Appendix A: Proofs

PROOF OF LEMMA 1:

To derive our expression, we solve for how the location of the indifferent consumer between firms 1 and 2, which is equal to firm 1's demand on the interval $[0, 1/n]$, changes with p_1 for a given p_2. Suppose a consumer's realized taste is $\chi \in [0, 1/n]$. The consumer's utility from buying good 1 at price p_1 is then

$$(9) \qquad u_1 = v - \chi t - p_1 - \lambda \int_0^{p_1} (p_1 - p)\, dF(p) + \int_{p_1}^{\infty} (p - p_1)\, dF(p)$$
$$- \lambda t \int_0^{\chi} (\chi - s)\, dG(s) + t \int_{\chi}^{1/n} (s - \chi)\, dG(s).$$

Replacing p_1 with p_2 and χ with $1/n - \chi$ in Equation (9), we get utility from buying good 2 at price p_2:

$$(10) \qquad u_2 = v - ((1/n) - \chi)t - p_2 - \lambda \int_0^{p_2} (p_2 - p)\, dF(p) + \int_{p_2}^{\infty} (p - p_2)\, dF(p)$$
$$- \lambda t \int_0^{1/n - \chi} ((1/n - \chi) - s)\, dG(s) + t \int_{1/n - \chi}^{1/n} (s - (1/n - \chi))\, dG(s).$$

Equations (9) and (10) are differentiable with respect to χ, and right and left differentiable with respect to p_1. Using this together with the fact that $u_1 = u_2$ for the indifferent consumer x^+ implies that

$$(11) \quad \left(\frac{dx^+}{dp_1}\right)_{\downarrow} = - \frac{(\partial u_1/\partial p_1)_{\downarrow} - (\partial u_2/\partial p_1)_{\downarrow}}{\partial u_1/\partial \chi - \partial u_2/\partial \chi} = - \frac{1}{2t}\left[\frac{2 + (\lambda - 1)\ F(p_1)}{2 + ((\lambda - 1)/2)[G(x^+) + G(1/n - x^+)]}\right],$$

and that $(dx^*/dp_1)_{\uparrow}$ is given by the expression in which $F_{\uparrow}(p_1)$ replaces $F(p_1)$ above. Similar calculations give the responsiveness of demand on the other side of the firm.

PROOF OF PROPOSITION 1:

If the condition in the proposition is satisfied, then there is a p^* satisfying $p^* - (t/n)((1 + \lambda)/2) \leq c \leq p^* - t/n$ for all $c \in [\underline{c}, \overline{c}]$. That this is a necessary and sufficient condition for local deviations to be unprofitable has been established in the text.

We now show that under the above condition, non-local deviations are also unprofitable. We start with increases in the price. First, note that the firm will never charge a price so high that it would be charging itself out of one market: if a deviating firm is charging itself out of one market, it is charging itself out of both, earning zero profits. Therefore, we only need to consider deviations for which $x \in (0, 1/2n)$. Recall Equation 11:

$$\frac{dx}{dp_1} = - \frac{1}{2t}\left[\frac{2 + (\lambda - 1)F(p_1)}{2 + ((\lambda - 1)/2)[G(x) + G((1/n) - x)]}\right].$$

Since $F(p_1) = G(1/n - x) = 1$ and $G(x)$ is increasing in x, in the range $x \in (0, 1/2n)$, firm 1's demand (as a function of p_1) is concave. This implies that if local deviations are unprofitable, non-local increases in the price are also unprofitable.

Next, we rule out the possibility that firm 1 might like to charge a price so that $x \in [1/2n, 1/n]$. In that case, Equations (9) and (10) imply that

$$-xt - p_1 + (p^* - p_1) - \lambda t\left(x - \frac{1}{4n}\right)$$

$$= -\left(\frac{1}{n} - x\right)t - p^* - \lambda t\, 2n\left(\frac{1}{n} - x\right)\frac{1/n - x}{2} + t\, 2n\left(x - \frac{1}{2n}\right)\frac{x - 1/2n}{2}.$$

Solving for p_1 gives

$$p_1 = p^* - \frac{1}{2}t\underbrace{\left[(\lambda + 1)\left(2x - \frac{1}{n}\right) + (\lambda - 1)\left(x - \frac{1}{4n} - nx^2\right)\right]}_{\equiv \kappa}.$$

To show that lowering the price to p_1 is not a profitable deviation, it is equivalent to show that

$$\frac{1}{n}(p^* - c) \geq 2x(p_1 - c) = 2x\left(p^* - c - \frac{1}{2}t\kappa\right).$$

Rearranging and using that $p^* - c \leq t(1 + \lambda)/2n$ gives that it is sufficient to show that

$$\left(2x - \frac{1}{n}\right)\frac{1 + \lambda}{n} \leq 2x\kappa, \tag{12}$$

or equivalently

$$(\lambda + 1)\left(2x - \frac{1}{n}\right)^2 \geq (\lambda - 1)\, 2x\left(nx^2 + \frac{1}{4n} - x\right) = (\lambda - 1)\; 2x\left(2x - \frac{1}{n}\right)\left(\frac{nx}{2} - \frac{1}{4}\right).$$

This simplifies to

$$(\lambda + 1)\left(2x - \frac{1}{n}\right) \geq (\lambda - 1)\; 2x\left(\frac{nx}{2} - \frac{1}{4}\right).$$

Notice that in the above inequality, the left-hand side is equal to the right-hand side for $x = 1/(2n)$ and greater for $x = 1/n$. Furthermore, the left-hand side is linear, while the right-hand side is quadratic and convex. This implies that the left-hand side is no less for all $1/2n \leq x \leq 1/n$.

For $n > 2$, we are left to rule out that firm 1 undercuts its rival and steals more than the entire adjacent market. We begin by ruling out deviations in which the firm captures less than two adjacent markets on each side. Let p_1' be the price at which the consumer located at $1/n$ is indifferent between buying from firm 1 and buying from firm 2. This consumers utility of buying from firm 1 is

$$v - \frac{1}{n}t - p_1' + (p^* - p_1') - \lambda t\left[\frac{1}{n} - \frac{1}{4n}\right].$$

In case she buys from firm 2, her utility is

$$v - p^* + t\frac{1}{4n}.$$

Thus, if the consumer is indifferent

$$p^* - p_1' = \frac{t}{2n}\left[2 + \frac{3}{4}(\lambda - 1)\right].$$

Consider the maximum price at which a local deviation is unprofitable; for this price $p^* - \underline{c} = (t/2n)[2 + \lambda - 1]$ and in this case $p_1' - \underline{c} = (t/2n)[(1/4)(\lambda - 1)]$. Thus even if firm 1 would get the entire two adjacent markets when setting p_1', this is unprofitable as $1/n\,(p^* - \underline{c}) > (4/n)(p_1 - \underline{c})$.[1] Obviously undercutting is (weakly) less profitable for any lower focal price or any higher level of marginal cost.

We are left to consider the case in which $n > 4$, and firm 1 steals more than two adjacent markets on each side. We show that this is unprofitable because it requires firm 1 to price below marginal cost. For the consumer located at $2/n$ to weakly prefer buying from firm 1 rather than firm 3, it must be that

$$v - \frac{2}{n}t - p_1 + (p^* - p_1) - \lambda t\left[\frac{2}{n} - \frac{1}{4n}\right] \geq v - p^* + t\frac{1}{4n}.$$

Hence, in this case $p^* - p_1 \geq (t/2n)[4 + (\lambda - 1)(7/4)] > (t/2n)[2 + \lambda - 1] \geq p^* - c$, which completes the proof.

PROOF OF PROPOSITION 2:

We have shown in the text that local deviations are unprofitable if and only if

$$p^* - \frac{t}{n}\frac{1 + \lambda}{2} \leq c_i \leq p^* - \frac{t}{n}$$

for all i and $c_i \in [\underline{c}_i, \overline{c}_i]$. It follows from the proof of Proposition 1 that if local deviations are unprofitable, so are global ones.

It remains to show the second part of the proposition. In the standard Salop model, for the consumer x between firms 1 and 2 who is indifferent between the two products,

$$x = \frac{t/n + p_2 - p_1}{2t}.$$

Hence, for realized cost c, firm 1's problem is

$$\max_{p_1} \frac{p_1 - c}{2t}\left(\frac{2t}{n} - 2p_1 + E[p_2 + p_n | c]\right).$$

[1] Clearly if $n = 3$, the firm cannot attract two adjacent markets on each side, as there are only three local markets. Nevertheless, the upper bound on profitability we use is still valid.

This implies that

$$P_1(c) = \frac{t}{2n} + \frac{E[p_2 + p_n | c]}{4} + \frac{c}{2}.$$

Suppose that the supremum of prices charged by firms 1, 2, and n are $\overline{p}_1$, $\overline{p}_2$, and $\overline{p}_n$, respectively. Suppose without loss of generality that $\overline{p}_1$ is the supremum of market-equilibrium prices of all firms. Then for any $c \in [\,\underline{c}_1, \overline{c}_1]$,

$$P_1(c) \leq \frac{t}{2n} + \frac{\overline{p}_2 + \overline{p}_n}{4} + \frac{c}{2}. \tag{13}$$

Taking the supremum of both sides implies

$$\overline{p}_1 \leq \frac{t}{2n} + \frac{\overline{p}_1 + \overline{p}_1}{4} + \frac{\overline{c}}{2}.$$

Rearranging gives the upper bound in the proposition.

Finally, we show that this upper bound can only be attained at $\overline{c}$. If no firm's price attains $\overline{p}_1$, we are done. Next, suppose that for a price $c < \overline{c}$, $P_1(c) = \overline{p}_1$. By Inequality (13), again we are done.

PROOF OF PROPOSITION 3:

Posit a candidate market equilibrium in which all firms set a deterministic price and in which the highest price p_H is strictly greater than the lowest price p_L. We prove that if the condition in the Proposition is satisfied, either (one of) the highest price firm(s) has a strict incentive to lower its price or (one of) the lowest price firm(s) has a strict incentive to raise its price, contradicting equilibrium.

We establish that the marginal profit of lowering the highest price is weakly greater than the marginal profit of raising the lowest price for all given cost realizations c. This is sufficient because it implies that the high-price firm has a strict incentive to lower its price when it has its lowest cost realization, or the low-price firm has a strict incentive to raise its price when it has its highest cost realization (which is higher than the high-price firm's lowest cost realization because the supports of the cost distributions overlap), contradicting equilibrium. Let x_H^+ and x_H^- be one of the highest cost firm's demands on its right and left, respectively. Define x_L^+ and x_L^- similarly. We want to establish that

$$(p_H - c)\Bigg[\underbrace{\frac{1}{2+((\lambda-1)/2)[G(x_H^+) + G(1/n - x_H^+)]}}_{\equiv 1/z_H^+} + \underbrace{\frac{1}{2+((\lambda-1)/2)[G(x_H^-) + G(1/n - x_H^-)]}}_{\equiv 1/z_H^-}\Bigg] \tag{14}$$
$$\times [2 + F_{\uparrow}(p_H)(\lambda - 1)] \geq (p_L - c)\left[\frac{1}{z_L^+} - \frac{1}{z_L^-}\right] \times [2 + F(p_L)(\lambda - 1)],$$

where z_L^+ and z_L^- are defined analogously to z_H^+ and z_H^-. For brevity, let $\eta_H \equiv [2 + F_{\uparrow}(p_H)(\lambda - 1)]$ and let $\eta_L \equiv [2 + F(p_L)(\lambda - 1)]$.

Notice that either $(z_L^+ (z_L^+/z_H^+) + z_L^- (z_L^-/z_H^-)) \leq ½ (z_L^+ + z_L^-) (1/z_H^+ + 1/z_H^-)$ or $(z_L^- (z_L^-/z_H^+) + z_L^+ (z_L^+/z_H^-)) \leq ½ (z_L^+ + z_L^-) (1/z_H^+ + 1/z_H^-)$. We distinguish two cases depending on whether the former (Case I) or the latter (Case II) holds.

Case I. We rewrite Equation 14 as

$$\eta_H \left(z_L^+ \frac{z_L^-}{z_H^+} + z_L^- \frac{z_L^+}{z_H^-} \right) \geq \left(1 - \frac{p_H - p_L}{p_H - c} \right) \eta_L (z_L^+ + z_L^-). \tag{15}$$

Equation 15 is equivalent to

$$\eta_H \left(z_L^+ \left(1 - \frac{z_H^+ - z_L^-}{z_H^+} \right) + z_L^- \left(1 - \frac{z_H^- - z_L^+}{z_H^-} \right) \right) \geq \left(1 - \frac{p_H - p_L}{p_H - c} \right) \eta_L (z_L^+ + z_L^-).$$

As $\eta_H > \eta_L$ a sufficient condition for Equation (14) to hold is that

$$\eta_H \left(z_L^+ \frac{z_H^+ - z_L^-}{z_H^+} + z_L^- \frac{z_H^- - z_L^+}{z_H^-} \right) \leq \frac{p_H - p_L}{p_H - c} \eta_L (z_L^+ + z_L^-). \tag{16}$$

Using that

$$|z_H^+ - z_L^-| = \frac{\lambda - 1}{2} \left| [G(x_H^+) - G(x_L^-)] - \left[G\left(\frac{1}{n} - x_L^-\right) - G\left(\frac{1}{n} - x_H^+\right) \right] \right|,$$

that $g(\cdot)$ is bounded by $2n$, and that for all $p < p_H$

$$\left| \frac{dx}{dp} \right|_{\downarrow}, \quad \left| \frac{dx}{dp} \right|_{\uparrow} \leq \frac{1}{2t} \frac{2 + (\lambda - 1) F_{\uparrow}(p_H)}{2 + (\lambda - 1)/2},$$

we get that

$$|z_H^+ - z_L^-| \leq \frac{\lambda - 1}{2} 2n |x_H^+ - x_L^-| \leq \frac{\lambda - 1}{2} 2n (p_H - p_L) \left(\frac{2 + (\lambda - 1) F_{\uparrow}(p_H)}{2 + (\lambda - 1)/2} \frac{1}{2t} \right),$$

and by a similar logic $|z_H^- - z_L^+|$ has the same upper bound. Combining these with Equation 16 implies that it is sufficient to prove

$$\frac{1}{p_H - c} \eta_L (z_L^+ + z_L^-) \geq (\eta_H)^2 \frac{\lambda - 1}{2} 2n \frac{1}{2 + (\lambda - 1)/2} \frac{1}{2t} \left(z_L^+ \frac{1}{z_H^+} + z_L^- \frac{1}{z_H^-} \right).$$

Using that $(z_L^+ (1/z_H^+) + z_L^- (1/z_H^-)) \leq ½ (z_L^+ + z_L^-)(1/z_H^+ + 1/z_H^-)$ it is sufficient to prove

$$\frac{1}{p_H - c} \eta_L \geq (\eta_H)^2 \frac{(\lambda - 1)/2}{2 + (\lambda - 1)/2} \frac{n}{2t} \left(\frac{1}{z_H^+} + \frac{1}{z_H^-} \right). \tag{17}$$

Since the high-price firm's demand is always less than or equal $1/n$, the fact that it does not want to lower its price implies

$$1 \geq \frac{n}{2t}(p_H - c)\ \eta_H\left(\frac{1}{z_H^+} + \frac{1}{z_H^-}\right).$$

Hence, a sufficient condition for Equation 17 to hold is that

$$\eta_L \geq \eta_H \frac{(\lambda - 1)/2}{2 + (\lambda - 1)/2}.$$

For $n = 2$, $F(p_L) = F_{\uparrow}(p_H)$, so the above is satisfied for any $\lambda > 1$. For $n > 2$, using that $F(p_L) \geq 1/n$ and $\eta_H \leq 1 + \lambda$, a sufficient condition for the above inequality to hold is that

$$(4 + \lambda - 1)\ (2n + \lambda - 1) \geq n(2 + \lambda - 1)\ (\lambda - 1).$$

Setting $a = \lambda - 1$, this can be rewritten as

$$0 \geq (n - 1)a^2 - 4a - 8n.$$

Since this quadratic has one positive and one negative root, if a is positive and

$$a \leq \frac{2}{n - 1}\left(1 + \sqrt{1 + 2n(n - 1)}\right),$$

the inequality is satisfied. This gives the bound in the proposition.

Case II. In this case, we rewrite Equation 14 as

$$\eta_H\left(z_L^- \frac{z_L^+}{z_H^+} + z_L^+ \frac{z_L^-}{z_H^-}\right) \geq \left(1 - \frac{p_H - p_L}{p_H - c}\right)\eta_L\,(z_L^+ + z_L^-).$$

The remaining steps are analogous to Case I and thus omitted.

PROOF OF PROPOSITION 4:

We begin by proving that each firm's pricing function is continuous in cost. This fact follows from the following lemma.

LEMMA 3: *Consider any interior equilibrium with* $\lambda < 38$ *and any cost realization* c_i *of firm i. Consider furthermore the range of prices* $p_i \geq c_i$ *such that for all equilibrium price vectors* $\mathbf{p}_{-i}$, *the indifferent consumers* $x^+(p_i, \mathbf{p}_{-i})$ *and* $x^-(p_i, \mathbf{p}_{-i})$ *are located within distance* $(0, 1/n)$ *of firm i's ideal product. Over this range of prices, firm i's expected profits are single peaked.*

PROOF:

Since for all price vectors under consideration, there exists indifferent consumers within distance of $1/n$ of firm i, Equation 11 implies that profits are differentiable wherever the market price distribution does not have an atom—which is almost everywhere—and continuous. Furthermore, at prices where the profit function is not differentiable, demand has a concave kink, and hence (as long as $p_i \geq c_i$) so do profits.

Suppose by contradiction that the profit function is not single-peaked in the relevant price region. This implies that the profit function must have a trough. At this trough, it obviously cannot have a concave kink, so it is differentiable. To arrive at a contradiction, we prove that if firm 1's first-order condition is satisfied at some price p_1, profits are lower slightly to the right of p_1.

Let the subscript 1 denote partial derivative with respect to firm 1's price of x^+ and x^- respectively. Note that for each $x(p_1, \mathbf{p}_{-1}) \in \{x^-(p_1, \mathbf{p}_{-1}), x^+(p_1, \mathbf{p}_{-1})\}$ one has

(18)

$$\limsup_{p_1' \searrow p_1} \frac{1}{p_1' - p_1}(x_1(p_1', \mathbf{p}_{-1}) - x_1(p_1, \mathbf{p}_{-1}))$$

$$= -\frac{1}{2t}\frac{(\lambda - 1)F'(p_1)}{2 + ((\lambda-1)/2)\,(G(x(p_1, \mathbf{p}_{-1})) + G(1/n - x(p_1, \mathbf{p}_{-1})))} - \frac{1}{2t}$$

$$\times \frac{\left\{\begin{array}{l}((\lambda-1)/2)(2 + (\lambda - 1)\ F(p_1)) \\ \times \limsup\limits_{p_1' \searrow p_1} \dfrac{G(x(p_1, \mathbf{p}_{-1})) - G(x(p_1', \mathbf{p}_{-1})) + G(1/n - x(p_1, \mathbf{p}_{-1})) - G(1/n - x(p_1', \mathbf{p}_{-1}))}{p_1' - p_1}\end{array}\right\}}{[2 + ((\lambda-1)/2)(G(x(p_1, \mathbf{p}_{-1})) + G(1/n - x(p_1, \mathbf{p}_{-1})))]^2}$$

$$\leq (x_1(p_1, \mathbf{p}_{-1}))^2 \frac{((\lambda-1)/2)\,2n}{2 + ((\lambda-1)/2)(G(x(p_1, \mathbf{p}_{-1})) + G(1/n - x(p_1, \mathbf{p}_{-1})))}$$

$$\leq (x_1(p_1, \mathbf{p}_{-1}))^2 \frac{((\lambda-1)/2)\,2n}{2 + (\lambda-1)/2}.$$

Let $\pi(p) = (p - c)E[x^+(p, \mathbf{p}_{-1}) + x^-(p, \mathbf{p}_{-1})]$. We will prove that

$$\limsup_{p_1' \searrow p_1} \frac{\pi'(p_1') - \pi'(p_1)}{p_1' - p_1} < 0.$$

This is sufficient because it shows that the derivative of the profit function is negative to the right of and sufficiently close to p_1, so that profits are smaller there.

By Equation 18, it is sufficient to prove

(19)

$$(p_1 - c)\underbrace{E[(x_1^-(p_1, \mathbf{p}_{-1}))^2 + (x_1^+(p_1, \mathbf{p}_{-1}))^2]}_{I}\frac{((\lambda-1)/2)\,2n}{2 + (\lambda-1)/2} + \underbrace{2E[x_1^-(p_1, \mathbf{p}_{-1}) + x_1^+(p_1, \mathbf{p}_{-1})]}_{II} < 0.$$

To bound the above, we begin showing that I divided by the square of II is less than or equal to ½$(k+1)^2/4k$, where

$$k \equiv \frac{2 + \lambda - 1}{2 + (\lambda-1)/2}.$$

Since $|x_1(p_1, \mathbf{p}_{-1})| \geq (1/2t)[2 + (\lambda - 1)F(p)]/[2 + (\lambda - 1)]$ and $|x_1(p_1, \mathbf{p}_{-1})| \geq (1/2t)[2 + (\lambda - 1)F(p)]/[2 + (\lambda - 1)/2]$, one has

$$\frac{\max\{|x_1^-(p_1, \mathbf{p}_{-1})|, |x_1^+(p_1, \mathbf{p}_{-1})|\}}{\min\{|x_1^-(p_1, \mathbf{p}_{-1})|, |x_1^+(p_1, \mathbf{p}_{-1})|\}} \leq \frac{2 + \lambda - 1}{2 + (\lambda - 1)/2} = k.$$

Now we use the following fact.

Fact 1. *Suppose* $\tilde{a}_+$ *and* $\tilde{a}_-$ *are positive random variables such that*

$$\frac{\sup\{\tilde{a}_+, \tilde{a}_-\}}{\inf\{\tilde{a}_+, \tilde{a}_-\}} \leq k$$

Then

$$\frac{E[\tilde{a}_+^2 + \tilde{a}_-^2]}{E[\tilde{a}_+ + \tilde{a}_-]^2} \leq \frac{1}{2}\,\frac{(k+1)^2}{4k}. \tag{20}$$

PROOF:

Suppose without loss of generality that $1 \leq \tilde{a}_+, \tilde{a}_- \leq k$. Since the quadratic function is convex, the ratio on the left-hand side of Inequality 20 is maximized if the support of $\tilde{a}_+, \tilde{a}_-$ consists of the extremal values 1, k. Thus, the left-hand side of the inequality is less than or equal to

$$\max_{b^+, b^- \in [0,1]} \frac{b^+(k^2 - 1) + 1 + b^-(k^2 - 1) + 1}{(b^+(k - 1) + 1 + b^-(k - 1) + 1)^2},$$

which is equivalent to maximizing

$$\max_{b^+, b^-} \frac{1}{2}\left[\frac{[(b^+ + b^-)/2](k^2 - 1) + 1}{([(b^+ + b^-)/2](k - 1) + 1)^2}\right].$$

For brevity, let $b = (b^+ + b^-)/2$. Then the first-order condition for the above maximization is satisfied if and only if

$$(b(k - 1) + 1)^2 (k^2 - 1) - (b(k^2 - 1) + 1)\, 2\,(k - 1)(b(k - 1) + 1) = 0,$$

which yields $b = 1/(k + 1)$. Substituting this into the maximand and rewriting gives the desired inequality.

Hence, for Inequality 19 to hold it is sufficient that

$$(p_1 - c)\left|E[x_1^-(p_1, \mathbf{p}_{-1}) + x_1^+(p_1, \mathbf{p}_{-1})]\right| \frac{1}{2}\,\frac{(k+1)^2}{4k}\,\frac{((\lambda - 1)/2)\,2n}{2 + (\lambda - 1)/2} < 2.$$

Since the firm prices its neighbors out of the market with probability zero, its first-order condition implies

$$(p_1 - c)\left|E[x_1^-(p_1, \mathbf{p}_{-1}) + x_1^+(p_1, \mathbf{p}_{-1})]\right| \leq \frac{2}{n}.$$

In this case, the above condition simplifies to

$$\frac{(k+1)^2}{4k}\,\frac{(\lambda-1)/2}{2+(\lambda-1)/2} < 1.$$

This condition holds for any $\lambda < 38$.

Since in an interior equilibrium, prices are above marginal costs and there exists an indifferent consumer between any two products for any marginal cost realization, the above lemma implies the following corollary:

COROLLARY 5: *In an interior equilibrium with $\lambda < 38$, the pricing function is continuous in cost for each firm.*

We are now ready to prove the statement of the proposition. We prove by contradiction; suppose that there exists (at least one) firm that does not charge a deterministic price. Corollary 5 implies that there must exist a nontrivial interval of prices, each of which the firm charges for some cost. On this interval, consider a price p^0 and a sequence of prices $p^i \searrow p^0$ such that i.) F is differentiable at p^i, p^0; ii.) the pricing function $p(\cdot)$ is differentiable at p^0 with a strictly positive derivative.[2] Let the corresponding costs be c^0 and $c^i \searrow c^0$.

Taking the difference between the first order condition for p^i and p^0, dividing it by $p^i - p^0$, and taking the limit as $p^i \to p^0$ while making use of the same calculation as in the proof Lemma 3, establishes that

$$p'(c^0) = \frac{E[x_1^+(p^0,\mathbf{p}_{-1}) + x_1^-(p^0,\mathbf{p}_{-1})]}{\left\{\begin{array}{l}(p^0 - c)\lim_{i\to\infty}\{E[x_1^+(p^i,\mathbf{p}_{-1}) + x_1^-(p^i,\mathbf{p}_{-1}) - x_1^+(p^0,\mathbf{p}_{-1}) - x_1^-(p^0,\mathbf{p}_{-1})]\}/(p^i - p^0) \\ +\, 2\,E[x_1^+(p^0,\mathbf{p}_{-1}) + x_1^-(p^0,\mathbf{p}_{-1})]\end{array}\right\}}$$

$$< \frac{1}{2 - [(k+1)^2/2k][((\lambda-1)/2)/(2+(\lambda-1)/2)]}.$$

By the firm's maximization problem,

$$(p^i - c^i)\; E[x_1^-(p^i,\mathbf{p}_{-1}) + x_1^+(p^i,\mathbf{p}_{-1})] + E[x^+(p^i,\mathbf{p}_{-1}) + x^-(p^i,\mathbf{p}_{-1})] = 0 \tag{21}$$

for each i, and a similar condition holds at p^0.

Fix any $\mathbf{p}_{-1}$. We find a condition under which for $x(\cdot,\cdot) \in \{x^+(\cdot,\cdot), x^-(\cdot,\cdot)\}$,

$$\limsup_{c^i \to c^0} \frac{[(p^i - c^i)x_1(p^i,\mathbf{p}_{-1}) + x(p^i,\mathbf{p}_{-1})] - [(p^0 - c^0)x_1(p^0,\mathbf{p}_{-1}) + x(p^0,\mathbf{p}_{-1})]}{c^i - c^0} < 0.$$

This is sufficient for a contradiction because it implies that the first-order condition 21 cannot hold for all p^i, p^0 (since for i sufficiently high, the difference between the left-hand sides of the first-order conditions is negative).

[2] Given our estimation in Lemma 3 (which we also use again below to bound the derivative of $p(\cdot)$), we can show that $p(\cdot)$ is Lipschitz continuous. Hence, we can apply the Fundamental Theorem of Calculus to conclude that its derivative must be strictly positive on a set of positive measure.

The above limsup is equal to

$$\text{(22)}\qquad \lim_{c^i\to c^0}\frac{x(p^i,\mathbf{p}_{-1})-x(p^0,\mathbf{p}_{-1})}{c^i-c^0}+\lim_{c^i\to c^0}\frac{[(p^i-p^0)-(c^i-c^0)]}{c^i-c^0}x_1(p^0,\mathbf{p}_{-1})$$

$$+\limsup_{c^i\to c^0}(p^i-c^i)\frac{x_1(p^i,\mathbf{p}_{-1})-x_1(p^0,\mathbf{p}_{-1})}{c^i-c^0}$$

$$=x_1(p^0,\mathbf{p}_{-1})(2p'(c^0)-1)+(p^0-c^0)\limsup_{c^i\to c^0}\frac{x_1(p^i,\mathbf{p}_{-1})-x_1(p^0,\mathbf{p}_{-1})}{c^i-c^0}.$$

Now we work on the last term above, which is equal to

$$-\frac{p^0-c^0}{2t}\limsup_{c^i\to c^0}\frac{1}{c^i-c^0}\Bigg\{\Bigg[\frac{(\lambda-1)(F(p^i)-F(p^0))}{2+((\lambda-1)/2)[G(x(p^0,\mathbf{p}_{-1}))+G((1/n)-x(p^0,\mathbf{p}_{-1}))]}\Bigg]$$

$$+\Bigg[\frac{(2+(\lambda-1)\ F(p^i))}{2+((\lambda-1)/2)[G(x(p^i,\mathbf{p}_{-1}))+G((1/n)-x(p^i,\mathbf{p}_{-1}))]}$$

$$-\frac{(2+(\lambda-1)\ F(p^i))}{2+((\lambda-1)/2)[G(x(p^0,\mathbf{p}_{-1}))+G((1/n)-x(p^0,\mathbf{p}_{-1}))]}\Bigg]\Bigg\}$$

$$=-\frac{p^0-c^0}{2t}\frac{(\lambda-1)F'(p^0)p'(c^0)}{2+((\lambda-1)/2)[G(x(p^0,\mathbf{p}_{-1}))+G((1/n)-x(p^0,\mathbf{p}_{-1}))]}$$

$$-\frac{p^0-c^0}{2t}\limsup_{c^i\to c^0}\frac{2+(\lambda-1)F(p^i)}{c^i-c^0}$$

$$\times\frac{((\lambda-1)/2)[G(x(p^0,\mathbf{p}_{-1}))-G(x(p^i,\mathbf{p}_{-1}))+G(1/n-x(p^0,\mathbf{p}_{-1}))-G(1/n-x(p^i,\mathbf{p}_{-1}))]}{\{2+((\lambda-1)/2)[G(x(p^0,\mathbf{p}_{-1}))+G(1/n-x(p^0,\mathbf{p}_{-1}))]\}\{2+((\lambda-1)/2)[G(x(p^i,\mathbf{p}_{-1}))+G(1/n-x(p^i,\mathbf{p}_{-1}))]\}}$$

$$\le(p^0-c^0)x_1(p^0,\mathbf{p}_{-1})\Bigg[\frac{\lambda-1}{\lambda+1}p'(c^0)F(p^0)$$

$$+\limsup_{c^i\to c^0}\frac{1}{c^i-c^0}\frac{((\lambda-1)/2)[G(x(p^0,\mathbf{p}_{-1}))-G(x(p^i,\mathbf{p}_{-1}))+G(1/n-x(p^0,\mathbf{p}_{-1}))-G(1/n-x(p^i,\mathbf{p}_{-1}))]}{\{2+((\lambda-1)/2)[G(x(p^0,\mathbf{p}_{-1}))+G(1/n-x(p^0,\mathbf{p}_{-1}))]\}}\Bigg].$$

Now, notice that only either $G(x(p^0,\mathbf{p}_{-1}))-G(x(p^i,\mathbf{p}_{-1}))$ or $G(1/n-x(p^0,\mathbf{p}_{-1}))-G(1/n-x(p^i,\mathbf{p}_{-1}))$ can be strictly greater than zero but not both, and since $G(s)-G(s')\le 2n(s-s')$ for any $s>s'$, the sum of these expressions is less than or equal to $2n|x(p^i,\mathbf{p}_{-1})-x(p^0,\mathbf{p}_{-1})|$. Using also $G(x(p^0,\mathbf{p}_{-1}))+G(1/n-x(p^0,\mathbf{p}_{-1}))\ge 1$, this implies that the above is less than or equal to

$$(p^0-c^0)p'(c^0)\left(x_1(p^0,\mathbf{p}_{-1})\frac{\lambda-1}{\lambda+1}F'(p^0)+(x_1(p^0,\mathbf{p}_{-1}))^2\frac{(\lambda-1)n}{2+(\lambda-1)/2}\right).$$

Substituting into Expression 22 and using that $|x_1(p^0,\ \mathbf{p}_{-1})|\le(1/(2t))\ [(1+\lambda)/(2+(\lambda-1)/2)]=k/(2t)$ implies that it is sufficient to prove

$$1+k^2\frac{(\lambda-1)}{(\lambda+1)}\frac{n}{2t}(p^0-c^0)p'(c^0)<2p'(c^0)+(p^0-c^0)p'(c^0)\frac{\lambda-1}{\lambda+1}F'(p^0).$$

Using that $F'(p^0) \geq D(p^0)[\theta_1(c^0)/p'(c^0)]$ and that

$$p^0 - c^0 = \frac{D(p^0)}{-D'(p^0)} \leq \frac{2/n}{(1/2t)(2/(1+\lambda))} = \frac{2t}{n}(1+\lambda),$$

the above becomes

$$1 + k^2(\lambda - 1)p'(c^0) < 2p'(c^0) + (p^0 - c^0)D(p^0)\frac{\lambda - 1}{\lambda + 1}\theta_1(c^0). \tag{23}$$

To finish our proof, we put a bound on the firm's profits $(p^0 - c^0)\ D(p^0)$. In a market equilibrium, no firm charges a price less than $\underline{c}$, so firm 1's profits are at least as much as it would make if both of its neighbors charge $\underline{c}$ with probability one. If firm 1 also charges $\underline{c}$, its demand in each of its two markets is $1/(2n)$. Now

$$|x_1(p_1, \mathbf{p}_{-1})| \leq \frac{1}{2t}\frac{1+\lambda}{2 + (\lambda-1)/2} = \frac{k}{2t}.$$

This implies that a sufficient condition for the firm to be able to sell profitably is

$$\overline{c} - \underline{c} < \frac{1/(2n)}{k/(2t)} = \frac{t}{n}\frac{1}{k} = \frac{t}{n}\frac{3+\lambda}{2(1+\lambda)}.$$

Furthermore, if this is the case, its profits are at least

$$2(p - c)\left(\frac{1}{2n} - \frac{k}{2t}(p - \underline{c})\right) = (p - c)\left(\frac{1}{n} - \frac{k}{t}(p - \underline{c})\right).$$

Maximizing this expression with respect to p and setting $c = \overline{c}$ gives

$$(p^0 - c^0)D(p^0) \geq \frac{k}{4t}\left(\frac{t}{n}\frac{1}{k} - (\overline{c} - \underline{c})\right)^2 = \frac{k}{4}\frac{t}{n^2}\left(\frac{1}{k} - \gamma\right)^2,$$

where $\gamma \equiv (\overline{c} - \underline{c})/(t/n)$.

Now we have two cases.

Case I: $k^2(\lambda - 1) \leq 2$. In this case, a sufficient condition for Inequality 23 to hold is

$$\frac{t}{n^2}\theta_1(c^0) > \frac{\lambda+1}{\lambda-1}\frac{4k}{(1-k\gamma)^2}.$$

Case II: $k^2(\lambda - 1) > 2$. Then, substituting our bound for $p'(c^0)$ into Inequality 23 and rearranging gives that a sufficient condition is

$$\frac{t}{n^2}\theta_1(c^0) > \frac{4k}{(1-k\gamma)^2}\frac{(1+\lambda)k^2 - (k+1)^2/4}{2 - [(\lambda-1)/(\lambda+1)][(k+1)^2/4]}.$$

This completes our proof.

CALCULATIONS FOR EXAMPLE 1: For the low-price firm 1 to have market share 3/4, it must be that in personal equilibrium consumers who are within a distance $\alpha = 3/8$ of firm 1's location $y_1 = 0$ buy from firm 1. Personal equilibrium requires that having expected this behavior, a consumer with realized taste $\chi = 3/8$ be indifferent between buying from the two firms. The above behavior induces expectations to pay p_1 with probability 3/4 and p_2 with probability 1/4, and (from Figure 1) also induces an expected distribution of the product's distance from ideal that is a step function with a density of 4 between 0 and 1/8, a density of 2 between 1/8 and 3/8, and a density of zero everywhere else. Given these expectations, a consumer's utility from buying good 1 at price p_1 if she has taste 3/8 is

$$u_1 = v - \frac{3}{8}t - p_1 + \frac{1}{4}(p_2 - p_1) - \lambda t \int_0^{1/8} \left(\frac{3}{8} - s\right) 4d(s) - \lambda t \int_{1/8}^{3/8} \left(\frac{3}{8} - s\right) 2d(s), \tag{24}$$

while the utility from buying good 2 at price p_2 is

$$u_2 = v - \frac{1}{8}t - p_2 - \lambda \frac{3}{4}(p_2 - p_1) - \lambda t \int_0^{1/8} \left(\frac{1}{8} - s\right) 4d(s) + t \int_{1/8}^{3/8} \left(s - \frac{1}{8}\right) 2d(s). \tag{25}$$

Setting $u_1 = u_2$ yields

$$\begin{aligned} v - \frac{3}{8}t - p_1 + \frac{1}{4}(p_2 - p_1) - \frac{\lambda t}{2}\left(\frac{3}{8} - \frac{1}{16}\right) - \frac{\lambda t}{2}\left(\frac{3}{8} - \frac{2}{8}\right) \\ = v - \frac{t}{8} - p_2 - \frac{3\lambda}{4}(p_2 - p_1) - \frac{\lambda t}{2}\left(\frac{1}{8} - \frac{1}{16}\right) + \frac{t}{2}\left(\frac{2}{8} - \frac{1}{8}\right). \end{aligned}$$

Rearranging gives

$$p_2 - p_1 = \frac{t(5/16 + \lambda 3/16)}{5/4 + \lambda 3/4} = \frac{t}{4}, \tag{26}$$

We now derive the range of marginal costs that can support the above prices, and the above personal equilibrium, as part of a market equilibrium. To do so, we take advantage of Lemma 1. For the indifferent consumer on either side of either firm, $G(x) + G(1/n - x) = G(3/8) + G(1/8) = 3/2$. Using that $\lambda = 5$, the responsiveness of demand to local deviations from the prices p_1, p_2 is

$$D_{2\downarrow}(p_2, p_1) = -\frac{6}{5},\ D_{2\uparrow}(p_2, p_1) = D_{1\downarrow}(p_1, p_2) = -1;\ D_{1\uparrow}(p_1, p_2) = -\frac{2}{5}.$$

Hence, for firm 2 not to want to deviate locally from the proposed equilibrium, p_2 must satisfy the following conditions:

$$(p_2 - c_2)\ D_{2\downarrow}(p_2, p_1) + D_2(p_2, p_1) = -(p_2 - c_2)\frac{6}{5} + \frac{1}{4} \leq 0$$

$$(p_2 - c_2)\ D_{2\uparrow}(p_2, p_1) + D_2(p_2, p_1) = -(p_2 - c_2)\,1 + \frac{1}{4} \geq 0.$$

This implies:

$$p_2 - \frac{1}{4} \leq c_2 \leq p_2 - \frac{5}{24} \tag{27}$$

By similar calculations, for firm 1 not to want to deviate locally, p_1 and c_1 must satisfy

$$p_1 - \frac{15}{8} \leq c_1 \leq p_1 - \frac{3}{4}. \tag{28}$$

Lemma 3 implies that if a a local deviation is unprofitable, a non-local deviation to an "interior" price (a price such that the indifferent consumer x is within distance $1/2$ of the firm) is also unprofitable. It is also clearly unprofitable to change one's price after capturing or losing the entire market, so that the above local conditions are sufficient for firms not to want to deviate.

PROOF OF LEMMA 2:

We begin with proving continuity. Suppose by contradiction that $c^i \to c$ but $P(c^i) \nrightarrow P(c)$. Then, since the pricing function is obviously bounded, we can choose the sequence so that $\mathrm{P}(c^i)$ converges; let $P(c^i) \to P' \neq P(c)$. Furthermore, suppose that $P' > P(c)$; the proof for the other case is analogous.

Since $P(c^i)$ is optimal when the marginal cost is c^i, a firm cannot benefit from marginally lowering its price. Using Equation 11 to express the firm's marginal profit from lowering its price, this implies that

$$\frac{1}{2n} - \big(P(c^i) - c^i\big)\frac{2 + (\lambda - 1)F_{\uparrow}(P(c^i))}{1 + \lambda} \geq 0. \tag{29}$$

Similarly, since $P(c)$ is optimal when the marginal cost is c, a firm cannot benefit from marginally raising its price. Using Equation 11, this implies that

$$\frac{1}{2n} - \big(P(c) - c\big)\frac{2 + (\lambda - 1)F(P(c))}{1 + \lambda} \leq 0. \tag{30}$$

Subtracting Inequality 29 from Inequality 30 gives

$$\big(P(c^i) - c^i\big)\frac{2 + (\lambda - 1)F_{\uparrow}(P(c^i))}{1 + \lambda} - \big(P(c) - c\big)\frac{2 + (\lambda - 1)F(P(c))}{1 + \lambda} \leq 0.$$

The limit of the left-hand side of this inequality as $i \to \infty$ is positive, a contradiction.

Next, we prove by contradiction that $P(c)$ is non-decreasing. Suppose that $c' > c$ and $P(c') < P(c)$. Since $P(c)$ is optimal when the marginal cost is c, a firm cannot benefit from marginally lowering its price. As above, this implies that

$$\frac{1}{2n} - \big(P(c) - c\big)\frac{2 + (\lambda - 1)F_{\uparrow}(P(c))}{1 + \lambda} \geq 0. \tag{31}$$

Similarly, since $P(c)$ is optimal when the marginal cost is c, a firm cannot benefit from marginally raising its price. Therefore,

$$\frac{1}{2n} - \big(P(c') - c'\big)\frac{2 + (\lambda - 1)F(P(c'))}{1 + \lambda} \leq 0. \tag{32}$$

Subtracting Inequality 31 from Inequality 32 gives

$$\big(P(c) - c\big)\frac{2 + (\lambda - 1)F_{\uparrow}(P(c))}{1 + \lambda} - \big(P(c') - c'\big)\frac{2 + (\lambda - 1)F(P(c'))}{1 + \lambda} \leq 0,$$

a contradiction.

PROOF OF PROPOSITION 5:

We first show that any symmetric equilibrium pricing function satisfies the above properties. Property 1 follows from Lemma 2. Since $P(\cdot)$ is increasing and continuous, $P^{-1}(p)$ is a closed interval for any p on the range of $P(\cdot)$. Let $p_1, p_2, \ldots$ be the (at most countable) set of prices p_i such that $P^{-1}(p_i)$ is a non-trivial interval, and let $[f_i, f_i'] = P^{-1}(p_i)$. These $[f_i, f_i']$ satisfy Property 2 by construction. Also, for any $c \notin [f_i, f_i']$, $P(c)$ is not an atom of the pricing distribution, so a firm's demand is differentiable, and hence $P(c)$ must satisfy Equation 7. This implies that Property 3 holds. Notice that $D_{1\uparrow}(P(c), P_{-1}(c)) = -(1/t)(2/(1 + \lambda))$, so firm 1 does not want to decrease its price at $\underline{c}$ only if $(P(\underline{c}) - \underline{c})(1/t)(2/(1 + \lambda)) \leq 1/n$, which implies the first part of Property 4. Also, $D_{1\downarrow}(P(c), P_{-1}(c)) = -1/t$. So for raising the price marginally to be unprofitable, we must have $(P(\overline{c}) - \overline{c})(1/t) \geq 1/n$, which implies the second part of Property 4.

We now argue that if $P(\cdot)$ satisfies the properties in the Proposition, it is an equilibrium pricing strategy. Notice that for any $c \in (\underline{c}, \overline{c})$, $c \notin [f_i, f_i')$, demand is differentiable from the right. Since $P(c) = \Phi(c)$ for all such c, our analysis in the text implies that there is no profitable local price increase. We are left to consider non-local price increases. Analogously to Proposition 1, since the demand curve is concave for price increases, the result is immediate.

Now for any $c \in (\underline{c}, \overline{c})$, $c \notin (f_i, f_i']$, demand is differentiable from the left. Furthermore, since $P(c) = \Phi(c)$ for all such c, our analysis in the text implies that local price decreases are unprofitable. We now consider non-local price decreases.

The proof mirrors the proof of Proposition 1. Suppose the realized cost is c, so that the firm's price in the posited equilibrium is $P(c)$. At this price, consumers' marginal utility in money from a price decrease is $2 + (\lambda - 1)F_{\uparrow}(P(c))$. We will use that as the price decreases, this marginal utility in money also decreases.

We first rule out the possibility that firm 1 might like to charge a price p_1 so that the indifferent consumer is $x \in [1/2n, 1/n]$. Equating Expressions (9) and (10), setting $p_2 = P(c)$, and replacing the difference in money utilities,

$$P(c) - p_1 + \left[-\lambda \int_0^{p_1} (p_1 - p)\, dF(p) + \int_{p_1}^{\infty} (p - p_1)\, dF(p)\right]$$
$$-\left[-\lambda \int_0^{P(c)} (P(c) - p)\, dF(p) + \int_{P(c)}^{\infty} (p - P(c))\, dF(p)\right],$$

with its upper bound $(2 + (\lambda - 1)F_{\uparrow}(P(c)))(P(c) - p_1)$, gives that for the indifferent consumer x

$$-xt + (P(c) - p_1)(2 + (\lambda - 1)F_{\uparrow}(P(c))) - \lambda t\left(x - \frac{1}{4n}\right)$$
$$\geq -\left(\frac{1}{n} - x\right)t - \lambda t\, 2n\left(\frac{1}{n} - x\right)\frac{1/n - x}{2} + t\, 2n\left(x - \frac{1}{2n}\right)\frac{x - 1/(2n)}{2},$$

so that

$$(33)\quad P(c) - p_1 \geq \frac{t}{2 + (\lambda - 1)F_{\uparrow}(P(c))} \underbrace{\left[(\lambda + 1)\left(2x - \frac{1}{n}\right) + (\lambda - 1)\left(x - \frac{1}{4n} - nx^2\right)\right]}_{\equiv \kappa}.$$

To show that lowering the price to p_1 is not a profitable deviation, it is sufficient to show that

$$\frac{1}{n}(P(c) - c) \geq 2x(p_1 - c).$$

Using Inequality (33), it is sufficient to show that

$$\frac{1}{n}(P(c) - c) \geq 2x\left(P(c) - c - \frac{t}{2 + (\lambda - 1)F_{\uparrow}(P(c))}\kappa\right).$$

Rearranging and using that $P(c) - c = t(1 + \lambda)/[n(2 + (\lambda - 1)F_{\uparrow}(P(c)))]$ gives

$$\left(2x - \frac{1}{n}\right)\frac{1 + \lambda}{n} \leq 2x\kappa,$$

which is equivalent to Inequality (12), which we verified in the proof of Proposition 1.

For $n > 2$, we are left to rule out that firm 1 undercuts its rival and steals more than the entire adjacent market. We begin by ruling out deviations in which the firm captures less than two adjacent markets. Let p_1 be the price at which the consumer located at $1/n$ is indifferent between buying from firm 1 and buying from firm 2. Substituting $x = 1/n$ into Equation 33 gives

$$P(c) - p_1 \geq \frac{t}{(2 + (\lambda - 1)F_{\uparrow}(P(c)))n}\left[2 + \frac{3}{4}(\lambda - 1)\right].$$

Using the expression for $P(c) - c$ we get $p_1 - c \leq [t/[(2 + (\lambda - 1)F_{\uparrow}(P(c)))n]][(1/4)(\lambda - 1)]$. Thus, even if firm 1 would get the entire two adjacent markets when setting p_1', this is unprofitable as $(1/n)(P(c) - c) > (4/n)(p_1 - c)$.

We are left to consider the case when $n > 4$ and firm 1 steals at least two adjacent markets on each side. We show that this is unprofitable because it requires firm 1 to price below marginal cost. For the consumer located at $2/n$ to weakly prefer buying from firm 1 rather than firm 3, it must be that

$$\begin{aligned} P(c) - p_1 &\geq \frac{t}{(2 + (\lambda - 1)F_{\uparrow}(P(c)))n}\left[4 + (\lambda - 1)\frac{7}{4}\right] \\ &> \frac{t}{(2 + (\lambda - 1)F_{\uparrow}(P(c)))n}[2 + \lambda - 1] = P(c) - c. \end{aligned}$$

This completes the proof that non-local price decreases are unprofitable.

We have established that there is no profitable deviation for $c \in (\underline{c}, \overline{c})$, $c \notin [f_i, f_i']$. For any $c \in (\underline{c}, \overline{c})$, $c \in [f_i, f_i']$, we have $P(f_i) = P(c) = P(f_i')$. Since it is not profitable to lower the price at f_i, it is also not profitable to lower it for c, and since it is not profitable to raise the price for f_i', it is also not profitable to raise it for c.

We are left to prove that there are no profitable deviations for $\underline{c}$ and $\overline{c}$. Our analysis of non-local deviations above (which only used that $P(c) = \Phi(c)$) implies that for $P(\underline{c}) = \Phi(\underline{c})$, there is no profitable deviation. Now suppose that $P(\underline{c}) < \Phi(\underline{c})$. Demand responsiveness to price decreases from $P(\underline{c})$ is then the same as when $P(\underline{c}) = \Phi(\underline{c})$. Hence, with the markup being lower, the incentive to lower the price is smaller than for $P(\underline{c}) = \Phi(\underline{c})$, so there is no profitable price decrease. Next, we deal with price increases from $\underline{c}$. Since $P(\underline{c}) < \Phi(\underline{c})$, we consider two cases. First, suppose that $P(c)$ is a constant p^*. Then, using that by Property 4 in the proposition $\Phi(\underline{c}) \geq p^* \geq \Phi(\overline{c})$, and Equation 7, the condition in Proposition 2 is satisfied. Hence, p^* is a market-equilibrium focal price. If *P(c)* is not constant, there is a largest interval $[\underline{c}, f_1']$ for which it is constant, and where $f_1' < \overline{c}$. In this case, our argument in the previous paragraph applies. Finally, a similar argument works for price deviations from $\overline{c}$.

PROOF OF COROLLARY 2:

Suppose by contradiction that there is a constant interval $[f_1, f_1']$. By Conditions 3 and 4 of Proposition 5, we must have $P(f_1) \leq \Phi(f_1)$. But by the same two conditions, we must also have $P(f_1') \geq \Phi(f_1')$, which is impossible since $\Phi(\cdot)$ is strictly increasing on the interval while $P(\cdot)$ is constant.

PROOF OF COROLLARY 3:

We first prove by contradiction that if $\Phi(c)$ is weakly decreasing, then any symmetric equilibrium is a focal-price one. Suppose the price is not deterministic. Then, by the continuity of the pricing function, there are cost levels c, $c' > c$ such that $P(c)$ and $P(c')$ are not atoms of the price distribution. Thus, for these cost levels, the chosen price must satisfy Equation 7. Using that $\Phi(c)$ is strictly decreasing, this means that $P(c') < P(c)$, contradicting that the pricing function is non-decreasing.

If $\Phi(c)$ is not weakly decreasing, then there are obviously non-constant $P(\cdot)$ satisfying Proposition 5.

PROOF OF COROLLARY 4:

The statement is true on both the constant and strictly increasing parts of the pricing function.