Online Appendix for "Reputation and School Competition"

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This appendix presents two extensions of the base model. Section 1 looks at the effect of increasing selection when there is imperfect selectivity. Section 2 provides an example of a hybrid equilibrium featuring both selective and non-selective schools.

Effect of Increased College Selectivity

In this section we show that if colleges are responsive to the preferences of students, then they have an incentive to become more selective in the sense of catering only to students with ability in a narrow range. Suppose that a college s selects students with admissions test scores in the range $\tau_s \in [\underline{\tau}_s, \bar{\tau}_s]$. Under the hypothesis that in equilibrium the amount of test prep is anticipated, this is equivalent to selecting students with ability conditional upon test score in a range $\hat{\alpha}_i \in [\underline{\alpha}_s, \bar{\alpha}_s]$. We ask how the payoffs of individuals in the college vary if $\underline{\alpha}_s$ is increased (by increasing $\underline{\tau}_s$).

PROPOSITION 1: *Suppose that college s admits only students with expected innate ability $\hat{\alpha}_i \in [\underline{\alpha}_s, \bar{\alpha}_s]$, with a continuous distribution over this interval given by $h_s(\hat{\alpha}_i)$; then increasing selectivity, $\underline{\alpha}_s$, raises the payoff of all students remaining in the school.*

PROOF:

Since we suppose that study effort is chosen optimally, then by the envelope theorem increasing selectivity has no marginal effect upon effort. Thus, we only need to derive the effect of increased selectivity upon the expected innate ability of a graduate. Consider individual i, with characteristics $(\alpha_i, \hat{\alpha}_i, t_i)$. where α_i is unobserved ability, $\hat{\alpha}_i$ is expected ability given the admissions exam score τ_i, and t_i is the graduation test. These characteristics have a multivariate normal distribution with strictly positive covariance matrix and zero means. Let $f(\alpha_i, \hat{\alpha}_i, t_i)$ be the corresponding probability density function, and $f(\hat{\alpha}_i, t_i)$ the density when α_i is integrated out. This implies:

$$E\left\{E\left\{\alpha_i | \hat{\alpha}_i \in [\underline{\alpha}_s, \bar{\alpha}_s], t_i\right\} | \hat{\alpha}_i\right\} = \frac{\int_{-\infty}^{\infty} \hat{\alpha}(t_i, s) f(\hat{\alpha}_i, t_i) dt_i}{\int_{-\infty}^{\infty} f(\hat{\alpha}_i, t_i) dt_i}$$

where:

$$\hat{\alpha}(t_i, s) = \left\{ \frac{\int_{\underline{\alpha}_s}^{\bar{\alpha}_s} \int_{-\infty}^{\infty} \alpha f(\alpha, \hat{\alpha}, t_i) h(\hat{\alpha}) \, d\alpha d\hat{\alpha}}{f_s(t_i)} \right\},$$

$$f_s(t_i) = \int_{\underline{\alpha}_s}^{\bar{\alpha}_s} \int_{-\infty}^{\infty} f(\alpha, \hat{\alpha}, t_i) h(\hat{\alpha}) \, d\alpha d\hat{\alpha}.$$

is the expected ability of an individual who attended school s and has graduation test score t_i.

When computing her future wage, the student has to take into account the correlation between $\hat{\alpha}_i$ and t_i. When selecting a school the student computes the expected utility given the future expected distribution of graduation test scores t_i. If for each realization of t_i the effect of increasing selectivity is positive, then so will be the effect of selectivity upon expected future wages at the time of admission.

Thus we have:

$$\frac{\partial \hat{\alpha}(t_i, s)}{\partial \underline{\alpha}_s} = \frac{f_s(\underline{\alpha}_s, t_i)}{f_s(t_i)} \left\{ -\frac{\int_{-\infty}^{\infty} \alpha f(\alpha, \underline{\alpha}_s, t_i) h(\underline{\alpha}_s) \, d\alpha}{f_s(\underline{\alpha}_s, t_i)} + \hat{\alpha}(t_i, s) \right\}$$

$$= \frac{f_s(\underline{\alpha}_s, t_i)}{f_s(t_i)} \left\{ \hat{\alpha}(t_i, s) - E\left\{ \alpha_i | t_i, \hat{\alpha}_i = \underline{\alpha}_s \right\} \right\} > 0,$$

where:

$$f_s(\underline{\alpha}_s, t_i) = \int_{-\infty}^{\infty} f(\alpha, \underline{\alpha}_s, t_i) h(\underline{\alpha}_s) \, d\alpha.$$

is the marginal distribution given $(\underline{\alpha}_s, t_i)$. The final inequality follows from $h(\underline{\alpha}_s) > 0$ and from the fact that $\hat{\alpha}_i$ is a good signal for α_i in the sense of Milgrom (1981).

Effect of Partial Selection

The previous section shows that once a school becomes selective, there is an incentive to be as selective as possible. In many markets, such as the U.S. college market, there is a mixture of highly selective schools and a large, non-selective sector (see Hoxby, 2009). Here we show that in our setup a competitive equilibrium with both types of schools is possible as long as there is a cost to entry by selective schools. In addition, when this happens the selective sector exerts a negative effort-related externality on the non-selective schools.

We begin by supposing that all schools have the same value added v, but there is a cost k to selection that students pay upon being accepted to a selective school (for instance, admissions systems entail costs). Schools in the selective sector set entrance requirements τ_s. This sector only attracts the better students, and thus

there is an endogenous cutoff $\bar{\tau}$ such that students with scores $\tau > \bar{\tau}$ enter the selective sector. The remainder, $\tau \leq \bar{\tau}$, are randomly assigned to non-selective schools.

We solve for the competitive equilibrium by backwards induction. We take $\bar{\tau}$ as given, and let r^{PS} be the test prep with partial selection and then work out the payoffs of students in the selective and non-selective sector. The final step is to determine a $\bar{\tau}$ that has two properties. First, students in the selective sector would never choose a non-selective school. Second, a school that cites an admission standard $\tau_s < \bar{\tau}$ would not have any demand.

Since students are the same *ex ante,* test prep is the same for all students. To simplify the analysis we set $r^{PS} = 0$ and then discuss the incentives for test prep once the equilibrium has been found. For students in the selective sector, study effort and wages are given by the free choice allocation for $\tau \geq \bar{\tau}$. The main task is to derive the effect of $\bar{\tau}$ upon the study effort and wages of workers in the non-selective sector. To simplify the computation, we suppose that students and the labor market are symmetric in the sense that each only uses school identity to compute future returns.[1] Thus, when a student from the non-selective sector enters the labor market his log wage is given by:

$$w_s(\bar{\tau}) = E\{\alpha | \tau \leq \bar{\tau}\} + e^{PS}(\bar{\tau}) + v,$$

where $e^{PS}(\bar{\tau})$ is the equilibrium effort in this sector given the cutoff $\bar{\tau}$. Given the normality of distributions we have the following result.

PROPOSITION 2: *The expected innate ability and wage, $w(\bar{\tau})$, of a student entering a non-selective school where $\tau \leq \bar{\tau}$ is:*

$$E\{\alpha | \tau \leq \bar{\tau}\} = -\frac{\sigma^2}{\sigma_\tau} IM\left(-\frac{\bar{\tau}}{\sigma_\tau}\right)$$

$$w(\bar{\tau}) = -\frac{\sigma^2}{\sigma_\tau} IM\left(-\frac{\bar{\tau}}{\sigma_\tau}\right) + e^{PS}(\bar{\tau}) + v,$$

where $e^{PS}(\bar{\tau})$ and v are the equilibrium effort and value added in this sector, σ^2 and σ_τ^2 are the variance of α and τ respectively, and $IM(z) = \frac{f(z)}{1-F(z)}$ is the inverse Mills ratio. The functions f, F are the p.d.f. and c.d.f. for the standard normal distribution, respectively.

PROOF:
The effect of selection upon expected ability can be computed using results

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[1] One could allow the future wages of students from the non-selective sector to vary with their private information τ. This would make the effect of effort a non-linear function of τ and hence greatly increase the complexity of the analysis without the addition of much insight.

on conditional expectations of normal random variates with truncation. From Birnbaum (1950) we have:

$$E\{X|Z \geq z\} = \mu IM(z),$$

where X and Z are standard normal random variables with zero mean and unit variance; $\mu = E\{XZ\}$. If X and Z have normal distributions with variances σ_X^2 and σ_Z^2, respectively, then:

(B1) $$E\{X|Z \leq z\} = E\{X\} - \frac{cov\{X,Z\}}{\sigma_Z} IM\left(\frac{E\{Z\} - z}{\sigma_Z}\right),$$

which after substitution implies the result.

We also have from Birnbaum (1950) that $x + \frac{1}{x} \geq IM(x) \geq x$ for $x \geq 0$, and $lim_{x \to -\infty} IM(x) = 0$. Thus we have:

PROPOSITION 3:   *(Wage in the non-selective sector) For $\bar{\tau} \leq 0$:*

$$\pi\bar{\tau} + e^{PS}(\bar{\tau}) + v \geq w(\bar{\tau}) \geq \pi(\bar{\tau}) + \frac{1}{\pi\bar{\tau}} + e^{PS}(\bar{\tau}) + v_{\bar{s}}.$$

where $\pi = \frac{\rho^\tau}{\rho + \rho^\tau}$.

Notice that as $\bar{\tau}$ falls, then so does the expected wage. Hence, as the selective sector grows in size, the expected income of individuals in the non-selective sector falls.

The equilibrium study effort in the non-selective sector depends upon how effort is rewarded when its students enter the labor market. An individual's choice does not affect the average effort in the sector, only her test graduation test score, t_i, upon leaving school. By construction we know that $\frac{\partial t_i}{\partial e_i} = 1$, so the next step is to work out the effect of test scores on future wages.

We make the simplifying assumption that once students enter the non-selective sector, they do not use the information about their entrance exam to compute the *marginal* effect of effort upon wages. This allows us to suppose that all students in the sector choose the same effort. Let $w(t, \bar{\tau})$ be the expected wage of an individual from the non-selective sector who enters the labor market with a test score of t. The level of effort, $e^{PS}(\bar{\tau})$, in this sector is the solution to:

$$-\Psi'(e^{PS}(\bar{\tau}), a) = \delta \frac{\partial w(\bar{\tau})}{\partial e} \equiv \delta E\left\{\frac{\partial w(t, \bar{\tau})}{\partial t}\right\},$$

Where $w(t, \bar{\tau})$ is the expected wage conditional upon t and the selection criteria $\bar{\tau}$. It is given by:

$$w(t, \bar{\tau}) = E\{\alpha|t, \tau \leq \bar{\tau}\} + e^{PS}(\bar{\tau}) + v.$$

Given that the last two terms do not depend upon t we need only work out the

expected ability:

$$
\begin{aligned}
E\left\{\alpha|t, \tau \leq \bar{\tau}\right\} &= E\left\{E\left\{\alpha|t,\tau\right\}|t,\tau\leq\bar{\tau}\right\} \\
&= E\left\{\pi^{t\tau}(t-v-e^{PS})+\pi^{\tau t}(\tau)|t,\tau\leq\bar{\tau}\right\} \\
&= \pi^{t\tau}(t-v-e^{PS})+\pi^{\tau t}E\left\{\tau|t,\tau\leq\bar{\tau}\right\}
\end{aligned}
$$

where the weights for the Bayesian learning rule are $\pi^{t\tau}=\frac{\rho^{t}}{\rho+\rho^{t}+\rho^{\tau}}$ and $\pi^{\tau t}=\frac{\rho^{\tau}}{\rho+\rho^{t}+\rho^{\tau}}$. To compute the final expectation notice that we can view τ as a random variable conditional upon the test score t, such that:

$$
\begin{aligned}
E\left\{\tau|t\right\} &= E\{\alpha|t\}+E\{\epsilon^{\tau}|t\} \\
&= \pi^{t}(t-v-e^{PS}\left(\bar{\tau}\right)),
\end{aligned}
$$

where $\pi^{t}=\frac{\rho^{t}}{\rho+\rho^{t}}=1-\pi^{NS}$ is the optimal weight when only the test score is known (see proposition 2 in text). Note also that the variance of τ is equal to the conditional variance given t plus the variance of the error term:

$$
\begin{aligned}
var\left\{\tau|t\right\} &= var\left\{\alpha|t\right\}+var\left\{\epsilon^{\tau}\right\} \\
&= \frac{1}{\rho^{\alpha}+\rho^{t}}+\sigma_{\tau}^{2} \\
&\equiv \sigma_{\tau|t}^{2}.
\end{aligned}
$$

We can now use formula (B1) with $X=Z=\tau$, to compute:

$$
E\{\tau|t,\tau<\bar{\tau}\}=\pi^{t}t-\sigma_{\tau|t}IM\left(\frac{\pi^{t}t-\bar{\tau}}{\sigma_{\tau|t}}\right).
$$

Using this result we have that the expected skill of an individual who obtains a test score t and who went to school in a non-selective sector with cutoff score $\bar{\tau}$ is:

(B2)
$$
w\left(t,\bar{\tau}\right)=E\{\theta^{NS}|t,\tau<\bar{\tau}\}=\pi^{t}t-\pi^{\tau t}\sigma_{\tau|t}IM\left(\frac{\pi^{t}(t-v-e^{PS}\left(\bar{\tau}\right))-\bar{\tau}}{\sigma_{\tau|t}}\right)+(1-\pi^{t})(v+e^{PS}\left(\bar{\tau}\right))
$$

We can compute a simple formula that captures the main effects on effort incentives by supposing that IM' is approximately linear locally, and observing that $E\left\{t-e^{PS}\left(\bar{\tau}\right)-v|\tau\leq\bar{\tau}\right\}=E\left\{\alpha|\tau\leq\bar{\tau}\right\}=-\frac{\sigma_{\alpha}^{2}}{\sigma_{\tau}}IM\left(-\frac{\bar{\tau}}{\sigma_{\tau}}\right)$. From this we get the following proposition.

PROPOSITION 4: *The marginal incentive for effort that students face in non-*

selective schools when these co-exist with a selective sector is approximately:

$$\frac{\partial w\left(\bar{\tau}\right)}{\partial e} = E\left\{\frac{\partial w(t,\bar{\tau})}{\partial t}\right\} \simeq \pi^t - \pi^{\tau t}\pi^t IM'\left(\frac{-\pi^{NS}\frac{\sigma_\alpha^2}{\sigma_\tau}IM\left(-\frac{\bar{\tau}}{\sigma_\tau}\right)-\bar{\tau}}{\sigma_{\tau|t}}\right)$$

$$\simeq \pi^t\left(1 - \pi^{\tau t}IM'\left(\frac{-\pi^{NS}\frac{\sigma_\alpha^2}{\sigma_\tau}IM\left(-\frac{\bar{\tau}-r^{PS}}{\sigma_\tau}\right)-\bar{\tau}}{\sigma_{\tau|t}}\right)\right)$$

$$< 1 - \pi^{NS}.$$

Thus the presence of selective schools reduces effort incentives for students at non-selective schools. We now have that the optimal effort for a student in the non-selective sector satisfies:

$$d'\left(e^{PS}\left(\bar{\tau}\right)\right) = \gamma\frac{\partial w\left(\bar{\tau}\right)}{\partial e}.$$

The effect of $\bar{\tau}$ on marginal effort rises, but is bounded above by the effect when there is no selection. If the cost of a selective school is k then the equilibrium size of the non-selective sector is given by the point at which a student with entrance score $\bar{\tau}$ is indifferent between a selective and a non-selective school:

$$\gamma w(\bar{\tau})+ \left(e^{PS}\left(\bar{\tau}\right) - d\left(e^{PS}\left(\bar{\tau}\right)\right)\right) = \left\{e^{FC} - d\left(e^{FC}\right)\right\} + \gamma\pi^{FC}\bar{\tau} - k.$$

For $k > 0$ there always exits at least one $\bar{\tau}$ that satisfies this equality—choose the smallest. This condition implies that if a selective school cites $\tau^s < \bar{\tau}$ as an admissions standard, students attending this school would be worse off than in the non-selective sector. Proposition 3 on effort in the non-selective sector implies $e^{NS} > e^{PS}\left(\bar{\tau}\right) > e^{FC}$. Hence we have shown:

PROPOSITION 5:   *Suppose non-selective schools have a cost advantage of $k > 0$ per student. Then there is a free-entry equilibrium characterized by $\bar{\tau}$ such that students with ability $\tau > \bar{\tau}$ attend selective schools and choose study effort e^{FC}; those with $\tau < \bar{\tau}$ attend non-selective schools and choose effort $e^{PS}\left(\bar{\tau}\right)$ such that:*

$$e^{NS} >   e^{PS}\left(\bar{\tau}\right)   > e^{FC}.$$

It is straightforward to add test prep to this case. Notice that test prep only has an effect if a student enters the selective sector, where the marginal effect conditional upon $\tau > \bar{\tau}$ is the same as in the free choice allocation. However, it has no effect for students in the non-selective sector, hence it follows that if test prep is allowed, it will be lower than in the free choice case $(r^{PS} < r^{FC})$.

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