Online Appendix: Endogenous Liquidity and the Business Cycle

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This document presents additional discussions and proofs not contained in the print version of "Endogenous Liquidity and the Business Cycle."

Discussions

B1. Properties of CD contracts

The set of competitive equilibrium CD has a continuum of contracts. For a particular example, Figure B1 depicts the entire set of equilibria. Each equilibrium is indexed by some ω^* corresponding to a participation threshold $\bar{\omega}^p$. The figure depicts the properties of the set. The upper panels display equilibrium liquidity and the implied interest rate for a participation cutoff ω^*. The bottom panels show the implied default rate, $F(\omega^p)/F(\bar{\omega}^p)$, and the loan size p^S for each equilibria. There are three equilibria of particular interest: the one for which, $\omega^p = \bar{\omega}^p$ —circle—, the equilibrium where $\bar{\omega}^p = 1$ —square— which corresponds to the optimal liquidity contract in DeMarzo and Duffie (1999), and the equilibrium with the largest loan size, p^S —diamond. It is worth discussing these properties.

Properties. The first property is that the CD for which $\bar{\omega}^p = \omega^p$, corresponds to the selling contracts of Section II. This is the case because, in equilibrium, defaulting or selling is the same. This is also the equilibrium with the lowest participation. Second, liquidity is increasing in the participation cutoff ω^*. The more collateralization, the higher the quality collateral pool and the lower the default rate. Third, because higher participation rates require greater incentives to participate, p^S may be decreasing in ω^*. As a consequence, p^S is possibly non-monotone in ω^*. In the quantitative section, I focus on the contract with the highest liquidity.

Observational Equivalence. Figure B2 follows the procedures to compute equilibria in Figure B1 and computes the highest liquidity contracts for different values of dispersion. In the top panel, one can observe that given an initial value of liquidity with sales, one can increase the dispersion in the equilibria with CD to obtain the same amount of liquidity. This figure illustrates the construction of observationally equivalent equilibria.

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Figure B1. Set of Equilibria CD Contracts.

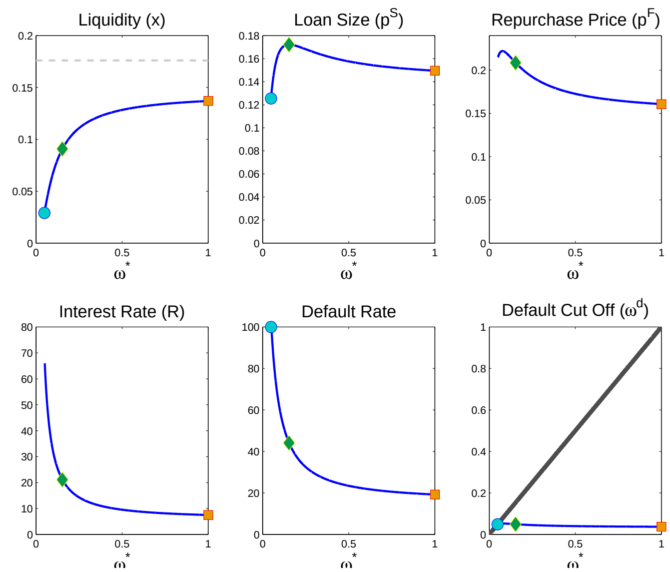
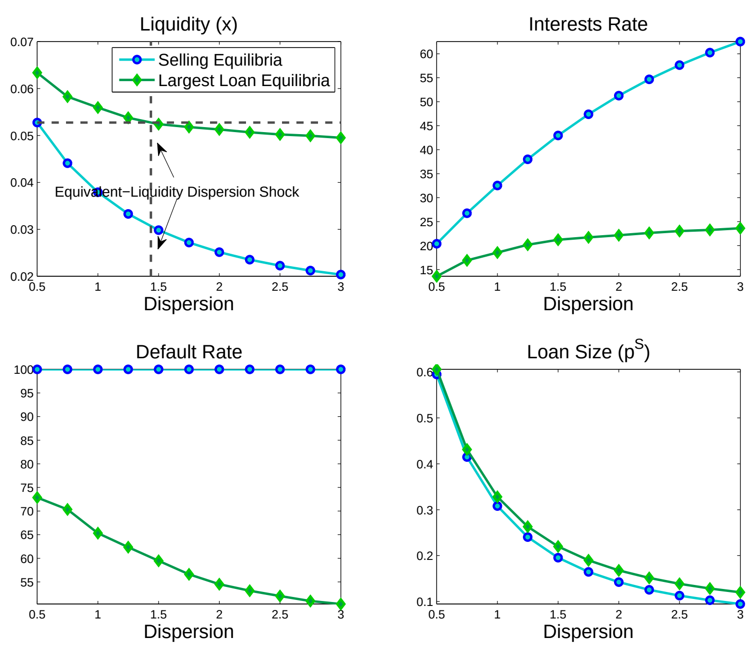


Figure B2. Observational Equivalence between Outright Sales and CD Contracts.



B2. A Glance at Recursive Competitive Equilibria

Endogenous liquidity. Figure B3 presents four equilibrium objects in each panel. Within each panel, the four curves correspond to combinations A (high and low) and ϕ (high and low). The x-axis of each panel is the aggregate capital stock, the endogenous state.

The top panels describe the equilibrium liquid funds per unit of capital, x, for both entrepreneur types. Given a combination of TFP and dispersion shocks, liquidity per unit of capital decreases with the aggregate capital stock (although its total value increases) for both types. For p-entrepreneurs, this negative relationship follows from decreasing marginal profits in the aggregate capital stock. With lower marginal benefits from increasing liquidity, p-entrepreneurs have less incentives to sell capital under asymmetric information. Comparing the curves that correspond to low and high dispersion shocks, we observe that liquidity falls with dispersion. As explained in Section II, increases in the quality dispersion increases the shadow cost of selling capital under asymmetric information. In contrast, TFP has the opposite effect. These results are clear from equation (5) which captures the tradeoffs in the choice of liquidity. An analogous pattern is found for i-entrepreneur's liquidity. The reason is that the demand for investment is weaker when the capital stock is greater or TFP is low.

Hours, consumption, investment, and output. As dispersion reduces the liquidity of producers, their effective demand for hours falls, causing a reduction in output. When TFP or the capital stock are high, hours and output are higher, as in any business cycle model. The figure also shows the negative effects of dispersion shocks on investment. With less liquidity available, the supply of investment claims shrinks. The reduction in the liquidity of p-entrepreneurs has ambiguous effects on their profits because this reduces the amount of labor hired but, wages also fall. This ambiguous wealth effect implies that the demand for capital may increase after liquidity shortages. Also, the ambiguous wealth effect could also increase consumption because of the increase in the cost of investment. For the calibration, the overall effect involves a strong reduction in investment, consumption, and hours together with an increase in the price of capital, q, as we should expect in a recession. The subsequent section discusses the ingredients that are needed for this result.

The analysis shows how the low correlation between Tobin's Q and investment is determined by two counterbalancing forces as in Lorenzoni and Walentin (2009). The first is TFP, which produces a positive correlation between Q and investment. The second is dispersion, which causes an increase in Tobin's Q together with a reduction in investment. This shows the connection among the six business cycle facts discussed in the Introduction.

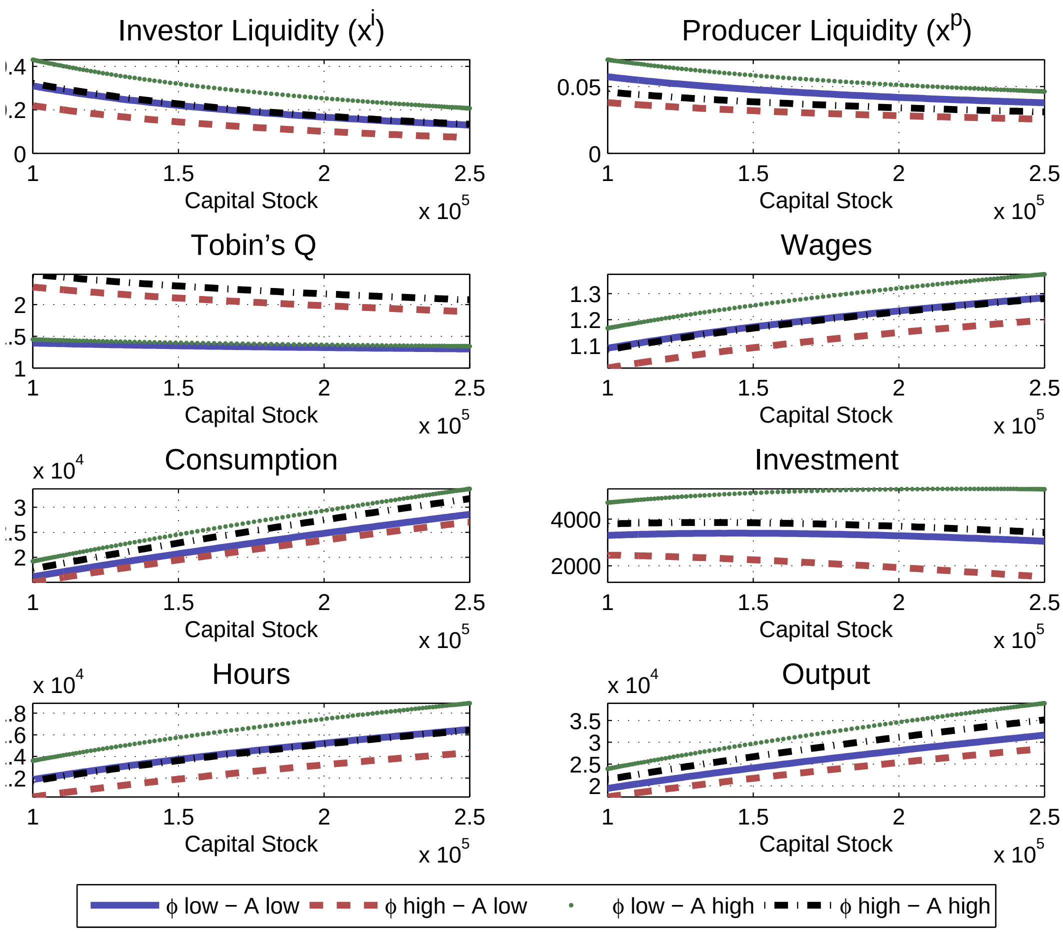


Figure B3. Equilibrium Variables across State-Space.

Additional Proofs

C1. Proof of Proposition 1

Rearranging the incentive compatibility constraints in the problem consists of solving:

$$
\begin{aligned}
r(x) &= \max_{l \geq 0, \sigma \in [0,1]} A l^{1-\alpha} - wl \\
& \quad \text{subject to} \\
\sigma w l &\leq \theta^L A l^{1-\alpha} \text{ and } (1-\sigma) wl \leq x.
\end{aligned}
$$

Denote the solutions to this problem by (l^*, σ^*). The unconstrained labor demand is $l^{unc} \equiv \left[\frac{A(1-\alpha)}{w}\right]^{\frac{1}{\alpha}}$. A simple manipulation of the constraints yields a pair of equations that characterize the constraint set:

$$
\begin{aligned}
l &\leq \left[A \frac{\theta^L}{\sigma w}\right]^{\frac{1}{\alpha}} \equiv l^1(\sigma) \qquad \text{(C1)} \\
l &\leq \frac{x}{(1-\sigma) w} \equiv l^2(\sigma) \qquad \text{(C2)} \\
\sigma &\in [0,1].
\end{aligned}
$$

As long as l^{unc} is not in the constraint set, at least one of the constraints will be active since the objective is increasing in l for $l \leq l^{unc}$. In particular, the tighter constraint will bind as long as $l \leq l^{unc}$. Thus, $l^* = \min\left\{l^1(\sigma^*), l^2(\sigma^*)\right\}$ if $\min\left\{l^1(\sigma^*), l^2(\sigma^*)\right\} \leq l^{unc}$ and $l^* = l^{unc}$ otherwise. Therefore, note that (C1) and (C2) impose a cap on l depending on the choice of σ. Hence, in order to solve for l^*, we need to know σ^* first. Observe that (C1) is a decreasing function of σ. The following properties can be verified immediately:

(C3)

$$
\lim_{\sigma \to 0} l^1(\sigma) = \infty \text{ and } l^1(1) = \left(\frac{\theta^L}{(1-\alpha)}\right)^{\frac{1}{\alpha}} \left[\frac{A}{w}(1-\alpha)\right]^{\frac{1}{\alpha}} = \left(\frac{\theta^L}{(1-\alpha)}\right)^{\frac{1}{\alpha}} l^{unc}.
$$

The second constraint curve (C2) presents the opposite behavior. It is increasing and has the following limits,

$$
l^2(0) = \frac{x}{\omega} \text{ and } \lim_{\sigma \to 1} l^2(\sigma) = \infty.
$$

These properties imply that $l^1(\sigma)$ and $l^2(\sigma)$ will cross at most once if $x > 0$. Because the objective is independent of σ, the entrepreneur is free to choose σ that makes l the largest value possible. Since $l^1(\sigma)$ is decreasing and $l^2(\sigma)$ increasing, the optimal choice of σ^* solves $l^1(\sigma^*) = l^2(\sigma^*)$ to make l as large as possible. This implies that both constraints will bind if one of them binds.

Adding them up, we find that $l^{cons}(x)$ is the largest solution to

$$\theta^L A l^{1-\alpha} - wl = -x. \tag{C4}$$

This equation defines $l^{cons}(x)$ as the largest solution of this implicit function. If $x = 0$, this function has two zeros. Restricting the solution to the largest root prevents us from picking $l = 0$. Thus, if $x = 0$, then $\sigma = 1$ and l solves $wl = \theta^L A l^{1-\alpha}$. This is the largest l within the constraint set of the problem.

Thus, we have that,

$$l^*(x) = \min\left\{l^{cons}(x), l^{unc}\right\}.$$

Since $l^1(\sigma)$ is monotone decreasing, if $\theta^L \geq (1-\alpha)$, then, $l^1(1) \geq l^{unc}$, by (C3). Because for $x > 0$, $l^1(\sigma)$ and $l^2(\sigma)$ cross at some $\sigma < 1$, then, $l^{cons} > l^{unc}$ and $l^* = l^{unc}$. Moreover, if $x = 0$, then the only possibility implied by the constraints of the problem is to set $\sigma = 1$. But since, $l^1(1) \geq l^{unc}$, then $l^* = l^{unc}$. Thus, we have shown that $\theta^L \geq (1-\alpha)$ is sufficient to guarantee that labor is efficient for any x. This proves the second claim in the proposition.

Assume now that $l^{unc} \leq \frac{x}{w}$. Then, the wage bill corresponding to the efficient employment can be guaranteed upfront by the entrepreneur. Obviously, $x \geq wl^{unc}$ is sufficient for optimal employment.

To pin down the necessary condition for the constraint to bind, observe that the profit function in (C4) is concave with a positive interior maximum. Thus, at $l^{cons}(x)$, the left-hand side of (C4) is decreasing. Therefore, if $l^{cons}(x) < l^{unc}$, then it should be the case that $\theta^L A\left(l^{unc}\right)^{1-\alpha} - wl^{unc} < -x$. Substituting the formula for l^{unc} yields the necessary condition for the constraints to be binding:

$$x < w^{1-\frac{1}{\alpha}}\left[A(1-\alpha)\right]^{\frac{1}{\alpha}}\left(1 - \frac{\theta^L}{(1-\alpha)}\right).$$

This shows that if $\theta^L < (1-\alpha)$, the amount of liquidity needed to have efficient employment is positive.

Figure C1 provides a graphical description of the arguments in this proof. The left panel plots l^1 and l^2 as functions of σ. It is clear from the figure that the constraint set is largest at the point where both curves meet. If l^{unc} is larger than the point where both curves meet, then, the optima is constrained. A necessary condition for constraints to be binding is that l^{unc} is above $1^2(1)$, otherwise l^{unc} will lie above. A sufficient condition for constraints to be binding is described in the right panel. The dashed line represents the left hand side of (C4) as a function of labor. The figure shows that when the function is evaluated at l^{unc}, and the result is below $-x$, then the constraints are binding.

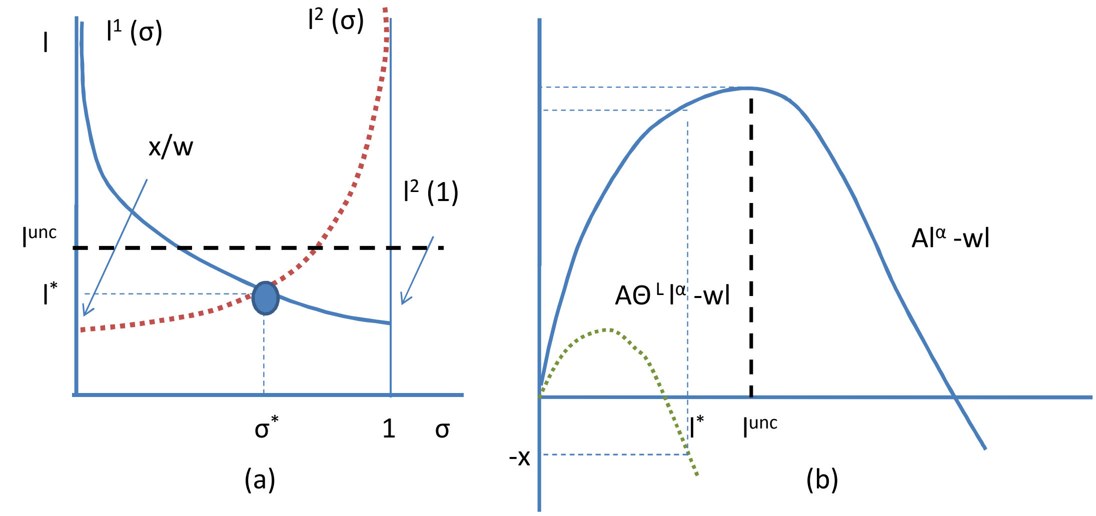


Figure C1. Derivation of Labor Constraint.

C2. Proof of Lemma 2

This Lemma is an application of the Principle of Optimality. By homogeneity, given a labor-capital ratio l/k, p-entrepreneur profits are linear in capital stock:

$$\left[A\left(l/k\right)^{1-\alpha}-w\left(l/k\right)+x\right]k. \tag{C5}$$

Observe that once x is determined by the choice of $\iota\left(\omega\right)$, the incentive compatibility constraint (1) and the working capital constraint (3) can be expressed in terms of the labor-capital ratio only:

$$A\left(l/k\right)^{1-\alpha}-\sigma w\left(l/k\right)\geq\left(1-\theta^{L}\right)A\left(l/k\right)^{1-\alpha} \tag{C6}$$

and

$$\left(1-\sigma\right)w\left(l/k\right)\leq x. \tag{C7}$$

l and σ don't enter the entrepreneur's problem anywhere else. Thus, optimally, the entrepreneur will maximize expected profits per unit of capital in (C5) subject to (C6) and (C7). This problem is identical to Problem 2. Thus, the value of profits for the entrepreneur considering the optimal labor-to-capital ratio is $r\left(x;w\right)k$.

Substituting this value into the objective of Problem 1 yields the following objective

$$W^{p}(k;p,q,w)=\max_{\iota(\omega)\geq 0} r\left(x;w\right)k+xk+qk\int\lambda\left(\omega\right)\left(1-\iota\left(\omega\right)\right)f_{\phi}\left(\omega\right)d\omega \tag{C8}$$

subject to:

$$x=p\int\iota\left(\omega\right)d\omega$$

where $r\left(x;w\right)$ is the value of Problem 2 which shows. Lemma 2.

C3. *Proof of Proposition 3*

The proof requires some preliminary computations. Note that the choice of ι determines x. In addition, Lemma 2 shows that the entrepreneur's profits are linear in the entrepreneur's capital stock. Thus, the following computations are normalized to the case when $k=1$.

Labor and liquidity. For any x such that $l^{*}\left(x\right)=l^{unc}$, the constraints (2) and (3) are not binding. Therefore, when x is sufficiently large to guarantee the efficient amount of labor per unit of capital, an additional unit of liquidity does not increase $r\left(x\right)$. For x below the amount that implements the efficient level of labor, both constraints are binding. Applying the Implicit Function Theorem to the pseudo-profit function (C4) yields an expression for the marginal increase in labor with a marginal increase in liquidity,

$$\frac{\partial l^{cons}}{\partial x}=-\frac{1}{\left(1-\alpha\right)\theta^{L}Al\left(x\right)^{-\alpha}-w}.$$

Note that the denominator satisfies,

$$\left(1-\alpha\right)\theta^{L}Al^{-\alpha}-w\leq\frac{\left[\theta^{L}Al^{1-\alpha}-wl\right]}{l}=\frac{-x}{l}<0,$$

which verifies that $\frac{\partial l^{cons}}{\partial x}>0$.

Marginal profit of labor. Let $\Pi\left(l\right)=Al^{1-\alpha}-wl$. The marginal product of labor is,

$$\Pi_{l}\left(l\right)=A\left(1-\alpha\right)l^{-\alpha}-w>0\text{ for any }l<l^{unc}.$$

Marginal profit of liquidity. Using the chain rule, we have an expression for the marginal profit obtained from an additional unit of liquidity.

$$r_{x}\left(x\right)=\Pi_{l}\left(l^{*}\left(x\right)\right)l^{*\prime}\left(x\right)=-\frac{A\left(1-\alpha\right)l^{*}\left(x\right)^{-\alpha}-w}{\left(1-\alpha\right)\theta^{L}Al^{*}\left(x\right)^{-\alpha}-w},\ l^{*}\left(x\right)\in\left(l^{cons}\left(0\right),l^{unc}\right)$$

and 0 otherwise.

Thus, liquidity has a marginal value for the entrepreneur whenever constraints are binding. Since $l^{*}\left(x\right)$ is the optimal labor choice, $\Pi\left(l^{*}\left(x\right)\right)=r\left(x\right)$, which explains the first equality $r_x(x) = \Pi_l(l^*(x))\, l^{*\prime}(x)$. Since $A(1-\alpha)\, l(x)^{-\alpha} - w$ approaches 0 as $l(x) \to l^{unc}$, $r_x(x) \to 0$, as x approaches its efficient level. Hence, $r_x(x)$ is continuous and $r(x)$ is everywhere differentiable. The marginal value of liquidity, $r_x(x)$, is decreasing in x ($r_{xx}(x) < 0$) since the numerator is decreasing and the denominator is increasing in x.

Equilibrium liquidity. To establish the result in Proposition 3, observe that as in the standard lemons problem in Akerlof (1970), if any capital unit of quality ω is sold in equilibrium, all the units of lower quality must be sold. Otherwise, the entrepreneur would be better off by substituting high-quality units and selling low-quality units instead. A formal argument requires dealing with jumps but the essence does not change.

Thus a cutoff rule defines a threshold quality ω^* for which all qualities below ω will be sold. Choosing the qualities to be sold is equivalent to choosing a threshold quality ω^* to sell. The entrepreneur chooses that threshold to maximize his objective function. Thus, ω^p solves:

$$\omega^p = \arg\max_{\omega^*} r(x)\, k + x + qk \int_{\omega^*}^{1} \lambda(\omega)\, f_\phi(\omega)\, d\omega$$

where

$$x = p^p \int_0^{\omega^*} \iota(\omega) f_\phi(\omega)\, d\omega.$$

The objective function is continuous and differentiable, as long as $f_\phi(\omega)$ is absolutely continuous. Thus, interior solutions are characterized by first order conditions. Substituting x, in $r(x)$ and taking derivatives yields the following first order condition:

$$(1 + r_x(x))\, p f_\phi(\omega^*) - q\lambda(\omega^*)\, f_\phi(\omega^*) \geq 0 \text{ with equality if } \omega^* \in (0,1). \tag{C9}$$

Qualities where $f_\phi(\omega^*) = 0$ are saddle points of the objective function, so without loss of generality $f_\phi(\omega^*)$ is canceled from both sides. There are three possibilities for equilibria: $\omega^* = 1, \omega^* \in (0,1)$, or $\omega^* \neq \varnothing$, where the latter case is interpreted as no qualities are sold. Thus, substituting the zero-profit condition for financial intermediaries, $pF(\omega^*) = q\mathbb{E}_\phi[\lambda(\omega) | \omega \leq \omega^*]\, F(\omega^*)$, we obtain that C9 becomes

$$(1 + r_x(x))\, \mathbb{E}_\phi[\lambda(\omega) | \omega \leq \omega^*] > \lambda(\omega^*).$$

In equilibrium, ω^* must belong to one of the following cases:

Full liquidity. If $\omega^* = 1$, then it must be the case that

$$\left(1 + r_x\left(q\bar{\lambda}\right)\right) \bar{\lambda} \geq \lambda(1). \tag{C10}$$

This condition is obtained by substituting $\omega^* = 1$ into C9. If this condition is violated, by continuity of r_x, the entrepreneur could find a lower threshold ω^* that maximizes the value of his wealth.

Interior solutions. For an interior solution $\omega^* \in [0, 1)$, it must be the case that

$$(1 + r_x(x))\,\mathbb{E}_\phi\left[\lambda(\omega)\,|\omega \le \omega^*\right] = \lambda(\omega^*) \tag{C11}$$

for $x = q\mathbb{E}_\phi\left[\lambda(\omega)\,|\omega \le \omega^*\right] F(\omega^*)$. Since $r_x(x)$ is continuous and decreasing, if the condition does not hold, the entrepreneur can be better of with a different cutoff.

Market Shutdowns. Finally, as in any lemons problem, there exists a trivial market shutdown equilibrium with $\omega^* = \emptyset$, and $p^p = 0$.

C4. Proof of Proposition 5

Since, we can factor k from the objective in (C8) to obtain

$$W^p(k; p, q, w) = k\left(\max_{\iota(\omega)\ge 0} r(x; w) + x + q\int \lambda(\omega)(1 - \iota(\omega)) f_\phi(\omega)\,d\omega\right). \tag{C12}$$

For the optimal choice of $\iota(\omega)$, call it $\iota^*(\omega)$, zero profits for the intermediary require:

$$p\int_0^1 \iota^*(\omega) f_\phi(\omega)d\omega = q\int_0^1 \lambda(\omega)\,\iota^*(\omega) f_\phi(\omega)d\omega.$$

Substituting this condition into (C12) the objective yields:

$$\begin{aligned}
W^p(k; p, q, w) &= k\left(r(x; w) + q\int_0^1 \lambda(\omega)\,\iota^*(\omega) f_\phi(\omega)d\omega + q\int \lambda(\omega)(1 - \iota^*(\omega)) f_\phi(\omega)\,d\omega\right)\\
&= k\left(r(x; w) + q\bar{\lambda}\right).
\end{aligned}$$

This shows that $W^p(k; p, q, w)$ can be written as $W^p(k; p, q, w) = \tilde{W}^p(p, q, w)k$ if

$$\tilde{W}^p(p, q, w) \equiv r(x; w) + q\bar{\lambda}.$$

Here, $r(x; w)$ is the solution to Problem 1 and x, p and ω^* are given by Proposition 3.

C5. Proof of Proposition 4

Note that $\frac{\lambda(\omega^*)}{\mathbb{E}_\phi[\lambda(\omega)|\omega\le\omega^*]}$ is increasing. Under the assumptions, the advantage rate is 1 when $\omega^* = 0$. At $\omega^* = 1$, the advantage rate is greater than 1. In contrast, $1 + r_x\left(q\mathbb{E}_\phi\left[\lambda(\omega)\,|\omega \le \omega^*\right]\right)$ is decreasing in ω^*, and starts at a number greater than 1. Thus, if the two curves cross, they must cross at a single point. Otherwise, if they don't cross, $\omega^* = 1$ is an admissible solution.

C6. Proof of Proposition 6

The proof of Proposition 6 is similar to one that appears in Bigio (2009) and relies on linear programming. Once $\iota(\omega)$ and x are determined, the problem of the i-entrepreneur becomes:

$$\hat{k}(x) = \max_{i^d, i^s} i - i^s + k^b$$

subject to,

$$i = i^d + qi^s$$
$$\theta^I i \geq i^s$$
$$qk^b + i^d \leq xk.$$

To solve this linear program we substitute for i to obtain an objective equal to:

$$\hat{k}(x) = \max_{k^b, i^d, i^s} i^d + (q-1)\, i^s + k^b$$
$$\theta^I i^d \geq \left(1 - q\theta^I\right) i^s$$
$$qk^b + i^d \leq xk.$$

Here, there are several cases: (i) When $q = 1$ the objective becomes $i^d + k^b$, and the working capital constraint becomes $k^b + i^d \leq xk$. Since i^s reduces the objective, $i^s = 0$. Hence, the value of the problem is $\hat{k}(x) = xk$, and policies are indeterminate. (ii) When $q > 1/\theta^I$, the value of the problem is indeterminate since $i^s \to \infty$ is feasible. This clearly is a solution that cannot be part of an equilibrium. (iii) If $q \in [0, 1)$, $i^s = 0, i^d = 0$ and $k^b = xk/q$. The value of the problem is $\hat{k}(x) = xk/q$. Finally, when $q \in (1, 1/\theta^I)$, we obtain that $i^d = xk$, $k^b = 0$, and $\theta^I i^d = \left(1 - q\theta^I\right) i^s$. Substituting for i^s, the objective of the problem becomes: $i^d + \frac{(q-1)\theta^I}{\left(1-q\theta^I\right)} i^d = \frac{\left(1-\theta^I\right)}{\left(1-q\theta^I\right)} i^d$. Hence, $\hat{k}(x) = \frac{\left(1-\theta^I\right)}{\left(1-q\theta^I\right)} xk$. Using the definition in the text we obtain $\hat{k}(x) = \left(q^R\right)^{-1} xk$. Thus, if $q \in [1, 1/\theta^I)$, $\hat{k}(x) = \left(q^R\right)^{-1} xk$.

C7. Proof of Proposition 7

The proof of Proposition 7 is similar to the proof of Proposition 3. Thus, I skip minor details. There is only one distinction. Due to the linearity in the production of capital and the constraints, in this case, the marginal value of an additional unit of liquidity is constant and equal to $\frac{q(x)}{q^R(x)}$, or Tobin's q. From Proposition 6 we know that for values of $q \in [1, 1/\theta)$ the value of the optimal financing problem is $\hat{k}(x) = \left(q^R\right)^{-1} xk$. Thus, the value of Problem 3 becomes:

$$W^i(k; p, q) = \max_{\iota(\omega)} \left(q^R\right)^{-1} xk + \int_0^1 (1 - \iota(\omega))\, \lambda(\omega)\, k f_\phi(\omega)\, d\omega$$

subject to:

$$x = p \int_0^1 \iota(\omega) f_\phi(\omega) d\omega.$$

Following the same steps as in the proof of steps of Proposition 3, we can argue that the equilibrium is determined by a threshold quality, ω^i. Substituting x:

$$W^i(k; p, q) = \max_{\omega^i} \left(q^R\right)^{-1} p \left(\int_0^{\omega^i} f_\phi(\omega) d\omega\right) k + \left(\int_{\omega^i}^1 \lambda(\omega) f_\phi(\omega) d\omega\right) k. \tag{C13}$$

Taking first order conditions yields:

$$\left(q^R\right)^{-1} p f_\phi\left(\omega^i\right) k \geq \lambda\left(\omega^i\right) f_\phi\left(\omega^i\right) k$$

and by substituting the zero-profit condition for intermediaries yields:

$$\left(q^R\right)^{-1} q \mathbb{E}_\phi\left[\lambda(\omega) | \omega \leq \omega^i\right] \geq \lambda\left(\omega^i\right)$$

which is the desired condition. The three cases in the statement of the proposition also follow from the proof of Proposition 3.

C8. Proof of Proposition 8

From equation (C13), the objective of the entrepreneur can be written as:

$$\begin{aligned}
& \left[\left(q^R\right)^{-1} p F\left(\omega^i\right) + \int_{\omega^i}^1 \lambda(\omega) k f_\phi(\omega) d\omega\right] k \\
= & \left[\left(q^R\right)^{-1} q \mathbb{E}_\phi\left[\lambda(\omega) | \omega \leq \omega^i\right] F\left(\omega^i\right) + \int_{\omega^i}^1 \lambda(\omega) k f_\phi(\omega) d\omega\right] k \\
= & \frac{1}{q^R}\left[q \int_0^{\omega^i} \lambda(\omega) k f_\phi(\omega) d\omega + q^R \int_{\omega^i}^1 \lambda(\omega) k f_\phi(\omega) d\omega\right] k \\
\equiv & \tilde{W}^i(q) k.
\end{aligned}$$

where the second line follows from the zero-profit condition for intermediaries.

C9. Proof of Proposition of 9

Given a set of prices (p^S, p^F, q) a p-entrepreneur maximizes,

$$\begin{aligned}
W^p(k) = & \max_{I(\omega), \iota(\omega)} r(x) k + xk + \ldots \\
& k \int_0^1 (1 - I(\omega)) \iota(\omega) \left(q\lambda(\omega) - p^F\right) + (1 - \iota(\omega)) q\lambda(\omega) f(\omega) d\omega
\end{aligned}$$

subject to:

$$x = p^S \int_0^1 \iota(\omega) f(\omega) d\omega.$$

Let $\Omega^D \equiv \{\omega : I(\omega) = 1, \iota(\omega) = 1\}$ be the set of qualities that feature a default in a CD market equilibrium. Let $\Omega^{ND} \equiv \{\omega : I(\omega) = 0, \iota(\omega) = 1\}$. Finally, let $\Omega \equiv \Omega^D \cup \Omega^{ND}$. The first step is to show that if a given quality is defaulted, all lower qualities will feature participation and default. This means that $I(\cdot)$ is decreasing almost everywhere. The second is to show that without loss of generality we can treat $\iota(\cdot)$ as decreasing almost everywhere. By an almost-everywhere decreasing function I mean that there exist two intervals $[0, \omega^o]$ and $[\omega^o, 1]$ such that the function is 1 almost everywhere in $[0, \omega^o]$ and $I = 0$ in $(\omega^o, 1]$.

The value of the objective of the entrepreneur can be expressed in terms for these sets:

$$V = x + r(x, X) + \int_{\Omega^{ND}} \left(q(X)\lambda(\omega) - p^F\right) f(\omega) d\omega + \int_{[0,1]\setminus\Omega} q\lambda(\omega) f(\omega) d\omega$$

with

$$x = \int_{\Omega^{ND}} p^S d\omega + \int_{\Omega^D} p^S d\omega.$$

Suppose $I(\cdot)$ is not decreasing almost everywhere. Then, we can find two intervals: $(\omega_{N_1}, \omega_{N_2})$ and $(\omega_{D_1}, \omega_{D_2})$ such that $I = 0$ almost everywhere in $(\omega_{N_1}, \omega_{N_2})$ and $I = 1$ almost everywhere in $(\omega_{D_1}, \omega_{D_2})$. Moreover, since $f(\omega)$ is continuous, we can find intervals of same measure. We want to show that if $I(\cdot)$ is non-monotone, the p-entrepreneur is not optimizing. The strategy consists of setting $I = 1$ in $(\omega_{D_1}, \omega_{D_2})$ and vice versa in $(\omega_{N_1}, \omega_{N_2})$ to show that this improves his value. Since both sets have the same measure, x remains invariant and only the first integral in the objective changes with the policy perturbation. The value of the integral terms in the objective is then:

$$\begin{aligned}
& \int_{\Omega^{ND}\setminus(\omega_{D_1},\omega_{D_2})} \left(q\left(X\right)\lambda\left(\omega\right)-p^F\left(\omega\right)\right) f\left(\omega\right)d\omega + \int_{(\omega_{N_1},\omega_{N_2})} \left(q\left(X\right)\lambda\left(\omega\right)-p^F\right) f\left(\omega\right)d\omega \\
= & \int_{\Omega^{ND}\setminus(\omega_{D_1},\omega_{D_2})} \left(q\left(X\right)\lambda\left(\omega\right)-p^F\left(\omega\right)\right) f\left(\omega\right)d\omega + \int_{(\omega_{N_1},\omega_{N_2})} q\left(X\right)\lambda\left(\omega\right) f\left(\omega\right)d\omega ... \\
& +p^F\left[F\left(\omega_{N_2}\right)-F\left(\omega_{N_1}\right)\right] \\
> & \int_{\Omega^{ND}\setminus(\omega_{D_1},\omega_{D_2})} \left(q\left(X\right)\lambda\left(\omega\right)-p^F\left(\omega\right)\right) f\left(\omega\right)d\omega + \int_{(\omega_{D_1},\omega_{D_2})} q\left(X\right)\lambda\left(\omega\right) f\left(\omega\right)d\omega + ... \\
& p^F\left[F\left(\omega_{N_2}\right)-F\left(\omega_{N_1}\right)\right] \\
= & \int_{\Omega^{ND}\setminus(\omega_{D_1},\omega_{D_2})} \left(q\left(X\right)\lambda\left(\omega\right)-p^F\left(\omega\right)\right) f\left(\omega\right)d\omega + \int_{(\omega_{D_1},\omega_{D_2})} q\left(X\right)\lambda\left(\omega\right) f\left(\omega\right)d\omega + ... \\
& p^F\left[F\left(\omega_{D_2}\right)-F\left(\omega_{D_1}\right)\right].
\end{aligned}$$

The first line is the value of the alternative strategy for the entrepreneur. The second line is an algebraic manipulation of the integral. The third follows from the monotonicity of λ, which holds by assumption. The third follows from the equivalence in the lengths of both intervals. The inequality shows that a non-monotone default strategy violates optimality.

We now turn to the non-monotonocity of $\iota\left(\omega\right)$. Observe that if $\iota\left(\omega\right)=1$ and $I\left(\omega\right)=0$, then the entrepreneur and the intermediary are indifferent between which qualities are brought to the contract. Collateral will be repurchased. Thus, without loss in generality, we can restrict attention to a decreasing $\iota\left(\omega\right)$. Thus, there are two threshold qualities: ω^p and $\bar{\omega}^p$. The first, defines a cutoff under which all qualities are defaulted. The second is a participation cutoff. An equilibrium where $\omega^p=\bar{\omega}^p$ is identical to the sales-only contract of Section II. Hence, $\omega^p\leq\bar{\omega}^p$. Thus, the objective for the entrepreneur becomes:

$$V = x + r\left(x\right) + \int_{\omega^p}^{\bar{\omega}^p} \left(q\lambda\left(\omega\right)-p^F\right)d\omega + \int_{\bar{\omega}^p}^{1} q\lambda\left(\omega\right)d\omega$$

subject to

$$x=\int_0^{\bar{\omega}^p} p^S d\omega.$$

The first-order conditions for ω^p is

$$q\left(X\right)\lambda\left(\omega^p\right)-p^F\geq 0, \tag{C14}$$

but since λ is continuous and ω^p interior, the equation holds with equality. The first-order condition for $\bar{\omega}^p$ is:

$$\begin{aligned}
(1 + r_x(x)) p^S &\geq \left(p^F - q\lambda(\bar{\omega}^p)\right) + q\lambda(\bar{\omega}^p) \rightarrow \\
r_x(x) p^S &\geq \left(p^F - p^S\right). \qquad \text{(C15)}
\end{aligned}$$

Finally, the zero-profit condition written in terms of ω^p and $\bar{\omega}^p$ yields:

$$p^F = \int_0^{\omega^p} q\lambda(\omega, \phi)\, d\omega + p^S \int_{\omega^p}^{\omega^*} d\omega. \qquad \text{(C16)}$$

Equations (C14), (C15) and (C16) correspond to the equations that characterize equilibria.

C10. Obtaining Equivalent Problems 7 and 8

By substituting the capital accumulation equation into the p-entrepreneur's budget constraint to obtain the following equivalent problem:

$$V^p(k, X) = \max_{c \geq 0, k' \geq 0, \iota(\omega), l, \sigma \in [0,1]} U(c) + \beta \mathbb{E}\left[V^j(k', X') | X\right], \ j \in \{i, p\}$$

subject to

$$c + q(X) k' = AF(k, l) - \sigma w(X) l + xk - (1-\sigma) w(X) l + q(X) \int_0^1 (1 - \iota(\omega)) \lambda(\omega) k f_\phi(\omega)\, d\omega$$

$$AF(k, l) - \sigma w l \geq \left(1 - \theta^L\right) A k^\alpha l^{1-\alpha}$$

$$(1 - \sigma) w l \leq xk$$

$$x = p^p(X) \int_0^1 \iota(\omega)\, d\omega.$$

His objective function is a function of c and k' and does not appear in the constraints below the budget constraint. In contrast, the choice of $\iota(\omega), l, \sigma$ only affects the right-hand side of the consolidated budget constraint and is constrained through the additional constraints. Thus, the entrepreneur maximizes his value function by choosing $\iota(\omega), l, \sigma$ to maximize the right-hand side of his budget constraint. This problem is identical to Problem 1. Therefore, we can re-write the p-entrepreneur's problem as:

$$V^p(k, X) = \max_{c \geq 0, k' \geq 0, \iota(\omega), l, \sigma \in [0,1]} U(c) + \beta \mathbb{E}\left[V^j(k', X') | X\right], \ j \in \{i, p\}$$

subject to

$$c + q(X) k' = \tilde{W}^p(X) k$$

where $\tilde{W}^p(X)$ is the marginal value of capital in Proposition 5 for prices $p(X), q(X)$

are $w(X)$. This is a consumption-savings problem with linear returns. Similar steps can be followed to obtain the value for i-entrepreneurs in Proposition 8.

C11. Proof of Proposition 11

Both statements of Proposition 11 follow from previous Propositions. I first prove the statements about labor inefficiency for any arbitrary state X. From Proposition 1, we know that if $\theta^L \geq (1-\alpha)$, then the labor-to-capital ratio of the individual entrepreneur is efficient for any choice of x. This proves the only if part. Instead, if $\theta^L < (1-\alpha)$, we know also from Proposition 1 that some positive amount of liquidity is needed to have the efficient labor-to-capital ratio. It is sufficient to show that amount is not obtained in equilibrium. From Proposition 3 we know that ω^p must satisfy

$$(1+r_x(x))\,\mathbb{E}_\phi\left[\lambda(\omega)\,|\omega \leq \omega^p\right] \geq \lambda(\omega^p)\,.$$

However, from Proposition 1 we also know that efficient employment implies that $r_x(x)=0$. Thus, the above condition becomes $\mathbb{E}_\phi\left[\lambda(\omega)\,|\omega \leq \omega^p\right] \geq \lambda(\omega^p)$ which by Assumption 2 implies that this is true only for $\omega^p=0$. This in turn implies that $x = q(X)\,\mathbb{E}_\phi\left[\lambda(\omega)\,|\omega \leq 0\right] F(0) = 0$. By Proposition 1 employment cannot be efficient as it requires some positive amount of liquidity.

I now prove the result for investment. Assume that $q(X)=1$ and, thus, $q^R(X)=1$. Therefore, by Proposition 7 we have that,

$$\mathbb{E}_\phi\left[\lambda(\omega)\,|\omega \leq \omega^i\right] = \lambda\left(\omega^i\right)$$

which implies that $\omega^i=0$. This in turn implies $x^i=0$ and, consequently, $i^d=0$ from Proposition 3. Since $i^d=0 \to i=0$, we have that aggregate investment cannot be positive.

C12. Proof of Proposition 12

Substitute the optimal policies described in Proposition 10 into the expression for $D(X)$ and $S(X)$ to obtain $I^s(X)=D(X)-S(X)$. Then use (37), (38) and (39) to clear out expressions for $I^s(X)$ and $I(X)$. In the proof the state X is fixed so I drop the arguments from the functions. Performing these substitutions, the aggregate version of the incentive compatibility condition becomes:

$$\frac{(1-\pi)\left(\varsigma^p\left(r+q\psi^p\right)/q-\psi^p\right)K-(1-\pi)\,\varphi^p K-\pi\varphi^i K}{\theta} \leq \frac{\pi\left[\varsigma^i\left(W^i\right)K-\psi^i K\right]}{(1-\theta)}.$$

I have introduced the following variables:

$$\begin{array}{ll} \varphi^p = \int_{\omega\leq\omega^p}\lambda(\omega)\,f_\phi(\omega)\,d\omega & \varphi^i = \int_{\omega\leq\omega^i}\lambda(\omega)\,f_\phi(\omega)\,d\omega \\ \psi^p = \int_{\omega>\omega^p}\lambda(\omega)\,f_\phi(\omega)\,d\omega & \psi^i = \int_{\omega>\omega^i}\lambda(\omega)\,f_\phi(\omega)\,d\omega \end{array}$$

that correspond to the expectations over the sold and unsold qualities of both groups. K clears out from both sides. I then use the definition of q^i and rearrange the expression to obtain:

$$\begin{aligned}\frac{(1-\pi)\varsigma^p r - \left((1-\pi)\left(1-\varsigma^p\right)\psi^p + (1-\pi)\varphi^p + \pi\varphi^i\right)q}{\theta q} &\leq \frac{\pi\left[\varsigma^i q\varphi^i - \left(1-\varsigma^i\right)\psi^i q^R\right]}{(1-\theta)\,q^R} \\ &\leq \frac{q\pi\varsigma^i\varphi^i}{(1-\theta q)} - \frac{\pi\left(1-\varsigma^i\right)\psi^i}{(1-\theta)}.\end{aligned}$$

I get rid of q from the denominators, rearrange terms and obtain,

$$\begin{aligned}& (1-\pi)\varsigma^p r\,(1-\theta q) - \left((1-\pi)\left(\left(1-\varsigma^p\right)\psi^p + \varphi^p\right) + \pi\varphi^i\right)q\,(1-\theta q) \\ \leq\ & \theta q^2\pi\varsigma^i\varphi^i - \theta q\,(1-\theta q)\,\pi\frac{\left(1-\varsigma^i\right)\psi^i}{(1-\theta)}.\end{aligned}$$

By arranging terms, the inequality includes linear and quadratic terms for q. This expression takes the form:

$$(q^*)^2 A + q^* B + C \geq 0 \tag{C17}$$

where the coefficients are:

$$A = -\theta\left((1-\pi)\left(\left(1-\varsigma^p\right)\psi^p + \varphi^p\right) + \pi\left(1-\varsigma^i\right)\varphi^i - \pi\theta\frac{\left(1-\varsigma^i\right)}{(1-\theta)}\psi^i\right)$$

$$B = \theta(1-\pi)\varsigma^p r + \left((1-\pi)\left(\left(1-\varsigma^p\right)\psi^p + \varphi^p\right) + \pi\varphi^i - \pi\theta\frac{\left(1-\varsigma^i\right)}{(1-\theta)}\psi^i\right).$$

$$C = -(1-\pi)\varsigma^p r$$

C is negative. Observe that

$$
\begin{aligned}
& (1-\pi)\left((1-\varsigma^p)\psi^p+\varphi^p\right)+\pi\varphi^i-\pi\frac{\left(1-\varsigma^i\right)}{(1-\theta)}\psi^i\theta \\
\geq\; & (1-\pi)\left((1-\varsigma^p)\psi^p+\varphi^p\right)+\pi\left(1-\varsigma^i\right)\varphi^i-\pi\frac{\left(1-\varsigma^i\right)}{(1-\theta)}\psi^i\theta \\
\geq\; & (1-\pi)\left((1-\varsigma^p)\psi^p+\varphi^p\right)+\pi\left(1-\varsigma^i\right)\varphi^i-(1-\pi)\left(1-\varsigma^i\right)\psi^i \\
\geq\; & (1-\pi)\bar{\lambda}-(1-\pi)\varsigma^p\psi^p+\pi\left(1-\varsigma^i\right)\bar{\lambda}-\pi\left(1-\varsigma^i\right)\psi^i-(1-\pi)\left(1-\varsigma^i\right)\psi^i \\
\geq\; & \bar{\lambda}-(1-\pi)\varsigma^p\psi^p-\pi\psi^i \\
\geq\; & 0
\end{aligned}
$$

where the second line follows from the assumption that $(1-\theta)\geq\pi$. The third line uses the identity $\bar{\lambda}=\psi^p+\varphi^p=\psi^i+\varphi^i$. The fourth line uses the fact that $\left(1-\varsigma^i\right)<1$ and the last line uses the fact that ψ^p and ψ^i are less than $\bar{\lambda}$. This shows that A is negative and B is positive. Evaluated at 0, (C17) is negative. It reaches a maximum at $-\frac{B}{2A}>0$. Thus, both roots of (C17) are positive. Let the roots be (q_1,q_2) where q_2 is the largest. There are three possible cases:

Case 1: If $1\in(q_1,q_2)$, then $q=1$ satisfies the constraint.

Case 2: If $1<q_1$, then $q=q_1$, since it is the lowest price such that the constraints bind with equality.

Case 3: If $q_2<1$, then there exists no incentive compatible price. Thus, $I=0$ and i-entrepreneurs consume part of their capital stock.

C13. Proof of Proposition 13

An identical proposition is shown in Bigio (2009). The proof is standard for consumption-savings problems with stochastic linear returns and homothetic preferences. The proof also implies that the economy admits aggregation.

*

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