# Random Walk Expectations and the Forward Discount Puzzle

Philippe Bacchetta and Eric van Wincoop*

Two well-known, but seemingly contradictory, features of exchange rates are that they are close to a random walk (RW) while at the same time exchange rate changes are predictable by interest rate differentials. The RW hypothesis received strong support from the work of Richard A. Meese and Kenneth Rogoff (1983) who were the first to show that macro models of exchange rate determination could not beat the RW in predicting exchange rates. On the other hand, Eugene F. Fama (1984) showed that high interest rate currencies tend to subsequently appreciate. This is known as the forward discount puzzle and stands in contrast to Uncovered Interest Parity (UIP), which says that a positive interest differential should lead to an expected depreciation of equal magnitude.

The RW hypothesis and the forward discount puzzle are not as contradictory as it seems since the predictability of exchange rate changes by interest differentials is limited. For example, Fama (1984) reports an average $R^2$ of 0.01 when regressing monthly exchange rate changes on beginning-of-period interest differentials. Instead of opposing these two features of the data, in this paper we investigate whether in fact they may be related to each other. In particular, we ask whether the predictability of exchange rates by interest differentials naturally results when participants in the FX market adopt RW expectations.

RW expectations in the FX market are quite common, in particular when using carry trade strategies, i.e. investing in high interest rate currencies and neglecting potential exchange rate movements. These strategies typically deliver significant excess returns (see Carlos Bazan et al. (2006), Craig Burnside et al. (2006), or Miguel Villanueva (2006) for recent evidence). Moreover, many observers argue that recent movements among major currencies are actually caused by carry trade strategies. The adoption of RW expectations may also be perfectly rational. Welfare gains that can be achieved from full information processing are likely to be small because the $R^2$ from exchange rate predictability regressions is so small. This needs to be weighed against the cost of full information processing.

It is sometimes argued informally that purchases of high interest rate currencies should lead to their appreciation. If correct, that would imply that trade based on RW expectations could indeed lead to the observed predictability of exchange rate changes by interest rates. However, we show that this simple intuition is misleading. With frequent trading based on RW expectations, we find that high interest rate currencies depreciate much more than what UIP would predict. However, when agents make infrequent FX portfolio decisions, we find that high interest rate currencies do indeed appreciate when investors adopt RW expectations. Thus, RW expectations can explain the forward premium puzzle, but only if FX trade is conducted infrequently.

This paper is closely related to Philippe Bacchetta and Eric van Wincoop (2006). We argue in that paper that less than 1% of global FX positions are actively managed. We therefore consider a model in which agents make infrequent FX portfolio decisions. We show that the welfare cost from making infrequent portfolio decisions is very small, especially in comparison with observed FX management fees. We also show that when agents make infrequent decisions about FX positions, high interest rate currencies tend to appreciate. This is particularly the case when agents process only partial information. In this paper we consider the particular case of partial information processing whereby agents simply adopt RW expectations. Apart from being realistic, the simple case of RW expectations also has the advantage that it leads to some precise analytical results.

The remainder of the paper is organized as follows. In Section I we examine the impact of frequent trading based on random walk expectations. In Section II we present the model with infrequent trading when the forward discount (interest differential) follows an autoregressive process. We particularly focus on an $AR(1)$ process, for which precise analytical results can be obtained. In section IV we take the general form of the model to the data and show that it can account for the forward discount puzzle only when investors make infrequent portfolio decisions. Section V concludes. Some technical details can be found in a Technical Appendix that is available on request.

# 1 Random Walk Expectations and Frequent Trading

In this section, we present a simple model assuming that investors trade each period expecting the exchange rate to follow a RW. We focus on the implications for the Fama regression $s_{t+1} - s_t = \beta_0 + \beta fd_t + e_t$. Here $s_t = \ln S_t$ is the log exchange rate and $fd_t$ is the forward discount. We show that frequent FX trading implies a positive and large Fama regression coefficient $\beta$, i.e., a bias opposite to the empirical evidence.

There are two countries, Home and Foreign. There is a single good with the same price in both countries, so that investors in each country face the same real return and make the same portfolio decisions. Agents can invest in nominal bonds of both countries. Asset returns, measured in the Home currency, are respectively $e^{i_t}$ and $e^{i_t^* + s_{t+1} - s_t}$ for Home and Foreign bonds. Here $i_t$ and $i_t^*$ are the log of one plus the nominal interest rates in Home and Foreign currencies. The forward discount is then $fd_t = i_t - i_t^*$. Real returns are assumed to be constant, which for simplicity we normalize at 0, as a result of a risk-free technology that investors also have access to.

There are overlapping generations of investors who live for two periods. They receive an endowment of the good when born that is worth one unit of the Home currency, invest it and consume the return the next period. Agents born at time $t$ maximize expected period $t+1$ utility $E_t C_{t+1}^{1-\gamma}/(1-\gamma)$ subject to $C_{t+1} = 1 + b_t(e^{q_{t+1}} - 1)$, where $b_t$ is the investment in Foreign bonds measured in terms of Home currency and $q_{t+1} = s_{t+1} - s_t + i_t^* - i_t$ is the log excess return on Foreign bonds from $t$ to $t+1$. The solution of this optimization is

$$b_t = \bar{b} + \frac{E_t q_{t+1}}{\gamma\sigma^2} \tag{1}$$

where $\bar{b}$ is a constant that depends on second moments and $\sigma^2 = var_t(q_{t+1})$ will be constant over time in equilibrium. Since we adopt a two-country model we assume that the steady-state supply of Foreign bonds is equal to half of total steady-state financial wealth. Assuming that the Foreign bond supply is fixed in terms of the Foreign currency, the log-linearized supply of Foreign bonds measured in the Home currency is $0.5s_t$. Here both the supply and $s_t$ are in deviation from their steady state. The Foreign bond market equilibrium condition in deviation from steady state then becomes

$$\frac{E_t q_{t+1}}{\gamma\sigma^2} = 0.5 s_t \tag{2}$$

The assumption of RW expectations implies that $E_t s_{t+1} = s_t$, so that $E_t q_{t+1} = -fd_t$. The one-period change in the equilibrium exchange rate is then

$$s_{t+1} - s_t = -\frac{2}{\gamma\sigma^2}(fd_{t+1} - fd_t) \tag{3}$$

so that the Fama regression coefficient is:

$$\beta = \frac{cov(s_{t+1} - s_t, fd_t)}{var(fd_t)} = -\frac{2}{\gamma\sigma^2}(\rho - 1)$$

where $\rho$ is the first-order autocorrelation coefficient of the forward discount. Since $\rho < 1$ if $fd_t$ is stationary, $\beta$ is positive so that the Fama regression has the wrong sign. The exchange rate is expected to depreciate, rather than appreciate as in the data, when the forward discount rises. Moreover, the Fama coefficient tends to be substantially larger than 1. For quarterly data discussed in section III, $\sigma$ and $\rho$ are about 0.05 and 0.8. Even when we set $\gamma = 10$ the implied Fama coefficient is $\beta = 16$.

The intuition for the wrong sign of the Fama coefficient comes from the stationarity of the forward discount. Stationarity implies that when the forward discount is above its long-run level, on average it will subsequently fall. Since a decrease in the forward discount means a decline in demand for the foreign currency, a large initial forward discount tends to be followed by a depreciation of the foreign currency. One might think that it is possible to get a negative Fama coefficient when the impulse response function for the forward discount is hump-shaped. In that case a shock that raises the interest differential continues to raise it for several periods before it starts to fall. This implies that an increase in the interest rate of a currency is followed by an appreciation in subsequent periods, which should lead to a negative Fama coefficient. But this reasoning is not fully correct. What matters most is that the interest rate will start to fall after it peaks so that the currency will depreciate when the interest differential is highest, leading to a positive Fama regression coefficient.

# 2 Infrequent Portfolio Adjustment

In this section, we present the model where investors make infrequent portfolio decisions. There are still overlapping generations of agents, but they now live $T+1$ periods and make only one portfolio decision for $T$ periods. Otherwise the model is the same as in Section I, which corresponds to the case $T = 1$. The crucial aspect is that portfolio holdings do not all respond to current information on interest rates. At any point in time there are $T$ generations of investors, only one of which makes a new portfolio decision. Information is therefore transmitted gradually into portfolio decisions and thus into prices. This corresponds to the fact that most FX positions are not actively managed.

Investors born at time $t$ invest $b_t$ in Foreign bonds, measured in the Home currency. They hold this Foreign bond investment constant for $T$ periods. Any positive or negative return on wealth leads holdings of the Home bond or the risk-free technology to adjust accordingly. An agent born at time $t$, starting with a wealth of one, accumulates a real wealth of $1 + b_t \sum_{i=1}^{T} (e^{q_{t+i}} - 1)$ at $t + T$, which is consumed at that time. End-of-life utility is the same as before. The optimal portfolio of investors born at time $t$ is then

$$b_t = \bar{b} + \frac{E_t q_{t,t+T}}{\gamma var_t(q_{t,t+T})} \tag{4}$$

where $q_{t,t+T} = q_{t+1} + ... + q_{t+T}$ is the cumulative excess return on Foreign bonds from $t$ to $t + T$.

The Foreign bond market equilibrium clearing condition (in deviation from steady state) then becomes

$$\sum_{j=1}^{T} \frac{1}{T} \frac{E_{t-j+1} q_{t-j+1,t-j+1+T}}{\gamma \sigma_T^2} = 0.5 s_t \tag{5}$$

where $\sigma_T^2 = var_t(q_{t,t+T})$ is constant over time in equilibrium. This equates the average holdings of the Foreign bond over the $T$ generations alive to the per capita Foreign bond supply.

Now adopt RW expectations, so that $E_t q_{t,t+T} = -\sum_{k=1}^{T} E_t fd_{t+k-1}$. Since investors have a multi-period horizon, we need to make an assumption about the statistical process of the forward discount. We assume that it follows an *AR(p)* process. This implies parameters $\alpha_i$ such that $\sum_{k=1}^{T} E_t fd_{t+k-1} = \sum_{i=1}^{p} \alpha_i fd_{t+1-i}$. The one-period change in the equilibrium exchange rate is

$$s_{t+1} - s_t = -\frac{2}{\gamma T \sigma_T^2} \sum_{i=1}^{p} \alpha_i (fd_{t-i+2} - fd_{t-i+2-T}) \tag{6}$$

The Fama regression of $s_{t+1} - s_t$ on $fd_t$ then yields the coefficient

$$\beta = -\frac{2}{\gamma T \sigma_T^2} \sum_{i=1}^{p} \alpha_i (\rho_{i-2} - \rho_{T+i-2}) \tag{7}$$

where $\rho_j = corr(fd_t, fd_{t-j})$ and $\rho_{-j} = \rho_j$. It is clear that when $T$ gets large, $\rho_{T+i-2}$ tends toward zero when the forward discount is a stationary process. Therefore the Fama coefficient becomes negative for $T$ large enough, assuming positive autocorrelations and positive $\alpha_i$.

A nice illustration of this is the special case of an $AR(1)$ process. Then $p = 1$ and $\alpha_1 = 1 + \rho + ... + \rho^{T-1}$, where $\rho$ is the autoregressive coefficient. The Fama regression coefficient becomes

$$\beta = -\frac{2}{\gamma T \sigma_T^2} \rho (1 - \rho^{T-2}) \sum_{i=1}^{T} \rho^{i-1} \tag{8}$$

The coefficient is positive for $T = 1$ (as shown in the previous section), zero for $T = 2$ and then turns negative for $T > 2$. The model can therefore account for the negative Fama coefficient in the data as long as agents make infrequent portfolio decisions. This is a result of delayed overshooting. When the Foreign interest rate falls (and therefore the forward discount rises), investors shift from Foreign to Home bonds and the Home currency appreciates. This continues for $T$ periods as new generations continue to reallocate their portfolio from Foreign to Home bonds due to the lower Foreign interest rate. Only after $T$ periods is this process reversed. Investors start buying Foreign bonds again and the Home currency depreciates. The reason is that the Foreign interest rate has increased by then and investors who sold Foreign bonds $T$ periods earlier are due to make a new portfolio decision. The continued appreciation for $T$ periods after the increase in the forward discount gives rise to a negative Fama coefficient.

When $T$ approaches infinity the Fama coefficient goes to zero. This implies that there is an intermediate value of $T$ for which the Fama coefficient is most negative. When $T$ is large the exchange rate response to interest rate shocks is small since only a small fraction of agents makes active portfolio decisions at any point in time. Both the initial appreciation and the subsequent appreciation for $T$ periods are then small.

# 3 Quantitative Illustration

We now quantify the Fama coefficient implied by the above model by estimating an autoregressive process for the forward discount. Moreover, we extend the model to allow for noise or liquidity traders. In the above model exchange rates are completely driven by interest rate shocks. It is well known though that interest rate shocks, or other observed macro fundamentals, account for only a small fraction of exchange rate volatility in the data. Therefore, instead of a per capita Foreign bond supply of 0.5 (in Foreign currency), we assume that it is $0.5X_t$, where $X_t$ represents shocks to net demand or supply associated with liquidity or noise traders. We assume that $x_t = ln(X_t)$ follows a random walk with innovation $\epsilon_t^x$ at time $t$ that is $N(0, \sigma_x^2)$ distributed. The only change to the expression (6) for $s_{t+1} - s_t$ is that $\epsilon_{t+1}^x$

is added on the right hand side. The expression for the Fama coefficient is unchanged. But the noise trade does affect the conditional variance $\sigma_T^2$ of the excess return that shows up in the expression for the Fama coefficient. Moreover, since the noise shocks are uncorrelated with interest rate shocks, they lower the $R^2$ of the Fama regression.

We estimate autoregressive processes for the forward discount using monthly data on 3-month interest rates for six currencies from December 1978 to December 2005. The currencies are the U.S. dollar, Deutsche mark-euro, British pound, Japanese yen, Canadian dollar and Swiss franc. The forward discount is equal to the U.S. interest rate minus the interest rate on one of the other currencies. Interest rates are 3-month rates quoted in the London Euromarket and obtained from Datastream. We use the simple average of the autoregressive coefficients and standard deviations of innovations estimated for the five forward discount series. While we have computed results for $p$ ranging from 1 to 5, for space considerations we only report results for an $AR(3)$ process. Results are not substantially different for lower or higher values of $p$.

We set $\gamma = 10$ (see Bacchetta and van Wincoop (2006) for a discussion). Figure 1 reports results for the Fama regression coefficient, the $R^2$ of the Fama regression, the autocorrelation of quarterly log exchange rate changes and the standard deviation of the quarterly log exchange rate change, with $T$ ranging from 1 to 15. Results are reported both for $\sigma_x = 0$ (previous section) and $\sigma_x = 0.04$. Results can be compared to the data, which yield an average Fama coefficient of -1.6, average $R^2$ of 0.02, average first-order autocorrelation of 0.055 and average quarterly standard deviation of 5.4%.

The model does well in accounting for the negative Fama coefficient. For $\sigma_x = 0.04$ the Fama coefficient remains close to -2 for $T \geq 3$. For $\sigma_x = 0$ it is even slightly more negative.

When $\sigma_x = 0.04$, the $R^2$ of the Fama regression is always less than 0.06 and less than 0.02 for $T \leq 4$. For $\sigma_x = 0$ it is less than 0.04 for $T \leq 4$ but gets much larger for higher values of $T$. The autocorrelation of quarterly changes in exchange rates is also small, less than 0.03 for both values of $\sigma_x$. These results indicate that the exchange rate behaves similar to a RW, with future exchange rate changes hard to predict by the forward discount and past exchange rate changes. The standard deviation of the quarterly log change in the exchange rate drops as $T$ rises, which weakens the portfolio response to interest rates. It becomes broadly consistent with the data for $T \geq 4$.

To summarize, when $T > 1$ (infrequent portfolio decision making) the model can account for a wide range of evidence about exchange rates, including the negative Fama coefficient as well as the near-RW behavior of the exchange rate. For example, when $T = 4$ the Fama regression coefficients are -1.6 and -1.4 for $\sigma_x$ respectively 0 and 0.04. The $R^2$ is respectively 0.04 and 0.02. The autocorrelations of quarterly exchange rate changes are respectively 0.03 and 0.02 and the standard deviations of quarterly exchange rate changes are 4.8% and 5.8%. These are all close to the data.

# 4 Conclusion

We have shown that even when the exchange rate is close to a RW, and investors therefore sensibly adopt RW expectations, exchange rate changes can be negatively predicted by the forward discount with a coefficient that is in line with the Fama or forward discount puzzle. This happens when investors make infrequent decisions about FX positions.

# References

[1] Bacchetta, Philippe and Eric van Wincoop (2006), "Incomplete Information Processing: A Solution to the Forward Discount Puzzle," working paper, University of Virginia.

[2] Bazan, Carlos, Kuntara Pukthuanthong, and Lee R. Thomas (2006), "Random Walk Currency Futures Profits Revisited," working paper, San Diego State University .

[3] Burnside, Craig, Martin Eichenbaum, Isaac Kleshchelski, and Sergio Rebelo (2006), "The Returns to Currency Speculation, " NBER Working Paper No. 12489.

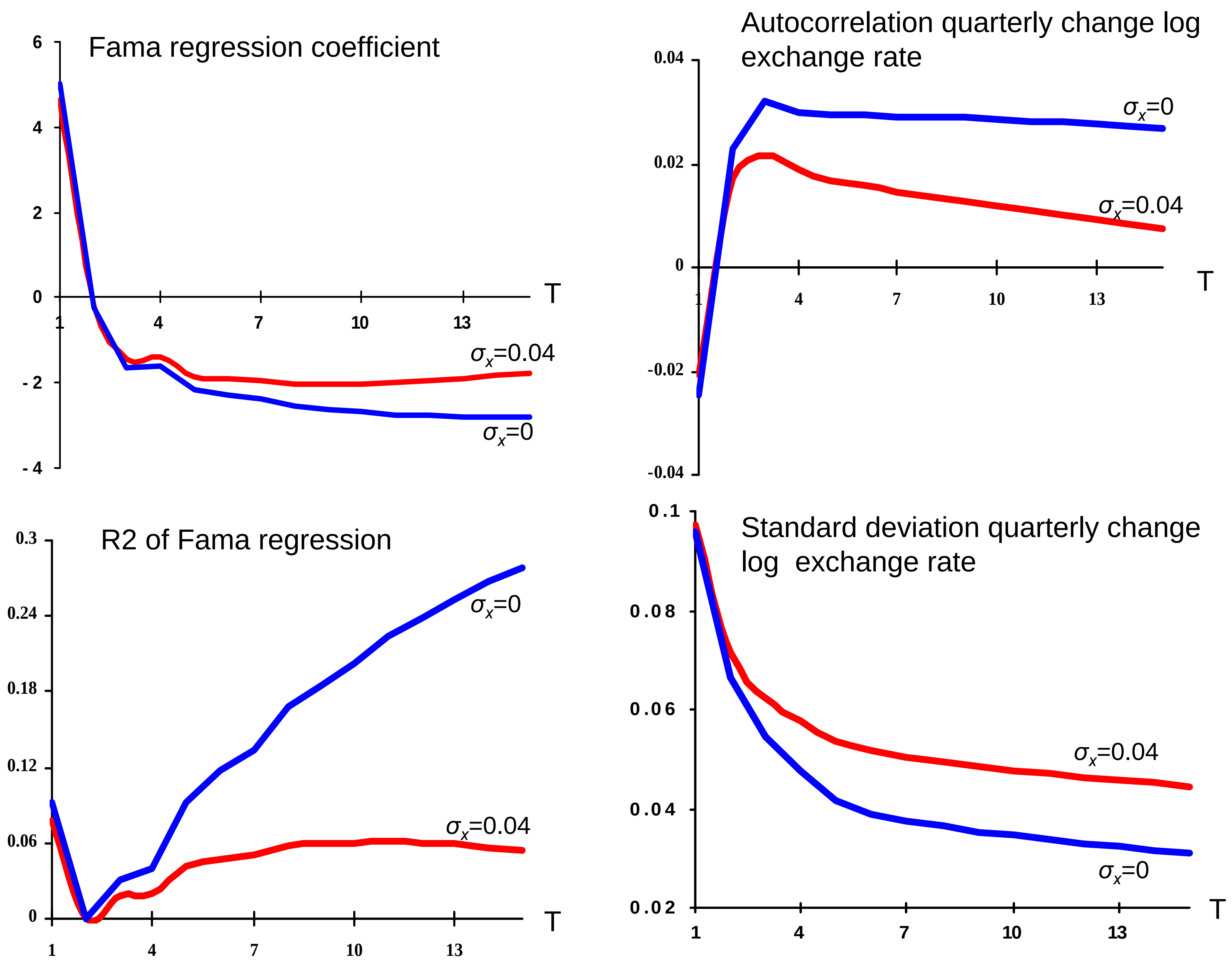
[4] Fama, Eugene F. (1984), "Forward and Spot Exchange Rates," *Journal of Monetary Economics* 14, 319-338.

[5] Meese, Richard A. and Kenneth Rogoff (1983), "Empirical Exchange Rate Models of the Seventies: Do They Fit Out of Sample?" *Journal of International Economics* 14, 345-373.

[6] Villanueva, Miguel (2006), "Forecasting Currency Excess Returns: Can the Forward Bias Be Exploited?" forthcoming, *Journal of Financial and Quantitative Analysis.*

*Bacchetta: Study Center Gerzensee, 3115 Gerzensee, Switzerland, University of Lausanne and Swiss Finance Institute (e-mail: phbacchetta@szgerzensee.ch); van Wincoop: University of Virginia, Department of Economics, P.O. Box 400182, Charlottesville, VA 22904-4182, (e-mail:vanwincoop@virginia.edu). Bacchetta acknowledges financial support by the National Centre of Competence in Research "Financial Valuation and Risk Management" (NCCR FINRISK). The paper was written while van Wincoop visited the Hong Kong Institute of Monetary Research. van Wincoop acknowledges support from the Institute.

# Figure 1 Model Moments



# Technical Appendix on "Random Walk Expectations and the Forward Discount Puzzle"

Philippe Bacchetta Eric van Wincoop

December 2006

This note provides some details on the derivation of the various equations and explains how this is implemented in Gauss. The Gauss code is available upon request.

## 1 Optimal Portfolio

The first-order condition for optimal portfolio choice for an agent born at $t$ is

$$\sum_{i=1}^{T} E_t e^{-\gamma c_{t+T}} (e^{q_{t+i}} - 1) = 0 \tag{1}$$

where $c_{t+T} = ln(C_{t+T})$ is log end-of-life consumption. A first-order approximation of log-wealth at zero-excess returns is $c_{t+T} = b_t q_{t,t+T}$. Substituting this into (1) and using that excess returns are normally distributed in equilibrium gives

$$\sum_{i=1}^{T} e^{E_t q_{t+i} + 0.5 var_t(q_{t+i}) - \gamma b_t cov(q_{t+i}, q_{t,t+T})} = T \tag{2}$$

Linearizing this expression around zero first and second moments equal to zero gives

$$b_t = \bar{b} + \frac{E_t q_{t,t+T}}{\gamma var_t(q_{t,t+T})} \tag{3}$$

where

$$\bar{b} = 0.5 \frac{\sum_{i=1}^{T} var_t(q_{t+i})}{\gamma var_t(q_{t,t+T})}$$

## 2 Excess return expectations

The forward discount follows an *AR(p)* process:

$$fd_t = \sum_{i=1}^{p} a_i fd_{t-i} + \epsilon_t \tag{4}$$

where $\epsilon_t \sim N(0, \sigma_f^2)$. We first derive $\alpha_i$ in the expression

$$\sum_{k=1}^{T} E_t fd_{t+k-1} = \sum_{i=1}^{p} \alpha_i fd_{t+1-i} \tag{5}$$

which is used in equations (7) and (8) in the text. Define

$$y_{s,t} = E_t fd_{t+s-p} \tag{6}$$

We need to compute a row vector $\boldsymbol{\beta}_s$ such that

$$y_{s,t} = \boldsymbol{\beta}_s \mathbf{y}_t \tag{7}$$

where

$$\mathbf{y}_t = \begin{pmatrix} fd_t \\ .. \\ fd_{t-p+1} \end{pmatrix} \tag{8}$$

For $1 \leq s \leq p$, $\boldsymbol{\beta}_s$ is a 1 by $p$ vector with 1 in element $p - s + 1$ and zeros otherwise. For $s > p$ we have

$$\begin{aligned} y_{s,t} &= E_t fd_{t+s-p} = a_1 E_t fd_{t+s-p-1} + ... + a_p E_t fd_{t+s-2p} = \\ a_1 y_{s-1,t} + ... + a_p y_{s-p,t} &= \left(a_1 \boldsymbol{\beta}_{s-1} + ... + a_p \boldsymbol{\beta}_{s-p}\right) \mathbf{y}_t \end{aligned}$$

It follows that for $s > p$

$$\boldsymbol{\beta}_s = a_1 \boldsymbol{\beta}_{s-1} + ... + a_p \boldsymbol{\beta}_{s-p} \tag{9}$$

This allows us to compute recursively any $\boldsymbol{\beta}_s$.

It follows that

$$E_t(fd_t + ...fd_{t+T-1}) = (\beta_p + ...\beta_{p+T-1})\mathbf{y}_t = \boldsymbol{\alpha} \begin{pmatrix} fd_t \\ .. \\ fd_{t-p+1} \end{pmatrix} \tag{10}$$

where $\boldsymbol{\alpha} = \boldsymbol{\beta}_p + ... + \boldsymbol{\beta}_{p+T-1}$. Denoting $\alpha_i$ as element $i$ of the vector $\boldsymbol{\alpha}$, this implies (5). In the Gauss code we first compute $\boldsymbol{\beta}_s$ in the `beta` vector. The `alpha` vector is computed in the subroutine `sigmatt`.

## 3 Forward discount autocorrelations and variance

The Fama coefficient is expressed in terms autocorrelations. $\rho_j$ is the autocorrelation of order $j$ ($\rho_j = corr(fd_t, fd_{t-j})$). It has the property that $\rho_j = \rho_{-j}$ so that $\rho_{-j} = \rho_{abs(-j)}$. These autocorrelations can be computed by using the Yule-Walker equations. Using the $AR$ process for $fd_t$ we get:

$$\rho_j = \frac{cov(fd_t, fd_{t-j})}{var(fd_t)} = \sum_{s=1}^{p} a_s \rho_{s-j} \tag{11}$$

Applying this jointly to $j = 1, ..., p$, and defining $\boldsymbol{\rho} = (\rho_1, .., \rho_p)'$, we have

$$\boldsymbol{\rho} = \mathbf{A}\boldsymbol{\rho} + \mathbf{d} \tag{12}$$

Matrix $\mathbf{A}$ is computed as follows. In row $j$ start with zeros and then for $s = 1, .., p$ add $a_s$ in column $abs(s - j)$ when $s \neq j$. Element $i$ of vector $\mathbf{d}$ is $a_i$. We can then solve

$$\boldsymbol{\rho} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{d} \tag{13}$$

where $\mathbf{I}$ is a $p$ by $p$ matrix. It also follows from from the $AR$ process that for $j > p$

$$\rho_j = a_1\rho_{j-1} + ... + a_p\rho_{j-p} \tag{14}$$

In the Gauss code the matrix $\mathbf{A}$ is amat and the vector $\mathbf{d}$ is bvec.

Using the $AR$ process for the forward discount, we find the variance:

$$var(fd) = var(fd) \sum_{i=1}^{p} \sum_{j=1}^{p} a_i a_j \rho_{abs(i-j)} + \sigma_f^2 \tag{15}$$

In the Gauss code, we compute tot=$\sum_{i=1}^{p} \sum_{j=1}^{p} a_i a_j \rho_{abs(i-j)}$ so that $var(fd) = \sigma_f^2/(1-$tot$)$. $\sigma_f^2$ is taken from the data (stanfd).

## 4 Conditional variance of excess return

We now describe how to compute the conditional variance of the excess return over $T$ periods, $\sigma_T^2 = var_t(q_{t,t+T})$. We start from

$$q_{t,t+T} = s_{t+T} - s_t - \sum_{i=1}^{T} fd_{t+i-1} \tag{16}$$

Introducing noise shocks, the equilibrium exchange rate can be written as

$$s_{t+T} = -\frac{2}{\gamma T\sigma_T^2}\sum_{j=1}^{T}\sum_{i=1}^{p}\alpha_i fd_{tT-j-i+2} + \sum_{i=1}^{T}\varepsilon_{t+i}^x = m\sum_{i=1}^{T}\phi_i fd_{t+i} + \sum_{i=1}^{T}\varepsilon_{t+i}^x + d_t \tag{17}$$

where $m = -\frac{2}{\gamma T\sigma_T^2}$, $\phi_i$ is a function of $\alpha_1, \alpha_2, ..., \alpha_p$ and $d_t$ is a variable known at time $t$ (we will ignore it, since we are computing the conditional variance). Thus, we have:

$$\sigma_T^2 = var_t\left(\sum_{i=1}^{T}(\phi_i m + \lambda_i) fd_{t+i}\right) + T\sigma_x^2 \tag{18}$$

where $\lambda_i = -1$ for $i = 1, ..., T-1$ and $\lambda_T = 0$. Now write the forward discount in terms of its MA representation:

$$fd_t = \sum_{j=1}^{\infty}\theta_j \epsilon_{t+1-j} \tag{19}$$

Define $\eta_k = \sum_{i=k}^{T}\theta_{i-k+1}\phi_i$ and $\mu_k = \sum_{i=k}^{T}\theta_{i-k+1}\lambda_i$. Then

$$\sigma_T^2 = c_1 m^2 + c_2 m + c_3 + T\sigma_x^2 \tag{20}$$

where $c_1 = \sigma_f^2\sum_{k=1}^{T}\eta_k^2$, $c_2 = 2\sigma_f^2\sum_{k=1}^{T}\eta_k\mu_k$ and $c_3 = \sigma_f^2\sum_{k=1}^{T}\mu_k^2$. This gives an implicit solution for $\sigma_T^2$, which is solved numerically with Gauss. There is a single root.

The coefficients of the MA representation, $\theta_i$, are computed from the impulse response function of the $AR$ process for $fd_t$; they are in the vector theta. The coefficients $\phi_i, \lambda_i$, $\eta_i$, and $\mu_i$ are computed in the subroutine sigmatt in the vectors phi, lambda, eta, and mu. The subroutine sigmatt uses the non-linear system solver nlsys.

# 5 Other parameters

The other statistics are straightforward. For example, from (7) in the text we can derive:

$$var(ds) = m^2\sum_i\sum_j \alpha_i\alpha_j cov(fd_{t-i+1} - fd_{t-i+1-T}, fd_{t-j+1} - fd_{t-j+1-T}) + \sigma_x^2 \tag{21}$$

Writing *abs* for absolute value, this becomes

$$var(ds) = m^2 var(fd) \sum_i \sum_j \alpha_i \alpha_j (2\rho_{abs(i-j)} - \rho_{abs(j+T-i)} - \rho_{abs(i+T-j)}) + \sigma_x^2 \tag{22}$$

The covariance $cov(ds, ds_{-1})$ is computed in a similar way.