Optimal Project Selection Mechanisms
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Online Appendix

In this Appendix there are four sub-sections supporting claims we make in the paper about the robustness of our results. In Appendix A we show that the optimal mechanism we derived is also optimal under Bayesian implementation. In Appendix B we derive the optimal mechanism when transfers can depend on the outcome of the project. In Appendix C we consider agents whose interests are partially aligned with those of the agency, so that they might prefer another high quality project to be selected. In Appendix D we allow for the possibility that an agent's benefit function $b_i(q_i)$ takes negative values for low qualities. To keep notation shorter we assume throughout this Appendix that the outside option is $q^\circ = 0$.

Appendix A – Bayesian Implementation

We show that our optimal mechanism also solve the problem under Bayesian implementation. We formulate the problem as follows:

$$\max_{\{p_i(\mathbf{q}),t_i(\mathbf{q})\}_i} \left\{ \mathbb{E}_{\mathbf{q}} \left[\sum_{i=1}^{n} [q_i p_i(\mathbf{q}) - t_i(\mathbf{q})] \right] \right\} \tag{1}$$

subject to

$$\text{feasibility} : \sum_{i=1}^{n} p_i(\mathbf{q}) \leq m \;\text{ and } 0 \leq p_j(\mathbf{q}) \leq 1 \text{ for all } \mathbf{q} \text{ and } j; \tag{2}$$

$$\text{limited liability } (LL_i) : t_i(\mathbf{q}) \geq 0 \text{ for all } i \text{ and for all } \mathbf{q}; \tag{3}$$

and the following interim incentive compatibility constraints for all i, and for all q_i and q_i',

$$(IC_i) : \mathbb{E}_{\mathbf{q}_{-i}} \left[p_i(\mathbf{q}) b_i(q_i) + t_i(\mathbf{q}) \right] \geq \mathbb{E}_{\mathbf{q}_{-i}} \left[p_i(q_i', \mathbf{q}_{-i}) b_i(q_i) + t_i(q_i', \mathbf{q}_{-i}) \right], \tag{4}$$

Recall that we defined for any quality of project i, q_i,

$$G_i\left(q_i\right) = q_i + b_i(q_i) + b_i'(q_i)\frac{F_i\left(q_i\right)}{f_i(q_i)} \tag{5}$$

to be the *virtual return* that is obtained from selecting project i and assumed that this function is strictly increasing.

Clearly, the derivations of the one project case remain unchanged. Recall from that section that we defined

$$a_i^*\left(q^\circ\right) = \begin{cases} \overline{q_i} & \text{if} \quad q^\circ > G_i\left(\overline{q_i}\right), \\ G_i^{-1}\left(q^\circ\right) & \text{if}\, q^\circ \in \left[G_i\left(\underline{q_i}\right), G_i\left(\overline{q_i}\right)\right], \\ \underline{q_i} & \text{if} \quad q^\circ < G_i\left(\underline{q_i}\right). \end{cases} \tag{6}$$

and showed that there exists a quality $g_i^\circ \in \left[\mathbb{E}\left[q_i\right], G_i\left(\overline{q_i}\right)\right]$, unique in $\mathbb{R}$, such that

$$V^*\left(g_i^\circ\right) - g_i^\circ = 0. \tag{7}$$

Also recall that for any (n-dimensional) vector of project qualities $\mathbf{q}$ we define an $(n+m)$-dimensional vector of *virtual project qualities*

$$\mathbf{x}(\mathbf{q}) = \left(\min\left\{G_1\left(q_1\right), g_1^\circ\right\}, ..., \min\left\{G_n\left(q_n\right), g_n^\circ\right\}, \underbrace{0, ..., 0}_{m \text{ times}}\right). \tag{8}$$

and the sum of its m largest components is $S_m\left(\mathbf{x}(\mathbf{q})\right)$.

We show that at any feasible mechanism in the Bayesian problem the firm profit is bounded by $\mathbb{E}_{\mathbf{q}}\left[S_m\left(\mathbf{x}(\mathbf{q})\right)\right]$ which is the optimal profit in the optimal dominant strategy mechanism. Since the optimal mechanism is feasible in the Bayesian problem, it is also optimal for the Bayesian problem. To proof this we follow the same steps as in the proof of theorem 1.

Lemma A1 *Let the expected payoff of manager i in the optimal mechanism be*

$$M_i\left(q_i\right) = \mathbb{E}_{\mathbf{q}_{-i}}\left[t_i(\mathbf{q}) + b_i(q_i)p_i(\mathbf{q})\right] \tag{9}$$

The target function in (1) can be written as

$$\mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n}\left[G_i(\mathbf{q}_i)p_i(\mathbf{q}) - M_i(\overline{q_i})\right]\right].$$

Proof of Lemma A1. We apply the "Mirlees trick" (see Mirlees, 1974 and Fudenberg and Tirole, 1992) to prove this lemma. The series of transformations in this proof are familiar for this type of models, but we include them here for completeness.

Step 1. Consider first the interim incentive compatibility constraints (4) which we refer to as IIC_i. The utility of project manager i under a mechanism when he reports truthfully is $M_i\left(q_i\right)$ which was defined in (9). His utility of misreporting a type q_i' equals

$$\begin{aligned}
&\mathbb{E}_{\mathbf{q}_{-i}}\left[t_i(q_i', \mathbf{q}_{-i}) + p_i(q_i', \mathbf{q}_{-i})b_i(q_i)\right] \\
=&\mathbb{E}_{\mathbf{q}_{-i}}\left[t_i(q_i', \mathbf{q}_{-i}) + b_i(q_i')p_i(q_i', \mathbf{q}_{-i}) + \left(b_i(q_i) - b_i\left(q_i'\right)\right)p_i(q_i', \mathbf{q}_{-i})\right] \\
=&M_i(q_i') + \mathbb{E}_{\mathbf{q}_{-i}}\left[\left(b_i(q_i) - b_i\left(q_i'\right)\right)p_i(q_i', \mathbf{q}_{-i})\right].
\end{aligned}$$

From the incentive compatibility constraint, we know that

$$M_i\left(q_i\right) \geq M_i(q_i') + \mathbb{E}_{\mathbf{q}_{-i}}\left[\left(b_i(q_i) - b_i\left(q_i'\right)\right)p_i(q_i', \mathbf{q}_{-i})\right]$$

i.e.

$$M_i\left(q_i'\right) - M_i\left(q_i\right) \leq \mathbb{E}_{\mathbf{q}_{-i}}\left[\left(b_i(q_i') - b_i\left(q_i\right)\right)p_i(q_i', \mathbf{q}_{-i})\right].$$

Using the last inequality twice (once switching the roles of q_i and q_i'), we get

$$\mathbb{E}_{\mathbf{q}_{-i}}\left[\left(b_i(q_i') - b_i\left(q_i\right)\right)p_i(\mathbf{q})\right] \leq M_i\left(q_i'\right) - M_i\left(q_i\right) \leq \mathbb{E}_{\mathbf{q}_{-i}}\left[\left(b_i(q_i') - b_i\left(q_i\right)\right)p_i(q_i', \mathbf{q}_{-i})\right]. \tag{10}$$

Step 2. Consider now two qualities q_i, q_i' such that $b_i(q_i) < b_i(q_i')$ then $\mathbb{E}_{\mathbf{q}_{-i}} p_i(q_i, \mathbf{q}_{-i}) \leq \mathbb{E}_{\mathbf{q}_{-i}} p_i(q_i', \mathbf{q}_{-i})$ holds. Indeed, by (10), we have

$$\mathbb{E}_{\mathbf{q}_{-i}} \left[\left(b_i(q_i') - b_i(q_i) \right) \left(p_i(q_i', \mathbf{q}_{-i}) - p_i(\mathbf{q}) \right) \right] \geq 0.$$

Because $b_i(q_i') - b_i(q_i) > 0$, this implies $\mathbb{E}_{\mathbf{q}_{-i}} p_i(\mathbf{q}) \leq \mathbb{E}_{\mathbf{q}_{-i}} p_i(q_i', \mathbf{q}_{-i})$.

Step 3. Dividing (10) by $q_i' - q_i$ and taking the limit as $q_i' \to q_i$ we find that

$$\frac{\partial M_i(q_i)}{\partial q_i} = b_i'(q_i) \, \mathbb{E}_{\mathbf{q}_{-i}} p_i(\mathbf{q}).$$

The equality holds almost everywhere, and the right-hand side is continuous in q_i almost everywhere. From the Fundamental Theorem of Calculus, we obtain

$$M_i(q_i) = M_i(\overline{q_i}) - \int_{q_i}^{\overline{q_i}} b_i'(\widetilde{q}_i) \mathbb{E}_{\mathbf{q}_{-i}} p_i(\widetilde{q}_i, \mathbf{q}_{-i}) d\widetilde{q}_i,$$

substituting this into (9) and rearranging yields the following expression.

$$M_i(\overline{q_i}) - \int_{q_i}^{\overline{q_i}} b_i'(\widetilde{q}_i) \mathbb{E}_{\mathbf{q}_{-i}} p_i(\widetilde{q}_i, \mathbf{q}_{-i}) d\widetilde{q}_i = \mathbb{E}_{\mathbf{q}_{-i}} \left[t_i(\mathbf{q}) + b_i(q_i) p_i(\mathbf{q}) \right]$$

$$\mathbb{E}_{\mathbf{q}_{-i}} t_i(\mathbf{q}) = - \left(\mathbb{E}_{\mathbf{q}_{-i}} b_i(q_i) p_i(\mathbf{q}) - M_i(\overline{q_i}) + \int_{q_i}^{\overline{q_i}} b_i'(\widetilde{q}_i) \mathbb{E}_{\mathbf{q}_{-i}} p_i(\widetilde{q}_i, \mathbf{q}_{-i}) d\widetilde{q}_i \right). \tag{11}$$

Step 4. We use integration by parts to show that

$$\int_{\underline{q_i}}^{\overline{q_i}} \left(\int_{q_i}^{\overline{q_i}} b_i'(\widetilde{q}_i) \mathbb{E}_{\mathbf{q}_{-i}} p_i(\widetilde{q}_i, \mathbf{q}_{-i}) d\widetilde{q}_i \right) f(q_i) dq_i = \int_{\underline{q_i}}^{\overline{q_i}} \left(b_i'(q_i) \mathbb{E}_{\mathbf{q}_{-i}} p_i(q_i, \mathbf{q}_{-i}) \right) F(q_i) dq_i \tag{12}$$

Step 5. Now, we substitute the transfers (11) into the expected value of the profit made from agent i :

$$
\begin{aligned}
& \mathbb{E}_{\mathbf{q}}\left[q_i p_i(\mathbf{q})-t_i(\mathbf{q})\right] \\
= & \mathbb{E}_{q_i}\left[\mathbb{E}_{\mathbf{q}_{-i}} q_i p_i(\mathbf{q})-\mathbb{E}_{\mathbf{q}_{-i}} t_i(\mathbf{q})\right] \\
= & \mathbb{E}_{q_i}\left[\mathbb{E}_{\mathbf{q}_{-i}} q_i p_i(\mathbf{q})+\left(\mathbb{E}_{\mathbf{q}_{-i}} b_i(q_i) p_i(\mathbf{q})-M_i(\overline{q_i})+\int_{q_i}^{\overline{q_i}} b_i'(\widetilde{q}_i) \mathbb{E}_{\mathbf{q}_{-i}} p_i(\widetilde{q}_i, \mathbf{q}_{-i}) d\widetilde{q}_i\right)\right] \\
= & \mathbb{E}_{q_i}\left[\mathbb{E}_{\mathbf{q}_{-i}}\left[q_i p_i(\mathbf{q})+b_i(q_i) p_i(\mathbf{q})-M_i(\overline{q_i})+\int_{q_i}^{\overline{q_i}} b_i'(\widetilde{q}_i) p_i(\widetilde{q}_i, \mathbf{q}_{-i}) d\widetilde{q}_i\right]\right] \\
= & \mathbb{E}_{q_i}\left[\mathbb{E}_{\mathbf{q}_{-i}}\left[q_i p_i(\mathbf{q})+b_i(q_i) p_i(\mathbf{q})-M_i(\overline{q_i})\right]\right]+\mathbb{E}_{q_i}\left[\int_{q_i}^{\overline{q_i}} b_i'(\widetilde{q}_i) \mathbb{E}_{\mathbf{q}_{-i}} p_i(\widetilde{q}_i, \mathbf{q}_{-i}) d\widetilde{q}_i\right]
\end{aligned}
$$

Using (12)

$$
\begin{aligned}
= & \mathbb{E}_{q_i}\left[\mathbb{E}_{\mathbf{q}_{-i}}\left[q_i p_i(\mathbf{q})+b_i(q_i) p_i(\mathbf{q})-M_i(\overline{q_i})\right]\right]+\int_{\underline{q_i}}^{\overline{q_i}}\left(b_i'(q_i) \mathbb{E}_{\mathbf{q}_{-i}} p_i(q_i, \mathbf{q}_{-i})\right) F(q_i) dq_i \\
= & \mathbb{E}_{q_i}\left[\mathbb{E}_{\mathbf{q}_{-i}}\left[q_i p_i(\mathbf{q})+b_i(q_i) p_i(\mathbf{q})-M_i(\overline{q_i})\right]\right]+\int_{\underline{q_i}}^{\overline{q_i}}\left(b_i'(q_i) \mathbb{E}_{\mathbf{q}_{-i}} p_i(q_i, \mathbf{q}_{-i})\right) \frac{F(q_i)}{f(q_i)} f(q_i) dq_i \\
= & \mathbb{E}_{q_i}\left\{\mathbb{E}_{\mathbf{q}_{-i}}\left[q_i p_i(\mathbf{q})+b_i(q_i) p_i(\mathbf{q})+\left(b_i'(q_i) p_i(q_i, \mathbf{q}_{-i})\right) \frac{F(q_i)}{f(q_i)}-M_i(\overline{q_i})\right]\right\} \\
& \mathbb{E}_{q_i}\left\{\mathbb{E}_{\mathbf{q}_{-i}}\left[G_i(q_i) p_i(\mathbf{q})-M_i(\overline{q_i})\right]\right\}
\end{aligned}
$$

which yields the desired equality.

$$
\mathbb{E}_{\mathbf{q}} \sum_{i=1}^n\left[q_i p_i(\mathbf{q})-t_i(\mathbf{q})\right]=\mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^n\left[G_i(q_i) p_i(\mathbf{q})-M_i(\overline{q_i})\right]\right].
$$

■

Lemma A2 *At any mechanism that satisfies the constraints, the following holds, for each agent* $i \in \mathcal{N}$

$$
\mathbb{E}_{\mathbf{q}}\left[\left[G_i(q_i) p_i(\mathbf{q})-M_i(\overline{q_i}, \mathbf{q}_{-i})\right]\right] \leq \mathbb{E}_{\mathbf{q}}\left[\min\left\{G_i(q_i), g_i^{\circ}\right\} p_i(\mathbf{q})\right]
$$

Proof of Lemma A2. Let $\mathbf{q}$ be a vector of qualities. First, by IIC_i we have

$$M_i\left(\overline{q_i}\right) \geq \mathbb{E}_{\mathbf{q}_{-i}}\left(t_i\left(\mathbf{q}\right) + b_i\left(\overline{q_i}\right) p_i\left(\mathbf{q}\right)\right),$$

otherwise agent i's type $\overline{q_i}$ would have an incentive to misreport as q_i. By LL_i, we have $t_i\left(\mathbf{q}\right) \geq 0$, thus for all $\mathbf{q}$,

$$\mathbb{E}_{\mathbf{q}_{-i}} b_i\left(\overline{q_i}\right) p_i\left(\mathbf{q}\right) \leq M_i(\overline{q_i}) \tag{13}$$

For all $q_i > a_i^*\left(g_i^\circ\right)$ we have $G_i\left(q_i\right) > g_i^\circ$, where $a_i^*\left(\cdot\right)$ is defined in (6). Therefore,

$$\begin{aligned}
&\left\{\int_{a_i^*\left(g_i^\circ\right)}^{\overline{q_i}} G_i(q_i)\mathbb{E}_{\mathbf{q}_{-i}} p_i(\mathbf{q}) f_i(q_i) dq_i - M_i(\overline{q_i})\right\} - \left\{\int_{a_i^*\left(g_i^\circ\right)}^{\overline{q_i}} g_i^\circ \mathbb{E}_{\mathbf{q}_{-i}} p_i(\mathbf{q}) f_i(q_i) dq_i\right\} \\
&= \int_{a_i^*\left(g_i^\circ\right)}^{\overline{q_i}} \left(G_i(q_i) - g_i^\circ\right)\mathbb{E}_{\mathbf{q}_{-i}} p_i(\mathbf{q}) f_i(q_i) dq_i - M_i(\overline{q_i}) \\
&\leq \left[\int_{a_i^*\left(g_i^\circ\right)}^{\overline{q_i}} \left(G_i(q_i) - g_i^\circ\right) f_i(q_i) dq_i - b_i\left(\overline{q_i}\right)\right] \frac{M_i(\overline{q_i})}{b_i\left(\overline{q_i}\right)} = 0.
\end{aligned}$$

The inequality between the second and third lines is an implication of the constraint (13). The last equality holds by definition of the cap g_i° (in (7)). Therefore,

$$\int_{a_i^*\left(g_i^\circ\right)}^{\overline{q_i}} G_i(q_i)\mathbb{E}_{\mathbf{q}_{-i}} p_i(\mathbf{q}) f_i(q_i) dq_i - M_i(\overline{q_i}) \leq \int_{a_i^*\left(g_i^\circ\right)}^{\overline{q_i}} g_i^\circ \mathbb{E}_{\mathbf{q}_{-i}} p_i(\mathbf{q}) f_i(q_i) dq_i.$$

which further implies

$$\begin{aligned}
&\int_{\underline{q_i}}^{\overline{q_i}} G_i(q_i)\mathbb{E}_{\mathbf{q}_{-i}} p_i(\mathbf{q}) f_i(q_i) dq_i - M_i(\overline{q_i}) \\
&\leq \int_{\underline{q_i}}^{a_i^*\left(g_i^\circ\right)} G_i(q_i)\mathbb{E}_{\mathbf{q}_{-i}} p_i(\mathbf{q}) f_i(q_i) dq_i + \int_{a_i^*\left(g_i^\circ\right)}^{\overline{q_i}} g_i^\circ \mathbb{E}_{\mathbf{q}_{-i}} p_i(\mathbf{q}) f_i(q_i) dq_i \\
&= \mathbb{E}_{\mathbf{q}_{-i}}\left[\int_{\underline{q_i}}^{a_i^*\left(g_i^\circ\right)} G_i(q_i) p_i(\mathbf{q}) f_i(q_i) dq_i + \int_{a_i^*\left(g_i^\circ\right)}^{\overline{q_i}} g_i^\circ p_i(\mathbf{q}) f_i(q_i) dq_i\right] \\
&= \mathbb{E}_{\mathbf{q}_{-i}}\mathbb{E}_{q_i}\left[\min\left\{G_i\left(q_i\right), g_i^\circ\right\} p_i\left(\mathbf{q}\right)\right]
\end{aligned}$$

where the last equality holds because, for all $q_i < a_i^* (g_i^\circ)$, we have $G_i(q_i) < g_i^\circ$, and for all $q_i > a_i^* (g_i^\circ)$, we have $g_i^\circ > G_i(q_i)$. ■

Proof of Theorem 1 in Bayesian version. The profit of the firm at some arbitrary mechanism satisfying (2)-(4) equals

$$\begin{aligned}
&\mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n}[G_i(q_i)p_i(\mathbf{q}) - M_i(\overline{q_i})]\right] \\
&\leq \mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n}\min\{G_i(q_i), g_i^\circ\} p_i(\mathbf{q})\right] \\
&\leq \mathbb{E}_{\mathbf{q}}\left[S_m\left(\min\{G_1(q_1), g_1^\circ\}, ..., \min\{G_n(q_n), g_n^\circ\}, \underbrace{0, ..., 0}_{m \text{ times}}\right)\right] \\
&= \mathbb{E}_{\mathbf{q}}[S_m(\mathbf{x}(\mathbf{q}))].
\end{aligned}$$

where the first expression for the profit was derived in Lemma A1, the first inequality holds by Lemma A2, and the second holds because of the feasibility constraints (2). Under the proposed mechanism, all the weak inequalities above hold as equalities everywhere, thus the mechanism achieves the optimal profit. The mechanism also satisfies the constraints (2)-(4), therefore it is optimal. ■

Appendix B – State Contingent Transfers

In this online appendix consider mechanisms with state contingent transfers. To do this we need to introduce a state space describing possible realized returns of the financed projects. Let the return of each project i be a nonnegative random variable R_i. We assume that, conditional on q_i, all R_i are independent and have the same support $\mathcal{R} \subset [0, +\infty)$. Each project i requires an initial investment I_i that does not depend on q_i. We assume that the quality q_i of project i is its expected net return conditional on q_i,

$$\mathbb{E}\left(R_i \mid q_i\right) - I_i = q_i.$$

That is, the project quality which is known to the managers is the expected return of the project. Under these assumptions, we provide an upper bound on the value that the firm can obtain when the firm can pay transfers contingent on the realization of both the project stochastic allocation, and the selected projects' outcomes. A state $\mathbf{s} = (\mathcal{M}, \mathbf{R})$ indicates the set of projects (if any) that were selected $\mathcal{M} \subset \mathcal{N}$ and the realization of the return of the projects which were selected $\mathbf{R} \in \mathbb{R}^{|\mathcal{M}|}$.

A mechanism is a function $(\widehat{\mathbf{q}}, \mathbf{s}) \mapsto (\mathbf{p}(\widehat{\mathbf{q}}), \mathbf{t}(\widehat{\mathbf{q}}, \mathbf{s}))$. To shorten notation let us denote the expected transfer conditional on true qualities by

$$t_i\left(\widehat{\mathbf{q}}, \mathbf{q}\right) = \mathbb{E}_s\left[t_i(\widehat{\mathbf{q}}, \mathbf{s}) \mid \mathbf{q}\right].$$

Incentive constraints are given by

$$p_i(\mathbf{q})b_i(q_i) + t_i\left(\mathbf{q}, \mathbf{q}\right) \geq p_i(q_i', \mathbf{q}_{-i})b_i(q_i) + t_i\left(\left(q_i', \mathbf{q}_{-i}\right), \mathbf{q}\right). \tag{14}$$

Limited liability constrains need to hold state-by-state, i.e.

$$t_i(\widehat{\mathbf{q}}, \mathbf{s}) \geq 0. \tag{15}$$

We can rewrite the optimization problem as follows.

$$\max_{\{p_i(\widehat{\mathbf{q}}),t_i(\widehat{\mathbf{q}},\mathbf{s})\}_{i=1,\dots,n}} \mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n}\left[q_i p_i(\mathbf{q}) - t_i(\mathbf{q},\mathbf{q})\right]\right] \tag{16}$$
$$s.t.: \text{ feasibility (2), state-by-state limited liability (15) and IC (14)}$$

The solution method for this problem follows similar steps as in sections III and IV in the paper. We derive an upper bound for the optimal value, and then propose a mechanism that satisfies the constraints and (here only approximately) achieves the upper bound.

Let V be the value of the target function at some arbitrary mechanism that satisfies all the constraints. Let $g_i^{\circ\circ}$ be this level of the outside option, which solves

$$\max_a \left\{\int_a^{\overline{q_i}} \left(q_i + b\left(q_i\right)\right) f_i(q_i) dq_i - \int_{\underline{q_i}}^{\overline{q_i}} b(q_i) f_i(q_i) dq_i + F\left(a\right) g_i^{\circ\circ}\right\} = g_i^{\circ\circ}. \tag{17}$$

For each project i, let the virtual quality of project i be redefined in the current context as $\widetilde{x}_i = \min\left\{q_i + b_i\left(q_i\right), g_i^{\circ\circ}\right\}$, where $g_i^{\circ\circ}$ is defined by (17). Let

$$\widetilde{\mathbf{x}}(\mathbf{q}) = \left(\min\left\{q_1 + b_1\left(q_1\right), g_1^{\circ\circ}\right\}, ..., \min\left\{q_n + b_n\left(q_n\right), g_n^{\circ\circ}\right\}, \underbrace{0, ..., 0}_{m \text{ times}}\right)$$

be the vector of virtual qualities, and $S_m(\widetilde{\mathbf{x}}(\mathbf{q}))$ the sum of the m largest virtual qualities.

Lemma 1 proves a preliminary result used in lemma 2 which derives the upper bound for the value.

Lemma B1 *At any mechanism that satisfies the constraints (2), limited liability (15) and (14) of the state dependent problem, the following holds, for each agent* $i \in \mathcal{N}$*.*

$$\mathbb{E}_{q_i}\left[\left(q_i + b_i(q_i)\right) p_i(\mathbf{q}) - U_i\left(\mathbf{q}\right)\right] \leq \mathbb{E}_{q_i}\left[\min\left\{q_i + b_i\left(q_i\right), g_i^{\circ\circ}\right\} p_i\left(\mathbf{q}\right)\right].$$

Proof of lemma B1. Let a_i^{**} be the unique quality threshold that maximizes the left-hand side in the equation (17). The first-order condition yields $a_i^{**}+b_i\left(a_i^{**}\right)=g_i^{\circ\circ}$, so that,

$$\text{for all } q_i < a_i^{**}, \text{we have } q_i+b\left(q_i\right)<g_i^{\circ\circ}, \text{ and for all } q_i > a_i^{**}, \text{ we have } q_i+b\left(q_i\right)>g_i^{\circ\circ}. \tag{18}$$

Rearranging (17), we obtain

$$\int_{a_i^{**}}^{\overline{q_i}}\left[\left(q_i+b_i\left(q_i\right)\right)-g_i^{\circ\circ}\right]f_i(q_i)dq_i-\int_{\underline{q_i}}^{\overline{q_i}}b_i(q_i)f_i(q_i)dq_i=0. \tag{19}$$

We first show the following inequality:

$$\begin{aligned}
&\left\{\int_{a_i^{**}}^{\overline{q_i}}\left(q_i+b_i\left(q_i\right)\right)p_i(\mathbf{q})f_i(q_i)dq_i-\mathbb{E}_{q_i}\left[U_i(\mathbf{q})\right]\right\}-\left\{g_i^{\circ\circ}\int_{a_i^{**}}^{\overline{q_i}}p_i(\mathbf{q})f_i(q_i)dq_i\right\}\\
=&\int_{a_i^{**}}^{\overline{q_i}}\left(\left(q_i+b_i\left(q_i\right)\right)-g_i^{\circ\circ}\right)p_i(\mathbf{q})f_i(q_i)dq_i-\mathbb{E}_{q_i}\left[U_i(\mathbf{q})\right]\\
\leq&\,\overline{p}_i(\mathbf{q})\left[\int_{a_i^{**}}^{\overline{q_i}}\left[\left(q_i+b\left(q_i\right)\right)-g_i^{\circ\circ}\right]f_i(q_i)dq_i-\int_{\underline{q_i}}^{\overline{q_i}}b(q_i)f_i(q_i)dq_i\right]=0.
\end{aligned}$$

Where $\overline{p}_i(\mathbf{q})=\sup_{q_i}\left\{p_i\left(\mathbf{q}\right)\right\}$. The inequality between the second and third lines follows from $p_i\left(\mathbf{q}\right)\leq\overline{p}_i\left(\mathbf{q}\right)$ and from (18) which implies that $\left(q_i+b\left(q_i\right)\right)-g_i^{\circ\circ}\geq 0$ in the integrated range, and from the constraints $\overline{p}_i\left(\mathbf{q}\right)b\left(q_i\right)\leq U_i\left(\mathbf{q}\right)$. The last equality holds by (19). Therefore,

$$\int_{a_i^{**}}^{\overline{q_i}}\left(q_i+b\left(q_i\right)\right)p_i(\mathbf{q})f_i(q_i)dq_i-\mathbb{E}_{q_i}\left[U_i(\mathbf{q})\right]\leq\int_{a_i^{**}}^{\overline{q_i}}g_i^{\circ\circ}p_i(\mathbf{q})f_i(q_i)dq_i. \tag{20}$$

which further implies

$$\begin{aligned}
&\int_{\underline{q_i}}^{\overline{q_i}} (q_i + b(q_i))\, p_i(\mathbf{q}) f_i(q_i) dq_i - \mathbb{E}_{q_i}[U_i(\mathbf{q})] \\
&= \int_{\underline{q_i}}^{a_i^{**}} (q_i + b(q_i))\, p_i(\mathbf{q}) f_i(q_i) dq_i + \int_{a_i^{**}}^{\overline{q_i}} (q_i + b(q_i))\, p_i(\mathbf{q}) f_i(q_i) dq_i - \mathbb{E}_{q_i}[U_i(\mathbf{q})] \\
&\leq \int_{\underline{q_i}}^{a_i^{**}} (q_i + b(q_i))\, p_i(\mathbf{q}) f_i(q_i) dq_i + \int_{a_i^{**}}^{\overline{q_i}} g_i^{\circ\circ} p_i(\mathbf{q}) f_i(q_i) dq_i \\
&= \mathbb{E}_{q_i}\left[\min\left\{q_i + b(q_i), g_i^{\circ\circ}\right\} p_i(\mathbf{q})\right]
\end{aligned}$$

where the inequality holds by (20) and the last equality holds by (18). ■

We now turn to the upper bound on the value of the problem.

Lemma B2 *For any mechanism with state-dependent transfers that satisfies feasibility (2), limited-liability (15) and incentive constraints (14), the value is bounded as follows.*

$$V \leq \mathbb{E}_{\mathbf{q}}\left[S_m(\widetilde{\mathbf{x}}(\mathbf{q}))\right] \equiv V_{\text{sup}}.$$

Proof of Lemma B2. We substitute $t_i(\mathbf{q}, \mathbf{q}) = U_i(\mathbf{q}) - b_i(q_i)\, p_i(\mathbf{q})$ in the objective function (16). The target function becomes

$$\max_{\{p_i(\mathbf{q}), U_i(q)\}_{i=1,\dots,n}} \mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n} \left[(q_i + b_i(q_i)) p_i(\mathbf{q}) - U_i(\mathbf{q})\right]\right] \tag{21}$$

The value of this target function at any mechanism that satisfies the constraints, V, will satisfy the following

$$\begin{aligned}
V &= \mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n} \left[(q_i + b(q_i))\, p_i(\mathbf{q}) - U_i(\mathbf{q})\right]\right] \\
&\leq \mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n} \min\left\{q_i + b(q_i), g_i^{\circ\circ}\right\} p_i(\mathbf{q})\right] \\
&\leq \mathbb{E}_{\mathbf{q}}\left[S_m\left(\min\left\{q_1 + b(q_1), g_1^{\circ\circ}\right\}, \dots, \min\left\{q_n + b(q_n), g_n^{\circ\circ}\right\}, \underbrace{0, \dots, 0}_{m \text{ times}}\right)\right].
\end{aligned}$$

where the first inequality holds by Lemma 1, and the second holds because of the feasibility constraints (2). ■

The lemma implies that the firm's profit is bounded above by V_{sup}. Under certain conditions which we soon list, V_{sup} is the supremum of the firm's profit for all mechanisms that satisfy the constraints (2), (15) and (14). It may not be achieved, but we can construct a mechanism that yields a profit that is ε-close to the bound V_{sup} in Lemma 2.

We assume from now that the benefit functions $b_i(.)$ are convex in the expected net return q_i and *less than unit elastic* with respect to the expected gross return $q_i + I_i$, i.e.

$$\frac{(q_i + I_i)\, b_i'(q_i)}{b_i(q_i)} \leq 1, \tag{22}$$

for all q_i. In particular this includes the case where the private benefit function is linear, $b_i(q_i) = \alpha_i + \beta_i q_i$ such that $I_i \geq \alpha_i \geq 0$ and $\beta_i \geq 0$.

We now propose a mechanism that gives the firm a profit close to the upper bound.[1] Consider the following allocation rule

$$p_i^{**}(\widehat{\mathbf{q}}) = \begin{cases} 1 & \text{if } \widetilde{x}_i(\widehat{\mathbf{q}}) > \widetilde{x}_{(m)}(\widehat{\mathbf{q}}) \\ \frac{m - \left|\left\{i \in \mathcal{N}:\, \widetilde{x}_i(\widehat{\mathbf{q}}) > \widetilde{x}_{(m)}(\widehat{\mathbf{q}})\right\}\right|}{\left|\left\{i \in \mathcal{N}:\, \widetilde{x}_i(\widehat{\mathbf{q}}) = \widetilde{x}_{(m)}(\widehat{\mathbf{q}})\right\}\right|} & \text{if } \widetilde{x}_i(\widehat{\mathbf{q}}) = \widetilde{x}_{(m)}(\widehat{\mathbf{q}}) \\ 0 & \text{if } \widetilde{x}_i(\widehat{\mathbf{q}}) < \widetilde{x}_{(m)}(\widehat{\mathbf{q}}) \end{cases}. \tag{23}$$

According to this allocation rule, a project that has a virtual quality lower than the m-th highest virtual quality is not selected. A project that has a virtual quality higher than the m-th highest virtual quality is selected for sure, and the rest of the

[1] To come up with this proposed mechanism we first solved the simpler problem in which transfers can depend on the true quality of all projects, not just the selected ones. The derivation of this mechanism follows similar steps to our previous analysis and we thus omit it.

projects are chosen at random from the set of projects that have exactly the m-th highest virtual quality.

The approximation allocates projects according to $\mathbf{p}^{**}$ with probability $1-\varepsilon$, for a given $\varepsilon > 0$ and selects projects at random (with equal probabilities) with probability ε. The resulting allocation rule is

$$p_i^{\varepsilon}(\widehat{\mathbf{q}}) = (1-\varepsilon)\, p_i^{**} + \varepsilon \frac{m}{n}. \tag{24}$$

Transfers are adjusted in order to satisfy the constraints. Let $\mathcal{X}$ be the event where project i is not selected, which occurs with probability $1 - p_i^{\varepsilon}(\widehat{\mathbf{q}})$. Consider the following transfers to manager i in any state in $s \in \mathcal{X}$, and in any state $s' \notin \mathcal{X}$ (i.e. when i is selected) in which $i's$ realized return is $r_i \in R_i$:

$$t_i^{\varepsilon}((\widehat{q}_i, \widehat{\mathbf{q}}_{-i}), s) = \frac{[p_i^{\varepsilon}(\overline{q_i}, \widehat{\mathbf{q}}_{-i}) - p_i^{\varepsilon}(\widehat{\mathbf{q}})]}{1 - p_i^{\varepsilon}(\widehat{\mathbf{q}})} [b_i(\widehat{q}_i) - b_i'(\widehat{q}_i)(\widehat{q}_i + I_i)] \tag{25}$$

$$t_i^{\varepsilon}(\widehat{q}_i, \widehat{\mathbf{q}}_{-i}, s') = \frac{[p_i^{\varepsilon}(\overline{q_i}, \widehat{\mathbf{q}}_{-i}) - p_i^{\varepsilon}(\widehat{\mathbf{q}})]}{p_i^{\varepsilon}(\widehat{\mathbf{q}})} b_i'(\widehat{q}_i) r_i. \tag{26}$$

Note that these transfers are well-defined only when $p_i^{\varepsilon}(\widehat{\mathbf{q}}) > 0$ for all $\widehat{\mathbf{q}} > 0$, which is the reason why we need $\varepsilon > 0$ and why the supremum V_{sup} cannot be achieved, but only approximated. The expected transfer to manager i, conditional on the vector $\mathbf{q}$ is

$$t_i^{\varepsilon}(\widehat{q}_i, \widehat{\mathbf{q}}_{-i}, \mathbf{q}) = [p_i^{\varepsilon}(\overline{q_i}, \widehat{\mathbf{q}}_{-i}) - p_i^{\varepsilon}(\widehat{\mathbf{q}})] [b_i(\widehat{q}_i) + b_i'(\widehat{q}_i)(q_i - \widehat{q}_i)] .$$

We show next that this mechanism satisfies all the constraints and delivers a value that is ε-close to the upper bound. Denote the payoff of project manager i when reported qualities are $(\widehat{q}_i, \widehat{\mathbf{q}}_{-i})$ and his true quality is q_i by $U_i(\widehat{q}_i, \widehat{\mathbf{q}}_{-i}, q_i)$. We now verify that incentive compatibility holds. Indeed, we have

$$U_i((\widehat{q}_i, \widehat{\mathbf{q}}_{-i}), \mathbf{q}) = [p_i^{\varepsilon}(\overline{q_i}, \widehat{\mathbf{q}}_{-i}) - p_i^{\varepsilon}(\widehat{q}_i, \widehat{\mathbf{q}}_{-i})] [b_i(\widehat{q}_i) - b_i(q_i) + b_i'(\widehat{q}_i)(q_i - \widehat{q}_i)] + p_i^{\varepsilon}(\overline{q_i}, \widehat{\mathbf{q}}_{-i})\, b_i(q_i)$$

The first term of this sum is always nonnegative. Indeed, $p_i^{\varepsilon}\left(\overline{q_i}, \widehat{\mathbf{q}}_{-i}\right) - p_i^{\varepsilon}\left(\widehat{q}_i, \widehat{\mathbf{q}}_{-i}\right) \geq 0$ and, because the functions $b_i\left(\cdot\right)$ are convex, then $b_i\left(\widehat{q}_i\right) - b_i\left(q_i\right) + b_i'(\widehat{q}_i)\left(q_i - \widehat{q}_i\right) \leq 0$. Thus, for all true quality q_i and vector of reported qualities $\widehat{\mathbf{q}}$,

$$U_i((\widehat{q}_i, \widehat{\mathbf{q}}_{-i}), \mathbf{q}) \leq p_i^{\varepsilon}\left(\overline{q_i}, \widehat{\mathbf{q}}_{-i}\right) b_i\left(q_i\right) = U_i((q_i, \widehat{\mathbf{q}}_{-i}), \mathbf{q}).$$

This ensures that the proposed mechanism is incentive-compatible. Because $b_i\left(q_i\right) - \left(q_i + I_i\right) b_i'\left(q_i\right) \geq 0$, the limited liability constraint is also satisfied. Moreover, evaluating the target function at the mechanism indexed by ε yields

$$\left(1-\varepsilon\right) V_{\max} + \varepsilon \frac{m}{n} \sum_{i=1}^{N} \mathbb{E}\left[q_i\right].$$

where $V_{\max}$ is the upper bound in Lemma 2. Since $\mathbb{E}\left(q_i\right) \leq V_{\max}$, this value is lower than but approximates $V_{\max}$, in the limit where ε goes to 0.

Theorem B *The project selection mechanism given by the allocation rule (24) and the transfers (25)-(26) is almost optimal for the problem stated in (16).*

Proof of Theorem B. Consider the allocation rule $\mathbf{p}^{**}$ defined in (23) and the expected transfers conditional on $\mathbf{q}$, $\mathbf{t}^{**}$ given by

$$t_i^{**}\left(\widehat{\mathbf{q}}, \mathbf{q}\right) = \left(p_i\left(\overline{q_i}, \widehat{\mathbf{q}}_{-i}\right) - p_i\left(\widehat{\mathbf{q}}\right)\right)\left[b_i\left(\widehat{q}_i\right) + b'(\widehat{q}_i)\left(q_i - \widehat{q}_i\right)\right].$$

Under the mechanism defined by $\mathbf{p}^{**}$ and $\mathbf{t}^{**}$, all the weak inequalities in Lemma 2 hold as equalities. Thus, $\mathbf{p}^{**}$ and $\mathbf{t}^{**}$ yield the upper bound V_{sup} from Lemma 2. One can easily verify that the mechanism defined by $\mathbf{p}^{\varepsilon}$ and $\mathbf{t}^{\varepsilon}$ satisfy the constraints (2), (15) and (14). Next, the allocation rule $\mathbf{p}^{\varepsilon}$ and the expected transfers $\mathbf{t}^{\varepsilon}$ are convex combinations of the allocation rule $\mathbf{p}^{**}$ and expected transfers $\mathbf{t}^{**}$ and the allocation rule and expected transfers of the random selection mechanism.

$$\begin{aligned} p_i^{\varepsilon} &= \left(1-\varepsilon\right) p_i^{**} + \varepsilon \frac{m}{N} \\ t_i^{\varepsilon} &= \left(1-\varepsilon\right) t_i^{**} + \varepsilon \cdot 0. \end{aligned}$$

The allocation rule $\mathbf{p}^{**}$ and transfers $\mathbf{t}^{**}$ yield $V_{\sup}$, whereas the random allocation mechanism yields the value $\frac{1}{n}\sum_{i=1}^{N}\mathbb{E}\left[q_i\right]$. Thus the value of the mechanism indexed by ε is $(1-\varepsilon)\,V_{\sup}+\varepsilon\frac{1}{n}\sum_{i=1}^{N}\mathbb{E}\left[q_i\right]$. This value is lower than $V_{\sup}$, but its limit is $V_{\sup}$ as ε goes to 0. ■

Appendix C – Partially aligned interests

In this appendix, we analyze the situation where managers have interests that are at least partially aligned with those of the firm in the sense that each manager i enjoys some private benefit, not only when his own project is selected, in which case he receives the benefit $b_i(q_i)$, but also when another manager's project is selected. In such a situation, it is natural to assume that the benefit manager i receives is an increasing function $d_{ij}(q_j)$ of the quality q_j of the selected project j, which could in principle also depend on the label of the selected project j. To fix ideas, consider the case of n agents and one project ($m = 1$). For all i, manager i's utility for all $\mathbf{q}$ is

$$b_i(q_i)p_i(\mathbf{q}) + \sum_{j:j\neq i} d_{ij}(q_j)p_j(\mathbf{q}) + t_i(\mathbf{q}),$$

so that we get the following incentive compatibility constraints: for all $\mathbf{q}$, for all i, and for all q_i',

$$b_i(q_i)p_i(\mathbf{q})+\sum_{j:j\neq i} d_{ij}(q_j)p_j(\mathbf{q})+t_i(\mathbf{q}) \geq b_i(q_i)p_i(q_i',\mathbf{q}_{-i})+\sum_{j:j\neq i} d_{ij}(q_j)p_j(q_i',\mathbf{q}_{-i})+t_i(q_i',\mathbf{q}_{-i}).$$

In the extreme case where manager i's benefit functions $b_i(\cdot)$ and $d_{ij}(\cdot)$ are the same function, increasing in quality, the agent prefers the highest quality project to be selected, regardless of whether it is his own project. In such a case, the interests of the agents and the financing agency are perfectly aligned. A mechanism that chooses the highest reported quality project and never offers transfers would be optimal. However, when at least for some qualities q_i, an agent enjoys his own project being financed more than someone else's project with the same quality, an adverse selection problem remains. To simplify the analysis, we will focus on the case where if the optimal mechanism selects manager i, then this manager would rather be selected than let another agent be selected. This condition requires that

$b_i(q_i) > d_{ij}(q_j)$ whenever $i's$ virtual quality is at least as high as $j's$. The condition can hold true when the benefit from a manager's own project from being selected is higher than the benefit of another being selected when their quality difference is not too high.[2] Our results in this section parallel the results we obtained for the main model.

Lemma C1 *Let the payoff of manager i in the optimal mechanism be*

$$M_i(\mathbf{q}) = t_i(\mathbf{q}) + b_i(q_i)p_i(\mathbf{q}) + \sum_{j \neq i} d_{ij}(q_j)p_j(\mathbf{q}). \tag{27}$$

The target function can be written as

$$\mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n}[H_i(q_i)p_i(\mathbf{q}) - M_i(\overline{q_i}, \mathbf{q}_{-i})]\right]$$

with

$$H_i(q_i) = q_i + b_i(q_i) + b_i'(q_i)\frac{F_i(q_i)}{f_i(q_i)} + \sum_{j:j\neq i} d_{ji}(q_i).$$

This generalizes the function $G_i(q_i)$ from the main model. We assume from now on that $H_i(q_i)$ is an increasing function.

Proof of Lemma C1. We can show that

$$M_i(q_i, \mathbf{q}_{-i}) = M_i(\overline{q_i}, \mathbf{q}_{-i}) - \int_{q_i}^{\overline{q_i}} b_i'(\widetilde{q_i})p_i(\widetilde{q_i}, \mathbf{q}_{-i})d\widetilde{q_i},$$

exactly like in the proof of Lemma A1 (main text) substituting this into (27) and

[2]When this assumption is not satisfied, further complications occur, similar to those arising when the agents may be unwilling to be selected, i.e. $b_i(q_i) < 0$ for low quality q_i values. This problem is analyzed in detail in the section D of this appendix.

rearranging yields the following expression.

$$t_i(\mathbf{q}) = -\left(b_i(q_i)p_i(\mathbf{q}) + \sum_{j:j\neq i} d_{ij}(q_j)p_j(\mathbf{q}) + \int_{\underline{q}_i}^{\overline{q_i}} b_i'(\widetilde{q_i})p_i(\widetilde{q_i}, \mathbf{q}_{-i})d\widetilde{q_i} - M_i(\overline{q_i}, \mathbf{q}_{-i})\right). \tag{28}$$

Fix $\mathbf{q}_{-i}$. Using integration by parts in the same way as in the proof of Lemma A1 in the main text, we can show that

$$\begin{aligned}
&\mathbb{E}_{q_i}\left[q_i p_i(\mathbf{q}) - t_i(\mathbf{q})\right] \\
=&\mathbb{E}_{q_i}\left[q_i p_i(\mathbf{q}) + b_i(q_i)p_i(\mathbf{q}) + \sum_{j:j\neq i} d_{ij}(q_j)p_j(\mathbf{q}) + b_i'(q_i)p_i(q_i, \mathbf{q}_{-i})\frac{F_i(q_i)}{f_i(q_i)} - M_i(\overline{q_i}, \mathbf{q}_{-i})\right].
\end{aligned}$$

Taking the expectation over $\mathbf{q}_{-i}$ and adding up over i yields

$$\begin{aligned}
&\mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n}\left[q_i p_i(\mathbf{q}) + b_i(q_i)p_i(\mathbf{q}) + \sum_{j:j\neq i} d_{ij}(q_j)p_j(\mathbf{q}) + b_i'(q_i)p_i(q_i, \mathbf{q}_{-i})\frac{F_i(q_i)}{f_i(q_i)} - M_i(\overline{q_i}, \mathbf{q}_{-i})\right]\right] \\
=&\mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n}\left[q_i p_i(\mathbf{q}) + b_i(q_i)p_i(\mathbf{q}) + b_i'(q_i)p_i(q_i, \mathbf{q}_{-i})\frac{F_i(q_i)}{f_i(q_i)} - M_i(\overline{q_i}, \mathbf{q}_{-i})\right]\right] \\
&+\mathbb{E}_{\mathbf{q}}\left[\sum_{i,j:i\neq j}\left[d_{ij}(q_j)\, p_j(\mathbf{q})\right]\right] \\
=&\mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n}\left[q_i p_i(\mathbf{q}) + b_i(q_i)p_i(\mathbf{q}) + b_i'(q_i)p_i(q_i, \mathbf{q}_{-i})\frac{F_i(q_i)}{f_i(q_i)} - M_i(\overline{q_i}, \mathbf{q}_{-i})\right]\right] \\
&+\mathbb{E}_{\mathbf{q}}\left[\sum_{i,j:i\neq j}\left[d_{ji}(q_i)\, p_i(\mathbf{q})\right]\right] \\
=&\mathbb{E}_{\mathbf{q}}\left[\sum_{i=1}^{n}\left[H_i(q_i)\, p_i(\mathbf{q}) - M_i(\overline{q_i}, \mathbf{q}_{-i})\right]\right].
\end{aligned}$$

■

Lemma C2 *Let h_i° be such that*

$$\int_{\underline{q}_i}^{\overline{q_i}} \max\left\{\left(H_i(q_i) - h_i^\circ\right), 0\right\} f_i(q_i)dq_i - b_i\left(\overline{q_i}\right) = 0.$$

At any mechanism that satisfies the constraints, the following holds, for each agent $i \in \mathcal{N}$.

$$\mathbb{E}_{\mathbf{q}}\left[H_i(q_i)p_i(\mathbf{q}) - M_i(\overline{q_i}, \mathbf{q}_{-i})\right] \leq \mathbb{E}_{\mathbf{q}}\left[\min\left\{H_i\left(q_i\right), h_i^{\circ}\right\} p_i\left(\mathbf{q}\right)\right].$$

Proof of Lemma C2. Let i be an agent and let $\mathbf{q}$ be a vector of qualities. By IC_i, we know that

$$b_i\left(\overline{q_i}\right) p_i\left(q_i, \mathbf{q}_{-i}\right) + \sum_{j:j\neq i} d_{ij}\left(q_j\right) p_j\left(q_i, \mathbf{q}_{-i}\right) + t_i\left(q_i, \mathbf{q}_{-i}\right) \leq M_i(\overline{q_i}, \mathbf{q}_{-i}),$$

otherwise agent i with quality $\overline{q_i}$ would have an incentive to misrepresent his quality by reporting q_i. By LL_i, we know .that $t_i\left(q_i, \mathbf{q}_{-i}\right) \geq 0$, and by feasibility, we know that $d_{ij}\left(q_j\right) p_j\left(q_i, \mathbf{q}_{-i}\right) \geq 0$, for all j, so that

$$b_i\left(\overline{q_i}\right) p_i\left(q_i, \mathbf{q}_{-i}\right) \leq M_i(\overline{q_i}, \mathbf{q}_{-i}).$$

Next, noting that H_i, and h_i° play exactly the same role as G_i and g_i°, the rest of the proof is analogous to the proof of Lemma A2 in the main text. It is therefore omitted. ■

Theorem C *The expected profit of the firm at the optimal mechanism equals*

$$\mathbb{E}_{\mathbf{q}}\left[\max_i\left[\min\left\{H_i\left(q_i\right), h_i^{\circ}\right\}, 0\right]\right].$$

The mechanism that selects the project i with the highest virtual quality redefined as

$$\min\left\{H_i\left(q_i\right), h_i^{\circ}\right\},$$

randomizes with equal probabilities between projects that achieve the maximum and compensate the managers according to the transfers

$$t_i(\mathbf{q}) = \begin{cases} b_i\left(\overline{q_i}\right) p_i\left(\overline{q}_i, \mathbf{q}_{-i}\right) + \sum_{j:j\neq i} d_{ij}\left(q_j\right) p_j\left(\overline{q}_i, \mathbf{q}_{-i}\right) \\ -b_i(q_i)p_i(\mathbf{q}) - \sum\limits_{j:j\neq i} d_{ij}(q_j)p_j(\mathbf{q}) - \int\limits_{q_i}^{\overline{q_i}} b_i'(\widetilde{q}_i)p_i(\widetilde{q}_i, \mathbf{q}_{-i})d\widetilde{q}_i. \end{cases} \tag{29}$$

is optimal.

The proof is identical to the proof of Theorem 1 in the main text, and is therefore omitted.[3] We would now like to compare the optimal profit achieved by the firm in this problem (partial interest alignment) to the profit that the firm achieves in the case without interest alignment, i.e. the case where $d_{ij}(q_i) = 0$ for all i and j, which is the one studied in the main model. The comparison is immediate from the expected profit expressions. Because for all i and all q_i, we have $G_i(q_i) < H_i(q_i)$, it follows that

$$\begin{aligned}&\int_{\bar{q}_i}^{\overline{q_i}} \max\{(G_i(q_i) - h_i^\circ), 0\} f_i(q_i) dq_i - b_i(\overline{q_i}) \\ < &\int_{\bar{q}_i}^{\overline{q_i}} \max\{(H_i(q_i) - h_i^\circ), 0\} f_i(q_i) dq_i - b_i(\overline{q_i}) \\ = &\,0\end{aligned}$$

where the equality is from the definition of h_i°. Then, by definition of g_i°, we obtain that $g_i^\circ < h_i^\circ$. This implies that the problem with partial alignment has a higher virtual quality for each agent and at each reported quality than the problem with no alignment:

$$\min\{G_i(q_i), g_i^\circ\} < \min\{H_i(q_i), h_i^\circ\}.$$

This lead to the following expected profit comparison.

Corollary C: *The firm's expected profit is higher under partial alignment than under nonalignment.*

[3] The assumption that b_i exceeds d_{ij} when i's virtualy quality is higher ensures that the proposed mechanism satisfies the constraints IC_i. Under the proposed mechanism, when a project is selected, its manager does not receive any transfer. Given the inequality and IC_i constraints, when a project is not selected, it receives a positive transfer, in agreement with limited liability.

One can easily see that the allocation of the optimal mechanism in the absence of partial alignment is implementable under partial alignment using transfers (29). These transfers are smaller than the ones that are used in the absence of alignment: the difference between the transfers under alignment and under nonalignment is for all i and all $\mathbf{q}$,

$$\Delta t_i(\mathbf{q}) = \sum_{j:j\neq i} d_{ij}(q_j)\left(p_j(\overline{q}_i, \mathbf{q}_{-i}) - p_j(\mathbf{q})\right) \leq 0$$

and the resulting expected profit is therefore higher in the alignment case. This observation immediately shows that the optimal profit in the alignment case is higher than the optimal profit in the no-alignment case.

Example 1 *Let $n = 2$, and let $b_i(q_i) = b_i q_i + b_i^\circ$ and $d_{ij}(q_{-i}) = d_{ij} q_{-i}$, where d_{ij}, b_i and b_i° are positive constants such that $d_{ij} < b_i^\circ$. Let f_i be uniform on $[0,1]$. Then*

$$H_i(q_i) = (1 + 2b_i + d_{ji})\, q_i + b_i^\circ$$

and h_i° is one of the solutions of the following equation in h

$$(1 + 2b_i + d_{ji})\frac{1}{2}\left(1 - \left(\frac{h - b_i^\circ}{1 + 2b_i + d_{ji}}\right)^2\right) + (b_i^\circ - h)\left(1 - \frac{h}{1 + 2b_i + d_{ji}}\right) = b_i + b_i^\circ.$$

The relevant solution to this quadratic equation is

$$h_i^\circ = (1 + 2b_i + d_{ji}) - \sqrt{(b_i^\circ)^2 + 2(1 + 2b_i + d_{ji})\, b_i}.$$

Appendix D: Possibly negative private benefits

In this Appendix, we relax the assumption that the managers necessarily want to be selected, that is we allow the functions $b_i(q_i)$ to take negative values (we still assume that this function is increasing in q_i). When $b_i(q_i)$ is negative for some quality q_i, then $-b(q_i) > 0$ can be interpreted as an opportunity cost of undertaking the project. In this case, the limited liability (LL) constraint $t_i \geq 0$ no longer implies the individual rationality (IR) constraint $t_i + b_i p_i \geq 0$, even in combination with the incentive compatibility (IC). For example, the constant mechanism that always selects project 1 regardless of the reports and without compensation satisfies LL and IC, but not IR for the lowest project 1 type if $b_1\left(\underline{q}_1\right) < 0$. Since it would be unnatural to consider mechanisms that violate individual rationality, where some managers are forced to implement their project against their own will, we will look for the optimal mechanism that satisfies LL, IR and IC.

Note that if $b_i(\overline{q}_i) < 0$, then by monotonicity, this implies that $b_i(q_i) < 0$ for all q_i, in which case IR implies LL. Therefore, in this case, LL can be ignored and the problem is reduced to the search for an optimal mechanism under IR and IC.[4] We consider here the case where $b_i\left(\underline{q}\right) < 0$ and $b_i(\overline{q}_i) > 0$, so that none of the constraints is implied by the other. We assume that the function $b_i(q_i)$ is strictly increasing and continuous. This implies that there exists a unique q_i^* such that $b_i(q_i^*) = 0$. This is the quality at which manager i is exactly indifferent between undertaking the project or not without compensation. We say that at quality q_i, agent i is *unwilling* if $q_i < q_i^*$ and *willing* if $q_i \geq q_i^*$. For simplicity, suppose that at most one project will

[4] In this case, the problem is mathematically very similar to the optimal mechanism problems where ironing is not needed, that arise in papers by Myerson (1981) and Manelli and Vincent (1995, section 6).

be selected, that is $m = 1$.

For each project i, let $G_i(q_i)$ be defined as in Section I, and let g_i° be defined as in Section II. For simplicity, assume that $G_i(q_i^*) \leq g_i^\circ$ for all i. We will show that the optimal mechanism selects the project which has the highest value of some project-specific transformation $K_i(q_i)$ of its quality q_i. Let $K_i(q_i)$ be the function such that

$$K_i(q_i) = \begin{cases} G_i(q_i) - \frac{b'(q_i)}{f(q_i)} & \text{for } q_i < q_i^* \\ q_i^* & \text{for } q_i = q_i^* \\ \min\{G_i(q_i), g_i^\circ\} & \text{for } q_i > q_i^* \end{cases}.$$

Also, for each i, we let $K_{-i}(\mathbf{q}_{-i}) := \max_{j \neq i}\{K_j(q_j), 0\}$ and we say that agent i is *competing* if the optimal allocation depends on his report, that is if $K_{-i}(\mathbf{q}_{-i}) \in \left[K_i\left(\underline{q}_i\right), K_i(\overline{q}_i)\right]$. For a competing agent, we let agent i's quality *threshold* be

$$z_i(\mathbf{q}_{-i}) = \inf\{q_i : K_i(q_i) \geq K_{-i}(\mathbf{q}_{-i})\} = \sup\{q_i : K_i(q_i) < K_{-i}(\mathbf{q}_{-i})\}.$$

This is the reported quality of agent i at which the allocation changes, holding the other agent's reports fixed. We show the following result.

Theorem D: *Let* $m = 1$. *The mechanism that randomly selects with equal probabilities among the projects with the highest value* $K_i(q_i)$ *above* 0 (the outside option), *selects the outside option if none exceeds this level, and pays the following transfers, is optimal under IR, LL and IC constraints. For each* $\mathbf{q}$, *and each* i*:*

- If *agent i is competing and selected with positive probability (i.e. $z_i \leq q_i$) and unwilling at his quality threshold (i.e. $z_i < q_i^*$), he receives a transfer*

$$t_i(\mathbf{q}) = \left(p_i\left(\underline{q}_i, \mathbf{q}_{-i}\right) - p_i(\mathbf{q})\right) b_i(z_i).$$

- *If agent i is competing, but he is either not selected or he is selected with probability less than one (i.e. $q_i \leq z_i$) and willing at his quality threshold (i.e. $q_i^* \leq z_i$), he receives a transfer*

$$t_i(\mathbf{q}) = (p_i(\overline{q}_i, \mathbf{q}_{-i}) - p_i(\mathbf{q}))\, b_i(z_i)\,.$$

- *If agent i is not competing, but is selected (i.e. $K_{-i}(\mathbf{q}_{-i}) < K_i\left(\underline{q}_i\right)$), he receives a transfer*

$$t_i(\mathbf{q}) = -b_i\left(\underline{q}_i\right).$$

- *In all other cases, agent i receives no transfer.*

Hence, if a competing agent is unwilling at his quality threshold, and regardless of whether he is willing or not at his true quality, he receives a positive transfer only if the mechanism allows him to decrease his probability to be selected by misreporting (a quality level lower than his true quality). In this case, if his true quality is slightly above the threshold (implying he is unwilling), the transfer keeps him from misreporting a lower quality level in order not to be selected. If his true quality is higher, the transfer keeps him from misreporting a quality level slightly above the threshold, in order to receive a positive transfer.

If a competing agent's is willing at his quality threshold, and regardless of whether he is willing or not at his true type, he receives a positive transfer only if the mechanism allows him to increase his probability of being selected by misreporting (a quality level higher than his true quality). In this case, if his type is slightly below his threshold type, the transfer keeps him from misreporting a high quality level in order to be selected. If his type is lower, the transfer keeps him from mimicking a type slightly below the threshold type, in order to receive a positive transfer.

It is helpful to understand the role of the constraints IC, IR and LL in determining the transfers, holding the allocation fixed at the optimum.

If only IC and LL are imposed, at the optimal transfer, four cases can occur. In the first case, if agent i is competing and unwilling at his quality threshold, he receives a positive transfer if he is selected and no transfer if he is not selected. The addition of the IR constraint makes no difference in this case. In the second case, if agent i is competing and willing at his quality threshold, he receives no transfer if he is selected and receives a positive transfer if he is not selected. The addition of the IR constraint makes no difference in this case either. In the third case, if agent i is not competing and not selected, he receives no transfer. The addition of IR makes no difference in this case either. In the fourth and last case, if agent i is not competing but is selected, his utility is below zero if his quality is near the lowest possible level: when the IR constraint is added, agent i is must now receive a positive transfer independent of his reported quality which must be large enough to maintain his utility at zero, when his quality is the lowest possible.

If only IC and IR are imposed, at the optimal transfer, three cases can occur. In the first case, if agent i is competing and unwilling at his quality threshold, he receives a positive transfer if he is selected and no transfer if he is not selected.[5] The addition of the LL constraint makes no difference in this case. In the second case, if agent i is competing and willing at his quality threshold, he receives no transfer if he is not selected and receives a negative transfer if he is selected: when the LL constraint is added, agent i must now receive a positive transfer when he is not selected and no transfer when he is selected, as in the optimal mechanism when only IC and LL are imposed. In the third and last case, if agent i is not competing, then

[5] The transfers in this case are identical to those of the optimal mechanism when only IC and LL are imposed.

regardless of his reported quality, he either receives no transfer (if he is not selected), or he receives a positive transfer (if he is selected). Either way, the addition of the LL constraint makes no difference in this case.

Proof of Theorem D. For all $\mathbf{q}_{-i}$, let $M_i(\mathbf{q}_{-i}) = p_i(\overline{q}_i, \mathbf{q}_{-i})\, b_i(\overline{q}_i) + t_i(\overline{q}_i, \mathbf{q}_{-i})$ be defined as the utility of the highest quality agent i, when he truthfully reports his quality $\overline{q}_i$ and other agents report qualities $\mathbf{q}_{-i}$. As in the main case considered in the paper, the agency's objective function equals

$$\mathbb{E}\left[\sum_i G_i(q_i)\, p_i(\mathbf{q}) - M_i(\mathbf{q}_{-i})\right].$$

We maximize this value, over $(p_i(\mathbf{q}))_i$ for all $\mathbf{q}$, and $(M_i(\mathbf{q}_{-i}))_i$ for all $\mathbf{q}_{-i}$, under feasibility, incentive-compatibility, limited liability and individual rationality. We only impose a subset of the constraints, the other constraints turn out not to be binding. Feasibility

$$\sum_i p_i(\mathbf{q}) \leq 1 \text{ and } p_i(\mathbf{q}) \geq 0 \text{ for all } \mathbf{q}.$$

No incentive to misreport for the highest type level:

$$p_i(\mathbf{q})\, b_i(\overline{q}_i) + t_i(\mathbf{q}) - M_i(\mathbf{q}_{-i}) \leq 0 \text{ for all } \mathbf{q}.$$

Limited liability:

$$t_i(\mathbf{q}) \geq 0 \text{ for all } \mathbf{q}.$$

The last two constraints imply

$$p_i(\mathbf{q})\, b_i(\overline{q}_i) - M_i(\mathbf{q}_{-i}) \leq 0 \text{ for all } \mathbf{q}.$$

Individual rationality for the lowest quality level $\underline{q}_i$:

$$\mathbb{E}_{q_i}\left[\frac{b'(q_i)}{f(q_i)} p_i(\mathbf{q})\right] - M_i(\mathbf{q}_{-i}) \leq 0 \text{ for all } \mathbf{q}_{-i}.$$

For all $\mathbf{q}_{-i}$ such that

$$K_{-i}(\mathbf{q}_{-i}) \in \left[G_i(q_i^*) - \frac{b'(q_i^*)}{f_i(q_i^*)}, \min\{G_i(q_i^*), g_i^\circ\} \right],$$

let $\gamma_i(\mathbf{q}_{-i}) \in [0,1]$ be a weight such that $K_{-i}(\mathbf{q}_{-i})$ is the convex combination of $G_i(q_i^*) - \frac{b'(q_i^*)}{f_i(q_i^*)}$ and $\min\{G_i(q_i^*), g_i^\circ\}$ with respective weights $\gamma_i(\mathbf{q}_{-i})$ and $1-\gamma_i(\mathbf{q}_{-i})$:

$$K_{-i}(\mathbf{q}_{-i}) = \gamma_i(\mathbf{q}_{-i}) \left(G_i(q_i^*) - \frac{b'(q_i^*)}{f_i(q_i^*)} \right) + (1-\gamma_i(\mathbf{q}_{-i})) \min\{G_i(q_i^*), g_i^\circ\}.$$

We will show that at any feasible solution, the objective function is bounded above by the value

$$\mathbb{E}_{\mathbf{q}} \left[\max_i \{v_i(\mathbf{q}), 0\} \right]$$

where

$$v_i(\mathbf{q}) = \begin{cases} \min\{G_i(q_i), g_i^\circ\} & \text{if } G_i(q_i^*) \leq K_{-i}(\mathbf{q}_{-i}), \\ \gamma_i(\mathbf{q}_{-i}) \left(G_i(q_i) - \frac{b'(q_i)}{f_i(q_i)} \right) & \\ + (1-\gamma_i(\mathbf{q}_{-i})) \min\{G_i(q_i), g_i^\circ\} & \text{if } G_i(q_i^*) - \frac{b'(q_i^*)}{f_i(q_i^*)} \leq K_{-i}(\mathbf{q}_{-i}) < G_i(q_i^*), \\ G_i(q_i) - \frac{b'(q_i)}{f_i(q_i)} & \text{if } K_{-i}(\mathbf{q}_{-i}) < G_i(q_i^*) - \frac{b'(q_i^*)}{f_i(q_i^*)}. \end{cases}$$

Note that the mechanism defined in the Theorem achieves this value, and that this mechanism satisfies all the feasibility, IC, IR and LL constraints.

We argue first that $\mathbb{E}_{q_i}[G_i(q_i) p_i(\mathbf{q})] - M_i(\mathbf{q}_{-i}) \leq \mathbb{E}_{q_i}[v_i(\mathbf{q}) p_i(\mathbf{q})]$ for all $\mathbf{q}_{-i}$. This is established in four steps.

Step 1: For all $\mathbf{q}_{-i}$,

$$\mathbb{E}_{q_i}[G_i(q_i) p_i(\mathbf{q})] - M_i(\mathbf{q}_{-i}) \leq \mathbb{E}_{q_i}[\min\{G_i(q_i), g_i^\circ\} p_i(\mathbf{q})].$$

The inequality is obtained from IC for all q_i, and from LL for all q_i such that $G_i(q_i) \geq g_i^\circ$. It was established in the proof of Theorem 1.

Step 2: For all $\mathbf{q}_{-i}$,

$$\begin{aligned}\mathbb{E}_{q_i}\left[G_i\left(q_i\right)p_i\left(\mathbf{q}\right)\right]-M_i\left(\mathbf{q}_{-i}\right) &\leq \mathbb{E}_{q_i}\left[G_i\left(q_i\right)p_i\left(\mathbf{q}\right)-p_i\left(\mathbf{q}\right)\frac{b'\left(q_i\right)}{f_i\left(q_i\right)}\right] \\ &= \mathbb{E}_{q_i}\left[\left(G_i\left(q_i\right)-\frac{b'\left(q_i\right)}{f_i\left(q_i\right)}\right)p_i\left(\mathbf{q}\right)\right].\end{aligned}$$

The inequality is obtained from IC for all q_i, and from IR for $q_i = \underline{q}_i$.

Step 3: From steps 1 and 2, from the definition of v_i and because $\gamma_i\left(\mathbf{q}_{-i}\right) \in [0,1]$ on its domain, it follows that for all $\mathbf{q}_{-i}$,

$$\mathbb{E}_{q_i}\left[G_i\left(q_i\right)p_i\left(\mathbf{q}\right)\right]-M_i\left(\mathbf{q}_{-i}\right) \leq \mathbb{E}_{q_i}\left[v_i\left(\mathbf{q}\right)p_i\left(\mathbf{q}\right)\right].$$

Step 4: We then obtain

$$\begin{aligned}&\mathbb{E}_{\mathbf{q}}\left[\sum_i G_i\left(q_i\right)p_i\left(\mathbf{q}\right)-M_i\left(\mathbf{q}_{-i}\right)\right] \\ \leq\, &\mathbb{E}_{\mathbf{q}}\left[\sum_i v_i\left(\mathbf{q}\right)p_i\left(\mathbf{q}\right)\right] \\ \leq\, &\mathbb{E}_{\mathbf{q}}\left[\max_i\left\{v_i\left(\mathbf{q}\right),0\right\}\right]\end{aligned}$$

where the last inequality holds because of the feasibility constraints. Finally, the mechanism proposed in the Theorem satisfies all the constraints, and reaches the bound because when we substitute it in the above inequalities, they all hold as equalities. This is because the proposed mechanism has the following properties.

- The IC constraints bind for all i and all q_i.
- For all i, and all $\mathbf{q}_{-i}$ with $G_i\left(q_i^*\right)-\frac{b'\left(q_i^*\right)}{f_i\left(q_i^*\right)} \leq K_{-i}\left(\mathbf{q}_{-i}\right)$, the LL constraint of agent i binds for all $q_i \geq q_i^*$, in particular for all q_i such that $G_i\left(q_i\right) \geq g_i^{\circ}$.
- For all i, and all $\mathbf{q}_{-i}$ with $K_{-i}\left(\mathbf{q}_{-i}\right) < G_i\left(q_i^*\right)$, the IR constraint of agent i binds for $q_i = \underline{q}_i$.

- For all $\mathbf{q}$ and all i such that $K_i(q_i) \geq \max\{K_{-i}(\mathbf{q}_{-i}), 0\}$, we have $v_i(\mathbf{q}) \geq \max_{j \neq i} v_j(\mathbf{q})$ and $v_i(\mathbf{q}) \geq 0$. Thus the selected project(s) under the proposed mechanism maximize $v_i(\mathbf{q})$.

We then conclude that the proposed mechanism is optimal. ■