

For Online Publication
Supplementary Appendix for ‘Collective Dynamic
Choice: The Necessity of Time Inconsistency’

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November 2014

1 Proof for More General Consumption Patterns

Proof of the Paper’s Theorem 4:

As in the proof of Theorem 1 in the paper, we apply a theorem by Koopmans (1960, Section 14). First note that by the fact that each u_i is nondecreasing and increasing in some dimension on $[0, 1]^\ell$ and V satisfies unanimity, his postulates 2 and 5 are satisfied. Next, his postulate 1 follows from continuity (in fact, differentiability) of V under the metric $d(C, C') = \sup_t \|c_t - c'_t\|$. Finally, time consistency implies his postulates 3, 3', and 4. Thus, there exists $0 < \delta < 1$ and a continuous u such that $V[\delta_1, u_1; \dots; \delta_n, u_n](C) = \sum_t \delta^{t-1} u(c_t)$ for all C .

By unanimity, it follows that u is nondecreasing and is increasing in some arguments.

Assume now that it is twice differentiable and that $\delta_i \neq \delta_j$ for all $i \neq j$.

Without loss of generality, let us normalize u so that $u(0) = 0$ and $u(1) = 1$ (where the inputs are now vectors).

Step 1: There exists i such that $\delta = \delta_i$.

Proof of Step 1: This proof proceeds as in the proof of Theorem 1 in the paper, simply restricting attention to the domain of consumptions such that $c_{t,k} = c_{t,k'}$ for all k, k' , which then reduces the domain to one dimension at each time. The previous proof then goes through (writing utility functions as functions of one dimension and noting that they are all strictly increasing when restricted to this domain of consumptions) and then implies the desired result. Therefore, there exists i such that $\delta = \delta_i$.

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Step 2: Given that all agents have different discount factors, it follows that u is an affine transformation of u_i , where i is an agent with discount factor of $\delta_i = \delta$.

Proof of Step 2: Suppose the contrary, so that $\delta_i = \delta$ and yet u is not an affine transformation of u_i

It follows that there exist $x \in [0, 1]^\ell$, $y \in [0, 1]^\ell$, $z_1 \in \mathbb{R}^\ell$, and $z_2 \in \mathbb{R}^\ell$, such that for all sufficiently small a : $x + az_1 \in [0, 1]^\ell$, $y + az_2 \in [0, 1]^\ell$, and

$$u(x + az_1) + \delta u(y + az_2) > u(x) + \delta u(y)$$

while

$$u_i(x + az_1) + \delta u_i(y + az_2) < u_i(x) + \delta u_i(y).$$

Let $w = (.5, .5, \dots, .5)$ be an ℓ dimensional vector with all entries being $.5$.

Set

$$C = (w, w, w, x, y, 0, 0 \dots)$$

and

$$C^{\varepsilon, a} = (w(1 + \varepsilon), w(1 - 2\frac{\varepsilon}{\delta}), w(1 + \frac{\varepsilon}{\delta^2}), x - az_1, y + az_2, 0, 0, \dots)$$

for $\varepsilon > 0$.

As before, for any j such that $\delta_j \neq \delta_i$,

$$U_j(C^{\varepsilon, a}) - U_j(C) = \varepsilon \left[\left(1 - \frac{\delta_j}{\delta}\right)^2 \right] \frac{du_j}{dh}(w) + \delta_j^3 [u_j(x + az_1) + \delta_j u_j(y + az_2) - u_j(x) - \delta_j u_j(y)] + O(\varepsilon^2),$$

where $\frac{du_j}{dh}(w)$ stands for the total derivative of u_j evaluated at w , which is strictly positive given that preferences are non-decreasing overall and increasing in at least one dimension.

Since $\delta_j \neq \delta$, for sufficiently small ε and $a = \varepsilon^{3/2}$, $U_j(C^{\varepsilon, a}) > U_j(C)$.

By a similar argument,

$$V(C^\varepsilon) - V(C) = \delta^3 [u(x + az_1) + \delta u(y + az_2) - u(x) - \delta u(y)] + O(\varepsilon^2),$$

while $U_i(C^{\varepsilon, a})$ can be written as:

$$U_i(C^{\varepsilon, a}) - U_i(C) = \delta^3 [u_i(x + az_1) + \delta u_i(y + az_2) - u_i(x) - \delta u_i(y)] + O(\varepsilon^2).$$

For sufficiently small ε , and $a = \varepsilon^{3/2}$ it follows that $V(C^{\varepsilon, a}) < V(C)$ and $U_i(C^{\varepsilon, a}) > U_i(C)$. This violates unanimity. Therefore, our supposition was incorrect and $u = u_i$. ■

2 When Voting is Transitive: Well-Ordered Alternatives

One situation in which there is enough structure on consumption streams to ensure transitive voting rules entails consumption streams that can be ordered so that the differences between any two streams of instantaneous utilities are monotonic. For simplicity, we assume in this section that individual instantaneous utilities coincide.

Well-ordered Alternatives *Consumption streams C and \widehat{C} are well-ordered relative to a society with discount factors $\delta_1, \dots, \delta_n$ and a utility function $u_i = u$ for all i if $u(c_t) - u(\widehat{c}_t)$ is monotone in t (either nonincreasing, or nondecreasing).*

As mentioned in the paper, well-ordering provides a strong linkage between the preferences of individuals. Intuitively, suppose that C and \widehat{C} are well-ordered and that, say, $u(c_t) - u(\widehat{c}_t)$ is increasing. When we consider the differences in net present values between C and \widehat{C} , we consider sums of the form

$$\sum_t \delta^{t-1} (u(c_t) - u(\widehat{c}_t)).$$

As we increase δ , more weight is put on elements further in the sequence $\{u(c_t) - u(\widehat{c}_t)\}_t$ and so whenever an agent with a discount factor of δ evaluates this expression as positive, so that C is preferred to \widehat{C} , so does any agent with a higher discount factor. In particular, there is a natural ordering of agents according to their discount factors. Consequently, restricting the set of consumption streams so that agents are well-ordered rules out voting cycles.

Before describing the structure well-ordering of alternatives imposes, we introduce several standard restrictions on the voting rules we consider.

A binary voting rule R is *neutral* if for any (C, C', U) and $(\widehat{C}, \widehat{C}', \widehat{U})$, whenever

$$p(C, C', U) = p(\widehat{C}, \widehat{C}', \widehat{U}) \text{ and } p(C', C, U) = p(\widehat{C}', \widehat{C}, \widehat{U})$$

then

$$CR[U]C' \text{ if and only if } \widehat{C}R[\widehat{U}]\widehat{C}'.$$

A binary voting rule R is *monotone* if for any (C, C', U) and \widehat{U} , whenever

$$p(C, C', U) \subset p(C, C', \widehat{U}) \text{ and } p(C', C, \widehat{U}) \subset p(C', C, U)$$

then

$$CP[U]C' \text{ implies } CP[\widehat{U}]C'.$$

Any weighted majority rule is a neutral and monotone binary voting rule.

When alternatives are well-ordered, neutrality and monotonicity are enough to rule out intransitivities in the strict societal voting rule. The intuition is the following. Suppose

that $CP[U]C'P[U]C''$ and, for simplicity, assume that $u(c_t) - u(c'_t)$ and $u(c'_t) - u(c''_t)$ are both increasing, so that their sum, $u(c_t) - u(c''_t)$, is also increasing. As discussed above, this type of well-ordering guarantees that if an agent with discount factor δ prefers C to C' , so would any individual who is more patient. Denote the threshold discount factor by δ' . Similarly, the threshold discount factor for preferring C' to C'' can be denoted by δ'' . Now, since individuals themselves are transitive, any agent preferring C to C' and C' to C'' also prefers C to C'' . It follows that the threshold discount factor for preferring C to C'' must be (weakly) lower than $\max\{\delta', \delta''\}$. So that in the society, the set of agents who prefer C to C'' is a super-set of either the set who prefer C to C' or the set who prefer C' to C'' . Neutrality and monotonicity then guarantee that $CP[U]C''$.

The following proposition formally summarizes the above discussion.¹

PROPOSITION 1

1. If C and \widehat{C} are well-ordered relative to a society with discount factors $\delta_1 \leq \dots \leq \delta_n$ and a utility function $u_i = u$ for all i , then if $i \leq j$ both (weakly) prefer C to \widehat{C} , then so does any k such that $i \leq k \leq j$.
2. If \mathcal{C} is a set of consumption streams such that any pair of consumption streams in \mathcal{C} is well-ordered relative to a society with discount factors $\delta_1 \leq \dots \leq \delta_n$ and a utility function $u_i = u$ for all i , then any neutral and monotone voting rule is quasi-transitive (so its strict ranking is transitive and therefore acyclic) over \mathcal{C} .

Proof of Proposition 1:

1. Consider the case where $u(c_t) - u(\widehat{c}_t)$ is nonincreasing, as the other case is similar, and suppose that there is some $t' > 0$ such that $u(c_t) - u(\widehat{c}_t) > 0$ for $t < t'$ and $u(c_t) - u(\widehat{c}_t) \leq 0$ for $t \geq t'$, as otherwise preferences are trivial.

Consider any j such that

$$\sum_t \delta_j^{t-1} (u(c_t) - u(\widehat{c}_t)) \geq 0. \tag{1}$$

It follows that if $\delta_k \leq \delta_j$, then

$$\delta_k^t \frac{\delta_j^{t'}}{\delta_k^{t'}} \geq \delta_j^t$$

for $t < t'$ and

$$\delta_k^t \frac{\delta_j^{t'}}{\delta_k^{t'}} \leq \delta_j^t$$

for $t \geq t'$.

¹We use the notion of *quasi-transitivity*: $CP[U]C'P[U]C''$ implies $CP[U]C''$. This notion is a restricted form of full transitivity, which allows for P to also be replaced by R .

Therefore, from (1), the above inequalities, and the definition of t' , it follows that

$$\sum_t \delta_k^{t-1} \frac{\delta_j^{t'-1}}{\delta_k^{t'-1}} (u(c_t) - u(\widehat{c}_t)) \geq 0.$$

This implies that

$$\sum_t \delta_k^{t-1} (u(c_t) - u(\widehat{c}_t)) \geq 0,$$

and so, if an agent j prefers C to \widehat{C} , then so do all agents with lower discount factors. Analogously, if an agent j prefers \widehat{C} to C , then so do all agents with higher discount factors and the claim follows.

2. From part 1 it follows that the set of agents preferring one stream to another (when well-ordered) is a connected interval which contains either agent 1 or n if it is nonempty. Consider streams $C, C', C'' \in \mathcal{C}$ such that $CP[U]C'$, and $CP[U]C''$. We now show that $CP[U]C''$.

Suppose first that both $u(c_t) - u(c'_t)$ and $u(c'_t) - u(c''_t)$ are increasing. Then, there exist δ' and δ'' such that any agent with discount $\delta > \delta'$ prefers C to C' and agent with discount $\delta > \delta''$ prefers C' to C'' (with indifference at δ' and δ'' , where $\delta' \in (0, 1]$ and $\delta'' \in (0, 1]$, respectively).

Now, $u(c_t) - u(c''_t) = (u(c_t) - u(c'_t)) + (u(c'_t) - u(c''_t))$ is increasing. Furthermore, for any δ ,

$$\sum_t \delta^{t-1} (u(c_t) - u(c''_t)) = \sum_t \delta^{t-1} (u(c_t) - u(c'_t)) + \sum_t \delta^{t-1} (u(c'_t) - u(c''_t)). \quad (2)$$

Therefore, there exists $\delta^* \leq \max\{\delta', \delta''\}$ such that any agent with discount $\delta > \delta^*$ prefers C to C'' (with indifference at δ^* if $\delta^* \in (0, 1]$). Suppose $\delta^* \leq \delta'$, then

$$p(C, C', U) \subset p(C, C'', U) \text{ and } p(C'', C, U) \subset p(C', C, U).$$

From neutrality and monotonicity it follows that $CP[U]C''$. A similar argument follows if $\delta \leq \delta''$. The case in which $u(c_t) - u(c'_t)$ and $u(c'_t) - u(c''_t)$ are decreasing also follows analogously.

Assume then that $u(c_t) - u(c'_t)$ is increasing, $u(c'_t) - u(c''_t)$ is decreasing, and $u(c_t) - u(c''_t)$ is increasing. As before, there exist δ' and δ'' such that any agent with discount $\delta > \delta'$ prefers C to C' and agent with discount $\delta < \delta''$ prefers C' to C'' (with indifference at δ' and δ'' , whenever $\delta' \in (0, 1]$ and $\delta'' \in [0, 1)$, respectively).

We now show that $\delta' \leq \delta''$. Indeed, suppose the contrary, so that $\delta' > \delta''$. In that case,

$$p(C, C', U) \subset p(C'', C', U) \text{ and } p(C', C'', U) \subset p(C', C, U),$$

and so $CP[U]C'$ and $C'P[U]C''$ would be inconsistent with neutrality and unanimity.

It follows that $\delta' \leq \delta''$. Note that for any $\delta' \leq \delta \leq \delta''$, C is weakly preferred to C' and C' is weakly preferred to C'' . From (2), C is weakly preferred to C'' . Since $u(c_t) - u(c''_t)$ is increasing, there exists $\delta^* \leq \delta' \leq \delta''$ such that any agent with $\delta > \delta^*$ prefers C to C'' (with

indifference at $\delta = \delta^*$ if $\delta^* \in (0, 1]$). The claim then follows as before. All other cases are analogous. ■

Quasi-transitivity rules out cycles in the strict relation, but is a weaker condition than full transitivity of R as it does not deal with cases of social indifference. The restriction to quasi-transitivity is necessary unless the set of alternatives is confined enough so that the individuals themselves are not indifferent between alternatives. The following example illustrates intransitivities involving societal indifferences.

Example (Intransitivities with Indifferences) Suppose society is composed of two agents and the voting rule is defined so that $CP[U]C'$ if both agents prefer C to C' and otherwise C and C' are declared to be indifferent (and $CR[U]C'R[U]C$). Suppose the agents have a linear instantaneous utility function, $u(c) \equiv c$, and discount factors $\delta_1 = 3/4$ and $\delta_2 = 1/2$. Consider now three consumption streams:

$$\begin{aligned} C &= (.9, 1, 0, 0\dots), \\ C' &= (1, .8, 0, 0, \dots), \quad \text{and} \\ C'' &= (.85, 1, 0, 0, \dots). \end{aligned}$$

Agent 1, with discount factor $\delta_1 = 3/4$, prefers C to C' and is indifferent between C' and C'' . Agent 2, with discount factor $\delta_2 = 1/2$, is indifferent between C and C' , and prefers C' to C'' . Thus, socially $C''R[U]C'R[U]CP[U]C''$, contradicting transitivity.