Allocating Security Expenditures under Knightian Uncertainty: an Info-Gap Approach

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November 16, 2006

Abstract

In this paper we demonstrate a methodology aimed at coping with resource allocation under Knightian (non-probabilistic) uncertainty by focusing on the example of competing security measures. The results of this application to security resource allocation also allow us to postulate a possible positivist explanation for the way governments are allocating these expenditures today. We explore the determination of the level and nature of government expenditures that affect security in different ways, and demonstrate that it is better to robust-satisfice the citizen's expected utility rather than to attempt to maximize it. Moreover, our analysis highlights one rationale for heightened spending on one set of defense measures, when there is less reliable information about threats to national security and their consequences.

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This research was supported by the Economics of National Security Program, Samuel Neaman Institute, grant no. 358.

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"*Reports that say that something hasn't happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know.*"

Donald Rumsfeld
U.S. Secretary of Defense

1 Introduction

When allocating resources, policy makers, like households, face trade-offs. Many of the trade-offs households encounter can reasonably be modeled in a deterministic setting—their incomes may be uncertain, but when deciding between a new refrigerator or a new television set, we generally assume that members of a household know the marginal utility they will derive from each. By contrast, policy makers nearly always confront decisions in which the connection between allocations and the desirability of outcomes are highly uncertain. No one can predict precisely by how much a dollar transferred between different components of the public health budget will affect an individual citizen's longevity. Nonetheless, policy makers know a great deal about the prevalence of infectious diseases, heart problems, and cancer in the population as a whole, and the efficacy of different treatments for large samples of patients. By combining the two, decision makers can derive fairly reliable probabilistic models that link different allocations of funding with the moments of a distribution of outcomes.

Unlike public health decisions, allocating resources for national security involves decisions where experiments are not possible (or at least unwise) and previous experience provides little or no useful data. Policy makers never know the exact probability distribution of possible damages to their citizens associated with alternative allocations of resources to government agencies. For instance, resources allocated to law enforcement agencies and department of defense may change the distribution of damages to an 'average' citizen from, say, criminal activities or traffic accidents on one hand, and those associated with an attack by terrorists or a belligerent neighbor on the other hand. Furthermore, terror attacks, or invasions by belligerent neighbors, are discrete, unique and relatively infrequent events that involve a small number of actors. When allocating resources to enhance the security of their citizens, policy makers have to decide on such appropriations without knowing the effects of their decisions on the ensuing distribution of possible damages, or at best have only some *qualitative* knowledge about these relationships. Experience simply does not supply enough information to derive the actual probability distribution of disutility from these different kinds of risks. Nor does it deliver a clear picture of the relationships between different levels and types of security expenditures to enhancing personal safety and protecting the nation from wars or terrorism, and the loss of human life and economic disruption such dangers entail.[1]

[1] Moreover, some of these risks are endogenous in the sense that their perpetrators react to the government

In this paper we demonstrate a methodology aimed at coping with resource allocation under Knightian (non-probabilistic) uncertainty by focusing on the example of competing security measures. The results of this application to security resource allocation also allow us to postulate a possible positivist explanation for the way governments are allocating these expenditures today. We explore the determination of the level and nature of government expenditures that affect security in different ways, and demonstrate that it is better to robust-satisfice the citizen's expected utility rather than to attempt to maximize it. Moreover, our analysis highlights one rationale for heightened spending on one set of measures, (defense), when there is much less information about its consequences or likelihood, when compared to those associated with crime and law-enforcement.

While our formulation in section 2 is generic, we illustrate it in section 3 with two examples. First, in the case of a terrorist threat, the trade-off is between non-security related expenditure, expenditure on counterterrorist operations abroad, and domestic security expenditures. In the case of a conventional threat, the security expenditures must be divided between intelligence gathering and war-fighting capabilities. Threats to national security of either type and their relationships to both defense and law-enforcement expenditures are particularly pertinent to the analysis of a highly uncertain world.

2 Formulation

Consider a country facing various threats to its national security. These threats may emanate from various sources including the threat of an invasion by an aggressive neighbor or a terrorist attack. We describe the threats to security from all sources as a bivariate distribution that includes both the event of being attacked, and the damage that the country will sustain, conditional on the attack taking place, recognizing and dealing with the fact that this distribution is highly uncertain. Policy makers must decide what portion of the economy's resources they will devote to countering these risks, as well as how to allocate this expenditure between different defense measures when each measure affects both the risk and the potential damage from an attack in a different way. This defense resource allocation dilemma is embedded within a standard economic framework in which the representative individual in this country derives utility $u(c)$ only from consumption, c, and that the threats we enumerated above are all expressed in terms of a drop in the value of this variable.

Normalizing the economy's resources to 1, the policy maker must choose the fraction of all resources it will devote to each of a number of different risk-mitigating expenditures $\chi = (\chi_1, \ldots, \chi_N)$. Without government debt, we require $\sum_i^N \chi_i \leq 1$.

Any government expenditure detracts from the resources available for consumption, so $c = 1 - \sum_i^N \chi_i$. On the other hand these government expenditures reduce the fractional loss ψ in resources resulting from the security risks citizens in the economy face, where $0 \leq \psi \leq 1$.

Let $\chi_{\rm c} = 1 - \sum_i^N \chi_i$, which is the fraction of GDP devoted to consumption. Denote the

efforts in different ways.

utility in the ideal situation where none of the risks materialize by $u_c = u(\chi_c)$.

The probability density function (pdf) of realized threats, conditioned on risk mitigating expenditures, is $p(\psi|\chi)$—a probability distribution unknown to policy makers. The best available estimate of $p(\psi|\chi)$ is denoted $\widetilde{p}(\psi|\chi)$ but it is incontrovertible that $\widetilde{p}(\psi|\chi)$ is highly unreliable.

We assume that the probability that a representative agent will suffer a loss in his welfare from an attack is P_w. The value of P_w is highly uncertain and its best estimate, $\widetilde{P}_w$, depends on both the level and distribution of defense expenditures.

Let $R(\chi|p, P_w)$ be the expected utility resulting from defense expenditure χ, when the probability of the threat being realized is P_w, and the pdf of the damage ψ is $p(\psi|\chi)$:

$$R(\chi|p, P_w) = \left(\int_0^1 u\left[(1-\psi)\chi_c\right] p(\psi|\chi)\, d\psi \right) P_w + (1-P_w)u_c \tag{1}$$

Higher expected utility is preferable over lower expected utility, but R_c is the lowest acceptable level of expected utility. It is a reward aspiration or a 'reservation reward'.

The info-gap model is a family of nested sets of probability models $p(\psi|\chi)$ and $P_w(\chi)$, indexed by α, which represents the degree of uncertainty in the policy maker's best estimate of the chances of any damage occurring and the conditional distribution of its level. We denote this info-gap model by $\mathcal{F}[\alpha, \widetilde{p}(\psi|\chi), \widetilde{P}_w(\chi)]$, where $\alpha \geq 0$.

Info-gap models obey two axioms:

(*i*) *Nesting* asserts that the range of possible pdfs increases as α increases:

$$\alpha < \alpha' \quad \Longrightarrow \quad \mathcal{F}[\alpha, \widetilde{p}, \widetilde{P}_w] \subset \mathcal{F}[\alpha', \widetilde{p}, \widetilde{P}_w] \tag{2}$$

(*ii*) *Contraction* asserts that, when $\alpha = 0$, the estimated models are the only possibilities:

$$\mathcal{F}[0, \widetilde{p}, \widetilde{P}_w] = \left\{ \widetilde{p}, \widetilde{P}_w \right\} \tag{3}$$

These two axioms endow α with its meaning of a *horizon of uncertainty.*

Let $\mu(\chi, \alpha)$ be the lowest expected reward given defense expenditures χ over an info-gap model α around $\widetilde{p}$ and $\widetilde{P}_w$. That is:

$$\mu(\alpha, \chi) = \min_{(p, P_W) \in \mathcal{F}(\alpha, \widetilde{p}, \widetilde{P}_w)} R(\chi|p, P_w) \tag{4}$$

The value $\widehat{\alpha}(\chi, R_c)$ is the robustness of expenditures χ with reward-aspiration R_c. It is the greatest range of Knightian uncertainty, α, up to which all probability models in $\mathcal{F}$ result in reward no less than R_c:

$$\widehat{\alpha}(\chi, R_c) = \max \{\alpha : \mu(\alpha, \chi) \geq R_c\} \tag{5}$$

$\widehat{\alpha}(\chi, R_c)$ is the robustness (to uncertainty in $\widetilde{p}$ and $\widetilde{P}_w$) of security expenditures χ which satisfice the expected utility at the level R_c.

The robustness function displays a fundamental trade-off between reward and robustness to uncertainty: robustness decreases as the aspired reward increases (Ben-Haim, 2006):

$$R_c > R'_c \quad \Longrightarrow \quad \widehat{\alpha}(\chi, R_c) \leq \widehat{\alpha}(\chi, R'_c) \tag{6}$$

Furthermore, if the aspiration is for the greatest reward expected with the estimated distribution, then the robustness is zero (Ben-Haim, 2005):

$$\chi^* = \arg\max_{\chi} R(\chi|\widetilde{p}, \widetilde{P}_w), \quad R_c = R(\chi^*|\widetilde{p}, \widetilde{P}_w) \quad \Longrightarrow \quad \widehat{\alpha}(\chi^*, R_c) = 0 \tag{7}$$

This means that if the estimated models $\widetilde{p}(\psi|\chi)$ and $\widetilde{P}_w$ are used to choose an expected-utility-maximizing allocation χ^*, then this aspiration has zero robustness to uncertainty in these models.

Since more robustness is preferable to less robustness, at the same level of satisficed utility, the decision maker may wish to choose χ to satisfice the utility and to maximize the robustness. This is an **info-gap robust-satisficing** decision approach, which is formally defined:

$$\widehat{\chi}(R_c) = \arg\max_{\chi} \widehat{\alpha}(\chi, R_c) \tag{8}$$

In this paper we adopt a variation of the classical info-gap approach in which the policy maker chooses the value of χ that maximizes the lowest possible value of expected utility over a set of probabilities and distribution functions indexed by α around the best estimates. For any desired level of robustness α, the decision maker will choose χ to maximize $\mu(\alpha, \chi)$ as defined in eq. 4. Note that for any aspiration level R_c, $\mu(\widehat{\chi}(R_c), \widehat{\alpha}(\widehat{\chi}(R_c), R_c)) \equiv R_c$.

3 An Illustration with Two Types of Security Expenditure

In this section we illustrate, hypothetically, how policy makers use fragmentary and highly uncertain evidence to allocate resources between two different types of expenditures devoted to security.

3.1 Background

How do we quantify security threats, and what are the different possible security expenditures that are meant to counter them? Consider first a threat from a conventional adversary. Table 1 presents the direct expenses of major U.S. wars along with U.S. fatalities. In the aftermath of World War I, Bogart (1920) began developing the tools to measure, compare, and aggregate all the different costs of war. These included both the direct costs in military expenditure and physical destruction, and the indirect costs associated with the capitalized values of losses in life and lost production.

According to Broadberry and Howlett (1998), the U.K. spent approximately half of its GDP fighting World War II during the years 1940 to 1944. In addition it suffered losses

Conflict	Total Direct Costs		People Mobilized		Fatalities	
	Millions of 2002$	*Percent of GDP*	*Thousands*	*Percent of Pop.*	*Numbers*	*Percent of Pop.*
Revolutionary War (1775 – 1783)	2.2	63%	200	5.70%	4,435	0.127%
War of 1812 (1812 – 1815)	1.1	13%	286	3.80%	2,260	0.030%
Mexican War (1846 – 1848)	1.6	3%	79	0.40%	1,733	0.008%
Civil War (1861 – 1865)	62	104%	3,868	11.10%	184,594	0.538%
Span. Amer. War (1898)	9.6	3%	307	0.40%	385	0.001%
World War I (1917 – 1918)	190.6	24%	4,744	4.60%	53,513	0.052%
World War II (1941 – 1945)	2,896.3	130%	16,354	12.20%	292,131	0.219%
Korea (1950 – 1953)	335.9	15%	5,764	3.80%	33,651	0.022%
Vietnam (1964 – 1972)	494.3	12%	8,744	4.30%	47,369	0.023%
First Gulf War (1990 – 1991)	76.1	1%	2,750	1.10%	148	0.000%

Table 1: American Costs and Casualties from Major Wars. *Source*: William D. Nordhaus, "The Economic Consequences of a War with Iraq" in *War with Iraq*, ed. Kaysen, *et. al.*

of physical capital that amounted to 89% of GDP in 1938 (see Mitchell 1980) and human capital losses (calculated rather conservatively in terms of just the schooling invested in those killed) of 2.5% of GDP in 1938. By any measure Soviet losses were far higher. During 1942 and 1943 defense expenditure in the Soviet Union reached 61% of GDP, losses of physical capital amounted to 223% of pre-war GDP and losses of human capital were 109% (Harrison (1998)). Of course these figures do not include the extraordinary privations suffered by those living under German occupation during much of this period. To study the cost of World War II for the United States, Rockoff (1998) employs a counterfactual approach developed by Goldin and Lewis (1975) to study the Civil War. According to his estimates the total present value of foregone consumption that can be attributed to both direct and indirect losses generated by the war equals 2.27 years of consumption in 1941.

As most of its effects are indirect, the impact of terror is not as well understood. Abadie and Gardeazabal (2003) estimate that terrorism in the Basque country of Northern Spain has reduced GDP by ten percent. Similarly, recent estimates of the loss in GDP that can be attributed to the impact terrorism after three years of recurring terrorism against Israel is also around ten percent (Eckstein and Tsiddon (2004) and Persitz (2005)). Estimates of the total cost of one incident, the September 11, 2001 attack on the World Trade Center in New York, including lost lifetime earnings of those killed, are between \$33 billion and \$36 billion (Bram, Orr and Rapaport (2002)). What is clear is that large-scale conventional warfare is far more costly than any losses associated with terror—Hess calculates that for countries that have experienced conflict between 1960-1992 (nearly all of it civil war or terrorism in this period) the loss in welfare associated with these conflicts is on average equivalent to a permanent eight percent drop in their consumption.

Fragmentary anecdotal and quantitative evidence exists about the impact of military expenditures on the probability distribution of war-related damage. Rohlfs (2005) estimates that the marginal effectiveness of a U.S. tank in Western Europe during World War II for 164 battles was twenty-four times the effectiveness of a single infantryman but eighty-seven times as expensive to use. The discrepancy is explained by the higher casualties associated with intensive use of infantry, implying that the U.S. government assigned a value of approximately one million dollars (in 2003 dollars) to each soldier's life saved on the battlefield. Although Rohlfs' study involves calculating ex-post a relatively simple trade-off in a single theater of a war in its fifth and sixth year, his estimates contain relatively large standard errors and vary across different sub-samples. By contrast, policy makers must determine ex-ante, both the overall effectiveness of defense expenditure and its optimal allocation at a stage when the nature and scale and even eventuality of a conflict may only be hypothetical. In the years prior to being attacked in 1941, the U.S.S.R. was engaged in a massive rearmament program that according to Bergson (1961) lowered per-capita consumption between 1937 and 1940 by as much as 8.4%. However not only did rearmament fail to deter Hitler's invasion as the Soviets had hoped, but because of serious military miscalculations, much of the arms and manpower was squandered during the summer and fall of 1941 without seriously slowing the German advance (Harrison (1985)).

3.2 Basic Structure

Consider an economy in which resources are allocated between civilian consumption, and two different types of security-related public expenditure. We divide security related public expenditures into two broad categories denoted χ_1 and χ_2. Both χ_1 and χ_2 are measured as shares of GDP, and we ignore the possibility of international borrowing. The allocation of resources influences in different ways the pdf $p(\psi|\chi)$ of potential harm to a representative member of society. The first effect on $p(\psi|\chi)$ is that of the overall amount spent on security, $\chi_1 + \chi_2$. The second influence on $p(\psi|\chi)$ comes from the composition of the security expenditures, (χ_1, χ_2). These two effects jointly determine the pdf of damage, ψ, given total security expenditures, χ, as explained below.

In order to illustrate our approach we focus on two alternative types of military expenditure. First, there are the expenditures associated with traditional armaments and military units such as armor, infantry, artillery, battle ships and bombers, etc. The second type incorporates new innovations in military technology and tactics that are based on the intensive use of information technology (IT) intelligence, high precision weaponry, command and control systems, etc, associated with what has in the last decade been termed the Revolution in Military Affairs (RMA). The choice of how much to devote to each type of expenditure brings with it different requirements in terms of recruiting and training. The first requires large numbers of soldiers that receive traditional military training. The second is associated with smaller sized units, but more highly trained. We refer to these expenditures as *traditional* and *RMA*, and denote them by χ_1 and χ_2, respectively.

Given the high degree of uncertainty in planning for conflict, we will make the reasonable assumption that policy makers do not know the true damage probability density function (pdf). Risks are characterized by a pdf over a range of possible damage, up to a maximum potential level which we denote by z. Given the long planning horizon necessary to prepare for armed conflict, we assume that for policymakers the relevant unit of time is a decade. Their best (but highly uncertain) estimate of the damage pdf conditional on being attacked and the chosen allocation of defense expenditure χ_1 and χ_2 is:

$$\widetilde{p}(\psi|\chi) = \frac{\psi^{a-1}(z-\psi)^{b-1}z^{1-a-b}\Gamma(a+b)}{\Gamma(a)\Gamma(b)} I_{(z-\psi)} \tag{9}$$

where the Γ function is given by $\Gamma(x) = \int_0^\infty t^{(x-1)}e^{-t}dt$, $I_{(z-\psi)}$ is the indicator function and:

$$a(\chi) \quad = 1 + \tfrac{\chi_1}{\chi_2} + \theta e^{\theta\chi_1} \ln(\chi_1+\chi_2)\,\chi_1 \tag{10}$$

$$b(\chi) \quad = 2 + \tfrac{\chi_1}{\chi_2} + e^{\chi_2} \ln(\chi_1+\chi_2)\,\chi_2 \tag{11}$$

The mean of this pdf is:

$$\mathrm{E}(\psi|\chi) = z\frac{\Gamma(1+a(\chi))\Gamma(a(\chi)+b(\chi))}{\Gamma(a(\chi))\Gamma(1+a(\chi)+b(\chi))} \tag{12}$$

This functional form for the best estimate of the damage density function reflects all available knowledge about possible damages from threats to national security, and how these risks are affected by both types of security expenditures. In particular, this functional form embeds the following underlying assumptions:

1. Mean damage generally declines as total security expenditure increases. That is, we expect behavior along the lines of:

$$\frac{\partial \mathrm{E}(\psi)}{\partial(\chi_1+\chi_2)} < 0 \tag{13}$$

2. Mean damage and extreme damage respond in opposite directions to increases in RMA-type expenditure, holding total security expenditure fixed. That is, we expect to generally observe:

$$\left(\left.\frac{\partial \mathrm{E}(\psi)}{\partial \chi_2}\right|_{\chi_1+\chi_2} \right) \times \left(\left.\frac{\partial \mathrm{Prob}(\psi > 0.4z)}{\partial \chi_2}\right|_{\chi_1+\chi_2} \right) < 0 \tag{14}$$

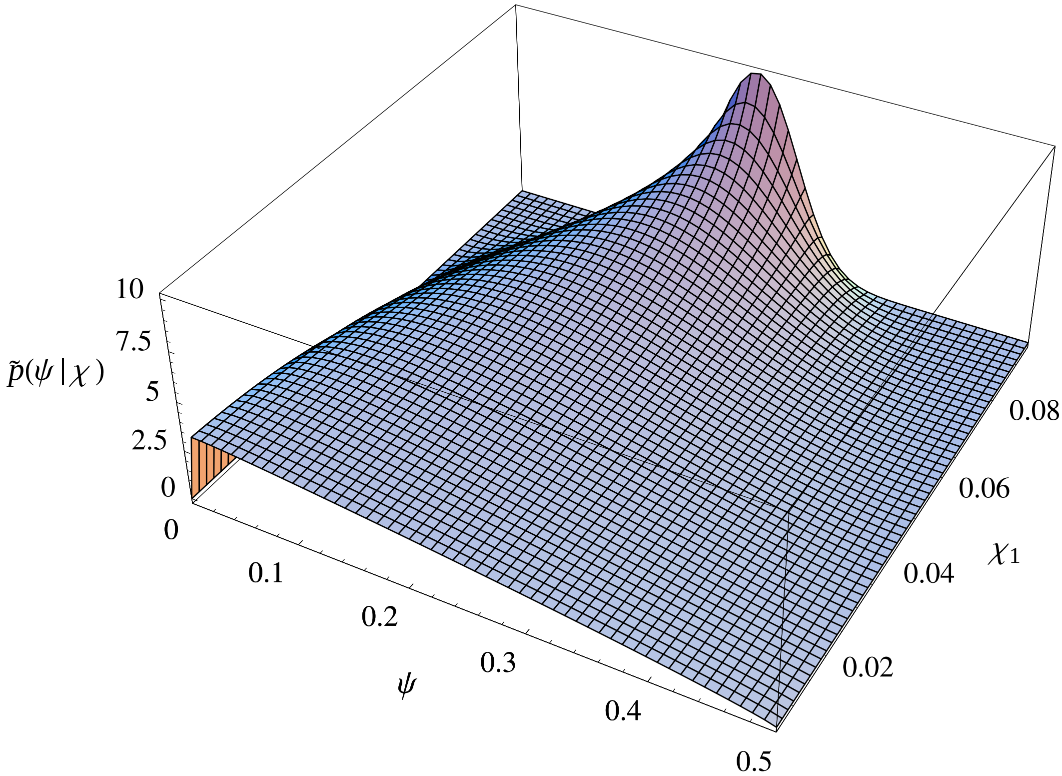

Figure 1: The probability density of damage, conditional on defense expenditure $\widetilde{p}(\psi|\chi) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\psi^{a-1}(z-\psi)^{b-1}$ for different values of χ_1, where total defense expenditure is 10%, $\chi_1 + \chi_2 = .1$, and $z = 1/2$, $\theta = 3$.

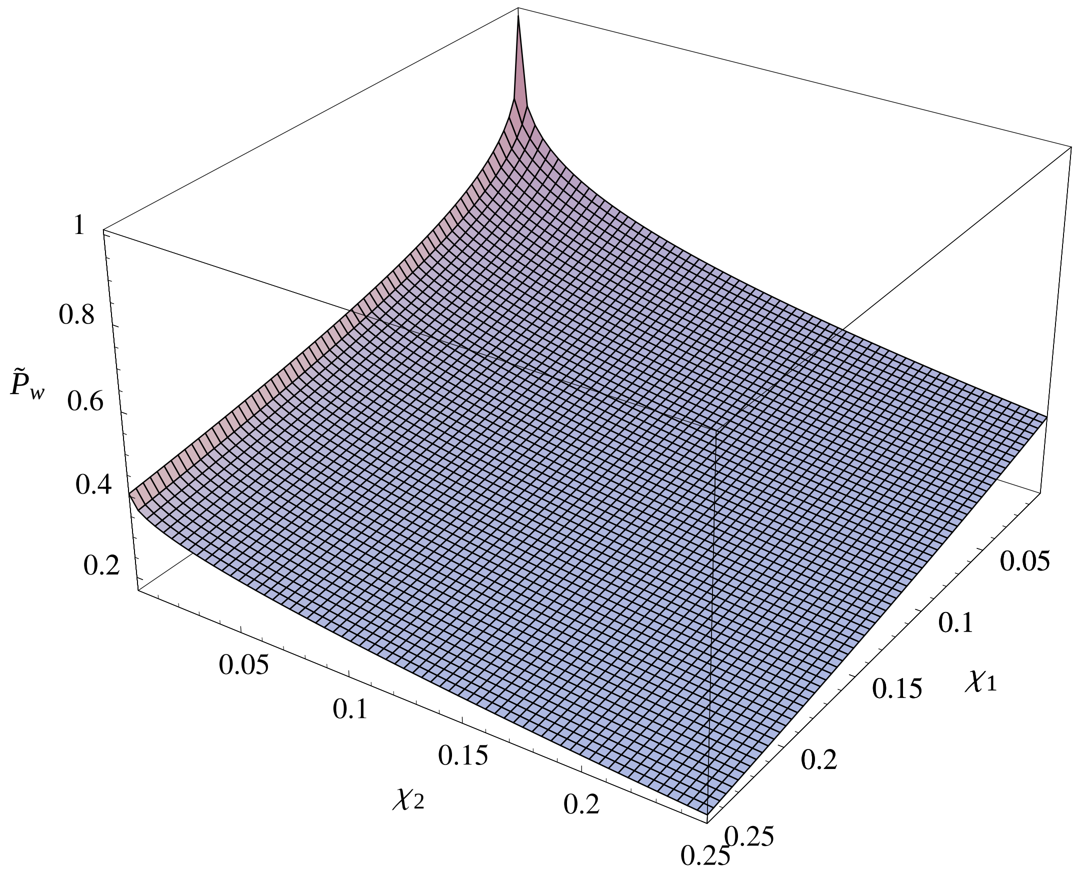

Figure 2: The probability distribution of being attacked $\widetilde{P}_{\rm w}(\chi)$ as in eq. 15, for different values of χ_1 and χ_2, where $z = 1/2$ and $\theta = 3$.

In this example we set $\theta = 3$. In Figure 1 we present the density function $\widetilde{p}(\psi|\chi)$ where the maximum damage $z = 1/2$, holding total defense expenditure constant at ten percent of GDP, and varying the value of the traditional military expenditure χ_1.

The probability of suffering an attack over the course of a decade is also subject to uncertainty, and is also a function of the overall size as well as distribution of defense expenditure. We denote the probability of attack by $P_{\rm w}$ and the best (but highly uncertain) estimate of this function is given by:

$$\widetilde{P}_{\rm w}(\chi) = 1 - \left(\frac{\beta_{\frac{1}{2}}(a(\chi), b(\chi))\Gamma(a(\chi) + b(\chi))(\chi_1 + \chi_2)}{\Gamma(a(\chi))\Gamma(b(\chi))} \right)^{\frac{1}{4}} \tag{15}$$

where $\beta_x(a, b) \equiv \int_0^x t^{a-1}(1-t)^{b-1}dt$ and $a(\chi)$ and $b(\chi)$ are defined in eq. 10 and 11.

The term $\chi_1 + \chi_2$ is the total military expenditure, and reflects its deterrent value. The term $[\beta_{1/2}(a, b)\Gamma(a + b)]/[\Gamma(a)\Gamma(b)]$ is the probability of an enemy attack successfully inflicting at most half of the maximal potential damage. The higher this number is, the lower the likelihood that an adversary will be tempted to launch an attack. The salient feature that we illustrate in Figure 2 is that $\widetilde{P}_{\rm w}(\chi)$ is a decreasing function of both types of expenditure.

3.3 Info-Gap Model of Uncertainty

The density function $\widetilde{p}(\psi|\chi)$ in eq.(9) is the best estimate of the pdf of damage of an attack given security allocations χ. However, this estimate is based on fragmentary and controversial evidence and generally contains serious but unidentifiable errors. The same is to be said for the estimated probability of attack, $\widetilde{P}_{\rm w}(\chi)$. The true values deviate from these estimates by unknown amounts.

We will use a fractional error info-gap model to represent the info-gaps in both the pdf of the damage and the probability of attack (Ben-Haim, 2006). Let $\mathcal{P}$ denote the set of all pdfs on [0, 1]. Our info-gap model is the following unbounded family of sets of pdfs $p(\psi)$ and probabilities $P_{\rm w}$:

$$\begin{aligned}\mathcal{F}(\alpha, \widetilde{p}, \widetilde{P}_{\rm w}) = \Big\{ \; p(\psi),\; P_{\rm w}: \quad & p(\psi) \in \mathcal{P},\; |p(\psi) - \widetilde{p}(\psi|\chi)| \le \alpha \widetilde{p}(\psi|\chi),\; \text{for all } \psi \\ & 0 \le P_{\rm w} \le 1,\; |P_{\rm w} - \widetilde{P}_{\rm w}(\chi)| \le \alpha \widetilde{P}_{\rm w}(\chi) \; \Big\}, \quad \alpha \ge 0 \end{aligned} \tag{16}$$

$\mathcal{F}(\alpha, \widetilde{p}, \widetilde{P}_{\rm w})$ includes every pdf $p(\psi)$ that deviates proportionally from the estimated density $\widetilde{p}(\psi|\chi)$, at any level of damage ψ, by no more than α. Similarly, $\mathcal{F}(\alpha, \widetilde{p}, \widetilde{P}_{\rm w})$ contains all attack-probabilities $P_{\rm w}$ which differ proportionally from the estimated value $\widetilde{P}_{\rm w}(\chi)$ by no more than α. The value of the fractional error, α, is unknown. Hence the info-gap model is not a single set, but rather an unbounded family of nested sets of possible pdfs and probabilities. Since α is unbounded there is no worst case.

3.4 Robustness Function

The expected utility if an attack occurs, based on the estimated pdf of damage, is:

$$\widetilde{r}(\chi) = \int_0^1 u\left[(1-\psi)\chi_{\rm c}\right] \widetilde{p}(\psi|\chi)\, {\rm d}\psi \tag{17}$$

Recall that the utility if an attack does not occur is $u_{\rm c}$, equal to $u(\chi_{\rm c})$. We assume that this is greater than the estimated expected utility in the case of attack:

$$\widetilde{r}(\chi) < u_{\rm c} \tag{18}$$

The total estimated expected utility, based on the estimated pdf of damage $\widetilde{p}(\psi|\chi)$ and the estimated probability of attack $\widetilde{P}_{\rm w}(\chi)$, is:

$$\widetilde{R}(\chi) = \widetilde{P}_{\rm w}(\chi)\widetilde{r} + (1 - \widetilde{P}_{\rm w}(\chi))u_{\rm c} \tag{19}$$

The expected utility for arbitrary $p(\psi|\chi)$ and $P_{\rm w}$ is specified in eq. (1), and the robustness function is defined in eq. (5). The robustness function, for values of robustness up to $\widehat{\alpha} = 1$ and based on the info-gap model of eq.(16) and on assumption (18), is derived in the appendix, section 5. Suppressing the notational dependence on χ, the result is:

$$\widehat{\alpha}(\chi, R_{\mathrm{c}}) = \begin{cases} \dfrac{(\widetilde{r} - u_{\mathrm{c}} - \delta_r)\widetilde{P}_{\mathrm{w}} + \sqrt{(\widetilde{r} - u_{\mathrm{c}} - \delta_r)^2 \widetilde{P}_{\mathrm{w}}^2 + 4\delta_r \widetilde{P}_{\mathrm{w}}(\widetilde{R} - R_{\mathrm{c}})}}{2\delta_r \widetilde{P}_{\mathrm{w}}} & \text{if } \widetilde{R} \geq R_{\mathrm{c}} \\ 0 & \text{else} \end{cases} \tag{20}$$

where:

$$\widetilde{r}_1 = \int_0^{\psi_{\mathrm{m}}} u\left[(1-\psi)\chi_{\mathrm{c}}\right] \widetilde{p}(\psi|\chi)\, \mathrm{d}\psi \tag{21}$$

$$\widetilde{r}_2 = \int_{\psi_{\mathrm{m}}}^{1} u\left[(1-\psi)\chi_{\mathrm{c}}\right] \widetilde{p}(\psi|\chi)\, \mathrm{d}\psi \tag{22}$$

$$\delta_r = \widetilde{r}_1 - \widetilde{r}_2 \tag{23}$$

and ψ_{m} is the median of $\widetilde{p}(\psi|\chi)$. Because we assume positive marginal utility we see that $\widetilde{r}_1 > \widetilde{r}_2$. Hence $\delta_r > 0$. This, together with eq.(18), implies that $\widetilde{r} - u_{\mathrm{c}} - \delta_r < 0$.

3.5 Numerical Results

We adopt the constant-risk-aversion utility function $u(c) = c^{1-\gamma}/(1-\gamma)$, where $\gamma > 0$. The nominal pdf of damage is defined in eqs.(9)–(11), and the probability of attack is defined in (15). The value of $\widetilde{R}$ in (19) is expressed in terms of utility. In our economy the representative individual has a maximum of a single unit of consumption, from which defense expenditure and damage are deducted. Therefore in our simulations we convert the values of $\widetilde{R}$ in (19) which are in units of utility into their consumption equivalents.

Figure 3 contains two examples corresponding to maximum potential damages $z = 1/2$ and $z = 1$ for different values of χ_1 and χ_2 in the grid $[0, 40\%] \times [0, 40\%]$. Frames are shown with robustness equal to 0, 1, 2 and 3, corresponding to immunity to 0, 100%, 200% and 300% error in the probability models. For the frame with robustness $\widehat{\alpha} = n$, each contour is indexed by an R_c value and consists of all defense expenditures (χ_1, χ_2) at which $\widehat{\alpha}(\chi, R_c) = n$.

Consider the four panels on the right-hand side of Figure 3 that correspond to maximum possible damage of 100%, $z = 1$. Recall from eq. 7 that a policy maker who wishes to maximize the expected utility must accept zero robustness to uncertainty. This is the case where an attack has the potential to drive consumption all the way to zero. Looking at the upper right-hand panel of Figure 3, we find that if policy makers maximize expected utility under $\widetilde{p}(\psi|\chi)$ and $\widetilde{P}_{\mathrm{w}}(\chi)$, the greatest level at which the expected utility can be satisficed is obtained if they choose to allocate 7.4% of GDP to defense and set $\chi_1 = .027$ and $\chi_2 = .047$. As illustrated in Table 2, under expected utility maximization the policymakers are willing to tolerate a high probability of being attacked ($\widetilde{P}_{\mathrm{w}}(\chi)$=.517), a relatively high level of expected damage if the attack occurs (E($\psi \mid \chi$)=.355), and a fairly large probability of suffering losses of over 40% of GDP, again if the attack occurs (Pr$\{\psi \geq .4z\}$=.392). On the other hand, if a policymaker is unsure of the reliability of their probabilistic estimates and require a robustness level of 100% ($\widehat{\alpha} = 1$), the optimal amount of defense expenditure increases to 28.8% of consumption, allocated such that $\chi_1 = .188$ and $\chi_2 = .1$. The probabilities of being

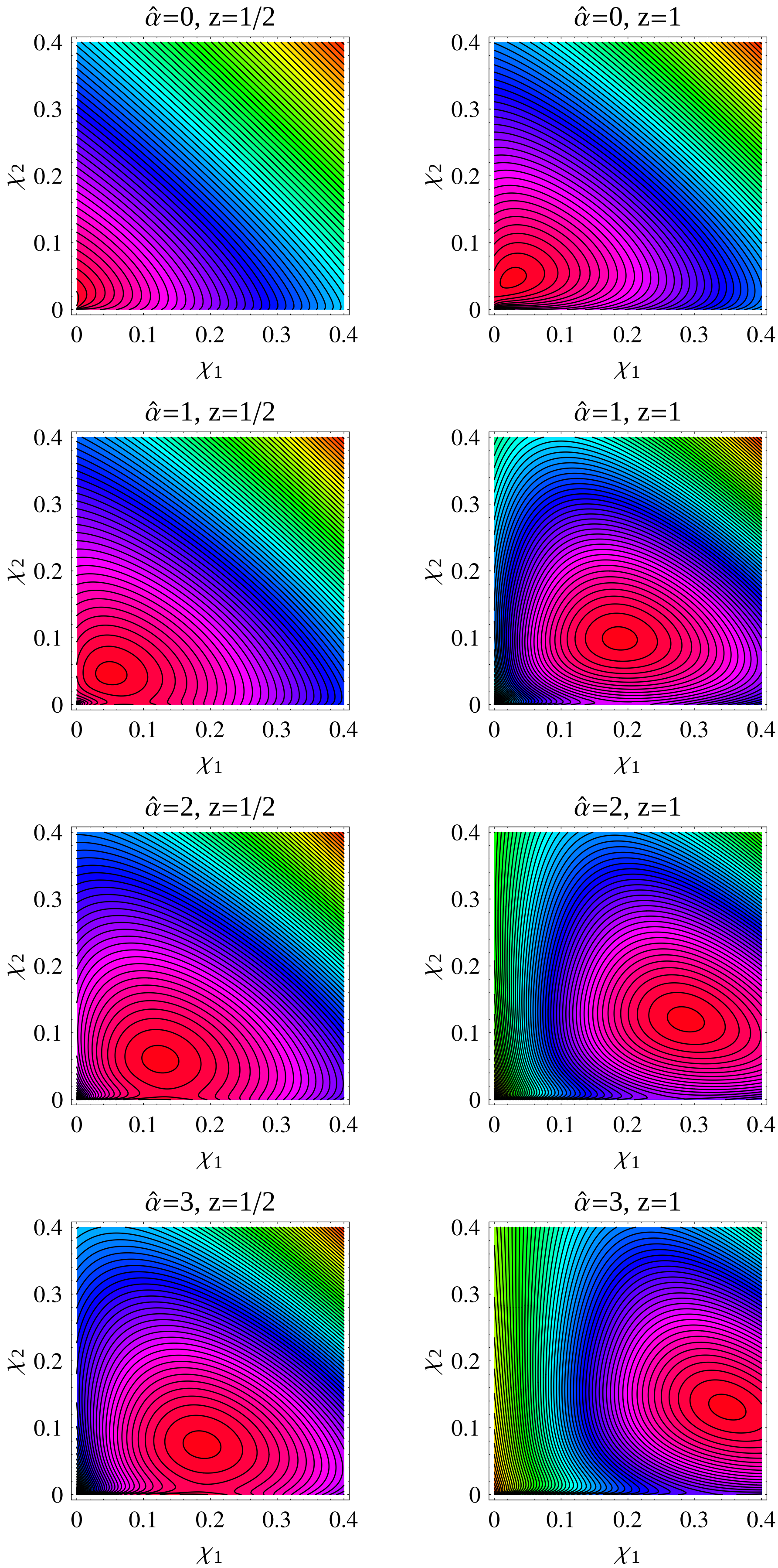

Figure 3: Contour plots for the value of $\widetilde{R}$ for different values of $\widehat{\alpha}$ and z

	χ_1	χ_2	$\tilde{P}_W$	$\mathrm{E}(\psi \mid \chi)$	$\Pr\{\psi \geq .4z\}$	c_E
			$z = 1/2$			
$\widehat{\alpha} = 0$	0	0.015	0.671	0.172	0.378	0.861
$\widehat{\alpha} = 1$	0.047	0.051	0.480	0.180	0.397	0.670
$\widehat{\alpha} = 2$	0.125	0.061	0.382	0.176	0.373	0.558
$\widehat{\alpha} = 3$	0.188	0.075	0.318	0.166	0.329	0.482
			$z = 1$			
$\widehat{\alpha} = 0$	0.027	0.047	0.517	0.355	0.392	0.710
$\widehat{\alpha} = 1$	0.188	0.100	0.299	0.305	0.289	0.486
$\widehat{\alpha} = 2$	0.287	0.121	0.223	0.267	0.215	0.383
$\widehat{\alpha} = 3$	0.349	0.132	0.185	0.244	0.175	0.328

Table 2: Values of χ_1, χ_2, $\tilde{P}_W$, $\mathrm{E}(\psi \mid \chi)$, $\Pr\{\psi \geq .4z\}$ and c_E at the highest levels at which expected utility can be satisficed with robustness equal to 0, 1, 2 and 3, and z=1/2 and 1.

attacked, of expected damage and the probability of suffering losses above 40% of GDP drop at these higher levels of expenditure.

Comparing the two different levels of defense expenditures and their allocations, we find that by demanding robustness against the unreliability of the probabilistic estimates of both damage and attack, a policymaker will substantially raise the amount of resources devoted to security. Furthermore rather than increasing the allocations to each by either the same amount or increasing them by the same proportions, we find that the allocation is now skewed in favor of the more traditional type of expenditures, χ_1. This property is even more emphasized when the required robustness increases to 200%, where the total amount of defense expenditure is 40.8% and $\chi_1 = .287$ and $\chi_2 = .121$, or 300% where the total amount of defense expenditure is 48.1% and $\chi_1 = .349$ and $\chi_2 = .132$.

This example implies that to achieve even 100% robustness requires devoting a large fraction of the economy's resources to defense. However this example was calculated with the maximum damage set to $z = 1$—so the threat involves some positive probability of complete annihilation.

When the maximum threat is set to $z = 1/2$ expected utility is maximized with a meager

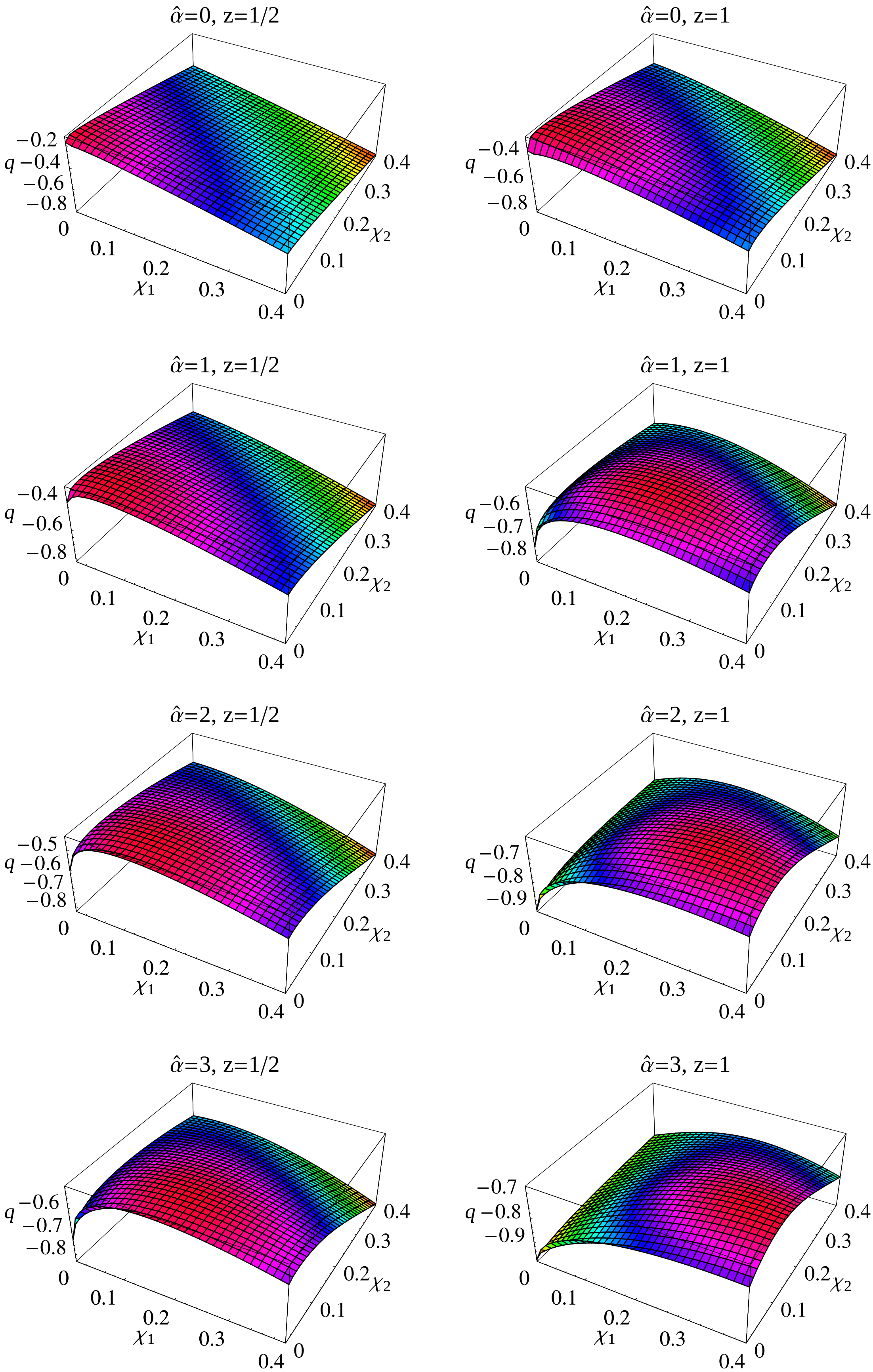

Figure 4: Three Dimensional plots for the value of $\widetilde{R}$ for different values of $\widehat{\alpha}$ and z

expenditure on defense of 1.5%, all of which is allocated to RMA-type expenditure. Forgoing robustness implies that expenditure on the traditional military is completely abandoned. When policymakers demand robustness at a level of 100%, the allocation between traditional and RMA-type expenditures is 4.7% and 5.1% respectively. If they require a robustness level of 200%, the total expenditure rises to 18.6% and traditional expenditures, χ_1, more than double to 12.5%. At 300% robustness, total expenditure is 26.3% more than two-thirds of which ($\chi_1 = 0.188$) is devoted to the traditional military.

In the last column in Table 2, we present the values of c_E, which reflect different levels of welfare associated with each combination of required robustness $\widehat{\alpha}$ and the defense allocation χ_1 and χ_2. First, we define $R_E(n)$ as the greatest value at which the expected utility can be satisficed with robustness equal to n. That is:

$$R_E(n) = \max_{\chi}\{R_c : \widehat{\alpha}(\chi, R_c) = n\} \tag{24}$$

The value of c_E is the consumption-equivalent to $R_E(n)$, defined through the utility function as: $R_E(n) = u(c_E)$. Figure 3 and its three dimensional companion Figure 4 represent the value of $q = c_E - 1$, the deviation in welfare (calculated in terms of consumption) from the utopian outcome associated with no war and no defense expenditure. The highest points in Figure 4 are attained by the values of χ_1 and χ_2 in Table 2.

The values of c_E in Table 2 demonstrate that achieving robustness comes with a cost. For example, consider the case where the maximum level of damage equals 1/2 and $\alpha = 0$. That is, the policy maker is willing put his complete trust in the estimated damage probability distribution and the probability of war, and how these depend on the level and allocation of defense expenditures. In this case as mentioned above, the policy maker chooses to devote only 1.5% of GDP to defense, all of which allocated to RMA-type expenditures. This generates a value of c_E=.861—hence the combined loss in welfare associated with the defense expenditure, the random effects of these expenditures on the probability of war, and its accompanying damage is worth 14% of GDP. If on the other hand the policy maker has less faith in the probability distribution and chooses to insulate himself against deviations of up to 100% from both the estimated distributions the policy he will adopt will be to set defense expenditures at 9.8% of GDP nearly equally divided between traditional and RMA-type expenditures. Achieving this level of robustness lowers the value of c_E to 0.67, and additional 19% of GDP, compared to the defense allocation associated with zero robustness.

4 Conclusion

The application of the information gap approach to the question of defense spending yields three main conclusions. First, the higher the robustness policy makers demand, the higher should be the overall level of defense expenditures. Second, the higher the robustness demanded, the more policy makers should favor those defense measures that are most likely to prevent extreme high levels of damage.

Beyond these normative conclusions the model also provides positive predictions. In a world with unreliable probabilistic information, we expect policy makers to favor higher expenditure on defense than would be appropriate if the only goal were expected utility maximization. Furthermore, it would seem that policy makers will favor expenditures on weapons systems and associated tactics and strategies that are both most effective in preventing worst case scenarios and also are better understood. These would suggest one possible *rationale* for military planners' tendency for conservatism.

5 Appendix: Derivation of the Robustness Function

In this appendix we derive the robustness function in eq.(20) based on assumption (18) and for values of the robustness not in excess of unity: $\widehat{\alpha} \le 1$. We make no assumptions about the utility function $u(c)$ other than that the marginal utility is positive: $\dot{u}(c) > 0$.

The robustness is defined in eq.(5). The main task is to find the pdf of the damage, $p(\psi|\chi)$, which, at horizon of uncertainty α, minimizes the expected utility $R(\chi|p, P_{\mathrm{w}})$ defined in eq.(1).

Because the marginal utility is positive it is evident that $R(\chi|p, P_{\mathrm{w}})$ is minimized by that pdf in $\mathcal{U}(\alpha, \widetilde{p}, \widetilde{P}_{\mathrm{w}})$ which puts as much weight as possible at large damage and as little weight as possible at low damage. For the fractional-error info-gap model in eq.(16) one readily shows that $\min_{\alpha} R(\chi|p, P_{\mathrm{w}})$ occurs with the following pdf:

$$p(\psi|\chi) = \begin{cases} (1-\alpha)\widetilde{p}(\psi|\chi) & \text{if } \psi \le \psi_{\mathrm{m}} \\ (1+\alpha)\widetilde{p}(\psi|\chi) & \text{else} \end{cases} \tag{25}$$

where ψ_{m} is the median of the estimated pdf $\widetilde{p}(\psi|\chi)$ and where $\alpha \le 1$.

If $\alpha > 1$ then $\min_{\alpha} R(\chi|p, P_{\mathrm{w}})$ occurs with the following pdf:

$$p(\psi|\chi) = \begin{cases} 0 & \text{if } \psi \le \psi_{\mathrm{s}} \\ (1+\alpha)\widetilde{p}(\psi|\chi) & \text{else} \end{cases} \tag{26}$$

where ψ_{s} satisfies:

$$(1+\alpha)\int_{\psi_{\mathrm{s}}}^{1} \widetilde{p}(\psi|\chi)\,\mathrm{d}\psi = 1 \tag{27}$$

In other words, ψ_{s} is the $1 - 1/(1+\alpha)$ quantile of $\widetilde{p}(\psi|\chi)$.

We will consider only the case $\alpha \le 1$. The derivation of robustness in excess of unity is analogous.

The utility $R(\chi|pP_{\mathrm{w}})$ in eq.(1), evaluated with the pdf in eq.(25), is:

$$R(\chi|pP_{\mathrm{w}}) = [\widetilde{r} - \delta_r\alpha - u_{\mathrm{c}}]\, P_{\mathrm{w}} + u_{\mathrm{c}} \tag{28}$$

where $\widetilde{r}$ and δ_r are defined in eqs.(17) and (23) and u_{c} is the utility if an attack does not occur, $u(\chi_{\mathrm{c}})$. The term $\widetilde{r} - \delta_r\alpha - u_{\mathrm{c}}$ is negative so the minimizing value of P_{w} in $\mathcal{U}(\alpha, \widetilde{p}, \widetilde{P}_{\mathrm{w}})$ is $(1+\alpha)\widetilde{P}_{\mathrm{w}}$. Thus the minimum expected utility, up to horizon of uncertainty α, is:

$$\min_{p, P_{\mathrm{W}} \in \mathcal{U}(\alpha, \widetilde{p}, \widetilde{P}_{\mathrm{w}})} R(\chi|p, P_{\mathrm{w}}) = [\widetilde{r} - \delta_r\alpha - u_{\mathrm{c}}]\,(1+\alpha)\widetilde{P}_{\mathrm{w}} + u_{\mathrm{c}} \tag{29}$$

Denote this minimum $\mu(\alpha)$, which decreases monotonically as α increases because $\delta_r > 0$. The robustness is, according to the definition in eq.(5), the greatest value of α up to which $\mu(\alpha)$ is no less than R_c. That is, the robustness is the lowest non-negative solution for $\widehat{\alpha}$ of:

$$\mu(\widehat{\alpha}) = R_c \tag{30}$$

This is a quadratic equation in α whose least non-negative root is eq.(20).

The derivation of the robustness function in eq.(30) is obtained by equating eq.(28) to R_c and solving for α, with $\widetilde{P}_w(\chi)$ instead of P_w.

6 References

Ben-Haim, Yakov, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press, London.

Ben-Haim, Yakov, 2005, Info-gap Decision Theory For Engineering Design. Or: Why 'Good' is Preferable to 'Best', in E. Nikolaidis, D. Ghiocel and S. Singhal, eds., *Engineering Design Reliability Handbook,* CRC Press, Boca Raton.

Bergson, Abram, 1961, *The Real National Income of Soviet Russia Since 1928,* Harvard University Press, Cambridge, MA.

Broadberry, Stephen and Peter Howlett, 1998, The United Kingdom: 'Victory at all Costs', in M. Harrison, ed., *The Economics of World War II,* Cambridge University Press, Cambridge, UK.

Eckstein, Zvi and Daniel Tsiddon, "Macroeconomic Consequences of Terror: Theory and the Case of Israel," *Journal of Monetary Economics*, 2004; 51: 971-1002.

Harrison, Mark, 1985, *Soviet Planning in Peace and War,* Cambridge University Press, Cambridge, UK.

Hess, Gregory D., "The Economic Welfare Cost of Conflict: An Empirical Assessment", 2003, Working Paper, Clarement Colleges.

Mitchell, Brian R., 1980, *European Historical Statistics, 1750-1975,* Macmillian Press Ltd., London.

Nordhaus, William D., 2002, The Economic Consequences of a War with Iraq, in C. Kaysen, S.E. Miller, M.B. Miller, W.D. Nordhaus and J.D. Steinbruner, eds., *War with Iraq: Costs, Consequences, and Alternatives,* American Academy of Arts and Sciencs, Cambridge, MA.

Persitz, Dotan, "Counterfactual Analysis of Terrorism Effect on the Economy: the Case of Israel", 2005, Working Paper, Tel-Aviv University.

Rockoff, Hugh, 1998, The United States: From Ploughshares to Swords, in M. Harrison, ed., *The Economics of World War II,* Cambridge University Press, Cambridge, UK.

Rohlfs, Chris, "How Much Did the U.S. Government Value its Troops' Lives in World War II?: Evidence from Dollar-Fatality Tradeoffs in Land Battles", 2005, Working Paper, University of Chicago.