Building Social Networks*

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Abstract

We examine the process of building social relationships as a non-cooperative game that requires mutual consent and involves reaching out to others at a cost. Players create their social network from amongst their set of acquaintances. Having acquaintances allows players to form naive beliefs about the feasibility of building direct relationships with their acquaintances. These myopic beliefs describe how the other players are expected to respond to the initiation of a link by a player. We introduce a stability concept called "monadic stability" where agents play a best response to their formed myopic beliefs such that these beliefs are self-confirming. The resulting equilibrium networks form subset of the set of pairwise stable networks.

Keywords: Social networks; network formation; pairwise stability; trust; self-confirming equilibrium.

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1 On link formation

The goal of this paper is to shed light on how individual links, the basic building blocks of social networks, are formed. We develop a model the captures stylized facts known about networks. We start by assuming that the creation of social ties requires some prior interaction, and therefore the process of link formation (under mutual consent) principally occurs between social acquaintances.[1] Individuals then "reach out" to some of their acquaintances to establish their social network. The process of reaching out is costly and hence requires deliberate choices. We assume that in the acquaintance set each player has knowledge about the payoffs of other players, and can formulate expectations about their behavior before undertaking costly actions. From this viewpoint our contribution fits very well with the model of friendship networks developed in Brueckner (2003). Individuals then choose who to include in their network based on their beliefs.

In the model we assume that players have simple, myopic beliefs about how their acquaintances will respond if a link is initiated. These beliefs only take into account the direct benefits that the addition or removal of a single link has for the payoffs of the other players. Hence, beliefs are based only on the first order marginal payoffs that can be assigned to links. The myopic nature of these beliefs is meant to capture the fact that are formed in the absence of substantial interaction among these individuals—between individual who are mere acquaintances.

Second we assume that players respond fully rationally to the beliefs that they have formed about the process of link formation. Hence, decision makers, after forming their myopic beliefs about other individuals and anticipating their actions, subsequently formulate their best response given these anticipations.[2] This implies that individual i initiates links with only those individuals that i thinks will benefit from those (direct) links. In doing so the initiating individual assumes that the respondent will consent to the link and, hence, the incurred link formation costs will not be wasted. Thus she will not have reached out in vain! This form of rationality that leads to the formation of the network constitutes the basis of a stability notion called *monadic stability*.

[1] It is well established that social networks do not emerge among random strangers, but are primarily formed between acquaintances. This literature is founded on Granovetter (1973) and confirmed empirically by Friedkin (1980), Wellman, Carrington, and Hall (1988), and Tyler, Wilkinson, and Huberman (2003). More recently new methodologies have been developed to detect community structures in social networks for testing such hypotheses. We refer to Newman and Girvan (2004) and Newman (2004) for the details of this methodology.

[2] In the model players are always rational individuals. Their beliefs are myopic because they are about acquaintances.

Formally in the paper we distinguish between two forms of monadic stability.[3] First we only consider the networks that are supported through a (myopic) belief system to which all players formulate a best response. These *weakly monadically stable* networks can be interpreted as the steady states of a learning process based on the formation of myopic belief systems. Weak monadic stability still leads to a relatively large and unappealing class of networks. Moreover, while these networks might be steady states of a learning process, they only become true equilibria if the anticipation behavior according to the myopic belief are confirmed in the actually played actions. Thus, weak monadic stability demonstrates the fact that networks amongst mere acquaintances may not always have very desirable properties.

These considerations lead to a second and stronger concept, which is simply called "monadic stability". Monadically stable networks are steady states of the myopic learning process in which the beliefs of the individual players are confirmed. Hence, these networks are supported through the self-confirming equilibria (Fudenberg and Levine 1993) of the learning process based on these myopic belief systems. It should be evident that the myopically stable networks are the only ones that can be seen as the equilibrium networks based on the introduced belief systems.

Third, we study the process of link formation and the creation of social networks using the consent model of network formation with two-sided costs first developed in Gilles, Chakrabarti, and Sarangi (2006). This simple model is based on *three simple and realistic principles* encompassing real-world networks: (1) Link formation should be based on a binary process of consent. (2) Link formation should in principle be costly. (3) The payoff structure of network formation should be as general as possible.

In the model a link between any two players i and j is only established when player j is willing to accept the link initiated by player i or vice versa. As suggested by the second principle, link formation is costly. Costs depend on the strategies chosen by the player in the link formation process and are incurred independent of the outcome, i.e., even if a link is not established the initiating player still has to pay for the act of trying to form that link.[4] In the model both players bear an individually determined cost of link formation. It is due to these two principles that beliefs play a significant role in the process of link formation.

Following our third modeling principle, we consider a very general payoff structure that has two components — an arbitrary benefit function and additive link for-

[3]The term monadic stability refers to the fact that it is stable from an individual's point of view as opposed to being stable from a pair or dyad's perspective.

[4]Like much of the networks literature we assume that these costs are exogenously given and not dependent on the network structure. Costs that are dependent on the network structure would be important for modeling congestion type effects.

mation costs.[5] Note that benefits depend on the resulting network, and the costs on the link formation strategies chosen by the players. The generality of the payoff functions adds to the strength of our results. For example, the connections model of (Jackson and Wolinsky 1996) which is one of the most popular network models can be written as a special case of the general payoff function used here.

In Gilles, Chakrabarti, and Sarangi (2006) we show that in general the consent model has a multitude of Nash equilibria and, consequentially, is not discerning. Moreover, the empty network is supported by a strict Nash equilibrium. Hence it is important to understand how link formation leading to a network occurs. In this paper we show that the introduction of a myopic form of *confidence* about the responses of other players in process of link formation suffices to reduce the number of equilibrium networks. Next, we give a complete characterization of the class of monadically stable networks. Our main result is that this class is exactly the family of *strictly* pairwise stable networks*. These networks form a strict subset of the class of pairwise stable networks (Jackson and Wolinsky 1996) and have very strong stability properties.

Another advantage of our model is that we differentiate between familiarity among individuals who can at best only be acquaintances, and the possibility of explicitly creating a mutually beneficial but costly relationship between these individuals. The literature on game theoretic approaches to network formation often allows links chosen by players to be interpreted as confirmations of already established relationships that occur in a non-modeled process prior to the formulated game.[6] Thus, our approach is more in line with Brueckner (2003), who categorically distinguishes the set of acquaintances a player has, from the friendship links she establishes between them. This also places our approach within the context of Granovetter's notion of strong social ties. (Granovetter 1973)

2 Preliminaries and notation

In this section we introduce the basic concepts and notation pertaining to non-cooperative games and networks. The section concludes with a brief overview of the consent model of network formation with two-sided costs. We follow the notation and terminology outlined in Jackson (2004) and Gilles, Chakrabarti, and Sarangi

[5] An arbitrary cost structure would require costs to be dependent on the outcome of the game or the network. The payoff specification then would become game dependent forcing us to give up generality in the results.

[6] This is, for example, the foundation for the notion of two-way flow Nash networks introduced by Bala and Goyal (2000) where a link initiated an agent functions like a telephone call.

(2006).

2.1 Non-cooperative games

A *non-cooperative game* on a fixed, finite player set $N = \{1, \dots, n\}$ is given by a list $(A_i, \pi_i)_{i \in N}$ where for every player $i \in N$, A_i denotes an action set. For every $a \in A$ and $i \in N$, we use $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \in A_{-i} = \prod_{j \neq i} A_j$ to represent the actions selected by the players other than i. Let $\pi_i \colon A \to \mathbb{R}$ denote player i's payoff function with $A = A_1 \times \cdots \times A_n$ being the set of all action tuples, and $\pi = (\pi_1, \dots, \pi_n) \colon A \to \mathbb{R}^N$ be the composite payoff function.

An action $a_i \in A_i$ for player $i \in N$ is called a *best response* to $a_{-i} \in A_{-i}$ if for every action $b_i \in A_i$ we have that $\pi_i(a_i, a_{-i}) \geqslant \pi_i(b_i, a_{-i})$. An action tuple $\hat{a} \in A$ is a *Nash equilibrium* of the game (A, π) if for every player $i \in N$

$$\pi_i(\hat{a}) \geqslant \pi_i(b_i, \hat{a}_{-i}) \text{ for every action } b_i \in A_i.$$

Hence, a Nash equilibrium $\hat{a} \in A$ satisfies the property that for every player $i \in N$ the action $\hat{a}_i$ is a best response to $\hat{a}_{-i}$.

2.2 Networks

In introducing the basic networks terminology we use established notation from Jackson and Wolinsky (1996), Dutta and Jackson (2003), and Jackson (2004). The reader may refer to these sources for a more elaborate discussion.

We limit our discussion to *non-directed networks* on the player set N. In such networks the two players making up a single link are both equally essential and the links have therefore a bi-directional nature. Formally, if two players $i, j \in N$ with $i \neq j$ are related we say that there exists a *link* between players them. We use the notion of a link to formalize the presence of some social relationship between individuals i and j. We use the notation ij to describe the binary link $\{i, j\}$. Let $g_N = \{ij \mid i, j \in N,\ i \neq j\}$ be the set of all potential links.

A *network* g on N is now introduced as any set of links $g \subset g_N$. In particular, the set of all feasible links g_N itself is called the *complete network* and $g_0 = \varnothing$ is known as the *empty network*. The collection of all networks is defined as

$$\mathbb{G}^N = \{g \mid g \subset g_N\}.$$

The set of (direct) *neighbors* of a player $i \in N$ in the network g is given by

$$N_i(g) = \{j \in N \mid ij \in g\} \subset N.$$

Similarly we introduce

$$L_i(g) = \{ij \in g_N \mid j \in N_i(g)\} \subset g$$

as the *link set* of player i in the network g. It only contains links with i's direct neighbors in g. We apply the convention that for every player $i \in N$, $L_i = L_i(g_N) = \{ij \mid i \neq j\}$ is the set of all potential links involving player i.

For every pair of players $i, j \in N$ with $i \neq j$ we denote by $g + ij = g \cup \{ij\}$ the network that results from adding the link ij to the network g. Similarly, $g - ij = g \backslash \{ij\}$ denotes the network obtained by removing the link ij from network g.

Relationship building—formalized in a link formation process—results into a network and within a network, benefits for the players are generated depending on how they are connected to each other. For every player $i \in N$, the function $\sigma_i : \mathbb{G}^N \to \mathbb{R}$ denotes her *network payoff function*. This function assigns to every network $g \subset g_N$ a value $\sigma_i(g)$ that is obtained by player i when she participates in network g. The payoffs obtained through the function $\sigma_i(g)$ should be interpreted as the *net* payoffs that player i realizes through participating in the network g, i.e., player i's gross benefits from network g minus all costs of participating in g induced by player i.

The composite network payoff function is now given by $\sigma = (\sigma_1, \ldots, \sigma_n) : \mathbb{G}^N \to \mathbb{R}^N$. Note that the empty network $g_0 = \varnothing$ generates (reservation) values $\sigma(g_0) \in \mathbb{R}^N$ that might be non-zero.

Several examples of standard network payoff functions for both noncooperative and cooperative games are reviewed in Jackson (2004). Additionally, in van den Nouweland (1993), Dutta, van den Nouweland, and Tijs (1998), Slikker (2000), Slikker and van den Nouweland (2000), and Garratt and Qin (2003) these network payoff functions are based on underlying cooperative games from where a lot of the networks literature originated. For a review of this strand of the literature we refer to van den Nouweland (2004).

2.3 Link-based stability concepts

We now present the definition of several stability conditions. We begin by introducing stability concepts that allow for adding and breaking links separately before considering them together. Note that the stability concepts introduced below are based on the properties of the network itself rather than strategic considerations of the players. This latter viewpoint has been introduced seminally by Jackson and Wolinsky (1996) and is further advocated in Jackson and Watts (2002), Jackson (2004), and Bloch and Jackson (2006).

First we introduce some auxiliary notation: Let $\sigma\colon \mathbb{G}^N \to \mathbb{R}^N$ be some network payoff function. For a given network $g \in \mathbb{G}^N$, we now define the following concepts for σ:

(a) For every $ij \in g$ the *marginal benefit* of this link for every player $i \in N$ is given by

$$D_i(g, ij) = \sigma_i(g) - \sigma_i(g - ij) \in \mathbb{R}. \tag{1}$$

(b) For every player $i \in N$ and link set $h \subset L_i(g)$ the *marginal benefit* to player i of the link set h in g is given by

$$D_i(g, h) = \sigma_i(g) - \sigma_i(g - h) \in \mathbb{R} \tag{2}$$

Using these additional tools we can give a precise description of the various link-based stability concepts.

Definition 2.1 *Let σ be a network payoff function on the player set N.*

(a) *A network $g \subset g_N$ is **link deletion proof** for σ if for every player $i \in N$ and every $j \in N_i(g)$ it holds that $D_i(g, ij) \geqslant 0$.*
Denote by $\mathcal{D}(\sigma) \subset \mathbb{G}^N$ the family of link deletion proof networks for σ.

(b) *A network $g \subset g_N$ is **strong link deletion proof** for σ if for every player $i \in N$ and every $h \subset L_i(g)$ it holds that $D_i(g, h) \geqslant 0$.*
Denote by $\mathcal{D}_s(\sigma) \subset \mathbb{G}^N$ the family of strong link deletion proof networks for σ.

(c) *A network $g \subset g_N$ is **link addition proof** if for all players $i, j \in N$: $\sigma_i(g + ij) > \sigma_i(g)$ implies $\sigma_j(g + ij) < \sigma_j(g)$.*
Denote by $\mathcal{A}(\sigma) \subset \mathbb{G}^N$ the family of link addition proof networks for σ.

(d) *A network $g \in \mathbb{G}^N$ is **strict link addition proof** for $\phi\colon \mathbb{G}^N \to \mathbb{R}$ if for all $i, j \in N$ it holds that $ij \notin g$ implies that $\sigma_i(g + ij) \leqslant \sigma_i(g)$.*
Denote by $\mathcal{A}_s(\sigma) \subset \mathbb{G}^N$ the family of strict link addition proof networks for σ.

(e) *A network $g \in \mathbb{G}^N$ is **strict* link addition proof** for $\phi\colon \mathbb{G}^N \to \mathbb{R}$ if for all $i, j \in N$ it holds that $ij \notin g$ implies that $\sigma_i(g + ij) < \sigma_i(g)$.*
Denote by $\mathcal{A}^\star_s(\sigma) \subset \mathbb{G}^N$ the family of strict link addition proof networks for σ.*

The two link deletion proofness notions are based on the severance of links in a network by individual players. In particular, the notion of link deletion proofness considers the stability of a network with regard to the deletion of a *single* link. This concept was seminally introduced in Jackson and Wolinsky (1996). Strong link deletion proofness considers the possibility that a player can delete any subset of her existing links. Clearly, strong link deletion proofness implies link deletion proofness. For further details on this concept we refer to Gilles, Chakrabarti, Sarangi, and Badasyan (2005) and Bloch and Jackson (2006).

Similarly, link addition proofness (Jackson and Wolinsky 1996) considers the addition of a single link by two consenting players to an existing network. A network is link addition proof if for every pair of non-linked players, at least one of these two players has negative benefits from the addition of a link between them. Hence, in a network requiring consent this link will never be added. Strict link addition proofness that when adding a non-existent link both players have non-positive payoffs.

Strict link addition proofness requires that for every pair of non-linked players, both of these players have non-positive benefits from adding the link between them, i.e., it requires that neither player wants to add a link. Clearly in a network requiring consent this link will never be added making it a significant strengthening of the link addition proofness requirement.

Strict* link addition proofness is a new concept, which has not yet been considered in the literature. It is a stronger condition than strict addition proofness in the sense that any link that is added to the network leads to strictly negative marginal benefits for the participating players. It is clear that $\mathcal{A}_s^\star(\sigma) \subset \mathcal{A}_s(\sigma) \subset \mathcal{A}(\sigma)$.

The simplest notion combining both addition and deletion proofness was seminally introduced by Jackson and Wolinsky (1996) as a stability concept called pairwise stability. It combines link deletion proofness with link addition proofness. Given that these two proofness conditions can be strengthened in various ways it is possible to define a variety of modifications of this concept. We now present some definitions below.

Definition 2.2 *Let σ be a network payoff function on the player set N.*

(a) *A network $g \in \mathbb{G}^N$ is **pairwise stable** for σ if g is link deletion proof as well as link addition proof.*
Denote by $\mathcal{P}(\sigma) = \mathcal{D}(\sigma) \cap \mathcal{A}(\sigma) \subset \mathbb{G}^N$ the family of pairwise stable networks for the payoff function σ.

(b) *A network $g \in \mathbb{G}^N$ is **strictly pairwise stable** for σ if g is strong link deletion proof as well as strict link addition proof.*
Denote by $\mathcal{P}_s(\sigma) = \mathcal{D}_s(\sigma) \cap \mathcal{A}_s(\sigma) \subset \mathbb{G}^N$ the family of strict pairwise stable networks for the payoff function σ.

(c) *A network* $g \in \mathbb{G}^N$ *is* ***strictly* pairwise stable*** *for* σ *if* g *is strong link deletion proof as well as strict* link addition proof.*
Denote by $\mathcal{P}^\star(\sigma) = \mathcal{D}_s(\sigma) \cap \mathcal{A}_s^\star(\sigma) \subset \mathbb{G}^N$ *the family of strict* pairwise stable networks for the payoff function* σ*.*

In the present paper we focus on the strongest combined requirement—strictly* pairwise stable networks. Given that players use the simplest myopic beliefs to activate links, it is reasonable to use such a strong requirement. Normally the class of strict* pairwise stable networks is a strict subset of the family of pairwise stable networks. It is possible that in many cases this class is empty. We therefore first address the question when these two classes of networks coincide. We identify three conditions under which the main proofness conditions result into the same networks.

First, the network payoff function σ is *network convex* if for every network $g \in \mathbb{G}^N$, every player $i \in N$ and every link set $h \subset L_i(g)$:

$$\sum_{ij \in h} D_i(g, ij) \geqslant 0 \text{ implies } D_i(g, h) \geqslant 0.$$

Second, the network payoff function σ is *link uniform* if for every network $g \in \mathbb{G}^N$, and all players $i, j \in N$ with $ij \notin g$ it holds that

$$\sigma_i(g) \leqslant \sigma_i(g + ij) \text{ implies } \sigma_j(g) \leqslant \sigma_j(g + ij).$$

With these properties we can now state an equivalence result.

Proposition 2.3 *Let the network payoff function* σ *be link uniform.*

(a) *Let* σ *be* ***discerning*** *in the sense that for every link addition proof network* $g \in \mathcal{A}(\sigma)$ *it holds that for all* $i, j \in N$ *with* $ij \notin g$ *it does not hold that* $\sigma_i(g) = \sigma_i(g + ij)$ *as well as* $\sigma_j(g) = \sigma_j(g + ij)$*. Then it holds that* $\mathcal{A}_s^\star(\sigma) = \mathcal{A}(\sigma)$*.*

(b) *If* σ *is discerning on* $\mathcal{A}(\sigma)$ *as well as network convex, then it holds that*

$$\mathcal{P}_s^\star(\sigma) = \mathcal{P}_s(\sigma) = \mathcal{P}(\sigma).$$

Proof. The proof of this equivalence result is fully based on the Equivalence Theorem in Gilles and Sarangi (2005).
To show assertion (a), we first refer to assertion (b) of the Equivalence Theorem in Gilles and Sarangi (2005) which concludes that under link uniformity it holds that $\mathcal{A}(\sigma) = \mathcal{A}_s(\sigma)$. Hence for any network $g \in A(\sigma)$ and $ij \notin g$ it holds that

$\sigma_i(g) \geqslant \sigma_i(g+ij)$ as well as $\sigma_j(g) \geqslant \sigma_j(g+ij)$. Also by link uniformity we deduce that

$$\sigma_i(g) = \sigma_i(g+ij) \text{ implies } \sigma_j(g) \leqslant \sigma_j(g+ij) \leqslant \sigma_j(g),$$

which in turn implies that $\sigma_j(g) = \sigma_j(g+ij)$. This contradicts that σ is discerning on $\mathcal{A}(\sigma)$. Hence, $\sigma_i(g) < \sigma_i(g+ij)$ as well as $\sigma_j(g) < \sigma_j(g+ij)$. This implies that $g \in \mathcal{A}^\star_s(\sigma)$. We now conclude that

$$\mathcal{A}^\star_s(\sigma) \subset \mathcal{A}_s(\sigma) = \mathcal{A}(\sigma) \subset \mathcal{A}^\star_s(\sigma).$$

which implies the assertion.

To show assertion (b) of Proposition 2.3 we note that this is an immediate consequence of assertion (a) above and assertion (a) of the Equivalence Theorem in Gilles and Sarangi (2005), which states that network convexity implies equivalence of strong link deletion proofness and link deletion proofness. ■

2.4 The consent model of network formation

We base our analysis of confidence in link formation in the setting of the "consent model of network formation" with two-sided link formation costs. In Gilles, Chakrabarti, and Sarangi (2006) we provide a non-cooperative model of network formation under consent based on Myerson's model of network formation under binary consent (Myerson 1991, page 448). Myerson's model incorporates the fundamental idea that pairs of players have to agree mutually on building links in any process of network formation. In Gilles, Chakrabarti, and Sarangi (2006) we extended this approach by introducing additive link formation costs. Here we provide a brief summary of this model.

Let $N = \{1, \dots, n\}$ be a given set of players and $\varphi \colon \mathbb{G}^N \to \mathbb{R}^N$ be a fixed, but arbitrary network payoff function representing the gross benefits that accrue to the players in a network. For every player $i \in N$, we introduce individualized link formation costs represented by $c_i = (c_{ij})_{j \neq i} \in \mathbb{R}^{N \setminus \{i\}}_+$. (Here, for some links $ij \in g_N$ it might hold that $c_{ij} \neq c_{ji}$.) Thus, the pair $\langle \varphi, c \rangle$ represents the basic benefits and costs of link formation to the individuals in N.

For every player $i \in N$ we introduce an action set

$$A_i = \{(\ell_{ij})_{j \neq i} \mid \ell_{ij} \in \{0, 1\}\} \tag{3}$$

Player i seeks contact with player j if $\ell_{ij} = 1$. A link is formed if both players seek contact, i.e., $\ell_{ij} = \ell_{ji} = 1$.

Let $A = \prod_{i \in N} A_i$ where $\ell \in A$. Then the resulting network is given by

$$g(\ell) = \{ij \in g_N \mid \ell_{ij} = \ell_{ji} = 1\}. \tag{4}$$

as stated, link formation is costly. Approaching player j to form a link costs player i an amount $c_{ij} \geqslant 0$. This results in the following game theoretic payoff function for player i:

$$\pi_i(\ell) = \varphi_i(g(\ell)) - \sum_{j \neq i} \ell_{ij} \cdot c_{ij} \tag{5}$$

where c is the link formation cost introduced at the beginning of this section.

The pair $\langle \varphi, c \rangle$ thus generates the non-cooperative game (A, π) as described above. We call this non-cooperative game the *consent model of network formation with two-sided link formation costs*, or for short the "consent model".[7] We summarize the characterization of the Nash equilibria of the consent model.

Lemma 2.4 (Gilles, Chakrabarti, and Sarangi 2006) *Let* φ *and* $c \geqslant 0$ *be given as above. A network* $g \subset g_N$ *is supported by a Nash equilibrium in the consent model* (A, π) *if and only if* g *is strong link deletion proof for the net payoff function* ψ *given by*

$$\psi_i(g) = \varphi_i(g) - \sum_{j \in N_i(g)} c_{ij}.$$

For a proof of this result we refer to Gilles, Chakrabarti, and Sarangi (2006).

A consequence of Lemma 2.4 is that the empty network $g_0 = \varnothing$ is supported as a Nash equilibrium in the consent model (A, π). Furthermore, g_0 can even be supported through a *strict* Nash equilibrium. Given the generality of the the consent model, this is a very undesirable result for network formation theory. It implies that equilibrium concepts based on different notions of stability have to be developed to explain the emergence of non-trivial social networks.

3 Monadic stability

In this section we introduce an equilibrium concept for network formation models that incorporates a (limited) form of boundedly rational anticipation or "myopic confidence" into the process of link formation. This equilibrium concept, called *monadic*

[7] While we limit our discussion to the two-sided cost setting in the current paper, Gilles, Chakrabarti, and Sarangi (2006) also discuss the consent model with one-sided link formation costs. Due to severe coordination problems this model performs even worse than the model with two-sided link formation costs.

stability, captures the idea that social networks are mainly formed between acquaintances who have already have some beliefs about each other. Hence, our main modeling assumption is that social networks arise only from links between *a priori* acquaintances and *not* among random strangers.

That social relations are mainly formed between acquaintances is confirmed empirically by Wellman, Carrington, and Hall (1988) using data from the East York area. This approach also forms the foundation of the model in Brueckner (2003), who models friendship based on links between players chosen from a given set of acquaintances. In the context of our model, it is assumed that all players in N are acquainted with each other without explicitly modeling how they get acquainted with each other. Moreover, we assume that each player has knowledge about the payoffs of the other players and formulates expectations about how the other players will respond to link proposals.

Under monadic stability, a player assumes that other players are likely to respond affirmatively to a proposal to form a link if the addition of this link is profitable for them, i.e., only the implications of direct links affect the expectations. Note that since further consequences are not taken into account, this form of behavior introduces a rather myopic form of forward looking behavior. This limited form of farsightedness thus models the anticipation of a player in a very specific manner—these beliefs assume that other players will do the "correct" thing when asked whether to form a link or not based only on that link. Also, this formulation of the belief structure retains a fair degree of realism in the model.

We now formalize these myopic belief systems for the case of two-sided link formation costs.

Let $\langle \varphi, c \rangle$ be a network payoff function and link formation cost. Let (A, π) be the consent model with two-sided link formation costs generated by $\langle \varphi, c \rangle$.

Within this setting we are now able to introduce myopic beliefs of players regarding the actions undertaken by the other players in the network formation process. This forms the foundation for the formulation of confidence in link formation.

Definition 3.1 *Let $\ell \in A$ be an arbitrary action tuple. For every player $i \in N$ we define i's **belief system** as expectations about direct links $\ell^{i\star} \in A$ based on ℓ by*

(i) *for every $j \neq i$ with $ij \in g(\ell)$ we let*

- $\ell^{i\star}_{ji} = 0$ *if* $\varphi_j(g(\ell) - ij) + c_{ji} > \varphi_j(g(\ell))$ *and*
- $\ell^{i\star}_{ji} = 1$ *if* $\varphi_j(g(\ell) - ij) + c_{ji} \leqslant \varphi_j(g(\ell))$,

(ii) *for every* $j \neq i$ *with* $ij \notin g(\ell)$ *we let*

- $\ell^{i\star}_{ji} = 0$ *if* $\varphi_j(g(\ell) + ij) - c_{ji} < \varphi_j(g(\ell))$ *and*
- $\ell^{i\star}_{ji} = 1$ *if* $\varphi_j(g(\ell) + ij) - c_{ji} \geqslant \varphi_j(g(\ell))$*,*

(iii) *and for all* $j, k \in N$ *with* $j \neq i$ *and* $k \neq i$ *we define* $\ell^{i\star}_{jk} = \ell_{jk}$.

In the myopic belief system introduced here each player assumes that other players will respond according to their direct incentives to form a link or not. Of course, these beliefs are rather limited since they may seem unreasonable if players can engage in some forward looking behavior. On the other hand, these beliefs are myopic and rather simple and can arise in the absence of substantial interaction among agents, i.e., even among mere acquaintances. Hence, these beliefs form an excellent starting point for link formation. The definition used allows for a sequential form of rationality in the reasoning of the players during the network formation process which is at the foundation of the following definitions of stability.

Definition 3.2 *Let* $\langle \varphi, c \rangle$ *be given.*

(a) *A network* $g \in \mathbb{G}^N$ *is* ***weakly monadically stable*** *if there exists some action tuple* $\hat{\ell} \in A$ *such that* $g = g(\hat{\ell})$ *and for every player* $i \in N$*:* $\hat{\ell}_i \in A_i$ *is a best response to* $\hat{\ell}^{i\star}_{-i} \in A_{-i}$ *for the payoff function* π*.*

(b) *A network* $g \in \mathbb{G}^N$ *is* ***monadically stable*** *if there exists some action tuple* $\hat{\ell} \in A$ *with* $g = g(\hat{\ell})$ *for which* g *is weakly monadically stable such that for every player* $i \in N$ *player* i*'s myopic beliefs* $\hat{\ell}^{i\star}$ *are confirmed, i.e., for every* $j \neq i$ *it holds that* $\hat{\ell}^{i\star}_{ji} = \hat{\ell}_{ji}$*.*

Weak monadic stability of a network is founded on the principle that every player $i \in N$ anticipates—as captured by her expectations about direct links—that other players will respond “correctly” to her attempts to form a link with them. Note that ℓ_{-i} is fully replaced by $\ell^{i\star}_{-i}$ in the standard best-response formulation of equilibrium for player i and is therefore irrelevant for the decision making process of i. Hence, a player will agree to form a link with i when it is myopically profitable to form this link. Similarly, unprofitable direct links initiated by i will be turned down.

Monadic stability strengthens the above concept by requiring that the beliefs of each player are *confirmed* in the resulting equilibrium. Hence, we impose a self-confirming condition on the equilibrium. This describes the situation that all players are fully satisfied with their beliefs; the observations that they make about the resulting network confirm their beliefs about the other players’ payoffs. This can be explained as the outcome of a process of updating the initial beliefs. The notion of self-confirming equilibrium was developed seminally by Fudenberg and Levine (1993).

To delineate the two monadic stability concepts for networks, we discuss a three player example. This example shows that the set of monadically stable networks is usually a strict subset of the weakly monadically stable networks.

Example 3.3 Consider three players $N = \{1,2,3\}$ and assume that $c_{ij} = 1$ for all players $i \in N$ and all potential links $ij \in L_i$, i.e., we assume uniform link formation costs. Let the network payoff function φ be given by the table below. This table identifies whether the network in question is weak monadically stable—indicated by M_w—or whether it is monadically stable—indicated by M.

Network	$\varphi_1(g)$	$\varphi_2(g)$	$\varphi_3(g)$	**Stability**
$g_0 = \varnothing$	0	0	0	M_w
$g_1 = \{12\}$	0	3	0	
$g_2 = \{13\}$	0	0	3	
$g_3 = \{23\}$	0	0	0	
$g_4 = \{12, 13\}$	3	0	0	
$g_5 = \{12, 23\}$	2	3	3	
$g_6 = \{13, 23\}$	2	2	5	M_w
$g_7 = g_N$	3	5	6	M

We consider four networks in this example explicitly, namely g_0, g_5, g_6 and $g_7 = g_N$.

Network g_0**:** First we claim that this network is strongly pairwise stable for the given payoff structure. Indeed, it is trivially SLDP and, given the network payoff function, it is link addition proof as well.

Second, we argue that this network is weakly monadically stable. We claim that it is supported by the strategy tuple $\ell_0 = ((1,1),(0,0),(0,0))$. Now we compute

$$\ell_0^{1\star} = (-,(1,0),(1,0))$$
$$\ell_0^{2\star} = ((0,1),-,(0,0))$$
$$\ell_0^{3\star} = ((1,0),(0,0),-)$$

We emphasize that in this case Player 1 believes that both other players are willing to make links with him, because there are direct benefits to forming such links. However, the other players believe that Player 1 will not attempt to make a link with them, because she has no direct (net) benefit from doing so.

Now we determine that

- $\beta_1\left(\ell_0^{1\star}\right) = (1,1)$ is the unique best response to $\ell_0^{1\star}$,
- $\beta_2\left(\ell_0^{2\star}\right) = (0,0)$ is the unique best response to $\ell_0^{2\star}$, and
- $\beta_3\left(\ell_0^{3\star}\right) = (0,0)$ is the unique best response to $\ell_0^{3\star}$.

Observe that Player 1 incurs link formation costs in this case and, hence, $\pi_1(\ell_0) = -2$ and $\pi_2(\ell_0) = \pi_3(\ell_0) = 0$.
Also, note that g_0 is *not* monadically stable. In the strategy tuple ℓ_0 player 1's belief system is not confirmed. He expects the other two players to form links with him, although they do not do so.

Network g_5**:** We argue that this network is not weakly monadically stable. The obvious candidate action tuple to support g_5 is given by $\ell_5 = ((1,0),(1,1),(0,1))$. We compute

$$\ell_5^{1\star} = (-,(1,1),(0,1))$$
$$\ell_5^{2\star} = ((1,0),-,(0,1))$$
$$\ell_5^{3\star} = ((1,1),(1,0),-)$$

We now derive that

- $\beta_1\left(\ell_5^{1\star}\right) = (1,0)$ is the unique best response to $\ell_5^{1\star}$,
- $\beta_2\left(\ell_5^{2\star}\right) = (1,0)$ is the unique best response to $\ell_5^{2\star}$, and
- $\beta_3\left(\ell_5^{3\star}\right) = (0,0)$ is the unique best response to $\ell_5^{3\star}$.

From this it is clear that g_5 cannot be supported by ℓ_5. This illustrates that weak monadic stability requires playing best response to a *specific* set of beliefs for each $i \in N$. Without such a restriction on the beliefs it would be possible to support any strategy as weakly monadic stable. Moreover, observe that agents only form beliefs about the behavior of their acquaintances with regard to direct links, making it myopic but realistic. In fact, because of this, it is possible that monadically stable equilibria do not exist. Finally, note that other action tuples can be ruled out in similar fashion.

Network g_6**:** First, we claim that this network is SLDP, but that it is not link addition proof. Strong link deletion proofness follows trivially from the zero payoffs listed for g_2 and g_3. Link addition proofness is not satisfied since adding the link 12 would make player 2 better off, while player 1 is indifferent.
Second, we argue that this network is weakly monadically stable. We show

that g_6 is supported by the action tuple $\ell_6 = ((0,1),(1,1,),(1,1))$. Again we compute

$$\ell_6^{1\star} = (-,(1,1),(1,1))$$
$$\ell_6^{2\star} = ((1,1),-,(1,1))$$
$$\ell_6^{3\star} = ((0,1),(1,1),-)$$

From this we conclude that

- $(0,1)$ and $(1,1)$ are both best responses to $\ell_6^{1\star}$, i.e., $\beta_1\left(\ell_6^{1\star}\right) = \{(0,1),(1,1)\}$,
- $\beta_2\left(\ell_6^{2\star}\right) = (1,1)$ is the unique best response to $\ell_6^{2\star}$, and
- $\beta_3\left(\ell_6^{3\star}\right) = (1,1)$ is the unique best response to $\ell_6^{3\star}$.

This shows that ℓ_6 is indeed a best response to the generated myopic beliefs. We therefore conclude that g_6 is weakly monadically stable. On the other hand, g_6 is *not* monadically stable. Indeed, in ℓ_6 the beliefs of player 2 are not confirmed.

Network g_7: First, we claim that this network is strictly* pairwise stable. Strong link deletion proofness follows trivially from the payoffs listed for all other networks. The net payoffs in these networks are at most the net payoff in g_7 for all players. The second condition is trivially satisfied since there are no links that are not part of $g_7 = g_N$.

Second, we argue that this network is weakly monadically stable. We show that g_7 is supported by the action tuple $\ell_7 = ((1,1),(1,1,),(1,1))$.[8] Again we compute

$$\ell_7^{1\star} = (-,(1,1),(1,1))$$
$$\ell_7^{2\star} = ((1,1),-,(1,1))$$
$$\ell_7^{3\star} = ((1,1),(1,1),-)$$

From this we conclude that

- $(1,0)$ and $(1,1)$ are both best responses to $\ell_7^{1\star}$, i.e., $\beta_1\left(\ell_7^{1\star}\right) = \{(1,0),(1,1)\}$,
- $\beta_2\left(\ell_7^{2\star}\right) = (1,1)$ is the unique best response to $\ell_7^{2\star}$, and
- $\beta_3\left(\ell_7^{3\star}\right) = (1,1)$ is the unique best response to $\ell_7^{3\star}$.

This shows that ℓ_7 is indeed a best response to the generated myopic beliefs. We therefore conclude that g_7 is weakly monadically stable.

[8] Obviously this is the only candidate action tuple for the complete network g_N.

Furthermore, all players' beliefs are confirmed in ℓ_7. Thus, we conclude that g_7 is monadically stable for ℓ_7.

This example clarifies the relation between weak monadic stability and the link based stability concepts introduced earlier. Using the insights from this example we now provide a more general characterization. ♦

3.1 Weak monadic stability

The following result gives a characterization of the relationship between weak monadically stability and other link stability concepts.

Theorem 3.4 *Let* $\langle \varphi, c \rangle$ *be such that* $c \gg 0$*, i.e.,* $c_{ij} > 0$ *and* $c_{ji} > 0$ *for all* $i, j \in N$.

(a) *Every weakly monadically stable network* $g \in \mathbb{G}^N$ *in the consent model* (A, π) *is link deletion proof for the network payoff function* ψ *given by*

$$\psi_i(g) = \varphi_i(g) - \sum_{j \in N_i(g)} c_{ij}. \tag{6}$$

(b) *Not every weakly monadically stable network in the consent model* (A, π) *is necessarily strongly link deletion proof or link addition proof for the network payoff function* ψ.

(c) *Not every strongly pairwise stable network for the network payoff function* ψ *is necessarily weakly monadically stable in the consent model* (A, π).

A proof of Theorem 3.4(a) is provided in Section 5 of the paper. This result is intuitive since weak monadic stability requires that each agent is playing a best response to their expectations about direct links with other players. Given that link formation is costly, in a best response a player will never initiate a link that will be turned down. Hence these networks are link deletion proof.

The proof of Theorem 3.4(b) is developed in Example 3.5 below. We recall that a network is not link addition proof if one of the players is better off while the other is no worse. Under weak monadic stability player i believes that player j will accept a link if she is not worse off. Hence it is possible to have a weakly monadically stable network that is not link addition proof. Moreover, since the beliefs only considers each pairs of players separately, a weakly monadically stable network need not be robust to the simultaneous deletion of multiple links. This is in contrast to the previous example where networks g_0 and g_6 are both weakly monadically stable as well as SLDP.

The proof of Theorem 3.4(c) is developed with the use of Example 3.6.

The next example shows the assertion stated in Theorem 3.4(b) and also shows some other interesting features of our approach.

Example 3.5 We consider four players, i.e., $N = \{1, 2, 3, 4\}$. Under the hypothesis of uniform link formation costs set at $c_{ij} = 1$ for all $i, j \in N$. The network benefits are described by $\varphi \colon \mathbb{G}^N \to \mathbb{R}^N_+$ with its values for some networks given in the following table:

Network	$\varphi_1(g)$	$\varphi_2(g)$	$\varphi_3(g)$	$\varphi_4(g)$
$g_0 = \varnothing$	0	0	0	0
$g_1 = \{12\}$	0	1	1	1
$g_2 = \{13\}$	0	1	1	1
$g_3 = \{14\}$	1	1	1	1
$g_4 = \{12, 13\}$	1	2	2	1
$g_5 = \{12, 14\}$	0	1	1	1
$g_6 = \{12, 13, 14\}$	5	0	0	2

For all remaining networks $g \in \mathbb{G}^N$: $\varphi(g) = (0, 0, 0, 0)$. This results in the following net payoffs ψ applying (6) to the networks given in the table above. For the remaining networks these net payoffs are negative.

Network	$\psi_1(g)$	$\psi_2(g)$	$\psi_3(g)$	$\psi_4(g)$	**Stability**
$g_0 = \varnothing$	0	0	0	0	
$g_1 = \{12\}$	-1	0	1	1	
$g_2 = \{13\}$	-1	1	0	1	
$g_3 = \{14\}$	0	1	1	0	
$g_4 = \{12, 13\}$	-1	1	1	1	M_w
$g_5 = \{12, 14\}$	-2	0	1	0	
$g_6 = \{12, 13, 14\}$	2	-1	-1	1	

Claim 1: g_4 *is link deletion proof, but neither strong link deletion proof nor link addition proof for* ψ.

Indeed, deleting either 12 or 13 from the network would not improve any of the involved player's payoff. Also, deleting both her links to establish g_0 is beneficial for player 1. Similarly, adding link 14 to g_4 would create g_6, which is strictly beneficial for player 1 and does not harm the net payoff for player 4.

Claim 2: g_4 *is weakly monadically stable.*

For that we consider the supporting strategy tuple

$$\ell = ((1, 1, 1), (1, 0, 0), (1, 0, 0), (0, 0, 0))$$

with $g(\ell) = g_4$. We claim that this strategy tuple is a best response to the myopic belief system $\ell^\star$ associated with g_4, given by

$$\ell^{1\star} = (-, (1,0,0), (1,0,0), (1,0,0))$$
$$\ell^{2\star} = ((1,1,1), -, (1,0,0), (0,0,0))$$
$$\ell^{3\star} = ((1,1,1), (1,0,0), -, (0,0,0))$$
$$\ell^{4\star} = ((1,1,1), (1,0,0), (1,0,0), -)$$

The best responses to these belief systems are given by

$$\beta_1\left(\ell^{1\star}\right) = \{(1,1,1)\}$$
$$\beta_2\left(\ell^{2\star}\right) = \{(1,0,0)\}$$
$$\beta_3\left(\ell^{3\star}\right) = \{(1,0,0)\}$$
$$\beta_4\left(\ell^{4\star}\right) = \{(0,0,0), (1,0,0)\}$$

This confirms that ℓ is indeed composed of best responses to the players' expectations about direct links $\ell^\star$.[9]

We conclude that network g_4 in this example is weakly monadically stable, but that it is neither strong link deletion proof nor link addition proof. Thus it is not strongly pairwise stable either. This shows the assertion stated as Theorem 3.4(b). ♦

The following example shows the assertion stated as Theorem 3.4(c). It also shows some interesting auxiliary properties. In this example, players are enticed to aim at the formation of networks that are very different from the network under consideration.

Example 3.6 Consider three players $N = \{1,2,3\}$ and assume that $c_{ij} = 1$ for all players $i \in N$ and all potential links ij, i.e., we assume uniform link formation costs. Let the network payoff function φ be given by the table below. This table also gives the modified network payoff function ψ.

[9]For completeness we remark that in this example the network g_6 is not weakly monadically stable, since it is not link deletion proof with respect to players 2 and 3.

Network	$\varphi_1(g)$	$\varphi_2(g)$	$\varphi_3(g)$	$\psi_1(g)$	$\psi_2(g)$	$\psi_3(g)$
$g_0 = \varnothing$	0	0	0	0	0	0
$g_1 = \{12\}$	2	2	2	1	1	2
$g_2 = \{13\}$	3	0	0	2	0	-1
$g_3 = \{23\}$	5	3	0	5	2	-1
$g_4 = \{12, 13\}$	0	0	4	-2	-1	3
$g_5 = \{12, 23\}$	0	2	0	-1	0	-1
$g_6 = \{13, 23\}$	0	0	0	-1	-1	-2
$g_7 = g_N$	5	5	5	3	3	3

First, we note that g_1 is strongly pairwise stable for ψ. Indeed, it is link deletion proof with regard to the link 12. Also, it is link addition proof as can easily be deduced from the given table.
Second, we remark that g_0 and g_1 are the only strong link deletion proof networks for ψ in this example.

Claim 1: g_0 *is strong link deletion proof, but not weakly monadically stable.*
The only plausible candidate strategy tuple to support g_0 is given by $\ell^0 = (\ell^0_1, \ell^0_2, \ell^0_3) = ((0,0),(0,0),(0,0))$. Now $g(\ell^0) = g_0$ and the myopic belief systems of all players for ℓ^0 are given by

$$\ell^{0,1\star} = (-,(1,0),(0,0))$$
$$\ell^{0,2\star} = ((1,0),-,(1,0))$$
$$\ell^{0,3\star} = ((0,1),(0,1),-)$$

The unique best responses to these expectations about direct links are given by $\beta\left(\ell^{0,\star}\right) = ((1,0),(0,1),(0,1))$ with $g\left(\beta\left(\ell^{0,\star}\right)\right) = g_3$. This shows clearly that g_0 is not weakly monadically stable.

Claim 2: g_1 *is strongly pairwise stable, but not weakly monadically stable.*
Consider the strategy given by

$$\ell = (\ell_1, \ell_2, \ell_3) = ((1,0),(1,0),(0,0))\,.$$

Obviously, $g(\ell) = g_1$. It is clear that ℓ is the only plausible candidate for a monadically stable strategy tuple supporting g_1. However, the belief systems of all players for ℓ are given by

$$\ell^{1\star} = (-,(1,0),(1,0)) \neq (-,\ell_2,\ell_3)$$
$$\ell^{2\star} = ((1,0),-,(0,0)) = (\ell_1,-,\ell_3)$$
$$\ell^{3\star} = ((1,0),(0,0),-) = (\ell_1,\ell_2,-)$$

The unique best responses to the myopic belief system above is given by[10]

$$\beta(\ell^\star) = \{((0,1),(1,0),(0,0))\}$$

From this we conclude that $g(\beta(\ell^\star)) = g_0$. This indeed shows that g_1 cannot be supported as weakly monadically stable. ♦

Example 3.6 also shows two other properties. First, there exist situations in which players aim for very different networks than the one under consideration. Indeed, in network g_1, Player 1 aims at forming network g_2 based on her myopic beliefs about the other players goals and payoffs. This is a deviation that is "lateral" in the sense that it is not a sub- or super-network of the network under consideration.

Second, Example 3.6 shows that the reverse of Theorem 3.4(a) does not hold. Namely, in the example we identified two strongly link deletion proof networks that are not weakly monadically stable under two-sided link formation costs. For network g_0, the reason is that it is not link addition proof and the players involved try to build these additional links through correct anticipation of the benefits to the other player involved.

For network g_1 the reason is more complex. Here the network under consideration is link addition proof as well. However, belief system allows for the type of lateral departures indicated in the discussion above. The exclusion of such lateral departures requires the further strengthening of the notion of weak monadic stability and beliefs that are not so simple.

3.2 Monadic stability and strict* pairwise stability

Next we turn to the analysis of regular monadic stability. The self-conformation requirement in this equilibrium concept strengthens the properties of the resulting equilibrium networks considerably. We first explore the relationship between the monadic stability condition and the weak monadic stability requirement.

Proposition 3.7 *Let $\langle \varphi, c\rangle$ be given.*

(a) *Every monadically stable network $g \in \mathbb{G}^N$ is weakly monadically stable for $\langle \varphi, c\rangle$ such that the supporting belief system $\hat{\ell}$ satisfies the property that $\hat{\ell}_{ij} = \hat{\ell}_{ji}$ for all pairs of players $i, j \in N$.*

[10] In particular, the best response of player 1 to $\ell_1^\star$ is unique and given by $\beta_1(\ell_1^\star) = (0,1)$ in which player 1 aims for the formation of network g_2 instead of g_1.

(b) *Not every weakly monadically stable network* g *for* $\langle\varphi, c\rangle$ *such that the supporting belief system* $\hat{\ell}$ *satisfies the property that* $\hat{\ell}_{ij} = \hat{\ell}_{ji}$ *for all pairs of players* $i, j \in N$ *is monadically stable.*

For a proof of Proposition 3.7(a) we refer to Section 5 of this paper. Assertion 3.7(b) is shown by Example 3.8 below.

Proposition 3.7 states that the self confirming condition in the monadic stability concept implies the bi-directionality of the steady state that results from updating the myopic belief systems. This updating process is described by the weak monadic stability condition.

Next we show that this conclusion cannot be reversed. Hence, there exist networks that are weakly monadically stable and satisfy the bi-directionality condition formulated in Proposition 3.7 but underlying the beliefs are not self confirming.

Example 3.8 Consider three players $N = \{1, 2, 3\}$ and as before assume that $c_{ij} = 1$ for all players $i \in N$ and all potential links $ij \in L_i$. We assume that the network payoff function φ is additive over the links. The basic payoff information is link-based and, thus, represented in the following table:

Link	φ_1	φ_2	φ_3
12	0	2	1
13	4	0	4
23	0	0	2

The table below provides the modified network payoff function ψ based on the assumption that benefits accrue in an additive fashion.

Network	$\psi_1(g)$	$\psi_2(g)$	$\psi_3(g)$	**Stability**
$g_0 = \varnothing$	0	0	0	
$g_1 = \{12\}$	-1	1	1	
$g_2 = \{13\}$	3	0	3	M_w
$g_3 = \{23\}$	0	-1	1	
$g_4 = \{12, 13\}$	2	1	4	
$g_5 = \{12, 23\}$	-1	0	2	
$g_6 = \{13, 23\}$	3	-1	4	
$g_7 = g_N$	2	0	5	

First, we claim that network g_2 is weakly monadically stable. Consider the strategy tuple given by

$$\ell = (\ell_1, \ell_2, \ell_3) = ((0, 1), (0, 0), (1, 0)).$$

Obviously, $g(\ell) = g_2$. The myopic belief systems of all players for ℓ are given by

$$\ell^{1\star} = (-,(1,0),(1,0))$$
$$\ell^{2\star} = ((0,1),-,(1,1))$$
$$\ell^{3\star} = ((0,1),(0,0),-)$$

The unique best responses to the myopic belief system above is given by

$$\beta(\ell) = \{((0,1),(0,0),(1,0))\}$$

From this we conclude that g_2 is indeed weakly monadically stable and that it satisfies the bi-directionality condition formulated in Proposition 3.7. However, g_2 is not monadically stable, since $\ell^{1\star}_{21} \neq \ell_{21}$ as well as $\ell^{2\star}_{32} \neq \ell_{32}$. ♦

Next we turn to the characterization of monadic stability in terms of link stability. The following result can be indicated as the main result of this paper. It states that monadic stability is equivalent to strict* pairwise stability. This allows us to conclude that monadic stability supports a non-trivial collection of equilibrium networks.

Theorem 3.9 *Let $\langle \varphi, c\rangle$ be such that $c \gg 0$. A network $g \in \mathbb{G}^N$ is monadically stable for $\langle \varphi, c\rangle$, if and only if g is strictly* pairwise stable for the network payoff function ψ given in Theorem 3.4.*

For a proof of Theorem 3.9 we refer to Section 5 of this paper in which we have collected the proofs of the main theorems. The proof first demonstrates that every strictly* pairwise stable network is weakly monadically stable. It then shows that the beliefs are also self confirming making it monadically stable.

4 Conclusion

We base our approach to link building—and through that to network formation—on the principle that links are formed between myopically rational acquaintances. We formulate myopic belief systems through which these players perceive the social setting in which they operate. The stable states of the resulting learning processes are identified as the class of weakly monadically stable networks. These networks are only truly stable if all players have their formed beliefs confirmed in the equilibrium state. This results into the subclass of monadically stable networks.

In this paper we show that—although the belief systems on which the players' learning processes are founded, are myopic and very simple—the resulting monadically stable networks have extremely strong and appealing properties; they exactly form the class of strictly* pairwise stable networks. In these networks no players has incentives to discard links and all pairs of non-linked players agree that building such a link only results into lower payoffs. Our main result, thus, unequivocally shows that behavior based on myopically formed beliefs can lead to very appealing and sensible social structures.

Furthermore, through auxiliary results we conclude that there is no relationship between weak monadic stability and strong link deletion proofness, i.e., one does not imply the other. Hence, there is also no relationship between weak monadic stability and strictly pairwise stable networks. We also conclude that weakly monadically stable networks satisfy link deletion proofness. However, they are not link addition proof. Hence, weakly monadically stable networks are not pairwise stable. Finally, we verify that the reverse implication does not hold.

With respect to monadic stability, we find that the set of monadically stable networks is a strict subset of the weakly monadically stable networks. Furthermore, not every weakly monadically stable network is strong link deletion proof indicating that they cannot be strictly* pairwise stable either.

In a recent paper Belleflamme and Bloch (2004) investigate a related stability concept. They look at reciprocal market sharing agreements by firms in oligopolistic markets and procurement auctions. They consider a finite number of firms, all of which are associated with a separate captive market. In the oligopolistic context this market is called the "home market". In the the procurement auction scenario this is the market in which the firm has bidding privileges. Firms may enter each others' markets unless they enter into reciprocal market sharing arrangements by forming links with each other. The stable networks identified in this context also satisfy monadic stability.

Finally, we turn to the relationship between monadic stability and the popular pairwise stable networks. Proposition 2.3 states that if the payoff function satisfies network convexity, link uniformity, and is discerning, then all strictly* pairwise stable networks are also pairwise stable. These two condition are satisfied for instance by the connections model of Jackson and Wolinsky (1996). Hence we can conclude that under these conditions, strictly* pairwise equilibria are pairwise stable. Hence the class of pairwise stable networks also satisfies monadic stability.

5 Proofs of the main theorems

In this section we address the proofs of Theorem 3.4(a), Proposition 3.7, and Theorem 3.9. We recall that the assertions stated as Theorem 3.4(b) and (c) are shown through the examples developed in Section 4 of this paper.

As a preliminary to the actual proofs, we develop some simple auxiliary insights for weakly monadically stable networks. Suppose that $g \in \mathbb{G}^N$ is weakly monadically stable relative to $\langle \varphi, c \rangle$.

Then there exists some action tuple $\hat{\ell} \in A$ such that $g = g(\hat{\ell})$ and for every player $i \in N$: $\hat{\ell}_i \in A_i$ is a best response to $\hat{\ell}^{i\star}_{-i} \in A_{-i}$ for the payoff function π.

For this setting we state three auxiliary results. In these results we do not assume that link formation costs are strictly positive; in each assertion the assumptions regarding these costs are stated explicitly.

Lemma 5.1 *If* $\hat{\ell}^{i\star}_{ji} = 0$ *then* $\ell_{ij} = 0$ *is a best response to* $\hat{\ell}^{i\star}$.

Proof. Clearly, if $\ell_{ij} = 1$ is selected, i incurs only costs $c_{ij} \geqslant 0$ and no benefits. This implies that player i does not benefit from trying to establish link ij. Hence, $\ell_{ij} = 0$ is a best response to $\hat{\ell}^{\star i}$. ■

Lemma 5.2 *If* $\hat{\ell}^{i\star}_{ji} = 0$ *and* $c_{ij} > 0$*, then* $\ell_{ij} = 0$ *is the unique best response to* $\hat{\ell}^{i\star}$.

Proof. Clearly, if $\ell_{ij} = 1$ is selected, i only incurs strictly positive costs $c_{ij} > 0$ and no benefits. This implies that player i makes a loss from trying to establish link ij. Hence, $\ell_{ij} = 0$ is the unique best response to $\hat{\ell}^{\star i}$. ■

Lemma 5.3 *If* $ij \in g(\hat{\ell})$ *with* $c_{ij} > 0$ *and* $c_{ji} > 0$*, then* $\hat{\ell}^{i\star}_{ji} = \hat{\ell}^{j\star}_{ij} = 1$.

Proof. First remark that $ij \in g(\hat{\ell})$ if and only if $\hat{\ell}_{ij} = \hat{\ell}_{ji} = 1$. The negation of the assertion stated in Lemma 5.2 applied to $\hat{\ell}_{ij} = 1$ and $\hat{\ell}_{ji} = 1$ independently now implies that $\hat{\ell}^{i\star}_{ji} = \hat{\ell}^{j\star}_{ij} = 1$. ■

Proof of Theorem 3.4(a)

We now proceed with the proof of Theorem 3.4(a) under the assumption that the cost structure c is strictly positive.

Suppose that $g \in \mathbb{G}^N$ is weakly monadically stable relative to $\langle \varphi, c \rangle$. Then there exists some action tuple $\hat{\ell} \in A$ such that $g = g(\hat{\ell})$ and for every player $i \in N$: $\hat{\ell}_i \in A_i$ is a best response to $\hat{\ell}^{i\star}_{-i} \in A_{-i}$ for the payoff function π. Of course $\hat{\ell}_i \in A_i$ is a best response to player i's myopic belief system $\hat{\ell}^{i\star}$.

Suppose that g is not link deletion proof. Then there exists a player $i \in N$ with $ij \in g$ for some $j \neq i$ and $\psi_i(g - ij) > \psi_i(g)$, or $\varphi_i(g - ij) + c_{ij} > \varphi_i(g)$. By definition, $\hat{\ell}^{j\star}_{ij} = 0$, and hence from Lemma 5.2 $\ell_{ji} = 0$ is the unique best response to $\hat{\ell}^{j\star}$. Since $ij \in g$ by assumption it has to hold that $\hat{\ell}_{ji} = 1$. This contradicts the hypothesis that

$\hat{\ell}_j$ is a best response to $\hat{\ell}^{j\star}$.
This contradiction indeed shows that g has to be link deletion proof relative to ψ.

Proof of Proposition 3.7(a)

Let $g \in \mathbb{G}^N$ be monadically stable and let action tuple $\hat{\ell} \in A$ support g as such.
Suppose that $ij \notin g$ with $\hat{\ell}_{ij} = 1$ and $\hat{\ell}_{ji} = 0$. Then by Lemma 5.1 $\hat{\ell}_{ij} = 1$ implies that $\hat{\ell}^{i\star}_{ji} = 1$. But this would then imply that $\hat{\ell}_{ji} \neq \hat{\ell}^{i\star}_{ji}$, violating the monadic stability condition.

Proof of Theorem 3.9

First we show that strict* pairwise stability for ψ implies monadic stability for $\langle \varphi, c \rangle$ under the hypothesis that $c \gg 0$.
Let $g \subset g_N$ be a network that is strictly* pairwise stable with regard to the net payoff function ψ. Then g is strong link deletion proof and satisfies the property that

$$ij \notin g \;\Rightarrow\; \psi_i(g + ij) < \psi_i(g) \text{ as well as } \psi_j(g + ij) < \psi_j(g).$$

Hence, this implies that

$$ij \notin g \;\Rightarrow\; \varphi_i(g + ij) - c_{ij} < \varphi_i(g) \text{ as well as } \varphi_j(g + ij) - c_{ji} < \varphi_j(g). \tag{7}$$

With g we now define for all $i \in N$:

- $\hat{\ell}_{ij} = 1$ if $ij \in g$, and
- $\hat{\ell}_{ij} = 0$ if $ij \notin g$.

We investigate whether the given strategy profile $\hat{\ell}$ is indeed a best response to $\hat{\ell}^\star$ as required by the definition of weak monadic stability.

Case A: $ij \notin g$.
From(7) it now follows immediately that $\hat{\ell}^{i\star}_{ji} = \hat{\ell}^{j\star}_{ij} = 0$. From the fact that $c_{ij} > 0$ and $c_{ji} > 0$ and the beliefs it follows from Lemma 5.2 that Case A implies that $\hat{\ell}_{ij} = 0$ is the unique best response to $\hat{\ell}^{i\star}$ as well as that $\hat{\ell}_{ji} = 0$ is the unique best response to $\hat{\ell}^{j\star}$.
Hence, for Case A the strategy satisfies the condition imposed by weak monadic stability.

Case B: $ij \in g$.
In this case $\hat{\ell}_{ij} = \hat{\ell}_{ji} = 1$.
Link deletion proofness of g now implies that $\hat{\ell}^{i\star}_{ji} = 1$ or else (7) is contradicted.

Cases A and B imply now that

$$ij \in g \text{ if and only if } \hat{\ell}^{i\star}_{ji} = \hat{\ell}^{j\star}_{ij} = 1 \tag{8}$$

Applying strong link deletion proofness and the conclusion from Case A leads us to the conclusion that $\hat{\ell}_i$ is the unique best response to $\hat{\ell}^{i\star}$. This in turn implies that $\hat{\ell}$ indeed supports g as a weakly monadically stable network.
Finally, it is immediately clear from (8) and the definition of $\hat{\ell}$ that for all $i, j \in N$: $\hat{\ell}^{i\star}_{ji} = \hat{\ell}_{ij}$. Thus, we conclude that $\hat{\ell}$ supports g as a monadically stable network. This completes the proof of the assertion.

Second, we show that monadic stability for $\langle \varphi, c \rangle$ implies strict* pairwise stability for ψ under the hypothesis that $c \gg 0$.
Let g be monadically stable for $\langle \varphi, c \rangle$. Then there exists some action tuple $\hat{\ell} \in A$ such that $g = g(\hat{\ell})$ and for every player $i \in N$: $\hat{\ell}_i \in A_i$ is a best response to $\hat{\ell}^{i\star}_{-i} \in A_{-i}$ for the payoff function π. Furthermore, $\hat{\ell}^{i\star} = \hat{\ell}_{-i}$.
From Theorem 3.4(a) we already know that g has to be link deletion proof for ψ since g is weakly monadically stable. Hence, for every $ij \in g$ we have that $\varphi_i(g - ij) + c_{ij} \geqslant \varphi_i(g)$. Now through the definition of the belief systems and the self-confirming condition of monadic stability we conclude that for every $ij \in g$:

$$\hat{\ell}_{ij} = \hat{\ell}^{j\star}_{ij} = \hat{\ell}_{ji} = \hat{\ell}^{i\star}_{ij} = 1.$$

Let $h \subset L_i(g)$. Define $\ell^h \in A_i$ by

$$\ell^h_{ij} = \begin{cases} \hat{\ell}_{ij} & \text{if } ij \notin h \\ 0 & \text{if } ij \in h \end{cases}$$

Then $g(\ell^h, \hat{\ell}_{-i}) = g \setminus h$. Since $\hat{\ell}_i$ is a best response to $\hat{\ell}^{i\star} = \hat{\ell}_{-i}$[11] it has to hold that $\pi_i(\ell^h, \hat{\ell}_{-i}) \leqslant \pi_i(\hat{\ell})$. Hence,

$$\varphi_i(g \setminus h) + \sum_{ij \in h} c_{ij} \leqslant \varphi_i(g).$$

This in turn implies that $\psi_i(g \setminus h) \leqslant \psi_i(g)$. Thus, since i and h were chosen arbitrarily, network g is indeed strong link deletion proof.
Next, let $ij \notin g$. Then $\hat{\ell}_{ij} = 0$ and/or $\hat{\ell}_{ji} = 0$. Suppose that $\hat{\ell}_{ji} = 0$. Then by the self-confirming condition of monadic stability it has to hold that $\hat{\ell}^{i\star}_{ji} = \hat{\ell}_{ji} = 0$. Hence by Lemma 5.2 $\hat{\ell}_{ij} = 0$. Thus we conclude that for every $ij \notin g$:

$$\hat{\ell}_{ij} = \hat{\ell}^{j\star}_{ij} = \hat{\ell}_{ji} = \hat{\ell}^{i\star}_{ij} = 0.$$

[11] Here we apply again the self-confirming condition that is satisfied by $\hat{\ell}$.

This in turn implies through the definition of the belief system that $\varphi_i(g+ij) - c_{ij} < \varphi_i(g)$ as well as $\varphi_j(g+ij) - c_{ji} < \varphi_j(g)$. Or $\psi_i(g+ij) < \psi_i(g)$ as well as $\psi_i(g+ij) < \psi_i(g)$. This is desired requirement for strict* pairwise stability.

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