the hazard without a bad observation at period 5



Note: Parameter values other than the process standard deviation are in Table 1, reference case 2.