Online Appendix
"Optimal Inference for Spot Regressions"

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Abstract

This supplemental appendix contains three separate sections. Section SA collects the proofs for all the theoretical results discussed in the main part of the paper. Section SB presents an additional theoretical result, establishing the optimality of the t-test for spot beta. Section SC presents various robustness checks related to the empirical analysis discussed in the main part of the paper.

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SA Proofs

Throughout the proofs, we use K to denote a generic positive constant which may change from line to line. We also sometimes write K_p to stress its dependence on some parameter p. We may further strengthen Assumptions 1 and 2 by assuming that the conditions hold for $T_1 = \infty$; this is without loss of generality due to the standard localization procedure as shown in Section 4.4.1 in Jacod and Protter (2012).

SA.1 Proofs for Section I

To prove Theorem 1, we need two lemmas that characterize the finite-sample behavior of the estimators of interest in the "limit" Gaussian linear regression model. Although the proofs for these lemmas are elementary, we provide them for completeness.

Lemma S1. *If $X = vW_1$ and $Y = \beta X + \varsigma^{1/2} W_2$ for some constants $v > 0$, $\beta \in \mathbb{R}$, and $\varsigma > 0$, then*

$$\frac{\sqrt{k-1}\big(\hat{\beta}_t - \beta_t\big)}{\sqrt{\hat{\varsigma}_t / \hat{v}_t}} = \xi_\beta. \tag{SA.1}$$

Proof. Under the parametric model considered in this lemma,

$$\Delta_i^n X = v^{1/2} \Delta_i^n W_1, \quad \Delta_i^n Y = \beta \Delta_i^n X + \varsigma^{1/2} \Delta_i^n W_2.$$

Therefore,

$$\begin{aligned}
\hat{c}_{12,t} &= \frac{1}{k\Delta_n} \sum_{i \in \mathcal{I}_{n,t}} (\Delta_i^n X)(\Delta_i^n Y) \\
&= \frac{1}{k\Delta_n} \sum_{i \in \mathcal{I}_{n,t}} (\Delta_i^n X)\Big(\beta \Delta_i^n X + \varsigma^{1/2} \Delta_i^n W_2\Big) \\
&= \beta \hat{c}_{11,t} + k^{-1} v^{1/2} \varsigma^{1/2} \xi_{12},
\end{aligned} \tag{SA.2}$$

and

$$\begin{aligned}
\hat{c}_{22,t} &= \frac{1}{k\Delta_n} \sum_{i \in \mathcal{I}_{n,t}} (\Delta_i^n Y)^2 \\
&= \frac{1}{k\Delta_n} \sum_{i \in \mathcal{I}_{n,t}} \Big(\beta \Delta_i^n X + \varsigma^{1/2} \Delta_i^n W_2\Big)^2 \\
&= \beta^2 \hat{c}_{11,t} + 2k^{-1} \beta v^{1/2} \varsigma^{1/2} \xi_{12} + k^{-1} \varsigma \xi_{22}.
\end{aligned} \tag{SA.3}$$

Combining $\hat{c}_{11,t} = k^{-1} v \xi_{11}$ with (SA.2) and (SA.3), we deduce that

$$\begin{aligned}\hat{c}_{11,t}\hat{c}_{22,t} - \hat{c}_{12,t}^2 &= \hat{c}_{11,t}\left(\beta^2 \hat{c}_{11,t} + 2k^{-1}\beta v^{1/2}\varsigma^{1/2}\xi_{12} + k^{-1}\varsigma\xi_{22}\right) - \left(\beta\hat{c}_{11,t} + k^{-1}v^{1/2}\varsigma^{1/2}\xi_{12}\right)^2 \\ &= k^{-2}v\varsigma\left(\xi_{11}\xi_{22} - \xi_{12}^2\right). \end{aligned} \tag{SA.4}$$

From the definitions of $\hat{v}_t$, $\hat{\beta}_t$, and $\hat{\varsigma}_t$ in the main text, it follows that

$$\frac{\hat{\beta}_t - \beta}{\sqrt{\hat{\varsigma}_t/\hat{v}_t}} = \frac{\frac{\hat{c}_{12,t}}{\hat{c}_{11,t}} - \beta}{\sqrt{\frac{\hat{c}_{22,t}}{\hat{c}_{11,t}} - \frac{\hat{c}_{12,t}^2}{\hat{c}_{11,t}^2}}} = \frac{\hat{c}_{12,t} - \beta\hat{c}_{11,t}}{\sqrt{\hat{c}_{11,t}\hat{c}_{22,t} - \hat{c}_{12,t}^2}}. \tag{SA.5}$$

Plugging (SA.2) and (SA.4) into (SA.5) yields

$$\frac{\hat{\beta}_t - \beta}{\sqrt{\hat{\varsigma}_t/\hat{v}_t}} = \frac{\xi_{12}}{\sqrt{\xi_{11}\xi_{22} - \xi_{12}^2}},$$

which readily implies the assertion of the lemma. *Q.E.D.*

Lemma S2. *ξ_β has a t-distribution with degree of freedom $k-1$.*

Proof. Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two generic independent k-dimensional standard normal vectors. It is easy to see that $(\xi_{11}, \xi_{12}, \xi_{22})$ equals to $(\|\boldsymbol{U}\|^2, \boldsymbol{U}^\top\boldsymbol{V}, \|\boldsymbol{V}\|^2)$ in distribution. Denote

$$\boldsymbol{M} = \boldsymbol{I}_k - \frac{\boldsymbol{U}\boldsymbol{U}^\top}{\|\boldsymbol{U}\|^2},$$

where $\boldsymbol{I}_k$ is the k-dimensional identity matrix. Observe that

$$\boldsymbol{V}^\top\boldsymbol{M}\boldsymbol{V} = \|\boldsymbol{V}\|^2 - \frac{\left(\boldsymbol{U}^\top\boldsymbol{V}\right)^2}{\|\boldsymbol{U}\|^2}.$$

Since the matrix $\boldsymbol{M}$ is idempotent with trace $k-1$ and $\boldsymbol{V} \sim \mathcal{N}(0, \boldsymbol{I}_k)$, the $\boldsymbol{U}$-conditional distribution of $\boldsymbol{V}^\top\boldsymbol{M}\boldsymbol{V}$ is χ^2_{k-1} almost surely.

Next, recall that by definition,

$$\xi_\beta = \frac{\xi_{12}}{\sqrt{\left(\xi_{11}\xi_{22} - \xi_{12}^2\right)/(k-1)}} \overset{d}{=} \frac{\boldsymbol{U}^\top\boldsymbol{V}/\|\boldsymbol{U}\|}{\sqrt{\boldsymbol{V}^\top\boldsymbol{M}\boldsymbol{V}/(k-1)}}. \tag{SA.6}$$

It is easy to see that the $\boldsymbol{U}$-conditional distribution of $\boldsymbol{U}^\top\boldsymbol{V}/\|\boldsymbol{U}\|$ is $\mathcal{N}(0,1)$. In addition, conditional on $\boldsymbol{U}$, $(\boldsymbol{U}^\top\boldsymbol{V}, \boldsymbol{M}\boldsymbol{V})$ are jointly normal; since $\boldsymbol{M}\boldsymbol{U} = 0$, $\boldsymbol{U}^\top\boldsymbol{V}$ and $\boldsymbol{M}\boldsymbol{V}$ are also conditionally independent. Hence, $\boldsymbol{U}^\top\boldsymbol{V}/\|\boldsymbol{U}\|$ is $\boldsymbol{U}$-conditionally independent of $(\boldsymbol{M}\boldsymbol{V})^\top(\boldsymbol{M}\boldsymbol{V}) = \boldsymbol{V}^\top\boldsymbol{M}\boldsymbol{V}$. We have shown that the $\boldsymbol{U}$-conditional distribution of $\boldsymbol{V}^\top\boldsymbol{M}\boldsymbol{V}$ is χ^2_{k-1}. Combining these facts, we see that, conditional on $\boldsymbol{U}$, the variable on the right-hand side of (SA.6) is t_{k-1}-distributed, which further implies that its unconditional distribution is also t_{k-1}. This proves the assertion of the lemma. *Q.E.D.*

Proof of Theorem 1. Under Assumption 1(i), the probability that the estimation block $\mathcal{I}_{n,t}$ contains at least one price jump is $O(\Delta_n)$. Therefore, with probability approaching 1, $\mathcal{I}_{n,t}$ does not contain any price jump. Since our calculation concentrates on this one block, we can and will without loss of generality assume in the subsequent analysis that there are no price jumps.

For each $i \in \mathcal{I}_{n,t}$, we set

$$x_{n,i} \equiv v_t^{1/2} \Delta_i^n W_1, \quad y_{n,i} \equiv \beta_t x_{n,i} + \varsigma_t^{1/2} \Delta_i^n W_2, \quad \boldsymbol{z}_{n,i} \equiv (x_{n,i}, y_{n,i})^\top .$$

We then define

$$\hat{\boldsymbol{c}}_t' = \begin{pmatrix} \hat{c}_{11,t}' & \hat{c}_{12,t}' \\ \hat{c}_{21,t}' & \hat{c}_{22,t}' \end{pmatrix} \equiv \frac{1}{k\Delta_n} \sum_{i \in \mathcal{I}_{n,t}} \boldsymbol{z}_{n,i} \boldsymbol{z}_{n,i}^\top,$$

which we further use to define $\hat{v}_t'$, $\hat{\beta}_t'$, and $\hat{\varsigma}_t'$ as

$$\hat{v}_t' \equiv \hat{c}_{11,t}', \quad \hat{\beta}_t' \equiv \frac{\hat{c}_{12,t}'}{\hat{c}_{11,t}'}, \quad \hat{\varsigma}_t' \equiv \hat{c}_{22,t}' - \frac{\left(\hat{c}_{12,t}'\right)^2}{\hat{c}_{11,t}'}.$$

Lemma S1 implies that

$$\frac{\sqrt{k-1}\left(\hat{\beta}_t' - \beta_t\right)}{\sqrt{\hat{\varsigma}_t'/\hat{v}_t'}} = \xi_\beta. \tag{SA.7}$$

Next, we show that $\hat{\boldsymbol{c}}_t - \hat{\boldsymbol{c}}_t' = o_p(1)$, for which we need some preliminary estimates. In particular, observe that for $i \in \mathcal{I}_{n,t}$,

$$\Delta_i^n X - x_{n,i} = \int_{(i-1)\Delta_n}^{i\Delta_n} b_{1,s} ds + \int_{(i-1)\Delta_n}^{i\Delta_n} \left(v_s^{1/2} - v_{(i-1)\Delta_n}^{1/2}\right) dW_{1,s} + \left(v_{(i-1)\Delta_n}^{1/2} - v_t^{1/2}\right) \Delta_i^n W_1. \tag{SA.8}$$

Also note that

$$E\left[\left|\int_{(i-1)\Delta_n}^{i\Delta_n} b_{1,s} ds\right|\right] \leq K\Delta_n = o(\Delta_n^{1/2}). \tag{SA.9}$$

By Itô isometry and the fact that $E[|v_t^{1/2} - v_{(i-1)\Delta_n}^{1/2}|^2] \leq K\Delta_n^{2\kappa}$, it follows that

$$\begin{aligned} E\left[\left|\int_{(i-1)\Delta_n}^{i\Delta_n} \left(v_s^{1/2} - v_{(i-1)\Delta_n}^{1/2}\right) dW_{1,s}\right|^2\right] &= E\left[\int_{(i-1)\Delta_n}^{i\Delta_n} \left(v_s^{1/2} - v_{(i-1)\Delta_n}^{1/2}\right)^2 ds\right] \\ &\leq K\Delta_n^{1+2\kappa}, \end{aligned}$$

and hence,

$$\int_{(i-1)\Delta_n}^{i\Delta_n} \left(v_s^{1/2} - v_{(i-1)\Delta_n}^{1/2} \right) dW_{1,s} = o_p(\Delta_n^{1/2}). \tag{SA.10}$$

Since $v_{(i-1)\Delta_n}^{1/2} - v_t^{1/2} = O_p(\Delta_n^{\kappa})$ and $\Delta_i^n W_1 = O_p(\Delta_n^{1/2})$, we also have

$$\left(v_{(i-1)\Delta_n}^{1/2} - v_t^{1/2} \right) \Delta_i^n W_1 = o_p(\Delta_n^{1/2}). \tag{SA.11}$$

Combining (SA.8), (SA.9), (SA.10), and (SA.11), we deduce that $\Delta_i^n X - x_{n,i} = o_p(\Delta_n^{1/2})$. By a similar argument, we can also show that $\Delta_i^n Y - y_{n,i} = o_p(\Delta_n^{1/2})$. Therefore,

$$\|\Delta_i^n \boldsymbol{Z} - \boldsymbol{z}_{n,i}\| = o_p(\Delta_n^{1/2}). \tag{SA.12}$$

It is also easy to see that $\boldsymbol{z}_{n,i} = O_p(\Delta_n^{1/2})$. Thus, by the triangle inequality,

$$\left\| (\Delta_i^n \boldsymbol{Z}) (\Delta_i^n \boldsymbol{Z})^\top - \boldsymbol{z}_{n,i} \boldsymbol{z}_{n,i}^\top \right\| \le 2 \left\| (\Delta_i^n \boldsymbol{Z} - \boldsymbol{z}_{n,i}) \boldsymbol{z}_{n,i}^\top \right\| + \|\Delta_i^n \boldsymbol{Z} - \boldsymbol{z}_{n,i}\|^2 = o_p(\Delta_n).$$

Since $\mathcal{I}_{n,t}$ contains a fixed number of elements, we further deduce that

$$\|\hat{\boldsymbol{c}}_t - \hat{\boldsymbol{c}}_t'\| \le \frac{1}{k\Delta_n} \sum_{i \in \mathcal{I}_{n,t}} \left\| (\Delta_i^n \boldsymbol{Z}) (\Delta_i^n \boldsymbol{Z})^\top - \boldsymbol{z}_{n,i} \boldsymbol{z}_{n,i}^\top \right\| = o_p(1). \tag{SA.13}$$

Finally, since $\hat{v}_t'$ and $\hat{\varsigma}_t'$ are strictly positive almost surely, we can use (SA.13) and the continuous mapping theorem to conclude that

$$\frac{\hat{\beta}_t - \beta_t}{\sqrt{\hat{\varsigma}_t / \hat{v}_t}} - \frac{\hat{\beta}_t' - \beta_t}{\sqrt{\hat{\varsigma}_t' / \hat{v}_t'}} = o_p(1).$$

The coupling claim of the theorem then follows from (SA.7) and the above display. The distributional claim is due to Lemma S2. *Q.E.D.*

Proof of Theorem 2. The assertions of the theorem follow from (SA.12) and the continuous mapping theorem. *Q.E.D.*

Proof of Theorem 3. For simplicity, we write $\hat{\beta}_{[1:k]}$ and $\hat{\varsigma}_{[1:k]}$ as $\hat{\beta}$ and $\hat{\varsigma}$, respectively. For each $i \in \{1, \ldots, k+h\}$, we set

$$x_{n,i} \equiv v_0^{1/2} \Delta_i^n W_1, \quad y_{n,i} \equiv \beta_0 x_{n,i} + \varsigma_0^{1/2} \Delta_i^n W_2, \quad \boldsymbol{z}_{n,i} \equiv (x_{n,i}, y_{n,i})^\top,$$

and use $(\boldsymbol{z}_{n,i})_{1 \le i \le k}$ to define $\hat{\beta}'$ and $\hat{\varsigma}'$ in the same way as described in the proof of Theorem 1. As shown in the proof of Theorem 1,

$$\|\Delta_i^n \boldsymbol{Z} - \boldsymbol{z}_{n,i}\| = o_p(\Delta_n^{1/2}), \quad \hat{\beta} - \hat{\beta}' = o_p(1), \quad \hat{\varsigma} - \hat{\varsigma}' = o_p(1). \tag{SA.14}$$

We further denote

$$\widehat{\mathrm{CAR}}_h' = \sum_{j=1}^{h} y_{n,k+j} - \left(\sum_{j=1}^{h} x_{n,k+j}\right)\hat{\beta}', \quad \widehat{\mathrm{se}}_h' = \sqrt{\left(h + \frac{(\sum_{j=1}^{h} x_{n,k+j})^2}{\sum_{i=1}^{k} x_{n,i}^2}\right)\frac{k\hat{\varsigma}'}{k-1}}.$$

The estimates in (SA.14) imply that

$$\widehat{\mathrm{CAR}}_h - \widehat{\mathrm{CAR}}_h' = o_p(\Delta_n^{1/2}), \quad \widehat{\mathrm{se}}_h - \widehat{\mathrm{se}}_h' = o_p(1),$$

yielding

$$\frac{\Delta_n^{-1/2}\widehat{\mathrm{CAR}}_h}{\widehat{\mathrm{se}}_h} = \tau_h + o_p(1), \quad \text{where} \quad \tau_h \equiv \frac{\Delta_n^{-1/2}\widehat{\mathrm{CAR}}_h'}{\widehat{\mathrm{se}}_h'}.$$

It remains to show that τ_h defined in the above display is t_{k-1}-distributed. Note that

$$\Delta_n^{-1/2}\widehat{\mathrm{CAR}}_h' = \varsigma_0^{1/2}\Delta_n^{-1/2}\left(W_{2,(k+h)\Delta_n} - W_{2,k\Delta_n}\right) - \left(\sum_{j=1}^{h} x_{n,k+j}\right)\Delta_n^{-1/2}\left(\hat{\beta}' - \beta_0\right).$$

Define k-dimensional random vectors $\boldsymbol{U} = (x_{n,i})_{1\leq i\leq k}$ and $\boldsymbol{V} = (\Delta_i^n W_2/\sqrt{\Delta_n})_{1\leq i\leq k}$. It is easy to show that

$$\Delta_n^{-1/2}\left(\hat{\beta}' - \beta_0\right) = \frac{\varsigma_0^{1/2}\boldsymbol{U}^\top\boldsymbol{V}}{\|\boldsymbol{U}\|^2}, \quad k\hat{\varsigma}' = \varsigma_0\left(\|\boldsymbol{V}\|^2 - \frac{\left(\boldsymbol{U}^\top\boldsymbol{V}\right)^2}{\|\boldsymbol{U}\|^2}\right).$$

Let $\mathcal{U}_0$ denote the information set generated by $\mathcal{F}_0$ and the W_1 process. Note that $\boldsymbol{U}$ is measurable with respect to $\mathcal{U}_0$. Moreover, conditional on $\mathcal{U}_0$, the variables $W_{2,(k+h)\Delta_n} - W_{2,k\Delta_n}$, $\Delta_n^{-1/2}(\hat{\beta}' - \beta_0)$, and $\hat{\varsigma}'$ are independent. This implies that $\Delta_n^{-1/2}\widehat{\mathrm{CAR}}_h'$ is independent of $k\hat{\varsigma}'$ conditional on $\mathcal{U}_0$. Moreover, the $\mathcal{U}_0$-conditional distributions of these variables satisfy

$$\left.\frac{\Delta_n^{-1/2}\widehat{\mathrm{CAR}}_h'}{\sqrt{\varsigma_0}}\right|\mathcal{U}_0 \sim \mathcal{N}\left(0, h + \frac{(\sum_{j=1}^{h} x_{n,k+j})^2}{\sum_{i=1}^{k} x_{n,i}^2}\right), \quad \left.\frac{k\hat{\varsigma}'}{\varsigma_0}\right|\mathcal{U}_0 \sim \chi_{k-1}^2.$$

These properties imply that τ_h is t_{k-1}-distributed conditional on $\mathcal{U}_0$. The unconditional distribution of τ_h is thus also t_{k-1} as asserted. *Q.E.D.*

SA.2 Proofs for Section II

Proof of Lemma 1. Since $\mathcal{I}_{n,t}$ contains a fixed number of elements, the estimate in (SA.12) holds jointly for all $i \in \mathcal{I}_{n,t}$. From here, the assertion of the lemma readily follows due to the continuous mapping theorem. *Q.E.D.*

Proof of Theorem 4. The spot beta estimator $\hat{\beta}_t$ may be rewritten as $f_\beta(\boldsymbol{r}_X, \boldsymbol{r}_Y) \equiv \boldsymbol{r}_X^\top\boldsymbol{r}_Y/\boldsymbol{r}_X^\top\boldsymbol{r}_X$. Let $f(\boldsymbol{r}_X, \boldsymbol{r}_Y)$ be a generic asymptotically unbiased regular estimator for β_t. Recall from the main text that the asymptotic risk function for f under the quadratic loss is given by, for $v>0$, $\beta\in\mathbb{R}$, and $\varsigma>0$,

$$R\left(f;v,\beta,\varsigma\right)=E\left[\left(f(v^{1/2}\boldsymbol{\eta},\beta v^{1/2}\boldsymbol{\eta}+\varsigma^{1/2}\boldsymbol{\epsilon})-\beta\right)^{2}\right],$$

where $\boldsymbol{\eta}$ and $\boldsymbol{\epsilon}$ are independent k-dimensional standard Gaussian random vectors. This is also the finite-sample risk of f in the limit experiment with observations being $(\boldsymbol{x},\boldsymbol{y})=(v^{1/2}\boldsymbol{\eta},\beta v^{1/2}\boldsymbol{\eta}+\varsigma^{1/2}\boldsymbol{\epsilon})$. In the limit Gaussian experiment (which belongs to the exponential family), the vector $(\boldsymbol{x}^\top\boldsymbol{x},\boldsymbol{x}^\top\boldsymbol{y},\boldsymbol{y}^\top\boldsymbol{y})$ forms the complete sufficient statistic for (v,β,ς), where we use the fact that the mean is known to be zero in the limit experiment. It is easy to see that $f_\beta(\boldsymbol{x},\boldsymbol{y})$ is unbiased and, because it is a function of the complete sufficient statistic, it is also uniformly minimum-variance unbiased (see Theorem 2.1.11 in Lehmann and Casella (1998)). That is, $R\left(f_\beta;\cdot\right)\leq R\left(f;\cdot\right)$ for any f that satisfies $E\left[f\left(\boldsymbol{x},\boldsymbol{y}\right)\right]=\beta$. This proves that $\hat{\beta}_t$ is the asymptotically uniformly minimum-variance unbiased estimator for β_t. *Q.E.D.*

Finally, we prove Theorem 5. We need some notation and preliminary estimates. For each $j\in\{1,\ldots,m_n\}$ and $i\in\mathcal{I}_{n,j}$, denote

$$\begin{aligned}
v_{n,j} &= v_{(j-1)k\Delta_n}, \quad \beta_{n,j}\equiv\beta_{(j-1)k\Delta_n}, \quad \varsigma_{n,j}\equiv\varsigma_{(j-1)k\Delta_n},\\
\tilde{x}_{n,i} &\equiv v_{n,j}^{1/2}\Delta_i^n W_1, \quad \tilde{y}_{n,i}\equiv\beta_{n,j}\tilde{x}_{n,i}+\varsigma_{n,j}^{1/2}\Delta_i^n W_2, \quad \tilde{\boldsymbol{z}}_{n,i}\equiv\left(\tilde{x}_{n,i},\tilde{y}_{n,i}\right)^\top.
\end{aligned}$$

We define

$$\tilde{\boldsymbol{c}}_j'=\begin{pmatrix}\tilde{c}_{11,j}' & \tilde{c}_{12,j}'\\ \tilde{c}_{21,j}' & \tilde{c}_{22,j}'\end{pmatrix}\equiv\frac{1}{k\Delta_n}\sum_{i\in\mathcal{I}_{n,j}}\tilde{\boldsymbol{z}}_{n,i}\tilde{\boldsymbol{z}}_{n,i}^\top,$$

and then set $\tilde{v}_j'$, $\tilde{\beta}_j'$, and $\tilde{\varsigma}_j'$ as

$$\tilde{v}_j'\equiv\tilde{c}_{11,j}', \quad \tilde{\beta}_j'\equiv\frac{\tilde{c}_{12,j}'}{\tilde{c}_{11,j}'}, \quad \tilde{\varsigma}_j'\equiv\tilde{c}_{22,j}'-\frac{(\tilde{c}_{12,j}')^2}{\tilde{c}_{11,j}'}.$$

The following lemma collects some useful uniform approximation results.

Lemma S3. *Suppose that* $\min\{1/2,\kappa\}>2/k$. *Under Assumption 2, the following statements holds for any fixed constant* $\iota>0$*:*

(a) $\max_{1\leq j\leq m_n}\left\|\tilde{\boldsymbol{c}}_j-\tilde{\boldsymbol{c}}_j'\right\|=o_p(\Delta_n^{\min\{1/2,\kappa\}-\iota})$;

(b) $\max_{1\leq j\leq m_n}\left(\left\|\tilde{\boldsymbol{c}}_j'\right\|+\left\|\tilde{\boldsymbol{c}}_j\right\|\right)=o_p\left(\Delta_n^{-\iota}\right)$;

(c) $\max_{1\leq j\leq m_n}(\tilde{c}_{11,j}')^{-1}=o_p(\Delta_n^{-2/k-\iota})$;

(d) for any constant $q\in(-\infty,1)$, $\max_{1\leq j\leq m_n}|\tilde{c}_{11,j}^q-\tilde{c}_{11,j}'^q|=o_p(\Delta_n^{\min\{1/2,\kappa\}-2(1-q)/k-\iota})$;

(e) $\max_{1\leq j\leq m_n}|\tilde{c}_{12,j}/(\tilde{c}_{11,j})^{1/2}-\tilde{c}_{12,j}'/(\tilde{c}_{11,j}')^{1/2}|=o_p(\Delta_n^{\min\{1/2,\kappa\}-3/k-\iota})$;

(f) if $\min\{1/2,\kappa\} > 3/k$, $\max_{1\le j\le m_n} \left|\tilde{\varsigma}_j - \tilde{\varsigma}'_j\right| = o_p(\Delta_n^{\min\{1/2,\kappa\}-3/k-\iota})$;

(g) if $\min\{1/2,\kappa\} > 3/k + 2/(k-1)$,

$$\max_{1\le j\le m_n} |(\tilde{\varsigma}_j)^{-1/2} - (\tilde{\varsigma}'_j)^{-1/2}| = o_p(\Delta_n^{\min\{1/2,\kappa\}-3/k-3/(k-1)-\iota}).$$

Proof. Part (a). For each $i \in \mathcal{I}_{n,j}$,

$$\Delta_i^n X - \tilde{x}_{n,i} = \int_{(i-1)\Delta_n}^{i\Delta_n} b_{1,s} ds + \int_{(i-1)\Delta_n}^{i\Delta_n} \left(v_s^{1/2} - v_{n,j}^{1/2}\right) dW_{1,s}. \tag{SA.15}$$

Since the drift process is bounded under the localized version of Assumption 2, we have

$$\max_{1\le j\le m_n} \left|\int_{(i-1)\Delta_n}^{i\Delta_n} b_{1,s} ds\right| \le K\Delta_n. \tag{SA.16}$$

Consider an arbitrary constant $p \ge 2$. Observe that

$$\begin{aligned}
E\left[\left|\int_{(i-1)\Delta_n}^{i\Delta_n} \left(v_s^{1/2} - v_{n,j}^{1/2}\right) dW_{1,s}\right|^p\right] &\le K_p E\left[\left|\int_{(i-1)\Delta_n}^{i\Delta_n} \left(v_s^{1/2} - v_{n,j}^{1/2}\right)^2 ds\right|^{p/2}\right] \\
&= K_p \Delta_n^{p/2} E\left[\left|\frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n} \left(v_s^{1/2} - v_{n,j}^{1/2}\right)^2 ds\right|^{p/2}\right] \\
&\le K_p \Delta_n^{p/2} E\left[\frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n} \left|v_s^{1/2} - v_{n,j}^{1/2}\right|^p ds\right] \\
&\le K_p \Delta_n^{(\kappa+1/2)p},
\end{aligned}$$

where the first inequality is by the Burkholder–Davis–Gundy inequality, the second inequality is by Jensen's inequality, and the last inequality follows from the assumed κ-Hölder continuity of the volatility process under the L_p-norm. This estimate further implies

$$\left\|\int_{(i-1)\Delta_n}^{i\Delta_n} \left(v_s^{1/2} - v_{n,j}^{1/2}\right) dW_{1,s}\right\|_p \le K_p \Delta_n^{\kappa+1/2}. \tag{SA.17}$$

Combining (SA.16) and (SA.17), we deduce

$$\|\Delta_i^n X - \tilde{x}_{n,i}\|_p \le K_p\left(\Delta_n + \Delta_n^{\kappa+1/2}\right). \tag{SA.18}$$

By the triangle inequality, Hölder's inequality, and the fact that $\|\tilde{x}_{n,i}\|_{2p} \le K_p \Delta_n^{1/2}$, the estimate in (SA.18) implies

$$\begin{aligned}
\frac{1}{k\Delta_n}\sum_{i\in\mathcal{I}_{n,j}}\left\|(\Delta_i^n X)^2-\tilde{x}_{n,i}^2\right\|_p &\le \frac{1}{k\Delta_n}\sum_{i\in\mathcal{I}_{n,j}}\left(\left\|2\tilde{x}_{n,i}\left(\Delta_i^n X-\tilde{x}_{n,i}\right)\right\|_p+\left\|(\Delta_i^n X-\tilde{x}_{n,i})^2\right\|_p\right)\\
&\le \frac{K_p}{k\Delta_n}\sum_{i\in\mathcal{I}_{n,j}}\left(\left\|\tilde{x}_{n,i}\right\|_{2p}\left\|\Delta_i^n X-\tilde{x}_{n,i}\right\|_{2p}+\left\|\Delta_i^n X-\tilde{x}_{n,i}\right\|_{2p}^2\right)\\
&\le \frac{K_p}{k\Delta_n}\sum_{i\in\mathcal{I}_{n,j}}\left(\Delta_n^{1/2}\left(\Delta_n+\Delta_n^{\kappa+1/2}\right)+\left(\Delta_n+\Delta_n^{\kappa+1/2}\right)^2\right)\\
&\le K_p\left(\Delta_n^{1/2}+\Delta_n^{\kappa}\right).
\end{aligned}$$

Using a similar argument, we can extend this estimate to the multivariate case:

$$\frac{1}{k\Delta_n}\sum_{i\in\mathcal{I}_{n,j}}\left\|\left(\Delta_i^n \boldsymbol{Z}\right)\left(\Delta_i^n \boldsymbol{Z}\right)^\top-\boldsymbol{z}_{n,i}\boldsymbol{z}_{n,i}^\top\right\|_p\le K_p\left(\Delta_n^{1/2}+\Delta_n^{\kappa}\right).$$

By the triangle inequality, this further implies that

$$\left\|\tilde{\boldsymbol{c}}_j-\tilde{\boldsymbol{c}}_j'\right\|_p\le K_p\left(\Delta_n^{1/2}+\Delta_n^{\kappa}\right).$$

Applying a maximal inequality under the L_p-norm, we then have

$$\left\|\max_{1\le j\le m_n}\left\|\tilde{\boldsymbol{c}}_j-\tilde{\boldsymbol{c}}_j'\right\|\right\|_p\le K_p m_n^{1/p}\Delta_n^{\min\{1/2,\kappa\}}\le K_p\Delta_n^{\min\{1/2,\kappa\}-1/p}.$$

Applying this estimate with p sufficiently large (i.e., $1/p<\iota$), we deduce the assertion in part (a).

Part (b). It is easy to see that for each p, $\left\|\tilde{\boldsymbol{c}}_j'\right\|_p$ is bounded uniformly across $j\in\{1,\ldots,m_n\}$. Applying a maximal inequality under the L_p-norm with $p>1/\iota$ yields $\max_{1\le j\le m_n}\left\|\tilde{\boldsymbol{c}}_j'\right\|=o_p\left(\Delta_n^{-\iota}\right)$. By the triangle inequality, this estimate and part (a) further imply $\max_{1\le j\le m_n}\left\|\tilde{\boldsymbol{c}}_j\right\|=o_p\left(\Delta_n^{-\iota}\right)$.

Part (c). Note that $\max_{1\le j\le m_n}(\tilde{c}_{11,j}')^{-1}\le K\max_{1\le j\le m_n}\xi_{11,j}^{-1}$ and $\xi_{11,j}$ are i.i.d. χ_k^2-distributed variables. Since $\xi_{11,j}^{-1}$ has finite pth moment for any $p<k/2$, we can use an L_p maximal inequality for such p to deduce that

$$\max_{1\le j\le m_n}\xi_{11,j}^{-1}=O_p\left(\Delta_n^{-1/p}\right).$$

By taking p sufficiently close to $k/2$, we prove the assertion of part (c).

Part (d). By the mean value theorem,

$$\left|\tilde{c}_{11,j}^{q}-\tilde{c}_{11,j}'^{q}\right|=\frac{|q|\left|\tilde{c}_{11,j}-\tilde{c}_{11,j}'\right|}{\left(\tilde{c}_{11,j}'+\lambda_{n,j}\left(\tilde{c}_{11,j}-\tilde{c}_{11,j}'\right)\right)^{1-q}},\tag{SA.19}$$

where the variable $\lambda_{n,j}$ takes values in $[0,1]$. By part (c), there exists a positive real sequence δ_{1n} such that $\delta_{1n}^{-1}=o(\Delta_n^{-(2/k)-\iota})$ and $\max_{1\le j\le m_n}(\tilde{c}_{11,j}')^{-1}\le\delta_{1n}^{-1}$ with probability approaching 1. In other words,

$$P\left(\min_{1\le j\le m_n}\tilde{c}'_{11,j}\ge\delta_{1n}\right)\to 1 \text{ and } \frac{\Delta_n^{(2/k)+\iota}}{\delta_{1n}}\to 0. \tag{SA.20}$$

By part (a), there exists a positive real sequence $\delta_{2n}=o(\Delta_n^{\min\{1/2,\kappa\}-\iota})$ such that

$$P\left(\max_{1\le j\le m_n}\left|\tilde{c}_{11,j}-\tilde{c}'_{11,j}\right|\le\delta_{2n}\right)\to 1. \tag{SA.21}$$

Under the assumption that $\min\{1/2,\kappa\}>2/k$, it is possible to find $\iota>0$ sufficiently small such that $\min\{1/2,\kappa\}-\iota>2/k+\iota$. Hence, $\delta_{1n}/\delta_{2n}\to\infty$. By (SA.20) and (SA.21), we deduce that, with probability approaching 1,

$$\min_{1\le j\le m_n}\tilde{c}'_{11,j}\ge 2\max_{1\le j\le m_n}\left|\tilde{c}_{11,j}-\tilde{c}'_{11,j}\right|,$$

which also implies that

$$\max_{1\le j\le m_n}\frac{1}{\tilde{c}'_{11,j}+\lambda_{n,j}\left(\tilde{c}_{11,j}-\tilde{c}'_{11,j}\right)}\le 2\max_{1\le j\le m_n}\frac{1}{\tilde{c}'_{11,j}}.$$

This estimate and part (c) imply that

$$\max_{1\le j\le m_n}\frac{1}{\left(\tilde{c}'_{11,j}+\lambda_{n,j}\left(\tilde{c}_{11,j}-\tilde{c}'_{11,j}\right)\right)^{1-q}}=o_p(\Delta_n^{-2(1-q)/k-\iota}). \tag{SA.22}$$

The assertion in part (d) then follows from (SA.19), (SA.22), and part (a).

Part (e). By the triangle inequality,

$$\left|\frac{\tilde{c}_{12,j}}{\sqrt{\tilde{c}_{11,j}}}-\frac{\tilde{c}'_{12,j}}{\sqrt{\tilde{c}'_{11,j}}}\right|\le\frac{\left|\tilde{c}_{12,j}-\tilde{c}'_{12,j}\right|}{\sqrt{\tilde{c}_{11,j}}}+\left|\tilde{c}'_{12,j}\right|\left|\frac{1}{\sqrt{\tilde{c}_{11,j}}}-\frac{1}{\sqrt{\tilde{c}'_{11,j}}}\right|. \tag{SA.23}$$

Applying the estimate in part (d) with $q=-1/2$, we deduce

$$\max_{1\le j\le m_n}\left|\frac{1}{\sqrt{\tilde{c}_{11,j}}}-\frac{1}{\sqrt{\tilde{c}'_{11,j}}}\right|=o_p\left(\Delta_n^{\min\{1/2,\kappa\}-3/k-\iota/2}\right). \tag{SA.24}$$

By part (a), part (c), and (SA.24),

$$\begin{aligned}\max_{1\le j\le m_n}\frac{\left|\tilde{c}_{12,j}-\tilde{c}'_{12,j}\right|}{\sqrt{\tilde{c}_{11,j}}} &\le o_p(\Delta_n^{\min\{1/2,\kappa\}-\iota/2})\cdot\left(o_p(\Delta_n^{-1/k-\iota/2})+o_p\left(\Delta_n^{\min\{1/2,\kappa\}-3/k-\iota/2}\right)\right)\\ &= o_p\left(\Delta_n^{\min\{1/2,\kappa\}-1/k-\iota}\right),\end{aligned} \tag{SA.25}$$

where the last line holds because $-1/k < \min\{1/2, \kappa\} - 3/k$ under the maintained assumption on k. By part (b) and (SA.24), we also have

$$\max_{1 \le j \le m_n} \left|\tilde{c}'_{12,j}\right| \left| \frac{1}{\sqrt{\tilde{c}_{11,j}}} - \frac{1}{\sqrt{\tilde{c}'_{11,j}}} \right| = o_p\left(\Delta_n^{\min\{1/2,\kappa\}-3/k-\iota}\right). \tag{SA.26}$$

The assertion of part (e) readily follows from (SA.23), (SA.25), and (SA.26).

Part (f). By the triangle inequality,

$$\left|\tilde{\varsigma}_j - \tilde{\varsigma}'_j\right| \le \left|\tilde{c}_{22,j} - \tilde{c}'_{22,j}\right| + \left| \frac{\tilde{c}^2_{12,j}}{\tilde{c}_{11,j}} - \frac{\tilde{c}'^2_{12,j}}{\tilde{c}'_{11,j}} \right|. \tag{SA.27}$$

Another use of the triangle inequality yields

$$\begin{aligned} \max_{1 \le j \le m_n} \left| \frac{\tilde{c}^2_{12,j}}{\tilde{c}_{11,j}} - \frac{\tilde{c}'^2_{12,j}}{\tilde{c}'_{11,j}} \right| \;\; &\le \;\; 2 \max_{1 \le j \le m_n} \left| \frac{\tilde{c}'_{12,j}}{\sqrt{\tilde{c}'_{11,j}}} \right| \max_{1 \le j \le m_n} \left| \frac{\tilde{c}_{12,j}}{\sqrt{\tilde{c}_{11,j}}} - \frac{\tilde{c}'_{12,j}}{\sqrt{\tilde{c}'_{11,j}}} \right| \\ &\quad + \max_{1 \le j \le m_n} \left| \frac{\tilde{c}_{12,j}}{\sqrt{\tilde{c}_{11,j}}} - \frac{\tilde{c}'_{12,j}}{\sqrt{\tilde{c}'_{11,j}}} \right|^2. \end{aligned} \tag{SA.28}$$

Note that

$$\left| \frac{\tilde{c}'_{12,j}}{\sqrt{\tilde{c}'_{11,j}}} \right| \le K \left(\sqrt{\xi_{11,j}} + \frac{|\xi_{12,j}|}{\sqrt{\xi_{11,j}}} \right)$$

and the variable $\xi_{12,j}/\sqrt{\xi_{11,j}}$ is $\mathcal{N}(0,1)$ distributed. It is then easy to see that for any $\iota > 0$,

$$\max_{1 \le j \le m_n} \left| \frac{\tilde{c}'_{12,j}}{\sqrt{\tilde{c}'_{11,j}}} \right| = o_p\left(\Delta_n^{-\iota}\right). \tag{SA.29}$$

By (SA.28), (SA.29), and part (e),

$$\max_{1 \le j \le m_n} \left| \frac{\tilde{c}^2_{12,j}}{\tilde{c}_{11,j}} - \frac{\tilde{c}'^2_{12,j}}{\tilde{c}'_{11,j}} \right| = o_p\left(\Delta_n^{\min\{1/2,\kappa\}-(3/k)-\iota}\right),$$

which together with (SA.27) and part (a) implies the assertion in part (f).

Part (g). Note that by Lemma S1 and Lemma S2, $k\tilde{\varsigma}'_j/\varsigma_{n,j}$ is χ^2_{k-1}-distributed. Similar to part (c), we have, for any $\iota > 0$,

$$\max_{1 \le j \le m_n} (\tilde{\varsigma}'_j)^{-1} = o_p(\Delta_n^{-2/(k-1)-\iota}). \tag{SA.30}$$

Hence, there exists a positive real sequence δ_{1n} such that

$$P\left(\min_{1 \le j \le m_n} \tilde{\varsigma}'_j \ge \delta_{1n}\right) \to 1 \text{ and } \frac{\Delta_n^{2/(k-1)+\iota}}{\delta_{1n}} \to 0.$$

By part (f), there exists a positive real sequence $\delta_{2n} = o(\Delta_n^{\min\{1/2,\kappa\}-3/k-\iota})$ such that

$$P\left(\max_{1\le j\le m_n}\left|\tilde{\varsigma}_j - \tilde{\varsigma}'_j\right| \le \delta_{2n}\right) \to 1.$$

Since $\min\{1/2,\kappa\} - 3/k > 2/(k-1)$, it is possible to find a sufficiently small $\iota > 0$ such that

$$\min\{1/2,\kappa\} - 3/k - \iota > 2/(k-1) + \iota,$$

which implies $\delta_{1n}/\delta_{2n} \to \infty$. These estimates imply that, with probability approaching 1,

$$\min_{1\le j\le m_n}\tilde{\varsigma}'_j \ge 2\max_{1\le j\le m_n}\left|\tilde{\varsigma}_j - \tilde{\varsigma}'_j\right|. \tag{SA.31}$$

By the mean value theorem,

$$\left|\frac{1}{\sqrt{\tilde{\varsigma}_j}} - \frac{1}{\sqrt{\tilde{\varsigma}'_j}}\right| \le \frac{\left|\tilde{\varsigma}_j - \tilde{\varsigma}'_j\right|}{2\left(\tilde{\varsigma}'_j + \lambda_{n,j}\left(\tilde{\varsigma}_j - \tilde{\varsigma}'_j\right)\right)^{3/2}}$$

for some variable $\lambda_{n,j}$ taking values in $[0,1]$, which together with (SA.31) implies that, with probability approaching 1,

$$\max_{1\le j\le m_n}\left|\frac{1}{\sqrt{\tilde{\varsigma}_j}} - \frac{1}{\sqrt{\tilde{\varsigma}'_j}}\right| \le K\max_{1\le j\le m_n}\left|\tilde{\varsigma}_j - \tilde{\varsigma}'_j\right| \cdot \max_{1\le j\le m_n}\frac{1}{\left(\tilde{\varsigma}'_j\right)^{3/2}}. \tag{SA.32}$$

The assertion of part (g) then follows from this estimate, (SA.30), and part (f). *Q.E.D.*

To prove the second assertion of Theorem 5, we also need to establish an anti-concentration property for the maximum of the absolute values of t-distributed random variables, given by Lemma S4 below.

Lemma S4. *Let $k \ge 2$ and $p^*_m(\cdot)$ denote the probability density function of $\max_{1\le j\le m}|\xi_{\beta,j}|$, where the $\xi_{\beta,j}$ variables are i.i.d. t-distributed with degree of freedom $k-1$. Then, there exists a finite constant $K^* > 0$ such that $p^*_m(x) \le K^*$ for all $m \ge 1$ and $x \ge 0$.*

Proof. Denote the probability density function and the cumulative distribution function of $|\xi_{\beta,j}|$ by $g(\cdot)$ and $G(\cdot)$, respectively. Since the $(\xi_{\beta,j})_{1\le j\le m}$ variables are i.i.d.,

$$P\left(\max_{1\le j\le m}|\xi_{\beta,j}| \le x\right) = \left(P\left(|\xi_{\beta,j}| \le x\right)\right)^m = G(x)^m,$$

which implies that

$$p^*_m(x) = mG(x)^{m-1}g(x). \tag{SA.33}$$

Since $G(x)$ is an increasing function,

$$mG(x)^{m-1}\int_x^\infty g(u)\,du \le \int_x^\infty p_m^*(u)\,du = P\left(\max_{1\le j\le m}|\xi_{\beta,j}|\ge x\right)\le 1.$$

Therefore,

$$mG(x)^{m-1}\le \frac{1}{\int_x^\infty g(u)\,du}. \tag{SA.34}$$

By (SA.33) and (SA.34),

$$p_m^*(x)\le H(x)\equiv \frac{g(x)}{\int_x^\infty g(u)\,du}. \tag{SA.35}$$

Let $p(\cdot)$ denote the probability density of the t-distribution with degree of freedom $k-1$ and note that $g(x)=2p(x)$ for $x\ge 0$. We can then rewrite

$$H(x)=\frac{p(x)}{\int_x^\infty p(u)\,du}.$$

Further recall that, by definition,

$$p(x)\equiv C_k\left(1+\frac{x^2}{k-1}\right)^{-k/2},\quad \text{where}\quad C_k\equiv \frac{\Gamma(k/2)}{\sqrt{(k-1)\pi}\Gamma((k-1)/2)}.$$

By applying L'Hôpital's rule, we have

$$\lim_{x\to\infty}H(x)=\lim_{x\to\infty}\frac{kx}{k-1+x^2}=0.$$

It is also easy to see that $H(\cdot)$ is continuous with $H(0)=2C_k$. This combined with the above convergence implies that $H(\cdot)$ is uniformly bounded. The assertion of the lemma then follows from (SA.35). *Q.E.D.*

Proof of Theorem 5. We first show that

$$\max_{1\le j\le m_n}\left|\frac{\tilde\beta_j-\beta_{n,j}}{\sqrt{\tilde\varsigma_j/\tilde v_j}}-\frac{\tilde\beta_j'-\beta_{n,j}}{\sqrt{\tilde\varsigma_j'/\tilde v_j'}}\right| = o_p(1). \tag{SA.36}$$

For this purpose, it is convenient to rewrite

$$\frac{\tilde\beta_j-\beta_{n,j}}{\sqrt{\tilde\varsigma_j/\tilde v_j}}=\frac{A_{n,j}}{\sqrt{\tilde\varsigma_j}},\quad \frac{\tilde\beta_j'-\beta_{n,j}}{\sqrt{\tilde\varsigma_j'/\tilde v_j'}}=\frac{A_{n,j}'}{\sqrt{\tilde\varsigma_j'}},$$

where

$$A_{n,j}\equiv \frac{\tilde c_{12,j}}{\sqrt{\tilde c_{11,j}}}-\beta_{n,j}\sqrt{\tilde c_{11,j}},\quad A_{n,j}'\equiv \frac{\tilde c_{12,j}'}{\sqrt{\tilde c_{11,j}'}}-\beta_{n,j}\sqrt{\tilde c_{11,j}'}.$$

Applying Lemma S3(d) with $q = 1/2$ yields that for any $\iota > 0$,

$$\max_{1\leq j\leq m_n} \left|\sqrt{\tilde{c}_{11,j}} - \sqrt{\tilde{c}'_{11,j}}\right| = o_p\left(\Delta_n^{\min\{1/2,\kappa\}-1/k-\iota}\right).$$

Since the β_t process is uniformly bounded under Assumption 2 after localization, the estimate above and Lemma S3(e) imply that for any $\iota > 0$,

$$\max_{1\leq j\leq m_n} \left|A_{n,j} - A'_{n,j}\right| = o_p\left(\Delta_n^{\min\{1/2,\kappa\}-3/k-\iota}\right). \tag{SA.37}$$

By the triangle inequality,

$$\max_{1\leq j\leq m_n} \left|\frac{A_{n,j}}{\sqrt{\tilde{\varsigma}_j}} - \frac{A'_{n,j}}{\sqrt{\tilde{\varsigma}'_j}}\right| \leq \max_{1\leq j\leq m_n} \frac{\left|A_{n,j} - A'_{n,j}\right|}{\sqrt{\tilde{\varsigma}'_j}} + \max_{1\leq j\leq m_n} |A_{n,j}| \left|\frac{1}{\sqrt{\tilde{\varsigma}_j}} - \frac{1}{\sqrt{\tilde{\varsigma}'_j}}\right|. \tag{SA.38}$$

Combining (SA.30) and (SA.37) yields

$$\max_{1\leq j\leq m_n} \frac{\left|A_{n,j} - A'_{n,j}\right|}{\sqrt{\tilde{\varsigma}'_j}} = o_p\left(\Delta_n^{\min\{1/2,\kappa\}-3/k-1/(k-1)-\iota}\right). \tag{SA.39}$$

By (SA.29) and Lemma S3(b), it is easy to see that $\max_{1\leq j\leq m_n} |A'_{n,j}| = o_p\left(\Delta_n^{-\iota}\right)$. Then, by (SA.37), we have for any $\iota > 0$,

$$\max_{1\leq j\leq m_n} |A_{n,j}| = o_p\left(\Delta_n^{-\iota}\right),$$

which together with Lemma S3(g) implies that for any $\iota > 0$,

$$\max_{1\leq j\leq m_n} |A_{n,j}| \left|\frac{1}{\sqrt{\tilde{\varsigma}_j}} - \frac{1}{\sqrt{\tilde{\varsigma}'_j}}\right| = o_p\left(\Delta_n^{\min\{1/2,\kappa\}-3/k-3/(k-1)-\iota}\right). \tag{SA.40}$$

Since $\min\{1/2,\kappa\} - 3/k - 3/(k-1) > 0$ under the maintained assumption on k, we can take ι sufficiently small so that the sequences in (SA.39) and (SA.40) are both $o_p(1)$. The assertion in (SA.36) then readily follows in view of (SA.38).

Next, we show that

$$\max_{1\leq j\leq m_n} \sup_{t\in[(j-1)k\Delta_n, jk\Delta_n]} \left|\frac{\beta_t - \beta_{n,j}}{\sqrt{\tilde{\varsigma}_j/\tilde{v}_j}}\right| = o_p(1). \tag{SA.41}$$

Rewrite

$$\frac{\beta_t - \beta_{n,j}}{\sqrt{\tilde{\varsigma}_j/\tilde{v}_j}} = \frac{\sqrt{\tilde{c}_{11,j}}\,(\beta_t - \beta_{n,j})}{\sqrt{\tilde{\varsigma}_j}},$$

and then observe that

$$\begin{aligned}\max_{1\leq j\leq m_n} \sup_{t\in[(j-1)k\Delta_n, jk\Delta_n]} &\left|\frac{\beta_t - \beta_{n,j}}{\sqrt{\tilde{\varsigma}_j/\tilde{v}_j}}\right| \\ &\leq \max_{1\leq j\leq m_n} \frac{\sqrt{\tilde{c}_{11,j}}}{\sqrt{\tilde{\varsigma}_j}} \cdot \max_{1\leq j\leq m_n} \sup_{t\in[(j-1)k\Delta_n, jk\Delta_n]} |\beta_t - \beta_{n,j}|.\end{aligned} \tag{SA.42}$$

By (SA.30) and Lemma S3(g), we have for any $\iota > 0$,

$$\max_{1\leq j\leq m_n} \frac{1}{\sqrt{\tilde{\varsigma}_j}} = o_p(\Delta_n^{-1/(k-1)-\iota}),$$

which together with Lemma S3(b) implies that for any $\iota > 0$,

$$\max_{1\leq j\leq m_n} \frac{\sqrt{\tilde{c}_{11,j}}}{\sqrt{\tilde{\varsigma}_j}} = o_p(\Delta_n^{-1/(k-1)-\iota/2}). \tag{SA.43}$$

Moreover, it is easy to see that for any $p \geq 2$,

$$\left\| \sup_{t\in[(j-1)k\Delta_n, jk\Delta_n]} |\beta_t - \beta_{n,j}| \right\|_p \leq K_p \Delta_n^{\kappa},$$

and by a maximal inequality, we can show that for any $\iota > 0$,

$$\max_{1\leq j\leq m_n} \sup_{t\in[(j-1)k\Delta_n, jk\Delta_n]} |\beta_t - \beta_{n,j}| = o_p\left(\Delta_n^{\kappa-\iota/2}\right). \tag{SA.44}$$

Combining (SA.42), (SA.43), and (SA.44) yields

$$\max_{1\leq j\leq m_n} \sup_{t\in[(j-1)k\Delta_n, jk\Delta_n]} \left| \frac{\beta_t - \beta_{n,j}}{\sqrt{\tilde{\varsigma}_j/\tilde{v}_j}} \right| = o_p\left(\Delta_n^{\kappa-1/(k-1)-\iota}\right). \tag{SA.45}$$

Since $\kappa - 1/(k-1) > 0$ under maintained assumptions, we can take $\iota > 0$ sufficiently small to make the random sequence in the above display $o_p(1)$. This proves (SA.41).

We are now ready to prove the first assertion of Theorem 5. By the definitions of $\tilde{\beta}_t$, $\tilde{v}_t$, and $\tilde{\varsigma}_t$, it is easy to see that

$$\sup_{t\in[0,T]} \left| \frac{\tilde{\beta}_t - \beta_t}{\sqrt{\tilde{\varsigma}_t/\tilde{v}_t}} \right| = \max_{1\leq j\leq m_n} \sup_{t\in[(j-1)k\Delta_n, jk\Delta_n)} \left| \frac{\tilde{\beta}_j - \beta_t}{\sqrt{\tilde{\varsigma}_j/\tilde{v}_j}} \right|,$$

which together with (SA.41) implies

$$\sup_{t\in[0,T]} \left| \frac{\tilde{\beta}_t - \beta_t}{\sqrt{\tilde{\varsigma}_t/\tilde{v}_t}} \right| = \max_{1\leq j\leq m_n} \left| \frac{\tilde{\beta}_j - \beta_{n,j}}{\sqrt{\tilde{\varsigma}_j/\tilde{v}_j}} \right| + o_p(1).$$

By (SA.36), the estimate displayed above further implies

$$\sup_{t\in[0,T]} \left| \frac{\tilde{\beta}_t - \beta_t}{\sqrt{\tilde{\varsigma}_t/\tilde{v}_t}} \right| = \max_{1\leq j\leq m_n} \left| \frac{\tilde{\beta}_j' - \beta_{n,j}}{\sqrt{\tilde{\varsigma}_j'/\tilde{v}_j'}} \right| + o_p(1).$$

The first assertion in Theorem 5 then follows from this estimate, Lemma S1, and Lemma S2.

To prove the second assertion of the theorem, we write for simplicity

$$\tilde{\tau}_n \equiv \sup_{t\in[0,T]} \left| \frac{\sqrt{k-1}\left(\tilde{\beta}_t - \beta_t\right)}{\sqrt{\tilde{\varsigma}_t/\tilde{v}_t}} \right|, \quad \tau_n^* \equiv \max_{1\leq j\leq m_n} |\xi_{\beta,j}|.$$

By the first assertion of the theorem, there exists a positive real sequence $\tilde{\delta}_n = o(1)$ such that $|\tilde{\tau}_n - \tau_n^*| \leq \tilde{\delta}_n$ with probability approaching 1. Therefore,

$$\begin{aligned} P\left(\tilde{\tau}_n > z_{n,1-\alpha}^*\right) &\leq P\left(\tau_n^* > z_{n,1-\alpha}^* - \tilde{\delta}_n\right) + o(1) \\ &= P\left(\tau_n^* > z_{n,1-\alpha}^*\right) + P\left(z_{n,1-\alpha}^* - \tilde{\delta}_n < \tau_n^* \leq z_{n,1-\alpha}^*\right) + o(1) \\ &= P\left(\tau_n^* > z_{n,1-\alpha}^*\right) + O(\tilde{\delta}_n) + o(1) = \alpha + o(1), \end{aligned} \tag{SA.46}$$

where the last line follows from Lemma S4. Similarly,

$$\begin{aligned} P\left(\tilde{\tau}_n > z_{n,1-\alpha}^*\right) &\geq P\left(\tau_n^* > z_{n,1-\alpha}^* + \tilde{\delta}_n\right) - o(1) \\ &= P\left(\tau_n^* > z_{n,1-\alpha}^*\right) - P\left(z_{n,1-\alpha}^* < \tau_n^* \leq z_{n,1-\alpha}^* + \tilde{\delta}_n\right) - o(1) \\ &= P\left(\tau_n^* > z_{n,1-\alpha}^*\right) - O(\tilde{\delta}_n) - o(1) = \alpha - o(1). \end{aligned} \tag{SA.47}$$

The second assertion of the theorem then follows from (SA.46) and (SA.47). *Q.E.D.*

SB Optimality of Spot Beta Testing

In this section, we establish an optimality result for the test based on the spot beta estimator. The testing problem concerns the null hypothesis

$$H_0 : \beta_t = \beta^*,$$

for some constant β^* against the alternative hypothesis

$$H_a : \beta_t \neq \beta^*.$$

Theorem 1 in the main text suggests rejecting the null hypothesis at significance level α if and only if

$$\frac{\sqrt{k-1}\left|\hat{\beta}_t - \beta^*\right|}{\sqrt{\hat{\varsigma}_t / \hat{v}_t}} > t_{1-\alpha/2,k-1}, \tag{SB.1}$$

where $t_{1-\alpha/2,k-1}$ denotes the $1-\alpha/2$ quantile of the t_{k-1} distribution. Below, we establish the optimality of this test.

Recall that the vector of observed high-frequency returns in the local estimation window is denoted by

$$\boldsymbol{r}_X \equiv \left(\Delta_i^n X / \sqrt{\Delta_n}\right)_{i \in \mathcal{I}_{n,t}}, \quad \boldsymbol{r}_Y \equiv \left(\Delta_i^n Y / \sqrt{\Delta_n}\right)_{i \in \mathcal{I}_{n,t}}.$$

Parallel to the definition of a regular estimator considered in Section II.B, we identify a test with its critical function $\phi(\boldsymbol{r}_X, \boldsymbol{r}_Y)$ and refer to it as being regular if $\phi(\cdot)$ is continuous (Lebesgue) almost everywhere. By Lemma 1, any regular test admits the following coupling

$$\phi\left(\boldsymbol{r}_X, \boldsymbol{r}_Y\right) = \phi\big(v_t^{1/2}\boldsymbol{\eta}, \beta_t v_t^{1/2}\boldsymbol{\eta} + \varsigma_t^{1/2}\boldsymbol{\epsilon}\big) + o_p(1), \tag{SB.2}$$

where the coupling variable on the right-hand side of (SB.2) can be viewed as a test in the limit Gaussian linear regression experiment. We then define the *asymptotic power function* of $\phi(\cdot)$ as

$$\psi\left(\phi; v_t, \beta_t, \varsigma_t\right) \equiv E\left[\phi\big(v_t^{1/2}\boldsymbol{\eta}, \beta_t v_t^{1/2}\boldsymbol{\eta} + \varsigma_t^{1/2}\boldsymbol{\epsilon}\big)\Big| \mathcal{F}_t\right]. \tag{SB.3}$$

Since $(\boldsymbol{\eta}, \boldsymbol{\epsilon})$ is independent of $\mathcal{F}_t$, the asymptotic power depends on the conditioning information only through $(v_t, \beta_t, \varsigma_t)$. As such, the asymptotic power function can be readily computed for any given critical function $\phi(\cdot)$.

We call a regular test $\phi(\cdot)$ *asymptotically unbiased* if it is unbiased in the limit experiment, that is, for any $v > 0$, $\beta \in \mathbb{R}$, and $\varsigma > 0$,

$$\psi\left(\phi; v, \beta, \varsigma\right) \leq \alpha \text{ under } H_0 : \beta = \beta^*,$$

and

$$\psi\left(\phi; v, \beta, \varsigma\right) \geq \alpha \text{ under } H_a : \beta \neq \beta^*.$$

Theorem S1, below, establishes the optimality of the test determined by (SB.1), namely,

$$\phi^*\left(\boldsymbol{r}_X, \boldsymbol{r}_Y\right) \equiv 1\left\{\frac{\sqrt{k-1}\left|\hat{\beta}_t - \beta^*\right|}{\sqrt{\hat{\varsigma}_t/\hat{v}_t}} > t_{1-\alpha/2,k-1}\right\}, \tag{SB.4}$$

among asymptotically unbiased tests.

Theorem S1. *The test ϕ^* defined by (SB.4) is asymptotically uniformly most powerful among asymptotically unbiased tests, that is,*

$$\psi\left(\phi^*; v, \beta, \varsigma\right) \geq \psi\left(\phi; v, \beta, \varsigma\right) \text{ for all } v > 0,\ \beta \in \mathbb{R},\ \varsigma > 0,$$

and any regular asymptotically unbiased test $\phi(\cdot)$.

Proof. The key step is to show that the critical function $\phi^*(\cdot)$ constitutes the uniformly most powerful (UMP) unbiased test in the limit Gaussian linear regression model given by

$$y_i = x_i\beta + \epsilon_i, \quad x_i \sim \mathcal{N}(0, v),\ \epsilon_i \sim \mathcal{N}(0, \varsigma),\ 1 \leq i \leq k,$$

where the variables x_i, ϵ_i, $1 \leq i \leq k$, are mutually independent. It is useful to note that, to test the null hypothesis $H_0 : \beta = \beta^*$ for any given β^*, we may replace y_i with $y_i - x_i\beta^*$ and equivalently test whether the slope coefficient is zero. Therefore, we may and will assume that $\beta^* = 0$ without loss of generality. Under the limit model, the ϕ^* test can be written explicitly as

$$\phi^*(\boldsymbol{x}, \boldsymbol{y}) = 1\left\{\frac{\sqrt{k-1}\,|\sum_i x_i y_i|}{\sqrt{\left(\sum_i x_i^2\right)\left(\sum_i y_i^2\right) - \left(\sum_i x_i y_i\right)^2}} > t_{1-\alpha/2,k-1}\right\},$$

where we write $\sum_i$ in place of $\sum_{i=1}^k$ for simplicity.

The joint density of $(x_i, y_i)_{1\leq i\leq k}$ belongs to the exponential family with the following representation:

$$\begin{aligned}
&\frac{1}{(2\pi)^k \varsigma^{k/2} v^{k/2}} \exp\left(-\frac{\sum_i (y_i - x_i\beta)^2}{2\varsigma} - \frac{\sum_i x_i^2}{2v}\right) \\
&= \frac{1}{(2\pi)^k \varsigma^{k/2} v^{k/2}} \exp\left(\frac{\beta}{\varsigma}\sum_i x_i y_i - \left(\frac{\beta^2}{2\varsigma} + \frac{1}{2v}\right)\sum_i x_i^2 - \frac{1}{2\varsigma}\sum_i y_i^2\right) \\
&= \frac{1}{(2\pi)^k \varsigma^{k/2} v^{k/2}} \exp\left(\theta U + \vartheta_1 U_1' + \vartheta_2 U_2'\right),
\end{aligned}$$

where

$$\begin{aligned}
U &\equiv \sum_i x_i y_i, \quad U_1' \equiv \sum_i x_i^2, \quad U_2' \equiv \sum_i y_i^2, \\
\theta &\equiv \frac{\beta}{\varsigma}, \quad \vartheta_1 \equiv -\left(\frac{\beta^2}{2\varsigma} + \frac{1}{2v}\right), \quad \vartheta_2 \equiv -\frac{1}{2\varsigma}.
\end{aligned}$$

Since $\varsigma > 0$, testing $H_0 : \beta = 0$ is equivalent to testing $H_0 : \theta = 0$. We can thus apply Theorem 4.4.1 in Lehmann and Romano (2005) to show that ϕ^* is the UMP unbiased test.

To proceed, we express the test $\phi^*(\boldsymbol{x}, \boldsymbol{y})$ using the sufficient statistics as

$$\phi^*(\boldsymbol{x}, \boldsymbol{y}) = 1\left\{\frac{\sqrt{k-1}\,|U|}{\sqrt{U_1' U_2' - U^2}} > t_{1-\alpha/2,k-1}\right\}.$$

From here, it is easy to see that the test rejects the null hypothesis if and only if U falls outside a closed interval determined by (U_1', U_2'), namely,

$$|U| > C\left(U_1', U_2'\right) \equiv \frac{t_{1-\alpha/2,k-1}\sqrt{U_1' U_2'}}{\sqrt{k - 1 + t_{1-\alpha/2,k-1}^2}}, \tag{SB.5}$$

which fulfills condition (4.16) in Lehmann and Romano (2005). To apply the said theorem, it remains to check the following conditions:

$$\begin{aligned}
E_0\left[\phi^*(\boldsymbol{x}, \boldsymbol{y})\,|U_1', U_2'\right] &= \alpha, & \text{(SB.6)} \\
E_0\left[U\phi^*(\boldsymbol{x}, \boldsymbol{y})\,|U_1', U_2'\right] &= \alpha E_0\left[U|U_1', U_2'\right], & \text{(SB.7)}
\end{aligned}$$

where E_0 denotes the expectation operator under the null hypothesis (i.e., $\beta = 0$).

To prove (SB.6), we note that the variable $\sqrt{k-1}U/\sqrt{U_1'U_2' - U^2}$ is t_{k-1}-distributed when $\beta = 0$, and so, its distribution does not depend on $(\vartheta_1, \vartheta_2)$. Therefore, by Corollary 5.1.1 in Lehmann and Romano (2005), $U/\sqrt{U_1'U_2' - U^2}$ is independent of (U_1', U_2'). Hence,

$$\begin{aligned} E_0\left[\phi^*\left(\boldsymbol{x}, \boldsymbol{y}\right)|U_1', U_2'\right] &= P_0\left(\left.\frac{\sqrt{k-1}\,|U|}{\sqrt{U_1'U_2' - U^2}} > t_{1-\alpha/2,k-1}\right| U_1', U_2'\right) \\ &= P_0\left(\frac{\sqrt{k-1}\,|U|}{\sqrt{U_1'U_2' - U^2}} > t_{1-\alpha/2,k-1}\right) = \alpha. \end{aligned}$$

This proves (SB.6) as desired. By symmetry, it is easy to see that the conditional distribution of U given (U_1', U_2') is symmetric around zero. From here, it follows that

$$E_0\left[U\phi^*\left(\boldsymbol{x}, \boldsymbol{y}\right)|U_1', U_2'\right] = E_0\left[U1\left\{|U| > C\left(U_1', U_2'\right)\right\}|U_1', U_2'\right] = \alpha E_0\left[U|U_1', U_2'\right] = 0,$$

which proves (SB.7).

By Theorem 4.4.1 in Lehmann and Romano (2005), $\phi^*\left(\boldsymbol{x}, \boldsymbol{y}\right)$ is the UMP unbiased test for $H_0 : \beta = \beta^*$ versus $H_a : \beta \neq \beta^*$ under the limit Gaussian linear regression model. Hence, the asymptotic power function of $\phi^*\left(\boldsymbol{r}_X, \boldsymbol{r}_Y\right)$ dominates the power functions of all unbiased tests in the limit experiment. Since the latter collection includes all asymptotic power functions of regular asymptotically unbiased tests, $\phi^*\left(\boldsymbol{r}_X, \boldsymbol{r}_Y\right)$ is asymptotically UMP unbiased. *Q.E.D.*

SC Empirical Robustness Checks

SC.1 Robustness checks for Section IV.A

Figures 2 and 3 in the main text present empirical results based on $k = 15$. Figures S1, S2, and S3 below provide analogous results for $k = 10$ and $k = 5$. As discussed in more detail in the main part of the paper, the adoption of a smaller window size helps mitigate nonparametric biases, while generally resulting in "noisier" inference. Meanwhile, underscoring the robustness of our empirical findings, as the figures show, all of our key results and corresponding conclusions remain intact to these alternative smaller choices of k.

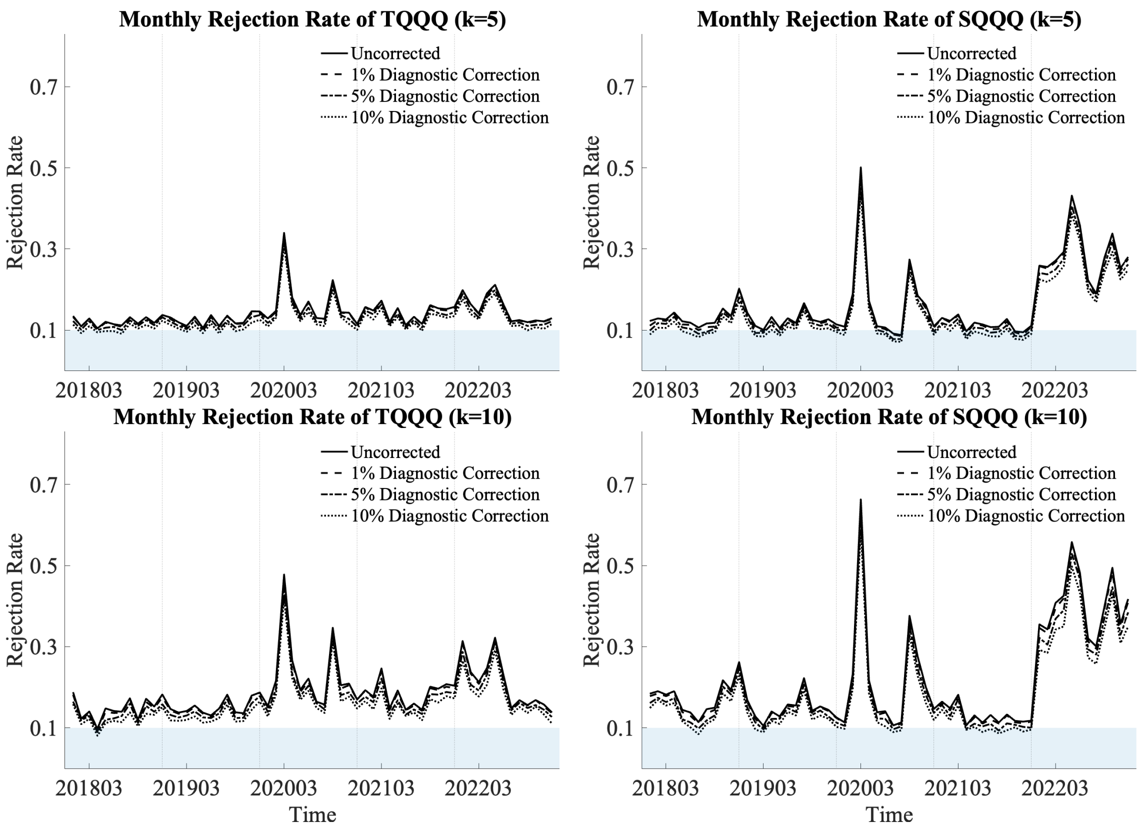

Figure S1: The figure is constructed in a similar way as Figure 2 in the main text, except that the window size is set to $k = 5$ (top) or $k = 10$ (bottom).

SC.2 Robustness checks for Section IV.B

Figure 4 in the main text shows the estimates obtained with $k = 15$. Figures S4 and S5 below provide robustness checks for $k = 10$ and $k = 5$, respectively. In parallel to the results discussed in the main part of the paper, these additional results are based on the use of the SPY ETF as the proxy for the market portfolio. As a further robustness check, we also repeat the same empirical analysis with the QQQ ETF in place of the SPY. These results are shown in Figures S6, S7, and S8. The key empirical findings and main conclusions from all of these additional robustness checks again remain qualitatively the same as the results discussed in the main part of the paper.

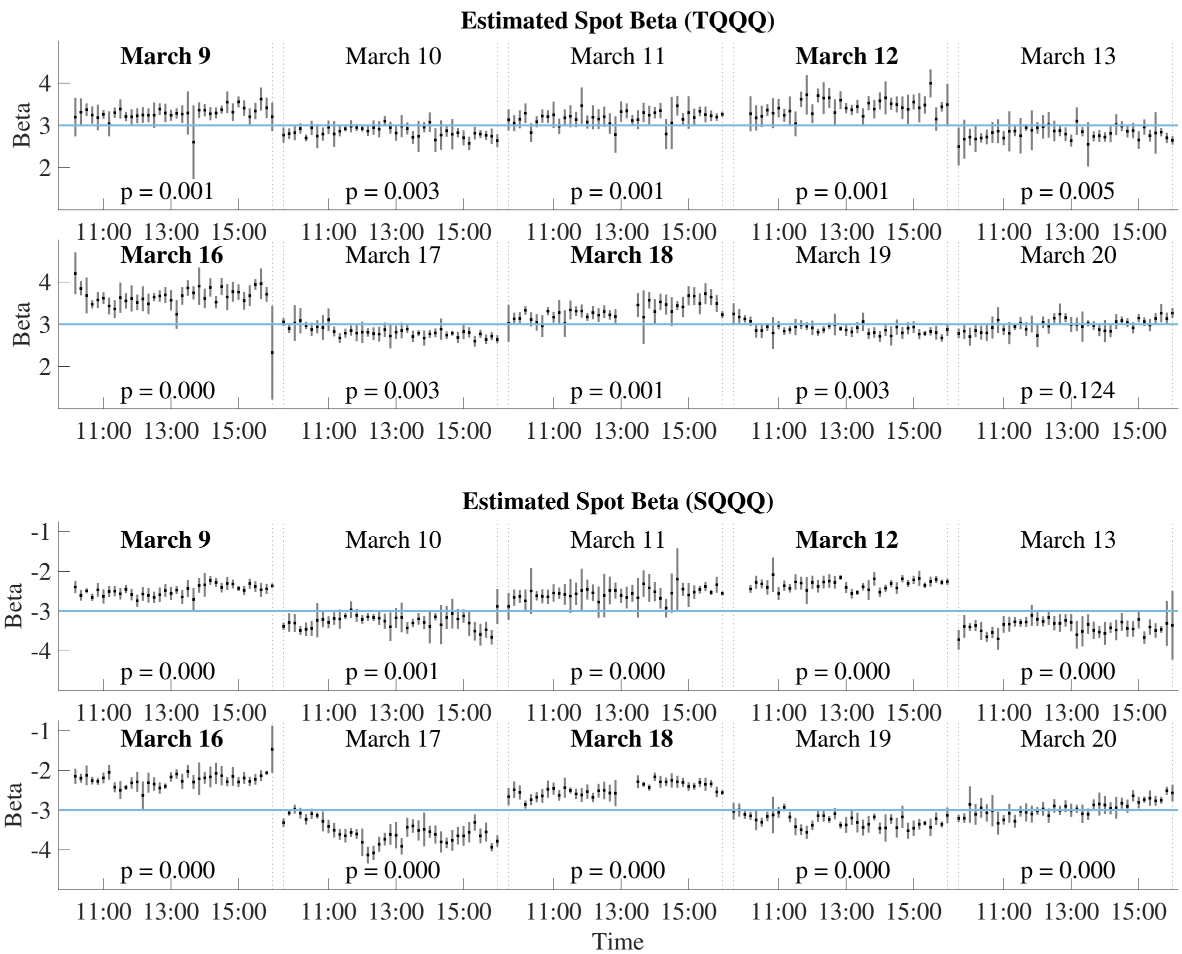

Figure S2: The figure is constructed in a similar way as Figure 3 in the main text, except that the window size is set to $k = 10$.

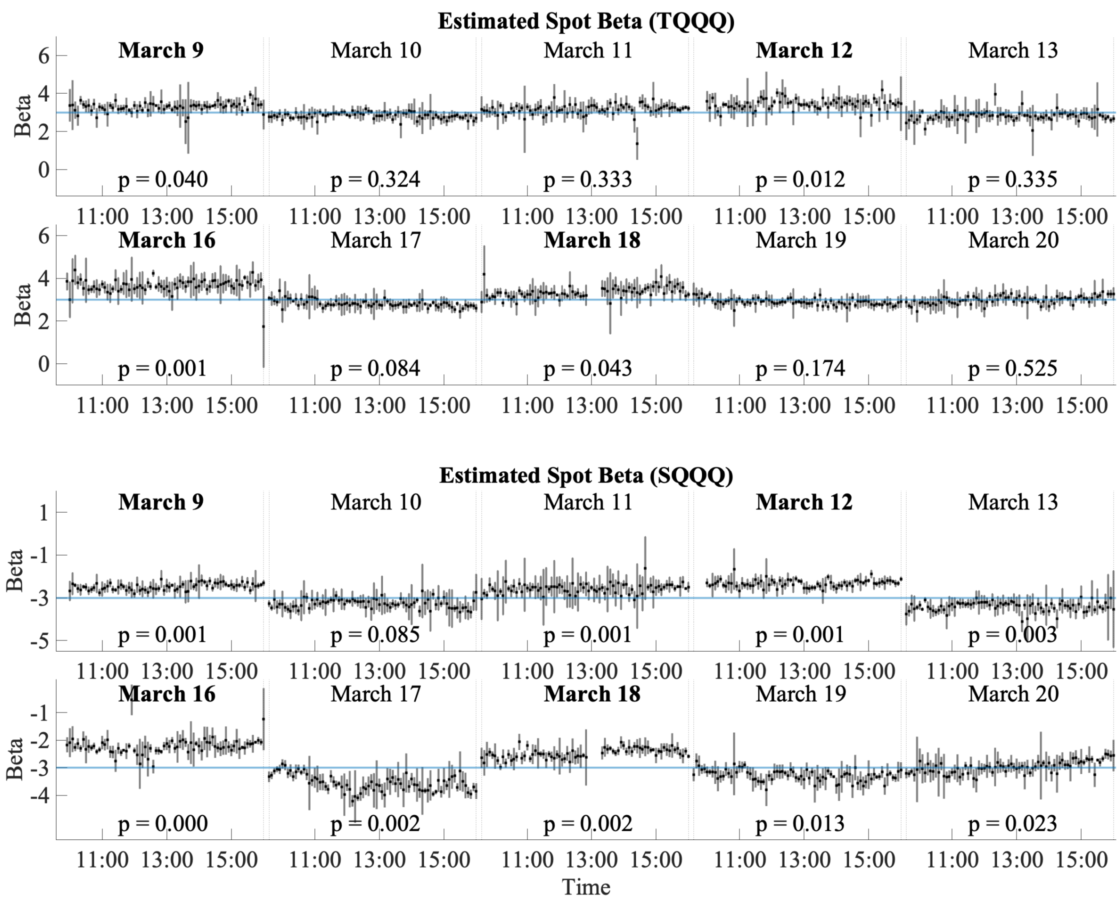

Figure S3: The figure is constructed in a similar way as Figure 3 in the main text, except that the window size is set to $k = 5$.

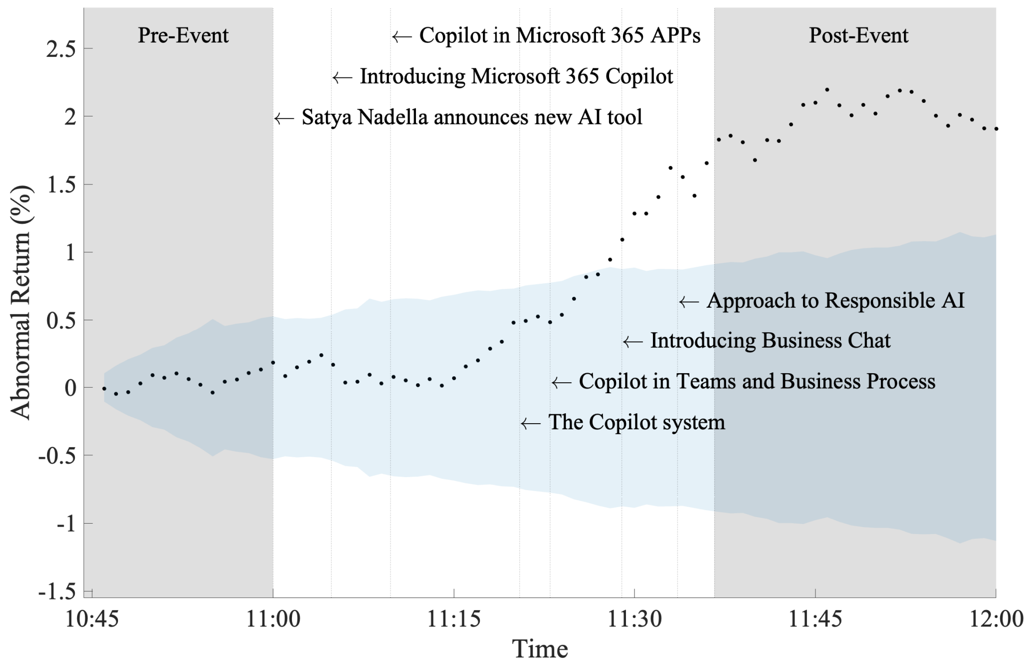

Figure S4: The figure is constructed in a similar way as Figure 4 in the main text, except that the window size is set to $k = 10$.

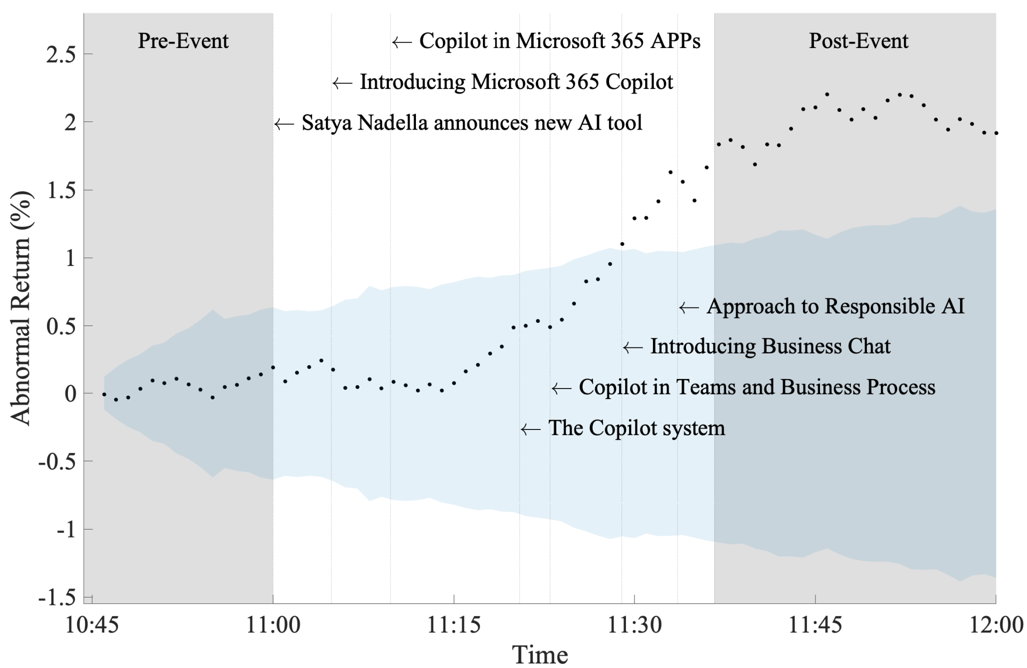

Figure S5: The figure is constructed in a similar way as Figure 4 in the main text, except that the window size is set to $k = 5$.

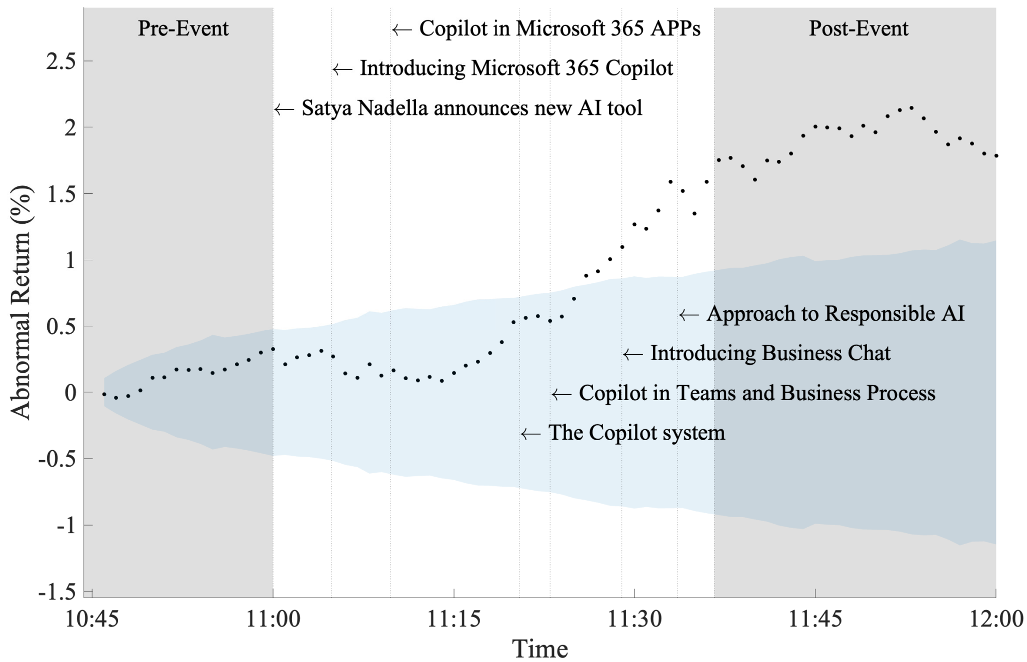

Figure S6: The figure is constructed in a similar way as Figure 4 in the main text, except that QQQ is used as the proxy for the market portfolio.

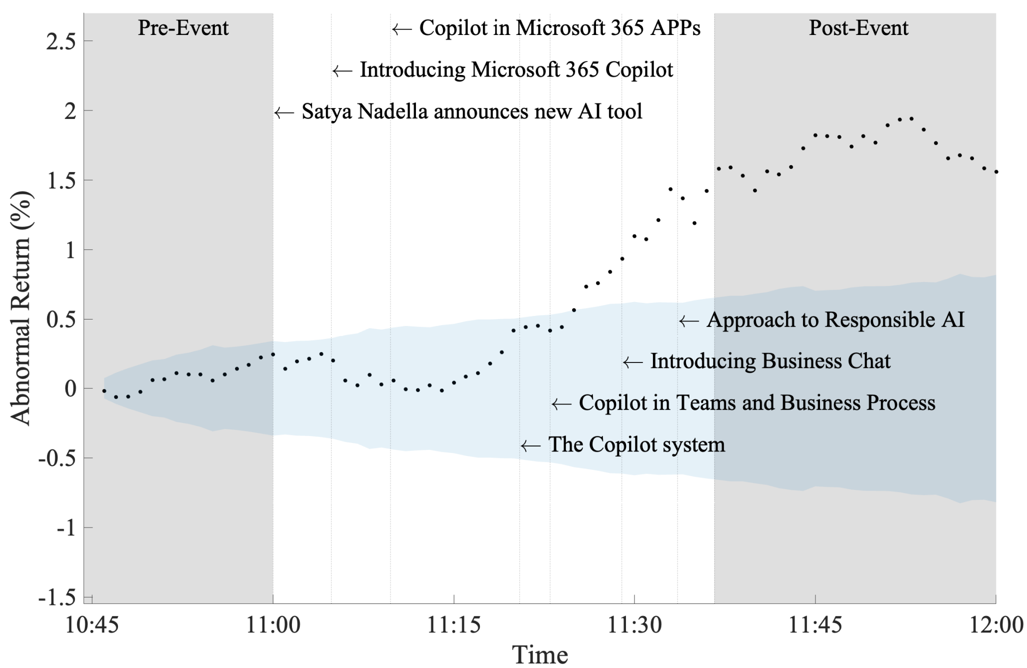

Figure S7: The figure is constructed in a similar way as Figure 4 in the main text, except that QQQ is used as the proxy for the market portfolio and $k = 10$.

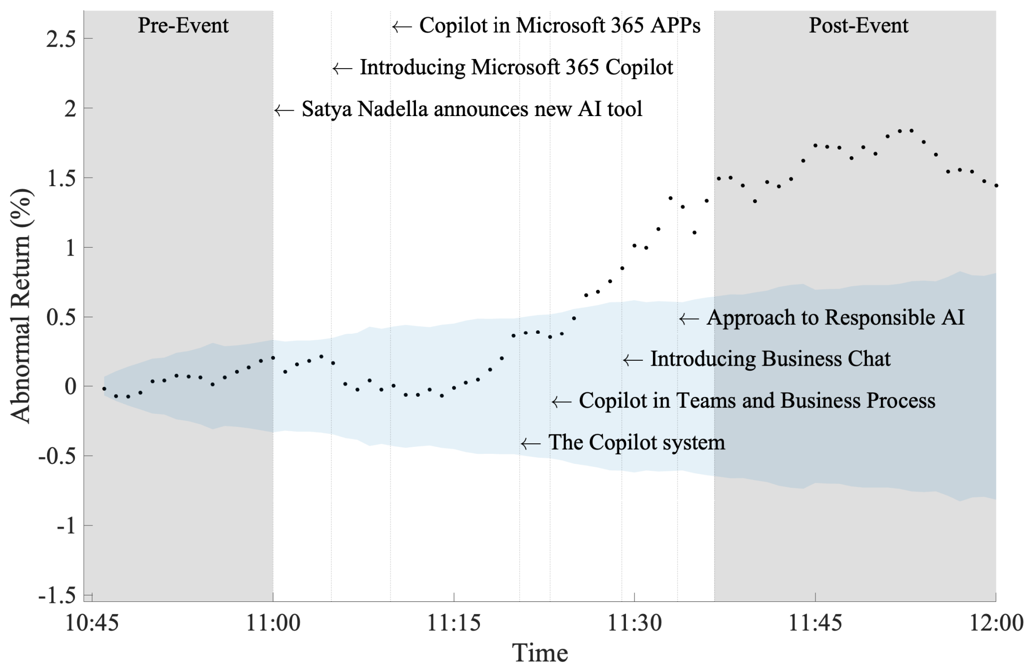

Figure S8: The figure is constructed in a similar way as Figure 4 in the main text, except that QQQ is used as the proxy for the market portfolio and $k = 5$.

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