Expanding "Choice" in School Choice

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Online Appendix

These notes consist of three sections. Section 1 defines and establishes the basic properties of the three algorithms in the large economy with a continuum of agents. Section 2 describes our simulation procedures in a greater detail than in the paper. Section 3 presents some omitted proofs of the paper.

1 Extensions of Algorithms to the Continuous Environment

It is convenient to explicitly model the randomizing device used to break the ties. For our purpose, it is sufficient to consider a vector $\omega = (\omega_1, ..., \omega_n) \in [0,1]^n =: \Omega$ of uniformly and independently generated numbers. (The vector of ω will be sufficiently rich enough to model the procedures we study.) Formally, we augment the type space by incorporating the random draw to $\mathcal{V} \times \Omega =: \Theta$, with its generic element denoted $\theta := (\mathbf{v}, \omega)$, and endow it with a product measure $\eta = \mu \times \xi_1 \times ... \times \xi_n$, where ξ_a is a uniform measure satisfying $\xi([0, \omega_a]) = \omega_a$ for each $\omega_a \in [0,1]$. This formalism avoids appealing to the law of large numbers (on the continuum of agents), by ensuring that a fraction ω_a of the student mass draws ω_a or less on each a-th random variable. A student of type $\theta = (\mathbf{v}, \omega)$ is then interpreted as having values $\mathbf{v}$ and drawing a vector ω. The student never observes ω, so her action required by the procedure will be measurable with respect to only $\mathbf{v}$; whereas (part or all of) ω component is "discovered" by the schools for their use in tie-breaking.

An *ex post allocation* is a measurable function $\psi := (\psi_1, ..., \psi_n) : \Theta \mapsto \Delta$ such that $\psi_a(\theta) \in \{0, 1\}$ and that $\int \psi_a(\theta)\eta(d\theta) = 1$ for each $a \in S$. Namely, ψ assigns a student with $\mathbf{v}$ to school a upon drawing ω such that $\psi_a(\mathbf{v}, \omega) = 1$. Let $\mathcal{Y}$ be the set of all ex post allocations. Later, we shall describe how each procedure generates an ex post allocation. Some procedures may not use the entire vector of ω, so the ex post allocation they produce may be measurable with respect to only some components of ω.

We define the alternative DA procedures here.

Ordinal preferences. In any DA algorithm, every student submits a ranking of schools. Formally, students' ordinal preferences are represented by a measurable function $P : \Theta \to \Pi$, where $P(\mathbf{v}, \omega) \in \Pi$ is an ordered list of n schools (ordered not necessarily according to true preferences). Since the ω is unobserved by the students (at least at the time of submitting the ordinal preferences), we require that $P(\mathbf{v}, \omega) = P(\mathbf{v}', \omega')$ whenever $\mathbf{v} = \mathbf{v}'$. We say a DA algorithm is *ordinally strategy-proof* if it is a (weak) dominant strategy for each student with $\mathbf{v}$ to choose $P(\mathbf{v}, \omega) = \pi(\mathbf{v})$.

School priorities (tie-breaking rules). We introduce a tie-breaker function which determines the priority of each student for each school as a function of the random draw (as well as their auxiliary message in the case of CADA), in the event of a tie. Formally, *tie-breaker function for school a* is a bounded measurable function $F_a : \Theta \mapsto \mathbb{R}$, such that a student θ' is interpreted as having a higher priority than student θ if $F_a(\theta') < F_a(\theta)$. A *tie-breaker* is a profile $\mathcal{F} = \{F_a : a \in S\}$ of tie-breaker functions. Specifically, the tie-breakers for DA-STB, DA-MTB, and CADA are determined as follows:

- *DA-STB:* The STB rule uses the same tie-breaker function for all schools. This is modeled by a tie-breaker with

  $$F_a(\mathbf{v}, \omega_1, ..., \omega_n) = \omega_1,$$

  $\theta = (\mathbf{v}, \omega_1, ..., \omega_n)$, for *every* school $a \in S$. In other words, a draw's draw ω_1 serves as a priority number for all schools. Heuristically, a real number ω_1 is drawn randomly from an interval $[0, 1]$, for each student, which then serves as her priority score.[1]

- *DA-MTB:* The MTB rule produce a randomly and independently drawn priority list for each school. This is modeled by a tie-breaker, with

  $$F_a(\mathbf{v}, \omega_1, ..., \omega_n) = \omega_a,$$

  for $a \in S$ and for each $\theta = (\mathbf{v}, \omega_1, ..., \omega_n)$. In other words, for each student, a vector $(\omega_1, ..., \omega_n)$ of independent draws determines her priority scores at different schools.

- *CADA:* In CADA, each student sends an auxiliary message of a target school (in addition to their ordinal preferences over schools). Given a (measurable) strategy profile $s : \mathcal{V} \to S$ determining the auxiliary message for each intrinsic type $\mathbf{v}$, the

[1] This heuristics invokes a law of large numbers, but our formal method does not rely on it for we assume a well-behaved randomization device.

tie-breaker function for school a is given by

$$F_a(\mathbf{v}, \omega_1, ..., \omega_n) = \begin{cases} \omega_1 & \text{if } s(\mathbf{v}) = a \\ 1 + \omega_2 & \text{if } s(\mathbf{v}) \neq a \end{cases}$$

That is, under F_a, ties are broken first in favor of students who target a, within them according to the random draw ω_1, and then ties among the rest are broken according to a random draw $\omega_2 + 1$ (where 1 act as a "penalty score" 1). Clearly, F_a is a measurable function since ω_1 and s are measurable.

Definition of DA algorithms: Given ordinal preferences P and a tie-breaker $\mathcal{F} = \{F_a : a \in S\}$, a DA algorithm is defined as follows. First, we define a measurable function Ch_{F_a} over subsets of Θ as the set of best ranked students for school $a \in S$ according to F_a from a given set up to the capacity. Formally, for any measurable $X \subset \Theta$, let

$$Ch_{F_a}(X) := \sup\{Y \subset X | \eta(Y) \leq 1, F_a(\theta) < F_a(\theta'), \forall \theta \in Y, \theta' \in X \backslash Y\}$$

denote the set of students chosen from X such that the set does not exceed the capacity and that the chosen students have a higher priority than those not chosen.

Next, we define the $DA_{\mathcal{F}}$ (deferred acceptance) mapping. Consider first a mapping $Q : \Theta \to \Pi$, where $Q(\theta)$ is an ordered list of any $k \leq n$ schools. (Recall $P(\theta)$ is a special case involving the full set of schools.) The DA mapping, $Q' = DA_{\mathcal{F}}(Q) \in \Pi$ is determined as follows. Every student with θ applies to her most preferred school in $Q(\theta)$. Every school a (tentatively) admits from its applicants in the order of F_a. If all of its seats are assigned, it rejects the remaining applicants. If a student θ is rejected by a, $Q'(\theta)$ is obtained from $Q(\theta)$ by deleting a in $Q(\theta)$. If a student θ is not rejected, then $Q'(\theta) = Q(\theta)$. More formally, let $T_a(Q) = \{\theta \in \Theta : a \text{ is ranked first in } Q(\theta)\}$ be the set of students that rank a as first choice. Note that $T_a(Q)$ is measurable. Then each school a admits students in $Ch_{F_a}(T_a(Q))$ and rejects students in $T_a(Q) \backslash Ch_{F_a}(T_a(Q))$. If $\theta \in T_a(Q) \backslash Ch_{F_a}(T_a(Q))$ for some $a \in S$, then $Q'(\theta)$ is obtained from $Q(\theta)$ by deleting a from the top of $Q(\theta)$; otherwise $Q'(\theta) = Q(\theta)$. Since Q is a measurable function, Q' is also measurable.

Repeated application of the $DA_{\mathcal{F}}$ mapping gives us the DA algorithm. That is, given a problem $(P, \mathcal{F})$, let $Q^0 = P$ and define $Q^t = DA_{\mathcal{F}}(Q^{t-1})$ for $t > 0$. Then Q^t converges almost everywhere to some measurable Q^* (Theorem 0 below). The matching can be then found by assigning θ to its top choice of $Q^*(\theta)$. Formally, define a mapping $\psi^{(P,\mathcal{F})} : \Theta \mapsto \Delta$ such that $\psi_a^{(P,\mathcal{F})}(\theta) = 1$ if a is the top choice of $Q^*(\theta)$, and $\psi_a^{(P,\mathcal{F})}(\theta) = 0$ otherwise. Since the schools' capacities are respected in each round and also in the limit, the mapping must be an ex post allocation.

We present two main results:

Well-definedness of the Procedure. The existence of $\psi^{(P,\mathcal{F})}$ follows from the next theorem.

Theorem 0. *For every $(P,\mathcal{F})$, $DA^t_{\mathcal{F}}(P)$ converges almost everywhere to some measurable $Q^*:\Theta\to\Pi$.*

Proof: Define the set of rejected students as $R^t=\{\theta:\theta\in T_a(Q^t)\backslash Ch_{F_a}(T_a(Q^t))$ for some $a\in S\}$. Then $\eta(R^t)$ goes to zero as t goes to infinity. Otherwise, if $\eta(R^t)\geq\kappa>0$ for all t, all the schools in every student's preference would be deleted in finite time because of finiteness of the number of schools, which in turn would imply that $\eta(R^t)$ goes to zero, a contradiction. Therefore, $DA^t_{\mathcal{F}}(P)$ converges almost everywhere to some Q^*. Since every $Q^t=DA^t_{\mathcal{F}}(P)$ is measurable, Q^* is also measurable. ▮

2 The Simulation Procedure

There are 5 schools each with a capacity of 20 seats and 100 students. Fix α. We independently draw 100 sets of vNM values for students. Let $\{\tilde{v}^s_{ia}\}$ denote a draw of vNM values, where superscript s denote the draw and $\tilde{v}^s_{ia}$ denotes student i's vNM value for school a. We normalize the level and the scale of each student's vNM utilities as follows:

$$v^s_{ia}=\frac{\tilde{v}^s_{ia}-\min_{a'}\tilde{v}^s_{ia'}}{\max_{a'}\tilde{v}^s_{ia'}-\min_{a'}\tilde{v}^s_{ia'}}$$

Given a normalized draw $\{v^s_{ia}\}$, fix the mechanism, define the following: p^s_{ia} is the probability that student i is assigned school a under the mechanism. $\pi^s_k(i)$ is the school that is ranked k-th in i's preference list. P^s is the set of popular schools. O^s is the set of oversubscribed schools in an equilibrium of CADA with no naive players.

A first best or utilitarian maximum solves

$$\overline{v}^s_{FB}=\frac{1}{100}\max_{\{\hat{p}^s_{ia}\}}\sum_i\sum_a\hat{p}^s_{ia}v^s_{ia}$$

Let $\{\bar{p}^s_{ia}\}$ denote a solution to the first best. There may be multiple solutions, we arbitrarily pick one.

Furthermore, we calculate

$$\delta^s_1=\frac{1}{100}\sum_i\sum_a p^s_{ia}\cdot\mathbf{1}(\pi^s_1(i)=a)$$

where δ^s_1 is the average probability of assigning a student to her first choice.

In the CADA experiments with naive players, we divide the set of students into two: N is the set of naive players who always target their first choice, and S is the set of strategically sophisticated players who play their best response strategies given others' strategies. We calculate utilitarian welfare as before. We also compute the number of students targeting their k-th choice in equilibrium, which we denote by T_k^s, $k \in \{1, 2, 3, 4\}$.

Given a draw $\{v_{ia}^s\}$, the set P^s is determined trivially. Next we describe how the other numbers are computed.

A *single tie breaker* is a list of 100 randomly drawn lottery numbers, one for each student. Under DA-STB the ties at a school are broken according to students' single random numbers. In CADA, we draw two single tie breakers, one to be used to break ties at one's target school, the other to be used at one's other schools. A *multiple tie breaker* is a list of $100 \times 5 = 500$ randomly drawn numbers, one for each student at each school. Under DA-MTB, the ties at a school are broken according to students' tie breaker numbers at that school.

For each draw $\{v_{ia}^s\}$, we independently draw 2,000 single tie breakers for DA-STB, and an additional set of 2,000 single tie breakers for CADA, and 2,000 multiple tie breakers for DA-MTB. Then p_{ia}^s for a mechanism is computed by

$$\frac{\text{Number of tie breakers at which } i \text{ is assigned } a}{2,000}.$$

The equilibrium of CADA is computed with single tie breakers being fixed. Given the strategies of other students, a student's best response is found by computing that student's expected utility over those tie breakers. Then O^s, the set of oversubscribed schools, is found by using students's equilibrium target schools. In experiments with naive players, naive players' target schools are fixed at their first choice.

Note that we are approximating the equilibrium by drawing (two sets of) 2,000 independent tie-breakers. The exact numbers are computed by considering 100! single tie-breakers and $(100!)^5$ multiple tie breakers, which is beyond the capabilities of our computational resources. Any further increase in the number of tie breakers beyond 2,000 does not increase the precision of our computations significantly.

For each $z^s \in \{\overline{v}^s, \overline{v}_{FB}^s, \pi_1^s, T_1^s, T_2^s, T_3^s, T_4^s, |P^s|, |O^s|\}$, we compute the average of z^s by

$$z = \frac{1}{100} \sum_{s=1}^{100} z^s.$$

Note that we drop all "s" from a variable to denote its mean over 100 iterations of an experiment. We report $100\frac{\overline{v}}{\overline{v}_{FB}}$ in our welfare figures.

3 Omitted Proofs

3.1 Ordinal Strategy-proofness.

Fix arbitrary ordinal preferences P. Let $P_{-\mathbf{v}} : \mathcal{V}\backslash\{\mathbf{v}\} \to \Pi$ denote the ordinal preferences of all students but $\mathbf{v}$ determined by P. Recall that $\pi(\mathbf{v}) \in \Pi$ represents the truthful ordinal preference induced by $\mathbf{v}$, that is $\pi(\mathbf{v})$ lists a before b if and only if $v_a > v_b$. To simplify the notation, let $\psi^P := \psi^{(P,\mathcal{F})}$, with $\mathcal{F}$ suppressed, and let $\psi^* := \psi^{(\pi[\cdot],P_{-[\cdot]},\mathcal{F})}$ denote the matching outcome for any given type when it submits its ordinal preferences truthfully and the others report P. When students report P, a student with type $\mathbf{v}$ receives expected utility of

$$\mathbb{E}_\omega \left[v \cdot \psi^P(\mathbf{v}, \omega)\right].$$

Theorem 2. *For every $(P, \mathcal{F})$, it is a (weak) dominant strategy for every student to submit her ordinal preferences truthfully to DA, that is, for all $\mathbf{v} \in \mathcal{V}$, P,*

$$\mathbb{E}_\omega \left[\mathbf{v} \cdot \psi^*(\mathbf{v}, \omega)\right] \geq \mathbb{E}_\omega \left[\mathbf{v} \cdot \psi^P(\mathbf{v}, \omega)\right].$$

Proof. It suffices to show that, for all $\theta = (\mathbf{v}, \omega)$,

$$\mathbf{v} \cdot \psi^*(\mathbf{v}, \omega) \geq \mathbf{v} \cdot \psi^P(\mathbf{v}, \omega).$$

Suppose to the contrary that

$$\mathbf{v} \cdot \psi^*(\mathbf{v}, \omega) < \mathbf{v} \cdot \psi^P(\mathbf{v}, \omega), \tag{1}$$

for some $\theta = (\mathbf{v}, \omega)$ and P. We show that there exists a finite many-to-one matching problem for which a DA algorithm fails strategy-proofness, which will then constitute a contradiction to the standard strategy-proofness result (Dubins and Friedman, 1981; Roth, 1982).

To begin, fix any $K \in \mathbb{N}_+$, and construct a discretization of (P, F) for θ as follows: For every $\mathbf{z} = (z_1, ..., z_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ where $z_a, y_b \in \{0, ..., K\}$, consider a set

$$\Theta_{\mathbf{z},\mathbf{y}} = \left\{(\tilde{\mathbf{v}}, \tilde{\omega}) \in \Theta : \frac{z_a}{K} \leq \tilde{v}_a \leq \frac{z_a + 1}{K}, \frac{ny_b}{K} \leq \tilde{\omega}_b \leq \frac{n(y_b + 1)}{K}, a, b \in S\right\}.$$

Let $\eta_{K,\min} = \min_{\Theta_{\mathbf{z},\mathbf{y}}} \eta(\Theta_{\mathbf{z},\mathbf{y}})$, and let $\#\Theta_{\mathbf{z},\mathbf{y}}$ be the integer part of $\frac{\eta(\Theta_{\mathbf{z},\mathbf{y}})}{\eta_{K,\min}}$.

Pick $\#\Theta_{\mathbf{z},\mathbf{y}}$ students in total from every set $\Theta_{\mathbf{z},\mathbf{y}}$ at random without repetition. Let $\{\theta^l\}$ denote the set of students that are picked. If $\frac{|\{\theta^l\}|}{n}$ is not an integer, pick additional students from the larger sets until obtaining an integer $\frac{|\{\theta^l\}|}{n}$. Note that the number of additional students to be picked this way is less than n and n is fixed, therefore this will be negligible in the limit as K goes to infinity. Now consider the problem in which the set $\{\theta^l\}$ of students are to be assigned to a set S of schools each with capacity $\frac{|\{\theta^l\}|}{n}$. Each student $\theta^l = (\mathbf{v}^l, \omega^l)$'s strict ordinal preference is given by $P(\theta^l)$. The schools' strict preferences are given by $\mathcal{F}$. Denote this problem by $(\{\theta^l\}, S, P, \mathcal{F})_K$, and the associated ex post allocation ψ_K^P. As K goes to infinity, $(\{\theta^l\}, S, P, \mathcal{F})_K$ approximate $(\Theta, S, P, \mathcal{F})$ arbitrarily closely. Hence, $\psi_K^P \rightarrow_{a.e} \psi^P$ and $\psi_K^{\pi(\mathbf{v}),P_{-\mathbf{v}}} \rightarrow_{a.e.} \psi^*$ as $K \rightarrow \infty$. Hence, if (1) holds, then there exists K such that

$$\mathbf{v} \cdot \psi_K^*(\mathbf{v}, \omega) < \mathbf{v} \cdot \psi_K^P(\mathbf{v}, \omega).$$

This contradicts the fact that, in every finite problem, submitting true preferences to the student-proposing deferred acceptance mechanism is a dominant strategy for every student (Dubins and Friedman, 1981; Roth, 1982). ▌

3.2 Welfare Performances of CADA with Naive Students

Theorem 8. *In the presence of naive students, the equilibrium allocation of CADA satisfies the following properties: (i) The allocation is OE, and is thus pairwise PE. (ii) The allocation is PE within the set K of oversubscribed schools. (iii) If every student is naive, then the allocation is PE within $K \cup \{l\}$ for any undersubscribed school $l \in J := S \backslash K$.*

Proof: Part (i) is precisely the same as Part (i) of Theorem 5 and is a consequence of Part (ii) below and of Part (ii) of Lemma 5 (which does not depend on whether the students are naive or not). Hence it is omitted. To prove Part (ii), it is useful to establish the following lemma. As before, let ϕ^* denote the ex ante allocation arising from the CADA game and let K and $J = S\backslash K$ be respectively the sets of oversubscribed and undersubscribed schools in equilibrium.

Lemma N. *Any reassignment of $\phi^*(\mathbf{v})$ within K will make a naive student with $\mathbf{v}$ strictly worse off, for almost every $\mathbf{v}$.*

Proof: Consider a naive student with $\mathbf{v}$. Assume without loss of generality that she prefers a strictly over all other schools (i.e., $v_a > v_b$, $\forall b \neq a$). (This is without loss of generality since the values are distinct for almost every student type.) Since the student is naive, she subscribes to school a with probability 1. If school a is undersubscribed, then the result is trivial since $\phi_k^*(\mathbf{v}) = 0$ for all $k \in K$. Hence, suppose school a is oversubscribed. Then, any reassignment $\mathbf{x} \in \Delta_{\phi^*(\mathbf{v})}^K := \{(x_1, ..., x_n) \in \Delta \,|\, x_a = \phi_a^*(\mathbf{v}), \forall a \in S \setminus K\}$ must satisfy

$$\sum_{b \in K} x_b = \phi_a^*(\mathbf{v}).$$

Since $v_a > v_b \ \forall b \neq a$, for any $\mathbf{x} \in \Delta^K_{\phi^*(\mathbf{v})}$, $\mathbf{x} \neq \phi^*_a(\mathbf{v})$, we must have

$$\sum_{b \in K} x_b v_b < \sum_{b \in K} x_b v_a = \phi^*_a(\mathbf{v}) v_a,$$

which implies that the student must be strictly worse off from any such reassignment. ‖

We are now ready to prove Parts (ii) and (iii):

Part (ii): We make use of the proof of Theorem 5. By Lemma N, a type-**v** naive student's assignment from the CADA, $\phi^*(\mathbf{v})$, is a unique solution to $[P(\mathbf{v})]$, for a.e. **v**, even without the constraint

$$\sum_{a \in K} p_a x_a \leq \sum_{a \in K} p_a \phi^*_a(\mathbf{v}). \tag{2}$$

Since $\phi^*(\mathbf{v})$ is feasible under (2), this must be a unique solution to $[P(\mathbf{v})]$.

For a non-naive student with a.e. **v**, the proof of Theorem 5 follows directly, so $\phi^*(\mathbf{v})$, is also a unique solution of $[P(\mathbf{v})]$. Since the equilibrium assignment of both types solves $[P(\mathbf{v})]$, the rest of the argument in the proof of Theorem 5 applies, proving that we ϕ^* is PE within K. ‖

Part (iii): Again let ϕ^* be the ex ante allocation arising from CADA. Suppose to the contrary that there exists a within-$K \cup \{l\}$ reallocation $\tilde{\phi}$ of ϕ^* that Pareto dominates ϕ^*. By Part (ii), ϕ^* is PE within K, so $\tilde{\phi}_l(\mathbf{v}) \neq \phi^*_l(\mathbf{v})$ for a positive measure of **v**, which in turn implies that there exists a set $A \subset \mathcal{V}$ with $\mu(A) > 0$ such that $\tilde{\phi}_l(\mathbf{v}) > \phi^*_l(\mathbf{v})$ for each $\mathbf{v} \in A$. Since $\tilde{\phi}(\mathbf{v}) \in \Delta^{K \cup \{l\}}_{\phi^*(\mathbf{v})}$, $\sum_{b \in K \cup \{l\}} \tilde{\phi}_b(\mathbf{v}) = \sum_{b \in K \cup \{l\}} \phi^*_b(\mathbf{v})$, so

$$\sum_{b \in K} \tilde{\phi}_b(\mathbf{v}) < \sum_{b \in K} \phi^*_b(\mathbf{v}) \quad \text{for all } \mathbf{v} \in A.$$

Assume without loss that **v** satisfies $v_a > v_b$ for all $a \in K$ and for all $b \neq a$. ($a \neq l$ since $\tilde{\phi}_l(\mathbf{v}) > \phi^*_l(\mathbf{v})$ is impossible if $a = l$.) Then, the type-**v** student's expected payoff from $\tilde{\phi}$ is

$$\begin{aligned}
\sum_{b \in S} \tilde{\phi}_b(\mathbf{v}) v_b &= \sum_{b \in K} \tilde{\phi}_b(\mathbf{v}) v_b + \tilde{\phi}_l(\mathbf{v}) v_l + \sum_{b \in J \setminus \{l\}} \phi^*_b(\mathbf{v}) v_b \\
&< \sum_{b \in K} \tilde{\phi}_b(\mathbf{v}) v_a + \left(\tilde{\phi}_l(\mathbf{v}) - \phi^*_l(\mathbf{v}) \right) v_a + \phi^*_l(\mathbf{v}) v_l + \sum_{b \in J \setminus \{l\}} \phi^*_b(\mathbf{v}) v_b \\
&= \left(\sum_{b \in K \cup \{l\}} \tilde{\phi}_b(\mathbf{v}) - \phi^*_l(\mathbf{v}) \right) v_a + \sum_{b \in J} \phi^*_b(\mathbf{v}) v_b \\
&= \sum_{b \in S} \phi^*_b(\mathbf{v}) v_b.
\end{aligned}$$

Since this inequality holds for almost every $\mathbf{v} \in A$, and since $\mu(A) > 0$, $\tilde{\phi}$ cannot Pareto dominate ϕ^*. ▌

4 Finite Market Approximation of the CADA Equilibria

We shall argue in several steps to show how the results by BC be applied in our setting to establish the finite market approximation of the main results.

4.1 Preliminaries.

Continuum Economy It is useful to develop a further formalism on the CADA. Consider the augmented type space $\Theta_0 := \mathcal{V} \times \Omega_R \times \Omega_T$, where its generic element $\theta_0 = (\mathbf{v}, \omega_R, \omega_T)$ represents a student's vNM values, his regular lottery draw and his target lottery draw. The type space is endowed with a product measure $\eta_0 := \mu \times \xi_R \times \xi_T$, where $\xi_i, i = R, T$ is a uniform measure, with associated distribution $U(\xi_i) = \xi_i$. Consider the target strategy $\alpha : \mathcal{V} \to S$, a measurable mapping from his values to a target school. Note that by Theorem 3 we can without loss focus on pure strategies. The original type θ_0 together with the target strategy α induces a "modified type" $\theta := \mathbf{v} \times \omega_1 \times ... \times \omega_n$, where ω_a is the priority score at school a that is derived using the target choices as well as the lottery draws. (Recall the lower the score is the better it is for a student, as before.) Let η denote the measure of the types. Note η is no longer a product measure, given the way ω_a's are constructed (see the F_a construction in Section 1). We shall sometimes say that (μ, α) induces a measure η of this modified types.

We now associate a continuum economy by a pair $E = (\eta_0, \alpha)$, and focus on η it induces. In the paper, we had focused on the outcome of the student proposing DA. But for our current purpose, we shall consider a stable matching of the economy E. Following Azevedo and Leshno (2013), a stable matching is characterized by the vector of cutoffs $c = (c_1, ..., c_n)$. We shall say *a sequence of continuum economies E^k converges to E, if the associated η_0^k converges to η_0 in the weak-* sense and the targeting strategies α^k converges pointwise to α, as $k \to \infty$.* Obviously, if E^k converges to E, then η^k weak-* converges to η.

Finite Economies We shall also study finite economies indexed by N, the number of students. An N-economy is described by an arbitrary sample of N drawings from Θ_0 such that the lottery numbers are distinct across all students, their target strategies $\bar{\alpha} : \mathcal{V} \to S$,[2] and the capacity vector $q = (q_1, ..., q_n)$. The student types are described by measure $\bar{\eta}_0$. As with the continuum economy, $\bar{E} := (\bar{\eta}_0, \bar{\alpha})$ induces the measure on the modified types,

[2]That is, we define the strategies for the entire domain of $\mathcal{V}$. Obviously, the values of $\bar{\alpha}$ outside the sampled $\mathbf{v}$'s will not be relevant. But this formalism is useful for defining the notion of convergence.

denoted by $\bar{\eta}$. A cutoff for a school $a \in S$ in a finite economy is the marginal (i.e., the highest) modified lottery number attained by any students assigned to that school. We shall say that *a sequence of finite economies* $(\bar{E}^N, q^N)$ *converges to a continuum economy* $E = (\eta_0, \alpha)$, *if the associated* $\bar{\eta}_0^N$ *converges to* η_0 *in the weak-* sense and the targeting strategies* α^N *converges pointwise to* α, *and* $nq_a/N \to 1$ *for all* $a \in S$, *as* $N \to \infty$. Again when $\bar{E}^N$ converges to E, then $\bar{\eta}^N$ weak-* converges to η.

4.2 Uniqueness of stable matching in the continuum economy.

Next we shall fix a continuum economy $E = (\eta_0, \alpha)$, and argue that it admits a unique stable matching.[3] To this end, note that E induces a set S_O of oversubscribed schools and a set S_U of undersubscribed schools.[4] Based on Lemma 5, we can without loss of generality assume that if $\alpha(v) \in S_O$, then the student type $\mathbf{v}$ prefers $\alpha(\mathbf{v})$ over any undersubscribed schools, and if $\alpha(v) \in S_U$, then that student applies to that school for sure. This restriction is without loss since they are implication of an equilibrium. Our focus on pure strategies also without restriction given our Theorem 3.

Given this, a student who targets an oversubscribed schools applies to that school before she applies to any undersubscribed school, and she is rejected by all other oversubscribed schools that the student didn't target. It then follows that the cutoff of any oversubscribed school $a \in S_O$ is uniquely pinned down by $U^{-1}(1/m_a) = 1/m_a$, where m_a is the measure of students who targeted a. For each undersubscribed school $b \in S_U$, given that all those students targeting b are guaranteed seats, all that matters is how the remaining seats are allocated to those who targeted oversubscribed schools but failed to be assigned to them. Hence, the assignments of these seats in any stable matching is characterized by a stable matching in a subeconomy consisting only of the remaining seats of the undersubscribed schools and those students who targeted but failed to get into the oversubscribed schools. The fact that the target lotteries are uniform means that the distribution of values, and thus ordinal preferences, of students who targeted but failed oversubscribed schools is uniquely determined. Further, by the construction of CADA priority rule, the priorities of these students are determined by the uniform "regular" lottery draws, so priorities of these students are common among all undersubscribed schools, and independent of their value types. Hence, the cutoffs of these schools are determined uniquely much in the same way as with DA-STB, following the arguments of Lemma 3.

[3]It is important to note that the uniqueness does not imply that the CADA equilibrium is unique since α is endogenous. We use the uniqueness of stable matching for the purpose of applying the results of Bodoh-Creed (2013).

[4]Formally, $a \in S_O$ if $m_a := \int_{\mathbf{v}} 1_{\{\alpha(\mathbf{v})=a\}} d\mu(\mathbf{v}) \geq 1$, and $a \in S_U$ if not.

Combining the results, we conclude that there is unique stable matching, characterized by unique "equilibrium" cutoffs.

4.3 Continuity and convergence

The uniqueness of stable matching admitted by our continuum economy E offers a couple of useful implications. Consider a sequence $\{E^k, c^k\}$ of continuum economies E^k and the associated market clearing cutoffs c^k. And suppose E^k converges to E and E admits unique cutoffs c. Then, Lemma B3 of Azevedo and Leshno (2013) shows that c^k converges to c. Let's call this **the limit-economy continuity**.

Next, suppose a sequence of finite economies $\bar{E}^N$ converges to a continuum economy E that admits a unique stable matching c.[5] And $\bar{E}^N$ admits a cutoff vector c^N. Then, by Lemma B4 of Azevedo and Leshno (2013), $c^N \to c$ as $N \to \infty$. We label this result **finite-economy convergence**.

4.4 Application of the properties

We now show how the two results from Subsection 4.3 allow us to apply Theorems 6 and 7 of Bodoh-Creed (2013). To this end, we need to check a couple of conditions. Let $u(\mathbf{v}, a, E) = \sum_{a \in S} v_a P_a(\mathbf{v}, a, E)$ denote the expected utility of the agents in the CADA game of the continuum economy E, when a student with type $\mathbf{v}$ targets a, where $P_a(\mathbf{v}, a, E)$ is the probability of assignment to school a. By the limit-economy continuity from Subsection 4.3, as E^k converges to E, the associate cutoff vectors c^k also converge to the unique cutoff vector c of E in the CADA game. Since a student's assignment probabilities $P_a(\mathbf{v}, a, E)$'s, given his target choice a and his ordinal preferences, are completely pinned down by the cutoffs, what this means is that $u(\mathbf{v}, a, E^k) \to u(\mathbf{v}, a, E)$ for all $(\mathbf{v}, a)$, since $\mathcal{V} \times S$ is compact, the continuity requirement of Theorem 6 of Bodoh-Creed (2013) holds.[6]

The second condition deals with the convergence of expected utility $u^N(\mathbf{v}, a, \bar{E}^N) = \sum_{a \in S} v_a P_a^N(\mathbf{v}, a, \bar{E}^N)$ to the continuum economy expected utility $u(\mathbf{v}, a, E)$ as $\bar{E}^N$ converges to E. This follows from the finite convergence property of Subsection 4.3. As a sequence of finite economies $\bar{E}^N$ converges to E (along with $nq_a/N \to 1$ for all $a \in S$), the

[5] Recall a stable matching is characterized by a cutoff vector.

[6] In the Bodoh-Creed's specification, the payoff is expressed as a function of profile of mixed strategies of agents in the economy. We can effectively describe this by the economy E. By the law of large numbers, mixed strategies entail a "deterministic" economy. We also proved that the equilibrium strategy must be effectively pure (Theorem 3), so it is without loss to focus on pure target choice. Finally, Bodoh-Creed (2013) focuses on ex post payoff function depending also on the state. The continuity in expected utility works also for his proof.

associated market-clearing cutoff vectors $\bar{c}^N$ converge also to the unique cutoff vector c of E in the CADA game. Since a student's assignment probabilities given his target school a and his ordinal preferences are completely pinned down by the cutoffs, it follows that $u^N(\mathbf{v}, a, \bar{E}^N) \to u(\mathbf{v}, a, E)$ as $\bar{E}^N \to E$, for all $(\mathbf{v}, a)$. Again the compactness of $\mathcal{V} \times S$ makes the convergence uniform. This satisfies the convergence requirement used in Theorems 6 and 7 of Bodoh-Creed (2013).