AEA Continuing Education Program

Matching Market Design

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Matching Market Design
Course Introduction

Parag Pathak

MIT and NBER

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What is this about?

"A theory of an intelligently guided invisible hand"
Economist, 2007

Market design is a field of economics which attempts to devise practical schemes for resource allocation problems

- Mechanism design and matching theories underpin the field as a general methodological framework. We will review some of the basics and then push ahead to the frontier.

  One of the strengths of this area is its tight connection to actual real-world systems, so there will be discussion of institutions ("rules of the game") and empirical issues.

  This course is intended to be an introduction and entryway to this literature.

  We would like to make this as interactive as possible.

Prominent examples of matching market design

- Labor market clearinghouses, especially in medicine [Roth, Niederle, etc]

- Student assignment systems
  - US K-12, Higher ed systems around world [Abdulkadiroğlu, Pathak, Roth, Sönmez, etc]

- Clearinghouses for organ exchange
  - Kidneys, now livers and lungs [Ashlagi, Roth, Sönmez, Unver, Ergin, etc]

- Hybrid matching/auction systems used for personnel management
  - US Military Academy, ROTC [Sönmez]

- Refugee resettlement systems
  - Not yet designed, but some attempts [Andersson, Kominers, Teytleboym, etc.]

Course Overview

- Co-taught with expert market designers
  - ▶ Atila Abdulkadiroğlu, Duke
  - ▶ Nikhil Agarwal, MIT
- Emphasis on link between theory and practice

- **Sunday**
  1 Overview and foundations
- **Monday**
  2 One-Sided Matching (Abdulkadiroğlu)
  3 Two-Sided Matching (Pathak)
  4/5 School assignment: one-sided meets two-sided (Abdulkadiroğlu/Pathak)
  6 Organ Markets (Agarwal)
- **Tuesday**
  7 Revealed Preference in Matching Markets (Agarwal)
  8 Matching in Richer Domains (Pathak)
  9 Causal Inference with Matching Markets (Abdulkadiroğlu)

Beyond F=ma: Economics to Engineering

Roth (EMA 2002): bridge building analogy

Consider the design of suspension bridges. Their simple physics, in which the only force is gravity, and all beams are perfectly rigid, is simple, beautiful and indispensable.

But bridge design also concerns metal fatigue, soil mechanics, and the sideways forces of waves and wind. Many questions concerning these complications cant be answered analytically, but must be explored using physical or computational models.

These complications, and how they interact with that part of the physics captured by the simple model, are the concern of the engineering literature. Some of this is less elegant than the simple model, but it allows bridges designed on the same basic model to be built longer and stronger over time, as the complexities and how to deal with them become better understood.

Why design markets?

- Invisible hand: markets can organize themselves efficiently

- Markets are efficient under broad set of conditions (1st welfare theorem)

  ✓ no externalities
  Key idea: markets may be incomplete due to lack of prices

  Remedies: Taxes, quotas, coasian solution

  ✓ perfect information

  ✓ perfect competition

  Note: akin to asking why we need regulation/gov't intervention

- Even Hayek argued that what makes a market free is that it has rules that allow it to work freely:

  "There is, in particular, all the difference between deliberately creating a system within which competition will work as beneficially as possible and passively accepting institutions as they are."   The Road to Serfdom

Some market design debates

- Practical considerations (or ad hoc restrictions?): Market vs. mechanism design

- Modeling, experimentation, computation, empirics tightly integrated (or are we missing the larger game?)

- Details matter (or do they?)

- Towards a general theory (or case studies?)

- Market design and economic theory vs. plumbing

Practical considerations: message spaces

Paul Milgrom (2009):

The resource allocation mechanisms used in practice often employ messages that are too simple to describe preferences completely. For example, in simultaneous first-price auctions of the sort utilized for wholesale trading of used cars to dealers, the auctioneer typically accepts individual bids on cars, and allows the bidder no opportunity to describe the extent to which it might be willing to substitute one car for another.

Similarly, the National Resident Matching Program (NRMP) uses a variant of the celebrated Gale-Shapley algorithm to assign doctors to hospitals, but accepts reports from hospitals that consist only of a number of positions and a rank order list of doctors, allowing a hospital only a meager opportunity to describe its preferences about the composition of its incoming class.

Practical considerations

Ilya Segal (2006, ES): A major theme in the *market design* literature is that the choice of mechanism is not determined by incentives alone. Full revelation of agents preferences is often impractical or undesirable for several reasons:

- Full revelation may requires prohibitive amounts of communication e.g., bidders valuations in combinatorial auctions valuations for all possible bundles is exponential in number of objects
- Agents may have to incur *evaluation costs* to learn their own preferences.
- The more information is revealed, the more deviations exploiting the revealed information become available to agents or the designer

  The market design literature has examined a variety of mechanisms that aim to achieve the desired goals without fully revealing agents preferences.

Practical considerations

Preston McAfee, John McMillan, and Simon Wilkie (2008): "Greatest Auction in History"

A second aspect of simplicity, and one harder to implement, requires that a simple strategy be optimal, or nearly optimal, behavior ... Economists were very much concerned that they could articulate simple bidding strategies for bidders that would perform well. It was expected that novice bidders would probably adopt such strategies.

Alvin Roth and Axel Ockenfels (AER 2002) on empirical patterns on last-minute bidding on online auctions (i.e. sniping):

In designing new markets, it will be important to consider not only the equilibrium behavior that we might expect experienced and sophisticated players eventually to exhibit, but also how the design will influence the behavior of inexperienced participants, and the interaction between sophisticated and unsophisticated players.

Practical considerations and the "real-world"

- Restrictions on mechanisms
  - ✓ Based on complexity / bounded rationality
  - ✓ Information revelation (privacy)
  - ✓ These can (and should) be modeled formally

- Policy recommendations: almost never Pareto dominance, so what criteria are we using?

- Political process involves compromises and timeline is often beyond our control: what do we do in these situations?

Market design and game theory

Roth (2002):

...in the long term, the real test of our success (as economists) will be not merely how well we understand the general principles that govern economic interactions, but how well we can bring this knowledge to bear on practical questions of microeconomic engineering. Just as chemical engineers are called upon not merely to understand the principles that govern chemical plants, but to design them, and just as physicians aim not merely to understand the biological causes of disease, but their treatment and prevention, a measure of the success of microeconomics will be the extent to which it becomes the source of practical advice, solidly grounded in well tested theory, on designing the institutions through which we interact with one another.

Duflo: Economist as Plumber (AER, 2017)

Is Plumbing Science?

- Plumbing experiments are useful by themselves
- Plumbing experiments have generated insights useful to pure science
  - ✓ Shine spotlight on understudied problems

"Scientists design general frames, engineers turn them into relevant machinery, and plumbers finally make them work in a complicated, messy policy environment. As a discipline, we are sometimes a little overwhelmed by "physics envy," searching for the ultimate scientific answer to all questions – and this will lead us to question the legitimacy of plumbing. This essay is an attempt to argue that plumbing should be an inherent part of our profession."

Peyton Young (1995), Equity: In Theory and Practice

Classification of Allocation schemes

- Forced equality

  - ✓ Leonardo's painting of Last Supper
  - ✓ Forced conscription, even during peace

- Lotteries

  - ✓ Immigration visas, jury duty
  - ✓ Fishing rights, emissions rights, import quotas
  - ✓ Land reforms (e.g., Oklahoma Land Rush, developing countries)
  - ✓ Draft during Vietnam War

- Rotation / Taking turns

  - ✓ Chore assignment in communes
  - ✓ Children are time-shared between

Oklahoma Land Rush of 1889

Young's taxonomy (cont.)

✓ Queuing: using time as in an auction; time in line is wasted

✓ Priority Lists (queue in advance)

  e.g., unions have criteria for determining who gets laid off first

  ▸ Need equity judgement about who deserves the good the most

✓ Compensation / transfers / prices

  Basic tradeoff

  - <u>Prices</u>: allow people to express preferences, but if market clearing price used then income determines everything
  - <u>Rationing</u>: may lead to over-delivery of goods to those who really do not value them, but allows "true needs" to be met (fairness/equity)

When to allow prices?

"Conscription to man the military services in peacetime: The appropriate free market arrangement is volunteer military forces; which is to say, hiring men to serve. There is no justification for not paying whatever price is necessary to attract the required number of men. Present arrangements are inequitable and arbitrary, seriously interfere with the freedom of young men to shape their lives, and probably are even more costly than the market alternative. (Universal military training to provide a reserve for war time is a different problem and may be justified on liberal grounds.)."

Milton Friedman, in *Capitalism and Freedom*

Weitzman (1977): Price System vs. Rationing

Heterogeneity: v is value for (divisible) commodity
λ is marginal utility of income

Under specific functional forms (quadratic utility, independent densities), and a loss function measuring how far we are giving good to those who value it the most, he obtains this relationship on the effectiveness of price system over uniform rationing:

$$\Delta = \underbrace{\text{var}(v)}_{\text{taste heterogeneity}} - \hat{p} \underbrace{\text{var}(\lambda)}_{\text{income heterogeneity}}$$

where $\hat{p}$ is the market-clearing price

- Prices preferred: when taste distribution is dispersed, or society has **more equal income distribution**

- Rationing preferred: when taste distribution is more uniform, or there is **greater income inequality**

Coase with Liquidity Constraints: Che-Gale-Kim
(RES 2013)

- A unit mass of risk-neutral buyers, each who demand one unit, mass $m \geq 0$ of non-buyers ("rest of population")

- Buyer has type (w, v)
  - ✓ $w =$ wealth $\in [0, 1] \sim G(w)$
  - ✓ $v =$ valuation $\in [0, 1] \sim F(v)$

- Non-buyer has same w distribution and $v = 0$

- Quasi-linear preferences: if consume good with probability x,

$$u(w, v) = vx + w - p,$$

cannot spend more than w; $w < v$ means wealth constrained

- Indivisible good is supplied elastically at zero marginal cost, up to capacity $S \in (0, 1)$

Welfare Criteria

- Utilitarian efficient: total realized value (or average value realized per unit)
    - ✓ Ex ante perspective ("Vickrey/Harsanyi test"): what would an individual choose should she have an equal chance of landing in the shoes of each member of society?

- First-best benchmark: buyers with $v \geq v^*$ are served, where v^* satisfies $S = 1 - F(v^*)$

- Average value realized:
$$E[v|v \geq v^*]$$

First-best Allocation

v

v^*

w
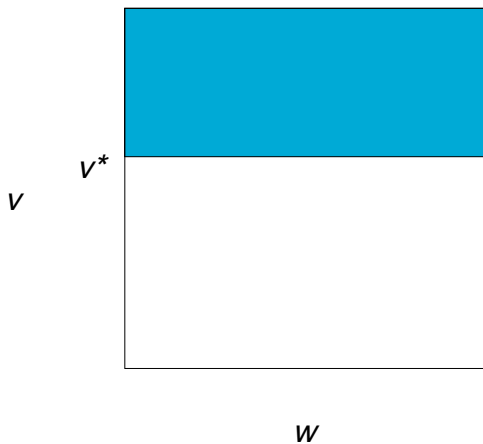
Three Mechanisms

1. Competitive market - resale right does not matter

2. Nonmarket (random) assignment without resale - price is capped and lottery assigns good; resale prohibited

3. Nonmarket with resale: same as above except resale is permitted after assignment

   Note: not solving mechanism design problem.. wait till recitation

- when budget constraint binds, optimal mechanism is *random assignment with regulated resale and cash subsidy*
- resale market is taxed to limit speculation for low-value agents, who accept a cash subsidy for not participating
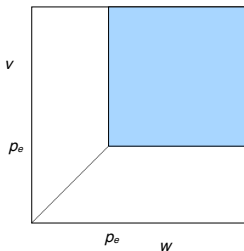
Competitive Market

- Demand = number of buyers **willing** and **able** to pay price

$$D(p) = [1 - F(p)][1 - G(p)]$$

- Supply = S
- Equilibrium price p_e satisfies

$$D(p_e) = S$$

- Average value realized: $E[v| v \geq p_e]$.

Nonmarket without Resale

- Price is capped at $q < p_e$ and excess demand is assigned randomly (i.e., lottery, with one entry per participating agent)

- Buyers with $(w, v) \geq (q, q)$ participate in the rationing and are successful with probability

$$\frac{S}{[(1 - F(q))(1 - G(q))]}$$

- Less efficient than market: random assignment allows buyers with low wealth to consume, but also with low valuations
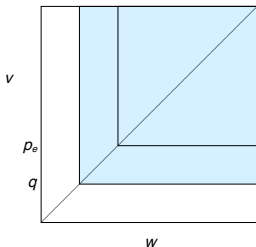
■ Average value realized: $E[v| \ v \geq q] < E[v| \ v \geq p_e]$.

Nonmarket (random) with resale

- Price is capped at $q < p_e$ and excess demand is rationed randomly; resale is permitted
- Suppose the resale price, r, is higher than q (if not, there would not be rationing)
- All buyers and even "non-buyers" with $w \geq q$ will participate in rationing
- All buyers with $(w, v) > (q, 0)$ participate; each gets good with probability

$$\rho(q, m) = \frac{S}{(1 + m)(1 - G(q))}$$

Note that $\rho(q, m) \to 0$ as $m \to \infty$

- Resale market:
  - ✓ Demand side: unsuccessful buyers purchase at the resale price r if $(w, v) \geq (r, r)$
  - ✓ Supply side: successful buyers/non-buyers with $v < r$ sell

Resale Market Equilibrium

- Measure of buyers: $[1 - F(r)][1 - G(r)](1 - \rho(q, m))$

- Measure of sellers: $S * (F(r) + m)/(1 + m)$

- Setting equal, its possible to show that the equilibrium resale price:

$$r^*(q, m) > p_e$$

- Average value comparison:

$$E[v|v \geq r^*] > E[v|v \geq p_e]$$

- Lower q and lower m raise average value realized

- As $m \to \infty$, $r^* \to p_e$ (asymptotic Coase theorem)

Summary

- Coase theorem doesn't apply if individuals are wealth constrained

- Allocating the good to the poor improves efficiency since only the wealthy can buy on the resale market

- Random assignment with resale does a better job than the market in allocating to poor

  Intuition:

  - ✓ Under market price, with liquidity constraints, good does not go to those who value it the most (high value, low wealth guys).
  - ✓ If rationed, then some high value and low wealth guys will get good.
  - ✓ With resale market, low value guys can resell, and this leads to more high value guys getting the good overall.

- Speculation limits this benefit and can wipe it out if there are many potential speculators

Two Related Questions

Q1: When to allow resale?

- **Speculators**: those with no intrinsic value for good

  Che-Gale: speculators in rationing phase will reduce the probability that low wealth and high value guys get the good. This, in turn, erodes the benefit of non-market assignment.

- **Frictions** in resale market

  Depends crucially on particular model of resale

  ### Q2: What determines the tradability/transferability of a good?

  - ▶ Roth (2007): Repugnant Transactions
  - ▶ Paternalism: e.g., people who wish to sell organs might not be making rational decisions
  - ▶ Che and Gale's argument is that allowing resale might be optimal on efficiency grounds if we have these wealth constrained agents and a utilitarian criterion

Why Centralize Markets?

- Much of economic theory about "decentralization" results – e.g., second welfare theorem

- Hayek (1945 AER): "*in a system in which the knowledge of the relevant facts is dispersed among many people, prices can act to cordinate the separate actions of different people.... The mere fact that there is one price for any commodity- or rather that local prices are connected in a manner determined by the cost of transport, etc.- brings about the solution which (it is just conceptually possible) might have been arrived at by one single mind possessing all the information which is in fact dispersed among all the people involved in the process.*"
  - Starting point in many matching applications is absence of prices
  - Heterogeneity of commodities means prices would be high-dimensional

- Without price signals, markets often clear using other means: timing

Unraveling

- Some entry-level labor markets in the US suffer from unraveling
  - ✓ Medical interns
  - ✓ Judges & lawyers
  - ✓ Sports (e.g. NBA, college bowls)
  - ✓ College admissions

- Policies trying to ban unraveling are often unsuccessful

- Seen as a rationale for centralization...

TABLE 1—A SELECTION OF MARKETS WITH TIMING PROBLEMS

Market	Organization	Stage
Postseason college football bowls	National Collegiate Athletic Association (NCAA)	1 and 3
Entry-level legal labor markets:		
Federal court clerkships	Judicial conferences	2, 1
American law firms	National Association for Law Placement (NALP)	1
Canadian articling positions	Articling Student Matching Program	
Toronto		4
Vancouver		3 or 4
Alberta (Calgary and Edmonton)		3
Entry-level business school markets		
New MBA's		1 (occasionally)
New marketing professors		1
Other entry-level labor markets:		
Japanese university graduates	Ministry of Labor; Nikkeiren	2
Clinical psychology internships	Association of Psychology Internship Centers	2
Dental residencies (three specialties and two general programs)	Postdoctoral Dental Matching Program	3
Optometry residencies	Optometric Residency Matching Services	1 and 3
Other two-sided matching:		
Fraternity rush		1
Sorority rush	National Panhellenic Conference	3
Entry-level medical labor markets:		
American first-year postgraduate (PGY1) positions	National Resident Matching Program (NRMP)	3
Canadian first-year positions	Canadian Intern and Resident Matching Service	3
U.K. regional markets for preregistration positions	Regional health authorities	
Edinburgh		3
Cardiff		3
Birmingham		4, 1
Newcastle		4, 1
Sheffield		3 or 4, 1
Cambridge		3
London Hospital		4
American specialty residencies:		
Neurosurgery	Neurological Surgery Matching Program	
Ophthalmology	Ophthalmology Matching Program	3
Otolaryngology	Otolaryngology Matching Program	3
Neurology	Neurology Matching Program	3
Urology	AUA Residency Matching Program	3
Radiation Oncology	Radiation Oncology Matching Program	1 and 3
Other specialties[a]	NRMP	3 and 4
Advanced speciality positions:		
12 (primarily surgical) specialities[b]	Specialties Matching Services	3
Three medical subspecialties[c]	Medical Specialities Matching Program	3
Four ophthalmology subspecialties	Ophthalmology Fellowship Match	3
Plastic surgery	Plastic Surgery Matching program	3

[a] Anesthesiology, emergency medicine, orthopedics, physical medicine, psychiatry, and diagnostic radiology.
[b] Colon/rectal surgery, dermatology, emergency medicine, foot/ankle surgery, hand surgery, ophthalmic plastic and reconstructive surgery, pediatric emergency medicine, pediatric orthopedics, pediatric surgery, reproductive endocrinology, sports medicine, and vascular surgery.
[c] Cardiovascular disease, gastroenterology, and pulmonary disease.

Li-Rosen (1998): "Unraveling in Matching Markets"

- Formalize incentives for unraveling by a 2 period matching model with
  - Uncertainty about productivities
  - Incomplete contractual market
- Clarify how unraveling responds to exogenous market structure

  Model
- Finite number of workers and firms
- 2 types of workers and firms: productive or not
  - (productive worker, productive firm) produce 1
  - The other pairs produce 0

Three market structures

- (# of productive workers)>(# of productive firms)
  →In any CE, every productive firm matched to a productive worker.
  CE payoff to productive workers=0 & to productive firms=1

- (# of productive firms)>(# of productive workers)
  →The opposite

- Balanced:
  In any CE, every productive participant matched to productive one.
  Any payoff division is OK

- Indivisibilities + Uncertainty about market structures (1)-(3)
  - → Participants face payoff risks and may want to contract early

Example

- Equal number of risk averse workers and firms

- $t = 2$: Each firm & worker turns out productive w.p. $\lambda \in (0, 1)$.
  A competitive equilibrium takes place as before; assume surplus split w.p. $1/2$

- $t = 1$: Firm and worker productivities unknown.
  Workers and firms can sign contracts of the form "worker gets r and firm gets $1 - r$ iff both are revealed productive; both get 0 otherwise."

- Definition: **Market unravels** if $\exists r$ such that some firm-worker pairs choose to sign contract rather than wait

- Expect utility of early contract for worker:

$$\underbrace{\lambda^2 u(r)}_{\text{payoff from proving productive}} + \underbrace{(1-\lambda^2)u(0)}_{\text{payoff from proving unproductive}}$$

- EU of early contract for firm: $\lambda^2 u(1-r) + (1-\lambda^2)u(0)$

- EU from waiting:

$$\lambda[u(1)/2 + u(0)/2] + (1-\lambda)u(0) = \frac{\lambda}{2}[u(1) - u(0)] + u(0)$$

- Let r^w & r^f be indifferent prices for worker and firm.

Equilibrium

- Equilibrium indifference condition:

$$u(r^w) = u(1 - r^f) = u(0) + \frac{1}{2\lambda}[u(1) - u(0)]$$

  - If λ is large, $r^w < r^f$ and unraveling (why?)
  - If λ is small, $r^f < r^w$ and no unraveling

- Intuition:
  - Early contract provides insurance against uncertain market structure
  - But leads to ex post assignment inefficiency because your partner may turn out to be unproductive

- Inefficiency decreasing in λ

Unraveling Driven by Market Incompleteness

- Optimal insurance arrangement would have all sign first period contract where
  - All wait until the 2nd period to achieve ex post efficient assignment
  - Each receives the same share of max total output

- Li-Rosen show how this can be implemented in a market with complete contingent claims (Arrow-Debreu)

- Why not?
  - Moral hazard (unmodelled)

- **Broader lesson**: market can be incomplete due to inadequate time to transact; centralized markets can sometimes mitigate timing issues

Social Choice Functions and Mechanisms

Formulation due to Leonid Hurwicz

- Central planner would like to choose an alternative in A

- Each agent i observes preferences $u_i(t_i, .)$ for type $t_i \in T_i$ over A, while the central planner does not

  Note: writing payoffs this way assumes private values, since payoffs depend only on your type, not on others' types

- Refer to

$$\omega \equiv t = (t_i)_{i \in N} \in \times_i T_i \equiv T$$

  as the **state of the world**, unknown by the central planner. A type can be a preference ordering or something more general.

- **Social choice function (SCF)**: $f : T \to A$, which the planner would like to implement; more generally we can think of social choice correspondence

- Recall a social *welfare* function maps to a preference ordering for society, while social *choice* function maps to a set of alternatives ("chosen" by society)

Historical context: Institution-free microeconomics

- Early debates between the comparative merits of economic systems tried to justify the market mechanism and prices

- Paradox of **Second Welfare theorem**: given a Pareto efficient allocation, we can construct prices to support that allocation as a price equilibrium

    ✓ To construct the supporting prices, we need to have full knowledge of the primitives (utilities, tastes, technologies) in the economy. . .

    ✓ Moreover, if we are already at a Pareto efficient allocation, why even bother with prices?

- Hayek (1945): prices may play an informational role

    ✓ If individuals have private information about their preferences, endowments, technologies, etc, this information is too enormous to be communicated to a central planner

    ✓ Market mechanism works by using prices to give people concise sufficient statistics allowing them to make coordinated choices and arrive at a socially optimal allocation

Mechanisms and Implementation

- Message space M_i is the set of messages a player can submit

- **Mechanism** $\varphi : \times_i M_i^\varphi \to A$ is a function, where we index the message space by the mechanism

- For $m = (m_i)$, the outcome of the mechanism implemented by the planner is

$$\varphi(m)$$

- We say that mechanism φ **implements** SCF f in dominant strategies if for all $t \in T$, there exists a weakly dominant strategy profile for t of the mechanism message sending game denoted by m such that

$$\varphi(m) = f(t)$$

Direct Mechanisms

- Formulation so far is very general; messages can be practically anything

- If each message space satisfies $M_i^\varphi = T_i$ for all $i \in I$, then we refer to φ as a **direct mechanism**

    ✓ Set of messages a player can send is simply their set of possible types

- Direct mechanism φ is **incentive compatible** if for all states of the world $t \in T$, t is an equilibrium of the mechanism game

    ✓ Revealing your type, or truth-telling, is an equilibrium

What are some properties of a mechanism?

- A direct mechanism is **strategy-proof** if it is dominant strategy incentive compatible

- A direct mechanism is **Pareto efficient**, if for any state of the world, it chooses a social outcome such that there exists no other social outcome that would make everybody weakly better off and at least one agent strictly better off

- A direct mechanism $\varphi : T \to A$ is **dictatorial**, if there exists an agent $i \in N$ such that for all states $t \in T$, $\varphi(t)$ is the top social alternative of i when his type is t_i

Theorem (Gibbard/Satterthwaite)

If $|A| \geq 3$ is finite, types of agents are such that the utility functions represent all strict preference rankings on A, then any Pareto-efficient and strategy-proof mechanism is necessarily dictatorial.

- Result is a close cousin of Arrow's theorem
- Pessimistic conclusion for strategy-proof mechanism design: how to get around?

Overcoming Gibbard-Satterthwaite

1) Stochastic mechanisms: Allow for mechanisms to be stochastic:

$$\varphi : T \to \Delta A,$$

where ΔA is probability distribution over alternatives

- ▶ Gibbard (1977): if mechanism only uses ordinal rankings and players evaluate outcomes using vNM utility, then only strategy-proof mechanisms are convex combinations of dictatorships and duple mechanisms

    - ✓ **Duple mechanism**: assign positive probability to at most two alternatives that do not depend on preferences

2) Relax dominant strategy requirement

    - ◇ Dominant strategies have the property that equilibrium behavior does not depend on beliefs, common knowledge of rationality, and the information structure, and therefore gives predictive power or robustness
    - ◇ Another practical motivation is that one can give advice

3) Restrict the preference domain to economic domains

Why strategy-proofness?

- Concerns about robustness came as early as Hurwicz (1972), who discussed the need for "non-parametric" mechanisms that are independent of assumptions regarding willingness to pay of agents

- Wilson (1985): argued for 'belief-free' trading rules by requiring that they "should not rely on features of agents' common knowledge such as their probability assessments."

- Axiomatic tradition treats strategy-proofness as a goal by itself
  - Some complain that objectives should only be consequentialist

Economic Domains

- Single-peaked preferences
- Indifferences:
    - An agent can be allocated a personalized resource and he only derives utility from the consumption of his own share
    - He would be indifferent among all social alternatives that assign him the same resource share
    - Sometimes known as the **hedonic** domain
- Made throughout quasi-linear mechanism design (Vickrey-Clarke-Groves)
    - Social outcome is a pair $(\delta, \chi) \in A$ where $\delta \in D$ is a social decision and $\chi \in \mathbb{R}^n$ is a vector of monetary transfers
    - Utility of agent i for his type $t_i \in T_i$ is

    $$u_i(t_i, (\delta, \chi)) = v_i(t_i, \delta) + \chi_i$$

    for a value function

    $$v_i : T_i \times D \to \mathbb{R}$$

- Nearly all of matching theory works in hedonic domain, opens door to strategy-proof mechanisms

One-Sided Matching Problems

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January 2018

Road Map

- How to allocate indivisible object in absence of monetary transfers?
- Allocation of indivisible objects when nobody owns them (or everybody owns everything collectively)
- Allocation of indivisible objects under individual ownership
- Allocation of indivisible objects under individual and collective ownership
- Lottery mechanisms

The West Lawn

Campus Housing at UVA
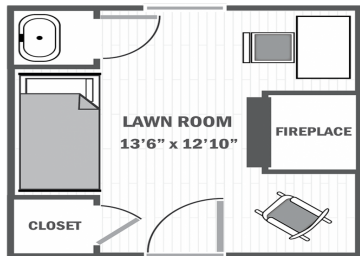
Campus Housing at UVA

Campus Housing at UVA

`http://uvamagazine.org/articles/how_lawnies_are_made`

Living on the Lawn is one of the University of Virginia's greatest honors and most enriching experiences for undergraduates. (Never mind the outdoor walk to the bathroom amid the elements and the tourists.) For Lawnies, the concept of the Academical Village isn't merely academic; they live it, in the same rooms that Thomas Jefferson designed for UVA's first students, only now with sinks and central heating in addition to the still-working fireplaces. To live on the Lawn makes you part of a community of similarly high-achieving students and of the several deans and other distinguished faculty members granted the privilege of living in the pavilions.

Bryanna Miller (Col '18) in the room she'd hoped for.

Campus Housing Allocation at UVA

`http://uvamagazine.org/articles/how_lawnies_are_made`:

- 47 of the Lawns 54 rooms allocated selectively.
- Lawn Selection Process Organizing Committee reviews and approves the selection criteria and the application form.
- Applications are collected in January
- The Selection Committee divides itself in two and splits the applications between the halves.
- Applications that survive the first step are discussed again, the group comes up with its list of 47
- Calibration Committee: a mechanism to review the composition of the list and allow the selectors to adjust for any unfair omissions of significant portions of the University community. "The concept arose in response to the situation two years ago, when the list included no African-American students, and to the instance one year ago, when the list had no one from the School of Nursing."

Outline

House Allocation Problems - Collective Ownership Economy

A **house allocation problem** (Hylland & Zeckhauser, 1979) is a triple $\langle I, H, \succ \rangle$.

I: set of agents

H: set of houses

$\succ$:   list of preferences over houses

            $A \succ B$: A is better than B

For simplicity assume:

1. $|H| = |I|$, and
2. the preferences are strict

            $A \succeq B$: A is better than B or A=B

The Outcome: A Matching

- A **(house) matching** $\mu : I \to H$ is a one-to-one and onto function from I to H.

  With everyday language it is an assignment of houses to agents such that
  1. every agent is assigned one house, and
  2. no house is assigned to more than one agent.

- A matching μ **Pareto dominates** another matching ν if
  1. $\mu(i) \succeq_i \nu(i)$ for all $i \in I$ and
  2. $\mu(i) \succ_i \nu(i)$ for some $i \in I$.

- A matching is **Pareto efficient** if it is not Pareto dominated by any other matching.

Serial Dictatorship

Simple serial dictatorship induced by f: Agent who is ordered first (by the ordering f) gets her top choice; agent ordered second gets his top choice among those remaining; and so on.

Theorem. A serial dictatorship is efficient and strategy-proof.

Serial dictatorship is Pareto efficient

Proof. Let f be the priority ordering and φ^f be the induced serial dictatorship.

- The first agent $f(1)$ is assigned her best most preferred house among all available houses, so we cannot make her better off

$\vdots$

- Agent $f(k)$ is assigned his best most preferred house among all remaining houses, so we cannot make him better off without making one of the first $k - 1$ agents worse off

$\diamond$

Serial dictatorship is strategy-proof

Proof. Let f be the priority ordering and φ^f be the induced serial dictatorship.

- The first agent $f(1)$ cannot do better than reporting any other preferences since she already receives her first choice house under her preferences,

$\vdots$

- Agent $f(k)$ cannot do better than reporting her true preferences, since the houses distributed until $f(k)$ is independent of $f(k)$'s preferences, and $f(k)$ receives her first choice among the remaining houses given her reported preferences.

$\diamond$

Characterizing Serial Dictatorship

- A mechanism is **nonbossy** if no agent can change another agent's assignment by changing her preferences and without changing her assignment
- A mechanism is **neutral** if the naming of houses does not matter, i.e. if we rename the houses without changing their physical qualities and without changing agents' preferences, each agent is assigned the same physical object (albeit with a new name)

  Theorem: (Svensson 1999) A mechanism is strategy-proof, nonbossy and neutral if and only if it is serially dictatorial.

Outline

Housing Markets - Individual Ownership Economy

A **housing market** (Shapley & Scarf 1974) is a 4-tuple $\langle I, H, \succ, \mu \rangle$.

I: set of agents

H: set of houses with $|H| = |I|$

$\succ$:   list of strict preferences over houses

μ: initial endowment matching

Let $h_i = \mu(i)$ denote the initial endowment of agent $i \in I$.

- A matching η is **individually rational** if

$$\eta(i) \succeq_i h_i \qquad \text{for all } i \in N.$$

- A matching η is **blocked** a coalition $T \subseteq I$ if there is a matching ν such that

  1. $\nu(i) \in \{h_i\}_{i \in T}$ for all $i \in T$,
  2. $\nu(i) \succeq_i \eta(i)$ for all $i \in T$,
  3. $\nu(i) \succ_i \eta(i)$ for some $i \in T$.

- A matching η is in the **core** of the housing market $(I, H, \succ, \mu)$ if there is no coalition that blocks it.

Gale's Top Trading Cycles Algorithm

(Described in Shapley & Scarf, attributed to David Gale)

Step 1: Each agent "points to" the owner of his favorite house. Since there are finite number of agents, there is at least one **cycle**.

Each agent in a cycle is assigned the house of the agent he points to and removed from the market with his assignment.

If there is at least one remaining agent, proceed with the next step.

Step t: Each remaining agent points to the owner of his favorite house among the remaining houses.

Every agent in a cycle is assigned the house of the agent he points to and removed from the market with his assignment.

If there is at least one remaining agent, proceed with the next step.

Q: Can you prove why there must be a cycle in each step?

A version of TTC (Abdulkadiroğlu and Sönmez 2003) was adopted for school assignment in New Orleans:

Properties of the Core

- **Theorem** (Roth & Postlewaite 1977): The outcome of Gale's TCC algorithm is the unique matching in the core of each housing market. Moreover, this matching is the unique competitive allocation.

- A **direct matching mechanism** is a systematic procedure to select a matching for each problem.

- A direct mechanism is **strategy-proof** if truth-telling is a dominant strategy in the resulting preference revelation game.

- **Theorem** (Roth 1982). Core (as a direct mechanism) is **strategy-proof**.

Properties of the Core - Proofs

Let μ be the matching obtained as the result of Gale's TTC algorithm.

First we prove that μ is in the core, then we prove that μ dominates any other matching through some coalition.

μ is in the core:

Let $C_1, C_2, ..., C_k$ be the agents in cycles (in the order they are removed) in Gale's TTC algorithm.

Note that no agent in C_1 can be in a blocking coalition, since they get their first choice under μ.

Given this, no agent in C_2 can be in a blocking coalition, since they get their first choice in $H \backslash \mu(C_1)$, iteratively we continue.

Properties of the Core - Proofs

There is no other matching in the core:

Consider a matching $\nu \neq \mu$. We will show it is dominated by μ.

Let b be the first agent who satisfies $\nu(b) \neq \mu(b)$ (according to the order of the cycles $C_1, ..., C_k$, if there are multiple agents in a cycle like b, then choose one of them arbitrarily).

Let b be in cycle C_ℓ. Note that for every agent a assigned before the cycle C_ℓ, we have $\nu(a) = \mu(a)$. (by definition)

Given this, for every agent $a \in C_\ell$, $\mu(a)R_a\nu(a)$ for all $a \in C_\ell$, as the agents in C_ℓ will be pointing to their most preferred alternative among those available.

Then we have for $a \in C_\ell$, $\mu(a)P_a\nu(a)$, by strictness of preferences.

Moreover for each agent $a \in A$, $\mu(a) = h_m$ which is owned by some $a_m \in A$ by construction of μ and C_ℓ. (so μ is feasible)

Hence μ dominates ν through coalition C_ℓ, concluding the proof.

Housing markets: competitive equilibrium

What about an exchange economy and competitive equilibrium?

The price mechanism will also achieve the core matching:

Prices of houses in a vector $p = (p_1, ..., p_n)$.

A house h_m is affordable for agent a_ℓ at p if $p_m \leq p_\ell$ (budget set).

A matching μ and price vector p is a **competitive equilibrium** if for any agent a, $\mu(a)$ is the best house she can afford at prices p.

Housing markets: competitive equilibrium

Proof of CE. Let P be a preference profile and let $C_1, C_2, ..., C_k$ be the cycles encountered in order in Gale's TTC algorithm for this market.

- Let price vector p be such that for any cycle C_m and for any agent $a_\ell \in A$ such that $p_\ell = q_m$ for some constant q_m (**each house in a cycle has the same price**) and let $q_m > q_{m+1}$ for any $m \in \{1, 2, ..., k-1\}$ (houses in earlier cycles have higher price).

Observe that (μ, p) is a competitive equilibrium. No agent a likes some house allocated in a later cycle more than $\mu(a)$. No agent can afford any house allocated in an earlier cycle. Hence every agent is allocated the best house she can afford. ⋄

Proof (Sketch) of Strategy-proofness:

Consider Gale's TTC algorithm.

Suppose an agent leaves the algorithm with her assignment in Step t.

She cannot stop the formation of cycles that form before Step t by misrepresenting her preferences.

(These cycles only depend on preferences of agents who are in those cycles.)

So she cannot receive a better assignment through a preference manipulation.

- **Theorem** (Ma 1994): Core is the only mechanism that is *Pareto efficient*, *individually rational*, and *strategy-proof*.

  Sketch of the Proof: Let ν be the matching in the core for housing market $\langle I, H, \succ, \mu \rangle$.

  Construct the preference relation $\succ_i'$ for each agent i, by elevating endowed house h_i to be just below $\nu(i)$ as follows:
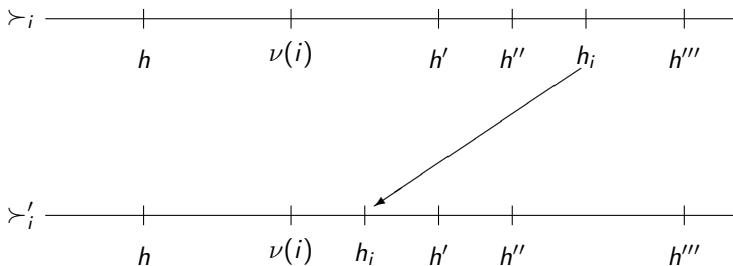
Let ϕ be a *PE, IR,* and *S-P* mechanism; ϕ_i is i's assignment.

Claim 1: $\phi(\succ') = \nu$ because ν is the only PE and IR matching under $\succ'$.

Let $C_1, ..., C_k$ be the cycles in TTC for $\succ$, and $\nu' = \phi(\succ')$.

Start with agents in first cycle C_1, and consider some agent $a_\ell \in C_1$,

$$\nu(a_\ell) \neq \nu'(a_\ell).$$

Since ϕ is IR and each agent receives top choice under in C_1 in ν,

$$\nu'(a_\ell) = h_\ell.$$

IR of ϕ implies that for each agent where $\nu(a_\ell) \neq \nu'(a_\ell)$,

$$\nu'(a_\ell) = h_\ell.$$

Hence, each of these agents obtains their endowment under ν', while under $\succ'$ they prefer their matching under ν. This implies ϕ is not Pareto efficient, unless $\nu'(a) = \nu(a)$ for all $a \in C_1$.

Iterating, similar argument applies for agents in C_2...

Claim 2: $\phi(\succ) = \nu$.

We will go from $\succ$ to $\succ'$ replacing one agent's preference at a time

Start with one agent i and use strategy-proofness to obtain:

$$\phi_i(\succ'_i, \succ_{-i}) \succeq'_i \phi_i(\succ)$$
$$\phi_i(\succ) \succeq_i \phi_i(\succ'_i, \succ_{-i})$$

IR means

$$\phi_i(\succ) \succeq_i h_i,$$

Since we have not altered anything above h_i in $\succeq'_i$, we have

$$\phi_i(\succ'_i, \succ_{-i}) \succeq_i \phi_i(\succ)$$

which implies that

$$\phi_i(\succ) = \phi_i(\succ'_i, \succ_{-i})$$

Then replace with true preferences one agent at a time

Efficient House Allocation

- Simple serial dictatorship induced by f: Agent who is ordered first (by the ordering f) gets her top choice; agent ordered second gets his top choice among those remaining; and so on.

- **Core from assigned endowments** μ: For any house allocation problem $\langle I, H, \succ \rangle$, select the core of the housing market $\langle I, H, \succ, \mu \rangle$.

  Theorem (Abdulkadiroğlu & Sönmez 1998): For any ordering f, and any matching μ, simple serial dictatorship induced by f and core from assigned endowments μ both yield *Pareto efficient* matchings. Moreover, for any *Pareto efficient* matching η, there is a *simple serial dictatorship* and a *core from assigned endowments* that yields it.

Proof

Let μ be a Pareto-efficient matching for a house allocation problem $\langle I, H, \succ \rangle$.

First, we will construct a priority order f so that $\varphi^f[\succ] = \mu$.

- Suppose that no agent receives his top choice in μ.

  ▶ We can construct a cycle as follows with two or more agents.

  ▶ Suppose each agent points to his top choice house and is pointed by the house he received in μ.

  ▶ Then, by finiteness of the problem there exists a cycle.

  ▶ Observe that there is no cycle with a single agent in it, as we assumed nobody receives his top choice in μ.

  ▶ Then, we can improve every agent in the cycle by assigning him the house he is pointing to and otherwise leaving μ unchanged.

- This contradicts μ being Pareto efficient.

- We showed that there exists an agent who receives his top choice in μ. Let him be $f(1)$. Let's exclude house $\mu(f(1))$ from the problem.
- Show in the same manner that there exists an agent who receives his top choice excluding $\mu(f(1))$, let him be $f(2)$
- We construct the remaining of f in an iterative manner similarly.
- It is straightforward to observe that $\varphi^f[\succ] = \mu$.

Next, we will construct an endowment matching ω for the same problem.

Let $\omega = \mu$. Then core from endowment $\omega = \mu$.

To see this, observe that the agent $f(1)$, constructed above, will point to his endowment in Gale's TTC algorithm in round 1, and will receive it. Similarly $f(2)$ will receive his endowment in round 1 or 2, and so on.

This proves that $\phi^\omega[\succ] = \mu$.

Outline

House allocation with existing tenants - Hybrid Ownership Structure

- Consider campus housing as an example
- Some agents – existing tenants – are grant fathered to their apartments
- Existing tenants as well as new comers can apply for housing
- **Individual rationality** requires that each existing tenant is given a house that she prefers as much as what she is currently occupying

House Allocation with Existing Tenants

A **house allocation problem with existing tenants** (Abdulkadiroğlu & Sönmez 1999) is a five-tuple $\langle I_E, I_N, H_O, H_V, \succ \rangle$ where

1. I_E is a finite set of existing tenants,
2. I_N is a finite set of newcomers,
3. $H_O = \{h_i\}_{i \in I_E}$ is a finite set of occupied houses,
4. H_V is a finite set of vacant houses where $h_0 \in H_V$ denotes the null house, and
5. $\succ = (\succ_i)_{i \in I_E \cup I_N}$ is a list of strict preference relations.

- Assume that the "null house" h_0 (i.e receiving nothing) is the last choice for each agent.

RSD with squatting rights

- Each existing tenant decides whether she will enter the housing lottery or keep her current house. Those who prefer keeping their houses are assigned their houses. All other houses become available for allocation.

- An ordering of agents in the lottery is randomly chosen from a given distribution of orderings. This distribution may be uniform or it may favor some groups.

- Once the agents are ordered, available houses are allocated using the induced *simple serial dictatorship*: The first agent receives her top choice, the next agent receives her top choice among the remaining houses and so on so forth.

  Major deficiency: It does not guarantee a better house to existing tenants, i.e. it is not individually rational, so existing tenants may refrain from entering lottery, which may result in inefficieny

A Real-life 'Individually Rational' Example: 'MIT-NH4' mechanism

- Start with ordering f
- First agent is *tentatively* assigned top choice among all houses, the next agent is *tenatively* assigned choice among the remaining houses, and so on, until a squatting conflict.
- *Squatting conflict*: when it is the turn of an existing tenant but every remaining house is worse than his current house. The *conflicting agent* is tentatively assigned the existing tenant's current house.

  Resolve a squatting conflict as follows:
  - ✓ Existing tenant is assigned his or her current house and removed from the process, and
  - ✓ All tentative assignments starting with the conflicting agent and up to the existing tenant are erased

  Start over again with the conflicting agent.
- Process is over when there are no houses or agents left. All tentative assignment are finalized.

Major deficiency: The 'MIT-NH4' mechanism is individually rational, but it is not efficient

Top Trading Cycles Mechanism

Fix an ordering f of agents. Interpret this as a **priority ordering**.

Step 1: Define the set of **available houses** for this step to be the set of vacant houses.

* Each agent points to his favorite house,
* each occupied house points to its occupant,
* each available house points to the agent with highest priority.

- There is at least one cycle. Every agent in a cycle is assigned the house that he points to and removed from the market with his assignment.
- If there is at least one remaining agent and one remaining house then we go to the next step.

TTC: Adjustment of Available Houses

- Whenever there is an available house in a cycle, the agent with the highest priority, i.e. agent $f(1)$, is also in the same cycle.
- If this agent is an existing tenant, then his house $h_{f(1)}$ cannot be in any cycle and it becomes available for the next step.
- All available houses that are not removed remain available.

Step t: The set of **available houses** for Step t is defined at the end of Step (t-1).

* Each remaining agent points to his favorite house among the remaining houses,
* each remaining occupied house points to its occupant,
* each available house points to the agent with highest priority among the remaining agents.

- Every agent in a cycle is assigned the house that he points to and removed from the market with his assignment.
- If the most senior (remaining) agent's house is vacated, then it is added to the set of available houses for the next step. All available houses that are not removed remain available.
- If there is at least one remaining agent and one remaining house then we go to the next step.

Efficiency, Individual Rationality, and Strategy-Proofness

- Natural generalization: TTC reduces to Gale's TTC for housing markets and Serial Dictatorship for house allocation problems.
- **Theorem** (Abdulkadiroğlu & Sönmez 1999): For any ordering f, the induced *top trading cycles mechanism* is
  * individually rational,
  * Pareto efficient, and
  * strategy-proof.

Lottery Mechanisms

Lottery mechanisms

- **lottery** λ: a probability distribution over matchings: $\lambda = (\lambda_\mu)_{\mu \in \mathcal{M}}$ such that $\sum_\mu \lambda_\mu = 1$.

- **lottery mechanism**: a procedure that assigns a lottery for each house allocation problem.

- **random assignment matrix**: $\varrho = [\rho_{i,h}]_{i \in A, h \in H}$ is a **bistochastic matrix** (a non-negative matrix with summation of elements of each row vector equals 1, and summation of elements of each column vector equals 1).

  Let $\varrho_i = (\rho_{i,h})_{h \in H}$ denote the random assignment vector for agent i under random assignment matrix ϱ.

Lottery mechanisms

- **random assignment matrix induced by lottery** λ is a non-negative n by n matrix $\varrho(\lambda) = [\rho_{i,h}]_{i \in A, h \in H}$ such that for any i and h, $\rho_{i,h} = \sum_{\mu(i)=h} \lambda_\mu$.

- Refer to λ as a **decomposition** of $\varrho(\lambda)$

- Note that $\varrho(\lambda)$ may have many decompositions

- **permutation matrix**: a random assignment matrix with entries 0 or 1. Note that each matching is equivalent to a permutation matrix and each permutation matrix is equivalent to a matching.

- **Birkhoff-von Neumann Theorem:** Every bistochastic matrix can be written as a convex combination of permutation matrices.

  Note: random assignment matrices may have many decompositions.

Example of two possible decompositions

	a	b	c	d
1	1/2	0	1/2	0
2	1/2	0	1/2	0
3	0	1/2	0	1/2
4	0	1/2	0	1/2

$=$

$\dfrac{1}{2}$

	a	b	c	d
1	1	0	0	0
2	0	0	1	0
3	0	1	0	0
4	0	0	0	1

$+\dfrac{1}{2}$

	a	b	c	d
1	0	0	1	0
2	1	0	0	0
3	0	1	0	0
4	0	0	0	1

	a	b	c	d
1	1/2	0	1/2	0
2	1/2	0	1/2	0
3	0	1/2	0	1/2
4	0	1/2	0	1/2

$= \dfrac{1}{4}$

	a	b	c	d
1	1	0	0	0
2	0	0	1	0
3	0	1	0	0
4	0	0	0	1

$+\dfrac{1}{4}$

	a	b	c	d
1	0	0	1	0
2	1	0	0	0
3	0	1	0	0
4	0	0	0	1

$+\dfrac{1}{4}$

	a	b	c	d
1	0	0	1	0
2	1	0	0	0
3	0	0	0	1
4	0	1	0	0

$+\dfrac{1}{4}$

	a	b	c	d
1	1	0	0	0
2	0	0	1	0
3	0	0	0	1
4	0	1	0	0

Implementing Random Assignments

- Implementing random assignments is nontrivial since assignments need to be "correlated."
- Consider assigning 3 goods a, b, c to 3 agents $1, 2, 3$.

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

- Birkhoff-von Neumann Theorem shows that any bistochastic matrix can be written as a convex combination of permutation matrices.
- Therefore, any random assignment can be implemented as a lottery over deterministic assignments when assigning n goods to n agents, with each agent getting exactly one good.

Lottery Mechanisms

A lottery λ is **ex-post efficient** if it gives positive probability to only Pareto-efficient matchings.

A lottery mechanism Φ is **strategy-proof** if for every agent, it is strategy-proof for all von Neumann-Morgenstern utility functions that represents her preferences.

That is, for any agent $a \in A$, for any two preference relations $\succ_a, \succ_a'$, any preference profile $\succ_{-a}$, and any utility function u compatible with $\succ_a$, we have

$$\sum_\mu \Phi\left[\succ_a, \succ_{-a}\right]_\mu u\left(\mu\left(a\right)\right) \geq \sum_\mu \Phi\left[\succ_a', \succ_{-a}\right]_\mu u\left(\mu\left(a\right)\right)$$

- Expected utility from lying cannot exceed expected utility from telling the truth
- Sometimes called **strongly strategy-proof**

A lottery mechanism Φ **treats equals equally** (**equal treatment of equals – ETE property**) if for any two agents $a, b \in A$ with $\succ_a = \succ_b$ we have

$$\varrho\left(\Phi\left[\succ\right]\right)_a = \varrho\left(\Phi\left[\succ\right]\right)_b$$

In words, it gives the same random assignment vector (i.e. the vector of allocation probabilities of houses) to two agents who have the same preferences

- in house allocation problems, ETE can only be satisfied by lottery mechanisms

Two mechanisms

- **Random serial dictatorship**: randomly choose an ordering of agents
  - RSD Π is strongly strategy-proof, ex-post efficient, neutral, and treats equals equally.

- **Core from random endowments**: Randomly distribute houses to agents. Then find the core of this induced housing market

- How many ways to randomly assign endowments?

An Equivalence of Lottery Mechanisms

Theorem (Abdulkadiroğlu and Sönmez 1998): Core from random endowments and random serial dictatorship are equivalent.

Sketch of the Proof: Both mechanisms induce a probability distribution on matchings.

Since there are an equal number of initial endowments as orderings, can we find a ONE-TO-ONE and ONTO function $f : \mathcal{M} \to \mathcal{F}$ such that the core found through endowment μ (denoted ψ^μ) is equal to serial dictatorship outcome π^f?

The bijective property of f makes sure that the two lottery mechanisms are equivalent

Lotteries in Hybrid Ownership Markets

- Consider the case where: $|H_V| = |I_N|$ (so that there are same number of agents and houses).
- Simpler PE, IR and S-P mechanism:
    1. Construct an initial allocation by
        1.1 assigning each existing tenant her own house and
        1.2 randomly assigning the vacant houses to newcomers with uniform distribution, and
    2. choose the core of the induced housing market to determine the final outcome.
- **Theorem** (Sönmez & Ünver 2005): The above mechanism is equivalent to an extreme case of TTC where newcomers are randomly ordered first and existing tenants are randomly ordered next.

Equivalence results

These results have been extended to richer domains, sometimes using different proof techniques

- Pathak and Sethuraman (2011): Multiseat object assignment, particular version of TTC vs. RSD; motivated by policy debate in school assignment

- Carroll (2014): closer to Papai's setting, where priority of objects can depend on agents assignment

- Ekici (2013): house allocation with existing tenants

- Lee and Sethuraman (2014): Generalizes all existing results, and extends to include inheritance tree priority structure (with simpler proof)

- What's missing here?

Example: $A = \{1, 2, 3, 4\}, H = \{a, b, c, d\}$

Preferences are given as:

$$\text{agents 1 and 2: } a \succ_i b \succ_i c \succ_i d$$

$$\text{agents 3 and 4: } b \succ_i a \succ_i d \succ_i c.$$

RSD yields:

	a	b	c	d
1	5/12	1/12	5/12	1/12
2	5/12	1/12	5/12	1/12
3	1/12	5/12	1/12	5/12
4	1/12	5/12	1/12	5/12

The following random assignment is preferred by every agent to the above random assignment for any compatible VNM utility functions:

	a	b	c	d
1	1/2	0	1/2	0
2	1/2	0	1/2	0
3	0	1/2	0	1/2
4	0	1/2	0	1/2

Other efficiency notions

Let u_i be a VNM utility function representing preferences $\succ_i$ for each agent i.
Expected utility of a lottery λ is simply denoted by

$$U_i(\lambda) = \sum_{\mu \in M} \lambda_\mu u_i(\mu(i)).$$

A lottery λ is **ex-ante efficient** if there is no lottery γ such that
$U_i(\gamma) \geq U_i(\lambda)$ for every agent i and $U_i(\gamma) > U_i(\lambda)$ for some agent i.

If we do not want to rely on the specific cardinal VNM utility function
defined, Bogomolnaia and Moulin defined the concept of ordinal efficiency

Ordinal Mechanisms

Q: Are there any issues with eliciting cardinal utilities in direct mechanisms in settings without transfers?

BM (2001): "The restriction to ordinal mechanisms is the central assumption in this paper. It can be justified by the limited rationality of the agents participating in the mechanism. There is convincing experimental evidence that the presentation of preferences over uncertain outcomes by vNM utility functions is inadequate (e.g., Kagel and Roth 1995). One interpretation of this literature is that the formulation of rational preferences over a given set of lotteries is a complex process that most agents do not engage into if they can avoid it. An ordinal mechanism allows the participants to formulate only part of their preferences that does not require to think about the choices over lotteries."

A random assignment matrix $\varrho = [\rho_{i,h}]_{i \in A, h \in H}$ **first-order stochastically dominates** another random assignment matrix $\varrho^* = \left[\rho_{i,h}^*\right]_{i \in A, h \in H}$ according to $\succ_i$ if for any $h \in H$,

$$\sum_{g \in H: g \succsim_i h} \rho_{i,g} \geq \sum_{g \in H: g \succsim_i h} \rho_{i,g}^*$$

and for some $h \in H$

$$\sum_{g \in H: g \succsim_i h} \rho_{i,g} > \sum_{g \in H: g \succsim_i h} \rho_{i,g}^*$$

Denote by

$$\rho \succ_i \rho^*$$

and $\succsim_i$ as weak counterpart.

A random assignment matrix ϱ is **ordinally efficient** if there is no random assignment ϱ^* such that $\rho^* \succsim_i \rho$ for every agent i and $\rho^* \succ_i \rho$ for some agent i.

- **Example** (Bogomolnaia & Moulin [2001]): $I = \{1, 2, 3, 4\}$, $H = \{a, b, c, d\}$. The preferences are as follows:

$$\text{agent } 1: \quad a \succ_1 b \succ_1 c \succ_1 d$$
$$\text{agent } 2: \quad a \succ_2 b \succ_2 c \succ_2 d$$
$$\text{agent } 3: \quad b \succ_3 a \succ_3 d \succ_3 c$$
$$\text{agent } 4: \quad b \succ_4 a \succ_4 d \succ_4 c$$

RSD induces the following random assignment:

	a	b	c	d
1	5/12	1/12	5/12	1/12
2	5/12	1/12	5/12	1/12
3	1/12	5/12	1/12	5/12
4	1/12	5/12	1/12	5/12

$P =$ (rows 1–4 above)

Next consider the lottery

$$\mathcal{L} = 0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & c & b & d \end{pmatrix} + 0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ c & a & d & b \end{pmatrix}$$

which induces the random assignment

		a	b	c	d
	1	1/2	0	1/2	0
$Q =$	2	1/2	0	1/2	0
	3	0	1/2	0	1/2
	4	0	1/2	0	1/2

* the random assignment P assigns everyone their 1st choices with $5/12$ probability, 2nd choices with $1/12$ probability, 3rd choices with $5/12$ probability and 4th choices with $1/12$ probability

* whereas Q assigns everyone their 1st choices with $1/2$ probability and 3rd choices with $1/2$ probability.

  Hence Q stochastically dominates P and therefore RSD is not ordinally efficient.

Relationships between Efficiency Notions

- Ordinal efficiency $\Rightarrow$ ex-post efficiency, but not vice versa
- Ex-ante efficiency under a particular a utility profile implies ordinal efficiency, but not all ordinally efficient lotteries are ex-ante efficient under a particular utility profile

- Birkhoff-von Neumann Theorem allows us to work in the domain of random allocations and not worry about the lotteries that implement them
  - Allows us to study mechanisms through the random assignment matrix (without loss of generality)
- Bogomolnania and Moulin define a class of random assignment mechanisms that are ordinally efficient

Probabilistic Serial Mechanism

(Cake) eating algorithm:

- An algorithm is identified by a speed function vector $\omega = (\omega_i(.))_{i \in A}$ and time horizon is 1.
- Each agent is represented by a mouse and each house is represented by a cake of size 1.
- Each mouse i simultaneously starts eating her most favorite cake at a speed $\omega_i(t)$ such that each mouse has the capacity of eating 1 whole cake i.e. $\int_0^1 \omega_i(t)\, dt = 1$.
- When one (or more) cake ends, each mouse i eats her remaining favorite cake using the speed function ω_i.
- The previous step is repeated until all cakes are consumed.

A random assignment matrix is found by determining the fraction of the each cake eaten by each mouse.

Example: Every mouse eats at the same speed at all times: $\omega_i(t) = 1$ for all $i \in H$ and all $t \in [0, 1]$.

Set of mice: $A = \{1, 2, 3, 4\}$, set of cakes: $H = \{a, b, c, d\}$, preferences

$$\text{agents 1 and 2: } a \succ_i b \succ_i c \succ_i d$$

$$\text{agents 3 and 4: } b \succ_i a \succ_i d \succ_i c.$$

	a	b	c	d
1	1/2	0	1/2	0
2	1/2	0	1/2	0
3	0	1/2	0	1/2
4	0	1/2	0	1/2

Example: For any $i \in \{1, 2\}$:

$$\text{agents 1 and 2: } a \succ_i b \succ_i c \succ_i d$$

$$\text{agent 3: } c \succ_3 a \succ_3 b \succ_3 d.$$

$$\text{agent 4: } a \succ_4 c \succ_4 d \succ_4 b$$

The resulting random assignment is

	a	b	c	d
1	1/3	4/9	0	2/9
2	1/3	4/9	0	2/9
3	0	1/9	2/3	2/9
4	1/3	0	1/3	1/3

- When eating speeds are the same $\omega_i(t) = 1$ for all i, we have the symmetric PS mechanism

- Satisfies equal treatment of equals

Properties of PS

Bogomolnania and Moulin establish the following:

1 Each PS mechanism is ordinally efficient. For every house allocation problem, and every ordinally efficient random assignment, there exists a PS mechanism that achieves it.

2 There is no mechanism that satisfies *ordinal efficiency, strategy-proofness* and *equal treatment of equals*.

3 PS mechanism is not strategy-proof.

   Intuition: If too many people like the second choice of an agent as their first choice, then this agent may be better off if he starts eating from his second choice instead of his first choice.

- Example (continued): if $i = 3$ submits

$$\text{agent } 3 : a \succ'_3 c \succ'_3 b \succ'_3 d$$

instead of $\succ_3$, then we obtain:

	a	b	c	d
1	1/4	1/2	0	1/4
2	1/4	1/2	0	1/4
3	1/4	0	1/2	1/4
4	1/4	0	1/2	1/4

Consider the following utility function for agent 3,

$$u_3(c) = 5, u_3(a) = 4, u_3(b) = 0.1, u_3(d) = 0,$$

then

$$U_3\left(\text{PS}\left[\succ\right]_3\right) = 5 \times 2/3 + 0.1 \times 1/9 = 3.3444$$
$$< U_3\left(\text{PS}\left[\succ'_3, \succ_{-3}\right]_3\right) = 4 \times 1/4 + 5 \times 1/2 = 3.5$$

Since there exists at least one such utility function, PS is not strategy-proof.

PS becomes strategy-proof in large markets

Theorem (Kojima and Manea 2009)

Fix agent i's utility function u_i, and assume u_i represents strict preferences. There is a finite bound M such that, if $q_a \geq M$ for all $a \in O$, then truthtelling is a dominant strategy for i under PS. The conclusion holds no matter how many other agents are participating in the market.

Remark Truthtelling is an <u>exact</u> dominant strategy in a <u>finitely</u> large markets.

The bound M can be reasonably small: Consider a school context, where a student finds only 10 schools acceptable, and her utility difference between any two consecutively ranked schools is constant. Then truthtelling is a dominant strategy for her in PS if each school has at least 18 seats.

Intuition of the Theorem

Manipulations have two effects:

(1) given the same set of available objects, reporting false preferences may prevent the agent from eating his most preferred available object

(2) reporting false preferences can affect **expiration dates** of each good.

   (1) always hurts the manipulating agent, while (2) can benefit the agent. Intuitively, the effect (2) becomes small as the market becomes large.

   A nontrivial part of the formal proof is that (2) becomes very small relative to (1) when the copies of each object type becomes large, so the agents hurt themselves in total.

Che-Kojima: Asymptotic Equivalence

Consider a sequence of economies, where a **q-economy** is composed of

- q copies of each (real) good and infinite copies of n, and
- Set of agents: only assume

$$\frac{(\text{number of agents with preference } \pi \text{ in } q\text{-economy})}{q}$$

converges as $q \to \infty$ for every preference π (the limit can be zero).

Theorem (Che and Kojima 2010). Fix the set of types of goods. The random assignments in RP and PS converge to each other as $q \to \infty$.

Formally, $\lim_{q \to \infty} \max_{\pi, a} |RP_a^q(\pi) - PS_a^q(\pi)| = 0$, where

$RP_a^q(\pi) := \Pr[\text{agents with preference } \pi \text{ get } a \text{ in } q\text{-economy in RP}]$,
$PS_a^q(\pi) := \Pr[\text{agents with preference } \pi \text{ get } a \text{ in } q\text{-economy in PS}]$.

Intuition of the Theorem (1)

- In PS, the random assignment is pinned down by the **expiration dates** of the goods. Expiration date T_a^q of good a is the time at which a is completely consumed away.

- The probability that an agent receives good a is duration of consuming good a, so

$$\max\{T_a^q - \max\{T_b^q | b \text{ is preferred to } a\}, 0\}.$$

probability of receiving a
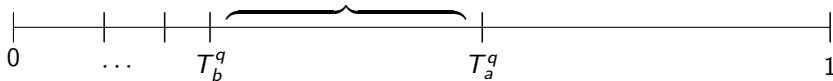
Intuition of the Theorem (2)

- Proof Idea: Find *RP*-analogues of expiration dates, and show that they converge to expiration dates in PS (in probability).
- Alternative formulation of RP.
    1. Each agent draws a number iid uniformly distributed in $[0, 1]$.
    2. The agent with the smallest draw receives her favorite good, and so on.
- Given realized draws, the cutoff $\hat{T}_a^q$ of good a under *RP* is the draw of the agent who receives the last copy of a.

- Since random draws are uniform over $[0,1]$, an agent will receive good a with probability

$$E[\max\{\hat{T}_a^q - \max\{\hat{T}_b^q | b \text{ is preferred to } a\}, 0\}].$$

draws such that the agent receives a

Intuition of the Theorem (3)

- Show cutoffs of RP converge to expiration dates of PS (in probability).
- They are different in general: In PS, a good is consumed proportionately to the number of agents who like it: In RP, a good may be consumed disproportionately to the number of agents who like it because of the randomness of draws.
- For RP in large markets, the law of large numbers kicks in: with a very high probability, a good is consumed almost proportionately to the number of agents who like that good best among available goods.
- The formal proof makes this intuition precise.

Nature of Ordinal Inefficiency in RP

- For any finite size, RP and PS may not be exactly equivalent.
- Consider a family of replica economies (i.e. agents of each preference type increase proportionately to q).

Proposition

In replica economies, if RP is ordinally efficient/inefficient in the base economy (i.e. $q = 1$), then RP is ordinally efficient/inefficient for all replicas.

- Thus, inefficiency of RP does not disappear completely in any finite replica economy, if RP is inefficient in the base economy.
- But the theorem says that the "magnitude" of ordinal inefficiency vanishes as markets become large.
- Manea (2008): Probability that RP fails exact ordinal efficiency goes to zero as the market becomes large.

Can we do better?

Theorem (Zhou 1990 JET): For $n \geq 3$, there is no mechanism that is anonymous, strategy-proof, and ex-ante efficient.

We say that a mechanism φ dominates ψ if for all preferences, it assigns a lottery that each agent weakly prefers in a first-order stochastic dominance sense, and there is some problem and agent for whom the preference is strict

Theorem (Erdil 2014 JET): The outcome of a random serial dictatorship is dominated within the class of ex-post efficient strategy-proof mechanisms which satisfy equal treatment of equals.

Erdil constructs a mechanism exploiting this basic intuition of the following problem. Preferences are given by:

$$1 : a \succ b$$
$$2 : c \succ a$$
$$3 : c \succ b$$
$$4 : c$$

If lottery is $4 - 3 - 2 - 1$ or $4 - 2 - 3 - 1$, then

$$\mu = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & a & b & c \end{array} \right)$$

If lottery is $3 - 1 - 2 - 4$ or $3 - 1 - 4 - 2$, then

$$\nu = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ a & 2 & c & 4 \end{array} \right)$$

Hence, with probability $\frac{1}{12}$, agent 1 is not assigned and b is not assigned to anyone

Replace μ and ν with

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ b & a & c & 4 \end{array} \right) + \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ a & 2 & b & c \end{array} \right)$$

with probability $\frac{1}{12}$. Assigns an extra $\frac{1}{12}$ of b to 1, holding everything else the same

- If 1's preference were $bP_1'a$, her assignment would be increased by $\frac{1}{12}$ of a due to symmetry of R_{-1}. Hence, 1 will not want to deviate

- Improvement treats agent 1 differently. To recover ETE, define analogous improvements for other agents when the names of agents are permuted, and randomize over such mechanisms with equal probability

- <u>Issue</u>: constructions of strategy-proof improvement like this are computationally demanding and possibly intractable for large markets

Lecture Wrap up

- Allocation of indivisible objects
- Collective Ownership
- Individual ownership
- Hybrid ownership structure
- Lottery mechanisms

Two-Sided Matching and Medical Labor Markets

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MIT and NBER

January 2018

From Theory to Practice

- Study of matching started as "pure" theory: first by David Gale and Lloyd Shapley (1962) who introduced DA

- Roth (1984) is a landmark paper, which observed
  - ▶ Since the 1950s, US hospitals have used a clearinghouse to assign graduating medical students to residencies.
  - ▶ Students apply and interview at hospitals in the fall, then students and hospitals submit rank-order preferences in February.
  - ▶ A computer algorithm is used to assign students to hospitals, and matches are all revealed on a single day: match day.
  - ▶ Roth realized that the doctors has independently discovered and were using exactly the Gale and Shapley DA algorithm!

History of NRMP

- 1900-1945: Unravelling of Appt. Dates

- 1945-1950: Chaotic Recontracting $\Rightarrow$ centralized mechanism introduced in response

- 1950-197x: High rates of orderly participation (95%) in centralized clearing house

- 197x-198x: Declining rates of participation, particularly among married couples

- 198x-present: Married couples return, following changes in algorithm to accomodate couples and other match variations

- Why might a centralized clearinghouse be valuable? Does its design matter?

TABLE I

Stable and Unstable (Centralized) Mechanisms

Market	Stable	Still in use (halted unraveling)
American medical markets		
NRMP	yes	yes (new design in '98)
Medical Specialties	yes	yes (about 30 markets)
British Regional Medical Markets		
Edinburgh ('69)	yes	yes
Cardiff	yes	yes
Birmingham	no	no
Edinburgh ('67)	no	no
Newcastle	no	no
Sheffield	no	no
Cambridge	no	yes
London Hospital	no	yes
Other healthcare markets		
Dental Residencies	yes	yes
Osteopaths (<'94)	no	no
Osteopaths (≥'94)	yes	yes
Pharmacists	yes	yes
Other markets and matching processes		
Canadian Lawyers	yes	yes (except in British Columbia since 1996)
Sororities	yes (at equilibrium)	yes

Figure: Roth (2002)

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One-to-One Matching: Marriage Problems

Marriage problem is a triple $\langle M, W, R \rangle$ where

$M = \{m_1, ..., m_p\}$ is a set of men

$W = \{w_1, ..., w_q\}$ is a set of women

$R = (R_{m_1}, \ldots, R_{m_p}, R_{w_1}, \ldots, R_{w_q})$ is a list of preferences

R_m: Preference relation over $W \cup \{m\}$
R_w: Preference relation over $M \cup \{w\}$

P_m, P_w: Strict preferences derived from R_m, R_w

Consider man m:

- wP_mw': man m prefers woman w to woman w'
- wP_mm: man m prefers woman w to remaining single
- mP_mw: woman w is **unacceptable** for man m

Similar notation for women.

Assumption. Unless otherwise mentioned all preferences are strict.

Matching

The outcome of a marriage problem is a **matching**.

Formally a matching is a function $\mu : M \cup W \to M \cup W$ such that:

1. $\mu(m) \notin W \Rightarrow \mu(m) = m$ for all $m \in M$,
2. $\mu(w) \notin M \Rightarrow \mu(w) = w$ for all $w \in W$, and
3. $\mu(m) = w \Leftrightarrow \mu(w) = m$ for all $m \in M$, $w \in W$.

Assumption. There are no **consumption externalities**: An individual prefers a matching μ to a matching ν if and only if he/she prefers $\mu(i)$ to $\nu(i)$.

A matching μ is **Pareto efficient** if there is no other matching ν such that $\nu(i)R_i\mu(i)$ for all $i \in M \cup W$ and $\nu(i)P_i\mu(i)$ for some $i \in M \cup W$.

Stability

A matching μ is **blocked by an individual** $i \in M \cup W$ if $iP_i\mu(i)$.

A matching is **individually rational** if it is not blocked by any individual.

A matching μ is **blocked by a pair** $(m, w) \in M \times W$ if they each prefer each other to their partners under μ, i.e.

$$wP_m\mu(m) \text{ and } mP_w\mu(w).$$

A matching is **stable** if it is not blocked by any individual or a pair.

Q: What is the relationship between stability and Pareto efficiency?

The **men-proposing deferred acceptance algorithm** gives a stable matching for each marriage problem:

1. Each man m proposes to his first choice (if he has any acceptable choices).

   Each woman rejects any offer except the best acceptable proposals and "holds" the most preferred acceptable proposal (if any).

k. Any man who was rejected at step $k - 1$ makes a new proposal to his most preferred acceptable potential mate who has not yet rejected him. (If no acceptable choices remain, he makes no proposal.)

   Each woman "holds" her most preferred acceptable proposal to date, and rejects the rest.

   Algorithm terminates when there are no more rejections. Each woman is matched with the man he has been holding in the last step. Any woman who has not been holding an offer or any who was rejected by all acceptable women remains single.

Proof that algorithm produces a matching:

- Women get (weakly) better off as the process goes on, and men get (weakly) worse off as the process goes on.

- The algorithm eventually stops producing a matching μ (since a woman never holds more than one offer).

Proof that matching μ is stable:

- It cannot be blocked by any individual, since men do not make any offers to unacceptable women, and women immediately reject unacceptable offers.

- Suppose pair (m, w) blocks μ. Then since $w P_m \mu(m)$, man m has made an offer to w in the algorithm and since they are not matched with each other w rejected m in favor of someone better.

  But w gets weakly better throughout the algorithm, contradicting pair (m, w) blocks μ. $\diamondsuit$

Core & Stability

Definition: A matching μ is in the **core** (by strong domination) if there exists no matching ν and coalition $T \subseteq M \cup W$ such that $\nu(i)P_i\mu(i)$ and $\nu(i) \in T$ for any $i \in T$.

Theorem: The set of stable matchings is equal to the core.

Stable Matchings: Three Main results

1) **Side-optimality**: There exists a man-optimal stable matching (μ^M) that every man likes at least as well as any other stable matching. Furthermore, the outcome of the man-proposing deferred acceptance algorithm gives the man-optimal stable matching.

2) **Opposing interests**: The man-optimal stable matching is the worst stable matching for each woman. Similarly, the woman-optimal stable matching is the worst stable matching for each man.

3) **Rural hospitals**: The set of agents who are matched is the same for all stable matchings.

Proof of Side Optimality

Terminology: w is **achievable** for m if there is some stable matching μ such that $\mu(m) = w$.

Inductive step: Suppose that up to Step k of the man-proposing deferred acceptance algorithm, no man has been rejected by an achievable partner. (This clearly holds for k=1.)

Claim: No man is rejected by an achievable partner at Step k.

Suppose that up to step k of the algorithm, no man has been rejected by an achievable partner, and that at step k woman w rejects man m (who is acceptable to w) and (therefore) holds on to some m'.

Then w is not achievable for m. Why? Suppose, to the contrary, μ is stable such that $\mu(m) = w$.

But m' has already proposed at this step, so he prefers w. Thus, μ can't be stable: (m', w) would be a blocking pair.

Therefore, every man is matched with best achievable partner under the outcome of the man proposing deferred acceptance algorithm μ^M, meaning that μ^M is the man-optimal stable matching. $\diamond$

Proof of Opposing Interests

Suppose there is a man m and stable matching μ such that $\mu^W(m)P_m\mu(m)$.

Then man m is not single under μ^W. Let $w \equiv \mu^W(m)$.

Since man m is not matched with the same woman under μ^W and μ, woman w is not matched with m under μ.

Therefore $\mu^W(w)P_w\mu(w)$ by the definition of woman-optimal stable matching. But then pair (m,w) block matching μ yielding the desired contradiction. $\diamond$

Proof of Rural Hospitals

Let μ be any stable matching. Observe that

1. $|\nu(W)| = |\nu(M)|$ at any matching ν (stable or not)

2. $|\mu^M(W)| \geq |\mu(W)| \geq |\mu^W(W)|$ since any man who is matched under μ^W should also be matched under μ and any man who is matched under μ should also be matched under μ^M

3. $|\mu^W(M)| \geq |\mu(M)| \geq |\mu^M(M)|$ since any woman who is matched under μ^M should also be matched under μ and any woman who is matched under μ should also be matched under μ^W

Hence

$$|\mu^M(W)| = |\mu(W)| = |\mu^W(W)| = |\mu^W(M)| = |\mu(M)| = |\mu^M(M)|$$

- Therefore, since $|\mu^M(W)| = |\mu(W)|$ and any man who is matched under μ should also be matched under μ^M, the same set of men should be matched under μ and μ^M.

- But μ is an arbitrary stable matching, so the set of men who are matched is the same for all stable matchings.

- A similar argument is valid for women. $\diamondsuit$

Let μ, μ' be two stable matchings.

- **Join**: $\mu \vee_M \mu' : M \cup W \to M \cup W$ assigns each man the more preferred of his two assignments under μ and μ' and each women the less preferred of his two assignments under μ and μ'.

- **Meet**: $\mu \wedge^M \mu' : M \cup W \to M \cup W$, defined analogously by reversing the preferences

  Lattice: partially ordered set where any two elements have unique join and meet

  Lattice Theorem (Conway): If μ and μ' are *stable* matchings, then not only are the functions $\mu \vee^M \mu'$ and $\mu \wedge^M \mu'$ both matchings, but they are also both stable.

Example:

There are 3 man and 3 woman with preferences:

$$
\begin{array}{ll}
P_{m_1}: \quad w_1 \ w_2 \ w_3 \ m_1 & \qquad P_{w_1}: \quad m_2 \ m_3 \ m_1 \ w_1 \\
P_{m_2}: \quad w_2 \ w_3 \ w_1 \ m_2 & \qquad P_{w_2}: \quad m_3 \ m_1 \ m_2 \ w_2 \\
P_{m_3}: \quad w_2 \ w_1 \ w_3 \ m_3 & \qquad P_{w_3}: \quad m_1 \ m_2 \ m_3 \ w_3
\end{array}
$$

Consider the following two matchings:

$$
\mu = \left(\begin{array}{ccc} m_1 & m_2 & m_3 \\ w_1 & w_2 & w_3 \end{array} \right) \quad \text{and} \quad \nu = \left(\begin{array}{ccc} m_1 & m_2 & m_3 \\ w_3 & w_1 & w_2 \end{array} \right)
$$

Then the join and meet of μ, ν are as follows:

$$
\begin{aligned}
\mu \vee^M \nu &= \left(\begin{array}{cccccc} m_1 & m_2 & m_3 & w_1 & w_2 & w_3 \\ w_1 & w_2 & w_2 & m_1 & m_2 & m_3 \end{array} \right) \\
\mu \wedge^M \nu &= \left(\begin{array}{cccccc} m_1 & m_2 & m_3 & w_1 & w_2 & w_3 \\ w_3 & w_1 & w_3 & m_2 & m_3 & m_1 \end{array} \right)
\end{aligned}
$$

Neither is a matching!

Can Men Receive Better Mates than at Man-Optimal Stable Matching?

If we can match a man with a woman who finds him unacceptable, then there may be a matching ν where all man receive better mates than under μ^M.

If that is not possible (i.e. if we are seeking an *individually rational matching*) then some man can receive better mates without hurting any man but not all man can receive better mates.

Theorem: There is no individually rational matching ν where $\nu(m)P_m\mu^M(m)$ for all $m \in M$.

See Roth and Sotomayor, Chapter 2 for a proof

Example. There are 3 men and 3 women with the following preferences:

$$
\begin{array}{ll}
P_{m_1}: \quad w_1 \ w_2 \ w_3 \ m_1 & \qquad P_{w_1}: \quad m_2 \ m_1 \ m_3 \ w_1 \\
P_{m_2}: \quad w_2 \ w_1 \ w_3 \ m_2 & \qquad P_{w_2}: \quad m_1 \ m_3 \ m_2 \ w_2 \\
P_{m_3}: \quad w_1 \ w_2 \ w_3 \ m_3 & \qquad P_{w_3}: \quad m_1 \ m_2 \ m_3 \ w_3
\end{array}
$$

Here

$$
\mu^M = \left(\begin{array}{ccc} m_1 & m_2 & m_3 \\ w_2 & w_1 & w_3 \end{array} \right)
$$

Although both m_1 and m_2 prefer and m_3 is indifferent for the following matching:

$$
\left(\begin{array}{ccc} m_1 & m_2 & m_3 \\ w_2 & w_1 & w_3 \end{array} \right)
$$

McVitie-Wilson on order-independence

Define two operations:

- PROPOSAL: make the next proposal of some man i. When complete, call REFUSAL for the woman to whom man i has just proposed
- REFUSAL: decides for woman j whether or not to refuse a new proposal from man i or to refuse the man she has temporarily held and replace him by man i. When complete, call PROPOSAL.

Perform PROPOSAL for each man in turn

This algorithm performs the same proposals and rejections although in general in a different order; since assignment not finalized until the end, outcome is the same as the simultaneous proposing version

Theorem. The outcome of the men-proposing deferred acceptance algorithm is independent of the order of proposals.

Direct Mechanism

Notation:

$\mathcal{R}_i$: Set of all preference relations for agent i

$\mathcal{R} = \mathcal{R}_{m_1} \times \cdots \times \mathcal{R}_{m_p} \times \mathcal{R}_{w_1} \times \cdots \times \mathcal{R}_{m_q}$: Set of all

preference profiles

$\mathcal{R}_{-i}$: Set of all preference profiles for all agents except agent i

$\mathcal{M}$: Set of all matchings

A **mechanism** is a systematic procedure which determines a matching for each marriage problem; a function $\varphi : \mathcal{R} \to \mathcal{M}$.

Stable Mechanism

A mechanism φ is **stable** if $\varphi(R)$ is stable for any $R \in \mathcal{R}$.

Similarly a mechanism is **Pareto efficient** if it always selects a Pareto efficient matching and it is **individually rational** if it always selects an individually rational matching.

Any stable mechanism is both Pareto efficient and also individually rational.

ϕ^M : Mechanism that selects the men-optimal stable

matching for each problem

ϕ^W : Mechanism that selects the women-optimal stable

matching for each problem

Incompatibility of Stability & Strategy-Proofness

Theorem (Roth 1982): There exists no mechanism that is both *stable* and **strategy-proof**.

Proof : Consider the following 2 men, 2 women problem with the following preferences.

$$R_{m_1} : \quad w_1 \ w_2 \ m_1 \qquad\qquad R_{w_1} : \quad m_2 \ m_1 \ w_1$$
$$R_{m_2} : \quad w_2 \ w_1 \ m_2 \qquad\qquad R_{w_2} : \quad m_1 \ m_2 \ w_2$$

In this problem there are only two stable matchings:

$$\mu^M = \left(\begin{array}{cc} m_1 & m_2 \\ w_1 & w_2 \end{array} \right) \quad \text{and} \quad \mu^W = \left(\begin{array}{cc} m_1 & m_2 \\ w_2 & w_1 \end{array} \right)$$

Let φ be any stable mechanism. Then $\varphi[R] = \mu^M$ or $\varphi[R] = \mu^W$.

If $\varphi[R] = \mu^M$ then woman w_1 can report a fake preference R'_{w_1} where only her top choice m_2 is acceptable and enforce her favorite stable matching μ^W to be selected by φ since it is the only stable matching for the manipulated economy (R_{-w_1}, R'_{w_1}).

If, on the other hand, $\varphi[R] = \mu^W$ then man m_1 can report a fake preference R'_{m_1} where only his top choice w_1 is acceptable and enforce his favorite stable matching μ^M to be selected by φ since it is the only stable matching for the manipulated economy (R_{-m_1}, R'_{m_1}). $\diamond$

Incentives Facing the Men Under the Men-Optimal Stable Mechanism ϕ^M

Theorem (Dubins & Freedman 1981, Roth 1982): Truth-telling is a dominant strategy for all men under the men-optimal stable mechanism.

- For any man, any strategy which agrees with truth-telling for the set of acceptable women as well as their relative ranking is also a dominant strategy. We consider any such strategy also as truth-telling (since the relative ranking of unacceptable women are irrelevant under any individually rational mechanism). Any other strategy is a dominated strategy.

- Truth-telling is a dominant strategy for all women under the women-optimal stable mechanism.

Proof of Strategy-proofness

- Fix the reports of all the women and all but one man.

  - <u>Plan</u>: Show that whatever report the man m starts with, he can make a series of (weak) improvements leading to a truthful report.

- Suppose man m is considering a strategy that leads to a match μ where he gets w. Each of the following changes improves his outcome:

1) Reporting that w is his only acceptable woman.

   - μ is still unblocked.
   - By rural hospitals, m must be matched, and so must get w.

2) Reporting honestly, but truncating at w.

   - m being unmatched is still blocked (because it was blocked if m reported just w), so m must do at least as well as w.

3) Reporting honestly with no truncation.

   - This won't affect DA relative to above strategy.

Many-to-One Matching: College Admissions Problems

College problem is a four-tuple $\langle C, S, q, R \rangle$ where

$C = \{c_1, ..., c_m\}$ is a set of colleges

$S = \{s_1, ..., s_n\}$ is a set of students

$q = (q_{c_1}, \ldots, q_{c_m})$ is a vector of college capacities

$R = (R_{c_1}, \ldots, R_{c_m}, R_{s_1}, \ldots, R_{s_n})$ is a list of preferences

R_s: Preference relation over $C \cup \{\emptyset\}$

R_c: Preference relation over 2^S (i.e. sets of students)

P_c, P_s: Strict preferences derived from R_c, R_s

College Preferences: Responsiveness

Assume that whether a student is acceptable for a college or not does not depend on other students in her class. Similarly, we will assume that the relative desirability of students does not depend on the composition of the class. This property is known as **responsiveness** (Roth 1985).

Formally, college preferences R_c is responsive iff

1. for any $T \subset S$ with $|T| < q_c$ and any $s \in S \setminus T$,

$$(T \cup \{s\}) \, P_c \, T \quad \Leftrightarrow \quad \{s\} \, P_c \, \emptyset, \text{ and}$$

2. for any $T \subset S$ with $|T| < q_c$ and any $s, s' \in S \setminus T$,

$$(T \cup \{s\}) \, P_c \, (T \cup \{s'\}) \quad \Leftrightarrow \quad \{s\} \, P_c \, \{s'\}.$$

Matching

The outcome of a college admissions problem is a **matching**.

Formally a matching is a *correspondence* $\mu : C \cup S \Longrightarrow C \cup S \cup \emptyset$ such that:
1. $\mu(c) \subseteq S$ such that $|\mu(c)| \leq q_c$ for all $c \in C$,
2. $\mu(s) \subseteq C$ such that $|\mu(s)| \leq 1$ for all $s \in S$, and
3. $s \in \mu(c)$ if and only if $\mu(s) = \{c\}$ for all $c \in C$ and $s \in S$.

Stability

A matching μ is **blocked by a college** $c \in C$ if there exists $s \in \mu(c)$ such that $\emptyset \, P_c \, s$.

A matching μ is **blocked by a student** $s \in S$ if $\emptyset \, P_s \, \mu(s)$.

A matching is **individually rational** if it is not blocked by any college or student.

A matching μ is **blocked by a pair** $(c, s) \in C \times S$ if

1. $c \, P_s \, \mu(s)$, and
2. 2.1 either there exists $s' \in \mu(c)$ such that $\{s\} \, P_c \, \{s'\}$, or
   2.2 $|\mu(c)| < q_c$ and $\{s\} \, P_c \, \emptyset$.

A matching is **stable** if it is not blocked by any agent or a pair.

College-Proposing Deferred Acceptance Algorithm

1. Each college c proposes to its top q_c acceptable students (and if it has less acceptable choices than q_c, then it proposes to all its acceptable students). Each student rejects any unacceptable proposals and, if more than one acceptable proposal is received, she "holds" the most preferred.

k. Any college c who was rejected at step $k-1$ by any student proposes to its most preferred q_c acceptable students who has not yet rejected it (and if among the remaining students there are fewer than q_c acceptable students, then it proposes to all).

Each students "holds" her most preferred acceptable offer to date, and rejects the rest.

Algorithm terminates when there are no more rejections.

Each student is matched with the college she has been holding in the last step.

Theorem (Gale & Shapley 1962): The college-proposing deferred acceptance algorithm gives a stable matching for each college admissions problem.

Many (although not all) results for marriage problems extend to college admissions problems. The following "trick" is often used to extend some of these results.

Given a college admissions problem $\langle C, S, q, R \rangle$, construct a **related marriage problem** as follows:

- "Divide" each college c_i into q_{c_i} seperate pieces $c_i^1, \ldots, c_i^{q_{c_i}}$ where each piece has a capacity of one; and let each piece have the same preferences over S as college c has. (Since college preferences are responsive, R_c is consistent with a unique ranking of students.)

  C^* : The resulting set of college "pieces" (or seats).

- For any student s, extend her preferences to C^* by replacing each college c_i in her original preferences R_s with the block $c_i^1, \ldots, c_i^{q_{c_i}}$ in that order.

So in the related marriage problem:

- each seat of a college c is an individual unit that has consistent preferences with college c,
- and students rank seats at different colleges as they rank the colleges whereas they rank seats at the same college based on the index of the seat.

Given a matching for a college admissions problem, it is straightforward to define a **corresponding matching** for its related marriage problem: Given any college c, assign students assigned to c in the original problem one at a time to pieces of c starting with lower index pieces.

Stability Lemma (Roth & Sotomayor 1989): A matching of a college admissions problem is stable if and only if the corresponding matching of its related marriage problem is stable.

Implications of the Stability Lemma

The Stability Lemma can be used to extend the following results for marriage problems to college admissions:

1. There exists a **student-optimal stable matching** μ^S that every student likes at least as well as any other stable matching. Furthermore, the outcome of the student-proposing deferred acceptance algorithm gives the student-optimal stable matching.

2. There exists a **college-optimal stable matching** μ^C that every college likes at least as well as any other stable matching. Furthermore, the outcome of the college-proposing deferred acceptance algorithm gives the college-optimal stable matching.

Indeed the following stronger result holds:

Theorem (Roth 1984): Consider any college $c \in C$. Among all students college c can be assigned under all stable matchings, it is assigned q_c of the best ones under μ^C.

Moreover,

1. The student-optimal stable matching is the worst stable matching for each college. Similarly the college-optimal stable matching is the worst stable matching for each student.
2. The set of students filled and the set of positions filled is the same at each stable matching.
3. The join as well as the meet of two stable matchings is each a stable matching.
4. There is no individually rational matching ν where $\nu(s)P_s\mu^S(s)$ for all $s \in S$.

New Results For College Admissions

Theorem (Roth 1986): Any college that does not fill all its positions at some stable matching is assigned precisely the same set of students at every stable matching.

Theorem (Roth & Sotomayor 1989): Let μ and ν be two stable matchings. For any college c,

- either $\{s\} \, P_c \, \{s'\}$ for all $s \in \mu(c) \setminus \nu(c)$ and $s' \in \nu(c) \setminus \mu(c)$, or
- $\{s'\} \, P_c \, \{s\}$ for all $s \in \mu(c) \setminus \nu(c)$ and $s' \in \nu(c) \setminus \mu(c)$.

A Different Result For College Admissions

As the following example shows, there can be an individually rational matching where each college receives a strictly better assignment than under the college-optimal stable matching.

Example: There are 2 colleges c_1, c_2 with $q_{c_1} = 2$, $q_{c_2} = 1$, and 2 students. The preferences are as follows:

$$
\begin{array}{llll}
R_{s_1}: & \{c_1\}\,\{c_2\} & R_{c_1}: & \{s_1, s_2\}\,\{s_2\}\,\{s_1\} \\
R_{s_2}: & \{c_2\}\,\{c_1\} & R_{c_2}: & \{s_1\}\,\{s_2\}
\end{array}
$$

Here both c_1 and c_2 strictly prefer ν to μ^C where:

$$
\mu^C = \left(\begin{array}{cc} c_1 & c_2 \\ s_1 & s_2 \end{array} \right) \quad \text{and} \quad \nu = \left(\begin{array}{cc} c_1 & c_2 \\ s_2 & s_1 \end{array} \right).
$$

College Admissions Problems: Incentives

Since any impossibility result obtained for a smaller class of problems immediately extends to a larger class. Therefore the following two results are immediate.

Theorem (Roth 1982): There exists no mechanism that is both *stable* and *strategy-proof*.

The following positive result is a direct implication of the Stability Lemma and the corresponding positive result for the marriage problem:

Theorem (Roth 1985): Truth-telling is a dominant strategy <u>for all students</u> under the student-optimal stable mechanism.

Incentives Facing Colleges under
Stable Mechanisms

For colleges, however, the situation is different.

Example: There are 2 colleges c_1, c_2 with $q_{c_1} = 2$, $q_{c_2} = 1$, and 2 students s_1, s_2. The preferences are as follows:

$$
\begin{array}{llll}
R_{s_1}: & \{c_1\} \, \{c_2\} & \quad R_{c_1}: & \{s_1, s_2\} \, \{s_2\} \, \{s_1\} \\
R_{s_2}: & \{c_2\} \, \{c_1\} & \quad R_{c_2}: & \{s_1\} \, \{s_2\}
\end{array}
$$

The only stable matching for this problem is:

$$
\mu^C = \left(\begin{array}{cc} c_1 & c_2 \\ s_1 & s_2 \end{array} \right)
$$

Now suppose college c_1 submits the manipulated preferences R'_{c_1} where only student s_2 is acceptable. For problem (R_{-c_1}, R'_{c_1}) the only stable matching is:

$$\left(\begin{array}{cc} c_1 & c_2 \\ s_2 & s_1 \end{array} \right).$$

Hence college c_1 benefits by manipulating its preferences under any stable mechanism (including the college-optimal stable mechanism).

Theorem (Roth 1985): There exists no stable mechanism where truth-telling is a dominant strategy for all colleges.

NRMP redesign: A new kind of core convergence

Roth and Peranson (1999):
Difference between hospital and doctor-proposing DA (no couples)

	1987	1993	1994	1995	1996
APPLICANTS					
# of Applicants Affected	20	16	20	14	21
Prefer Doctor-Proposing	12	16	11	14	12
Prefer Hospital-Proposing	8	0	9	0	9
PROGRAMS					
# of Programs Affected	20	15	23	15	19
Prefer Doctor-Proposing	8	0	12	1	10
Prefer Hospital-Proposing	12	15	11	14	9

Magnitude of conflict of interest/manipulations

Simulation on randomly generated data.

Simple model: n hospital programs, n doctors, (no couples).

Preferences are drawn independently and uniformly. Each doctor applies to k hospitals.

$C(n)$ = number of doctors matched differently at hospital-proposing and doctor-proposing DAs.

"Core non-convergence": $k = n \to \infty$
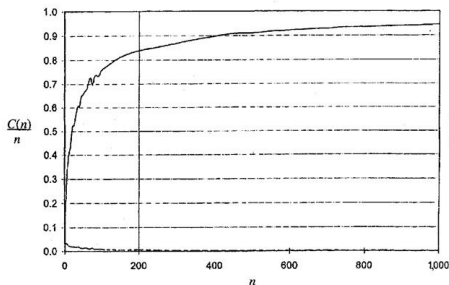
Figure: Roth and Peranson

$C(n)/n$: proportion of workers who receive different matches at different stable matchings, in a simple market with n workers and n firms, when each worker applies to k firms, each firm ranks all workers who apply, and preferences are uncorrelated.

As market grows, so does the set of stable matchings

"Core convergence": k fixed, $n \to \infty$
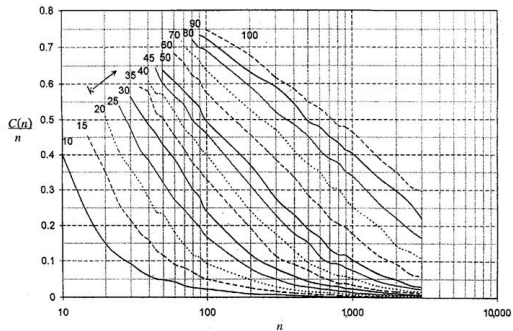
Figure: Roth and Peranson

As market grows for fixed k, the set of stable matchings shrinks.

Magnitude of possible manipulations by students

Upper limit of the number of applicants who could benefit by truncating their lists at one above their original match point.

Note: for students, truncation is known to be "exhaustive" (Roth and Vande Vate 1991).

	1987	1993	1994	1995	1996
Program-Proposing	12	22	13	16	11
Applicant-Proposing	0	0	2	2	9

- Applicants can manipulate due to match variations
- But, more applicants can manipulate program-proposing algorithm than the applicant-proposing algorithm.
- Both numbers are very small.

Magnitude of possible manipulations by hospitals

Upper limit of the number of hospital programs that could benefit by truncating their lists at one above their original match point (for hospitals, truncation is <u>not</u> exhaustive)

	1987	1993	1994	1995	1996
Program-Proposing	15	12	15	23	14
Applicant-Proposing	27	28	27	36	18

More programs are able to manipulate applicant-proposing algorithm.

Model of Large Matching Markets

There are constants $\bar{q}, \tilde{q}, k$ (independent of n). G^n is a game of incomplete information such that

- there are n colleges, with quota at most $\bar{q}$.
- there are at most $\tilde{q}n$ students.
- Preferences of colleges are common knowledge (the result holds under incomplete information as well). Utility $u_c(S')$ for college c of being matched with a set of students S' is additive:

$$u_c(S') \begin{cases} = \sum_{s \in S'} u_c(\{s\}) \text{ if } |S'| \leq q_c, \\ < 0 \text{ otherwise.} \end{cases}$$

$u_c(\{s\})$ is always positive (every student is acceptable). The value $\sup\{u_c(\{s\}) | n \in \mathbb{N}, s, c \text{ are in } G^n\}$ is finite.

- Preferences of students are private information. A student's preference list is drawn from a uniform distribution over preference lists of length k, independently across students (more general cases are analyzed in the paper.)

- Timing of the game: Students and colleges submit their preference lists and quotas simultaneously. DA is applied under the reported preferences.

Kojima and Pathak (2009)

Given $\varepsilon > 0$, a strategy profile is an ε-**Nash equilibrium** if no player gains more than ε by unilateral deviation.

Theorem

For any $\varepsilon > 0$, there exists n such that truth-telling by every agent is an ε-Nash equilibrium for any game with more than n colleges.

Kojima and Pathak (2009)

There is also a result regarding the "counting analysis" by Roth and Peranson.

Theorem

The expected proportion of colleges that can manipulate DA when others are truthful goes to zero as the number of colleges goes to infinity. The expected proportion of colleges that are matched to the same set of students in all stable matchings goes to one as the number of colleges goes to infinity.

DA is strategy-proof for students, so truthtelling is an optimal strategy for students.

Strategic rejection by a college causes a chain of application and rejections. Some of the rejected students may apply to the manipulating college, and the college may be made better off if these new applicants are desirable.

In a large market, there is a high probability that there will be many colleges with vacant positions. So the students who are strategically rejected (or those who are rejected by them and so on) are likely to apply to those vacant positions and be accepted. So the manipulating college is unlikely to be made better off.

Large Market Matching Models

- Azevedo-Leshno (2016): A Supply Demand Framework for Two-Sided Matching Markets, *Journal of Political Economy*, 124(5), 1235-1268.
  - ▶ Continuum model, tractable comparative statics

- Ashlagi-Kanoria-Leshno (2017): Unbalanced Random Matching Markets, *Journal of Political Economy*, 125(1), 69-98.
  - ▶ Random preference model, with improved proof; comparative statics by supply vs. demand

- Lee (2017): Incentive Compatibility of Large Centralized Matching Markets (random bipartite graphs), *Review of Economic Studies*, 84(1).

- Working papers by Che and Tercieux (2017); Lee and Yariv (2017): Stability vs. Efficiency

Complementarities: Couples in the labor market

- Major post-war transformation of US labor market

- Dual-career households raise issues of job coordination

- Nagging problem for medical clearinghouses

  - ✓ 1940s: Timing of transactions lead to creation of centralized market, most graduates were men

  - ✓ 1970s: More women (e.g., 21% in 1977), so not uncommon for women to be married to another doctor

  - ✓ Married doctors felt that existing clearinghouses did not serve them well and increasing numbers began finding jobs outside the process

  - ✓ Roth (1984): Problem was that mechanism did not allow preferences over pairs to be expressed

Couples in two-sided matching markets

- Currently, many married couples in the medical match (1,000 out of 25,000 in NRMP) and they usually want to coordinate jobs

- DA does not accommodate couples: possible that one partner assigned to Boston, the other to LA

- Participation of medical students in NRMP dropped in 1970s, especially among couples

- NRMP adopted an initial procedure to handle couples in the 1970s

Initial procedure in the 1970s

- Couples (after being certified by their dean) could register for the match as a couple

    - ✓ They had to specify one member of the couple as the leading member
    - ✓ They submitted a separate rank order list of positions for each member of the couple

- The leading member went through the match as if single

- The other member then had his/her rank order list edited to remove positions not in the 'same community' as the one the leading member had matched to

    - ✓ Initially the NRMP determined communities; in a later version, when couples were still defecting, couples could specify this themselves.

- Note: this is similar to the current rules in Scotland

Handling couples

- Did this work?

    - ✓ Violated the **iron law of marriage**: you can't be happier than your spouse!
    - ✓ Couples consume pairs of jobs. So an algorithm that only asks for their preference orderings over individual jobs can't hope to avoid instabilities (appropriately redefined to include couples preferences)

- The NRMP allowed couples to submit preferences over pairs of hospitals and participation recovered

- The Roth-Peranson Algorithm (1998-)

Non-existence

- Allowing couples to rank pairs of programs is still not sufficient
- A stable matching does not necessarily exist when there are couples (Roth and Sotomayor)

- Example (based on Klaus and Klijn, 2005)
  - ✓ There are hospitals h_1, h_2 with one position each, one single doctor s and one couple $c = (f, m)$
  - ✓ Their preferences are

$$s : h_1, h_2$$
$$c : (h_1, h_2)$$
$$h_1 : m, s$$
$$h_2 : s, f$$

  - ✓ If couple is matched, $\mu(c) = (h_1, h_2)$. Then s is unmatched, and together with h_2 can block μ
  - ✓ If couple is unmatched, $\mu(c) = c$, then if $\mu(s) = h_1$, then (c, h_1, h_2) block or if $\mu(s) = h_2$ or $\mu(s) = s$, then (s, h_1) block

Non-existence (cont.)

- Klaus and Klijn (2005) find "maximal domain" results: identify condition known as **weak responsiveness** which effectively rules out complementarity in preferences such as due to geography

  ✓ Only 1 out of 167 couples in APPIC dataset satisfy this assumption

- Using stated preference data from NRMP, Roth and Peranson (1999) run a number of matching algorithms using submitted preferences from 1993, 1994 and 1995

  ✓ Algorithms always produce stable matching

- What about the market for clinical psychologists?

  ✓ There is a stable matching for all years 1999-2007

Matching with couples: Theory

Kojima, Pathak and Roth (QJE 2013) consider a model similar to Kojima and Pathak but assume there are a small number of couples.

Theorem

The probability that there exists a stable matching converges to one, as the size of the market (number of colleges) goes to infinity with the number of couples being fixed.

Theorem

For any $\varepsilon > 0$, there exists n such that truth-telling by every agent is an ε-Nash equilibrium under the Roth-Peranson algorithm for any game with more than n colleges.

Rough Intuition

Roth-Peranson algorithm will find a stable matching if couples are not displaced by another couple or single doctors.

In a large market, there is a high probability that there will be many colleges with vacant positions. So couples and singles are unlikely to apply and displace a couple in a hospital. So the algorithm is likely to terminate, producing a stable matching.

2002 Antitrust Lawsuit

- 16 law firms filed class action lawsuit representing 3 former residents, arguing that NRMP violated antitrust laws and was conspiracy to depress resident's wages

  - "Defendants and others have illegally contracted, combined and conspired among themselves to displace competition in the recruitment, hiring, employment and compensation of resident physicians, and to impose a scheme of restraints which have the purpose and effect of fixing, artificially depressing, standardizing and stabilizing resident physician compensation and other terms of employment."

- Suit was dismissed in August 2004

- Milton Friedman (1962): the American Medical Association is the "strongest trade union in America... licensure has reduced both the quantity and quality of medical practice... It has retarded technological development both in medicine itself and in the organization of medical practice."

  - But is the match the problem?

Academic work related to Antitrust suit

- Niederle and Roth (JAMA 2003): compare wages of 14 internal medicine specialities, where 4 use a match and 10 do not; Report no effect of match on salaries

- Bulow and Levin (AER 2006): develop model where firms set salary levels before matching with workers, and show that the wage distribution is compressed compared to competitive equilibrium, because of inflexible wage schedule

- Agarwal (AER 2016): empirical model of the match, "implicit tuition"

Student Assignment in Public School Choice - I

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January 2018

Road Map

- How to think about choosing a mechanism?
  - ▶ Trade-offs among stability, efficiency and strategy-proofness
- Single vs multiple (i.e. school-specific) tie breaking in DA
- DA with various tie breaking rules vs Student Optimal Stable matching vs Efficient matching
- TTC and minimizing envy with efficient matchings

Public School Choice

- *Brown vs. Board* (1954): "separate educational facilities are inherently unequal"

- Court-ordered desegregation plans

- Many urban areas have abandoned solely residential based systems in favor of open enrollment or school choice
  - ✓ People already have school choices because they can "vote with their feet"
  - ✓ Alternative schooling models: charters, vouchers, theme oriented small schools
    - Difficulty of clearing supply and demand via residential assignment

- Cited rationales include equity considerations; desire to break link between housing market and school options; introduce quasi-market forces into education

Recent literature thinking about the problem of assigning students to schools in public school choice plans in the US.

2003: New York City adopts a new centralized mechanism

2005: Boston changes the rules of their existing centralized mechanism

2007: England bans class of 'First Preference First' mechanisms nationwide

2009: Chicago abandons mechanism midstream

2012: Denver and NOLA adopt new mechanisms

2013: Boston adopts Home-Based Plan

2014: Washington DC and Newark adopt new mechanisms

Experience with mechanisms in the field has inspired new theoretical and empirical questions

How to think about Student Assignment?

Key questions:

- What is taken as exogenous and at what stage does the mechanism designer enter?

- How should we interpret school preferences?
    - ✓ What role for incentives of schools and schools in efficiency calculus?
    - ✓ Are we concerned that schools may want to operate outside of system (instability)?
    - ✓ More generally, is the market one or two-sided? (Stability vs. justified envy)

- What type of information can participants report?

- Are there consumption externalities?

- Do these mechanisms generate real allocative efficiencies? Productive efficiencies?

A few weeks after Abdulkadiroğlu and Sönmez (2003) was published in the June 2003 AER, it was described in the Boston Globe

School assignment flaws detailed

Two economists study problem, offer relief

By Gareth Cook
GLOBE STAFF

Boston uses a deeply flawed system for assigning students to its public schools, pushing more students out of their top-choice schools than necessary and giving parents a reason to lie about which schools they want, according to a pair of researchers who recently published their findings in a leading economics journal.

A new system, they say, could greatly reduce the anxiety in the city's annual school-choice process, in which thousands of parents submit lists of their top choices and await the computer-generated decision that will affect the next year to five years of their child's education.

The researchers found that once the parents submit their lists, they are subject to a poorly designed method of allocating spots in the top schools. By using a different technique, they say, the city could get more students into one of their top-choice schools while also making the system fairer. The alternate

technique, which the researchers outline in the paper, could be put in place with relatively simple, inexpensive changes and would not require the city to change any of its broader policies, according to the researchers and other academics who have seen the paper.

"Once all this is known, I don't see how they can keep the Boston mechanism," said Turkish economist Tayfun Sonmez, one of the researchers who studied Boston's system.

For more than two decades, policy-makers have devoted enormous amounts of attention to various ways to assign students to schools, sparking

philosophical debates, charges of racial and economic discrimination, and tangled court battles — all of which have played out with particular drama in Boston. But the authors say their work, which also examined districts in Columbus, Minneapolis, and Seattle, is the first rigorous examination of how best to do the actual matching once the policy is decided.

The research has broader implications as well. If more parents were happier with their school assignments, it would help keep them from fleeing for the suburbs and bolster the fortunes of the school district — and the city. Officials with the Boston public schools and the Boston School Committee readily acknowledge that parents are frustrated with

Boston Globe, September 12, 2003

Parental Incentives in Boston

- Evidence from BPS School Guide (2004, p3, quotes in original):

  For a better choice of your "first choice" school...
  consider choosing less popular schools.

- Advice from the West Zone Parent's Group:

  One school choice strategy is to find a school you like
  that is undersubscribed and put it as a top choice, OR,
  find a school that you like that is popular and put it as
  a first choice and find a school that is less popular for a
  "safe" second choice.

Gale and Shapley (1962)'s **Deferred Acceptance Algorithm** in Boston

Redesign of NYC High School Match

- Some schools actively rank students ⇒ Incentives to circumvent the match

  Deputy Chancellor: Before you might have a situation where a school was going to take 100 new children for 9th grade, they might have declared 40 seats, and then placed the other 60 outside the process.

  ⇒ Stable matching algorithm

Redesign of NYC High School Match

- Promote student welfare to the extend possible

  It [was] not unusual for up to 45 percent of students who apply to schools outside their neighborhood to be rejected by all their choices (New York Times, 03/11/03), after which they would be assigned without regard for their stated choices

  ⇒ Student optimal stable matching - student proposing deferred acceptance algorithm

Redesign of NYC High School Match

- Further gaming by schools:

  Schools gave higher priority to students who ranked them as first or second choice (public info) Students were faced to make strategic decisions while ranking schools

  $\Rightarrow$ No stable mechanisms that is strategy-proof for schools, however stability removes some school incentives to manipulate

  Student proposing deferred acceptance algorithm is strategy-proof for students

Tie breaking in matching mechanisms

- Well-known and widely applied matching algorithms use strict student preferences over schools and strict rankings of students at schools
- Most of the matching literature is built on the assumption of strict preferences, mainly because strict preferences have been more appropriate in earlier applications.
- However, as opposed to earlier applications, indifferences in school preferences are a primary feature of school choice problems.
- A typical example: Sort students at schools in the following order
  - ▶ Students with siblings attending the school
  - ▶ Students living with 2 miles radius of school
  - ▶ All other students
- Districts use lotteries to break ties among equal priority students?
- How to break ties? A single lottery that applies to all school? A different lottery at each school?

Example 1: Tie-breaking is consequential

Three students; three schools, each with one seat and each is indifferent among all students.

$$
\begin{aligned}
&\text{True Preferences} \\
i_1: \quad & s_2 - s_1 - s_3 \\
i_2: \quad & s_1 - s_2 - s_3 \\
i_3: \quad & s_1 - s_2 - s_3
\end{aligned}
$$

Assign a random number to each student and then break ties at schools accordingly:

Ties broken as	$s_1 - s_2 - s_3$
$i_1 - i_2 - i_3$	$i_2 - i_1 - i_3$
$i_1 - i_3 - i_2$	$i_3 - i_1 - i_2$
$i_2 - i_1 - i_3$	$i_2 - i_1 - i_3$
$i_2 - i_3 - i_1$	$i_2 - i_3 - i_1$
$i_3 - i_1 - i_2$	$i_3 - i_1 - i_2$
$i_3 - i_2 - i_1$	$i_3 - i_2 - i_1$

Each matching is **efficient**

Example 2: Multiple tie breaking can be inefficient

	True Preferences
$i_1:$	$s_2 - s_1 - s_3$
$i_2:$	$s_1 - s_2 - s_3$
$i_3:$	$s_1 - s_2 - s_3$

Ties broken as

$s_1:$	$i_1 - i_3 - i_2$
$s_2:$	$i_2 - i_1 - i_3$
$s_3:$	$i_3 - i_1 - i_2$

The student proposing DA produces

$$\left(\begin{array}{ccc} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{array} \right).$$

which is dominated by

$$\left(\begin{array}{ccc} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{array} \right).$$

NYC DOE argued that a more equitable approach would be to draw a new random order for each school:

"I believe that the equitable approach is for a child to have a new chance... This might result in both students getting their second choices, the fact is that each child had a chance. If we use only one random number, and I had the bad luck to be the last student in line this would be repeated 12 times and I never get a chance. I do not know how we could explain that to a student and parent."

"When I answered questions about this at training sessions, (It did come up!) people reacted that the only fair approach was to do multiple runs."

Single vs Multiple Tie Breaking

TABLE 1—TIEBREAKING FOR GRADE 8 APPLICANTS IN NYC IN 2006–2007

Choice	Deferred acceptance single tiebreaking DA-STB (1)	Deferred acceptance multiple tiebreaking DA-MTB (2)	Student-optimal stable matching (3)	Improvement from DA-STB to student-optimal	Number of students (4)
1	32,105.3 (62.2)	29,849.9 (67.7)	32,701.5 (58.4)	+1	633.2 (32.1)
2	14,296.0 (53.2)	14,562.3 (59.0)	14,382.6 (50.9)	+2	338.6 (22.0)
3	9,279.4 (47.4)	9,859.7 (52.5)	9,208.6 (46.0)	+3	198.3 (15.5)
4	6,112.8 (43.5)	6,653.3 (47.5)	5,999.8 (41.4)	+4	125.6 (11.0)
5	3,988.2 (34.4)	4,386.8 (39.4)	3,883.4 (33.8)	+5	79.4 (8.9)
6	2,628.8 (29.6)	2,910.1 (33.5)	2,519.5 (28.4)	+6	51.7 (6.9)
7	1,732.7 (26.0)	1,919.1 (28.0)	1,654.6 (24.1)	+7	26.9 (5.1)
8	1,099.1 (23.3)	1,212.2 (26.8)	1,034.8 (22.1)	+8	17.0 (4.1)
9	761.9 (17.8)	817.1 (21.7)	716.7 (17.4)	+9	10.2 (3.1)
10	526.4 (15.4)	548.4 (19.4)	485.6 (15.1)	+10	4.7 (2.0)
11	348.0 (13.2)	353.2 (12.8)	316.3 (12.3)	+11	2.0 (1.1)
12	236.0 (10.9)	229.3 (10.5)	211.2 (10.4)		
Unassigned	5,613.4 (26.5)	5,426.7 (21.4)	5,613.4 (26.5)	Total:	1,487.5

Are we missing anything good with single tiebreaking?

Single tiebreaking can produce all student-optimal stable matchings

Proposition:(Abdulkadiroglu, Pathak and Roth (2009))

μ is produced by a multiple tie breaking
μ cannot be produced by any single tie breaking.
Then μ involves inefficiency.
That is, there is another stable matching that Pareto dominates μ.

In other words

- Take a preference profile and
- a student-optimal matching μ for that preference profile,
- Then there is a single-tie breaking rule that produces μ

Example 3: Even single tie breaking may be inefficient

True Preferences

i_1 : $s_2 - s_1 - s_3$
i_2 : $s_1 - s_2 - s_3$
i_3 : $s_1 - s_2 - s_3$

Priorities

s_1 :
s_2 : $i_2 - i_1 - i_3$
s_3 : $i_3 - i_1 - i_2$

Break ties at s_1 as $s_1 : i_1 - i_3 - i_2$

Example 3: Even single tie breaking may be inefficient

True Preferences

$i_1:$ $s_2 - s_1 - s_3$
$i_2:$ $s_1 - s_2 - s_3$
$i_3:$ $s_1 - s_2 - s_3$

Priorities

$s_1:$ $i_1 - i_3 - i_2$
$s_2:$ $i_2 - i_1 - i_3$
$s_3:$ $i_3 - i_1 - i_2$

The student proposing DA produces

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

which is dominated by

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{pmatrix}$$

Stable Improvements Cycle (Erdil and Ergin 2008)

- Single tiebreaking produce all student-optimal stable matchings
- But it does not guarantee a student-optimal stable matching, i.e. it may involve inefficiencies
- Erdil and Ergin's (2008) SIC algorithm:
  - Start with a stable (not necessarily student optimal) matching
  - A SIC consists of a cycle of students, such that each student in the cycle prefers the school that the student to the right of him is assigned, and he is the highest ranked student at that school among all students who prefer that school to their match
  - If the stable matching is not student optimal, it admits a SIC (Theorem 1, Erdil and Ergin 2008)
  - If a SIC exists, assign each student in the cycle tthe school that the student to right of him is assigned
  - Repeat the process until no more SIC exists
- SIC produces a student-optimal stable matching
- However, SIC is not strategy-proof (Erdil and Ergin 2008)

SIC and Student Optimal Stable Matching

Table 1—Tiebreaking for Grade 8 Applicants in NYC in 2006–2007

Choice	Deferred acceptance single tiebreaking DA-STB (1)	Deferred acceptance multiple tiebreaking DA-MTB (2)	Student-optimal stable matching (3)	Improvement from DA-STB to student-optimal	Number of students (4)
1	32,105.3 (62.2)	29,849.9 (67.7)	32,701.5 (58.4)	+1	633.2 (32.1)
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3	9,279.4 (47.4)	9,859.7 (52.5)	9,208.6 (46.0)	+3	198.3 (15.5)
4	6,112.8 (43.5)	6,653.3 (47.5)	5,999.8 (41.4)	+4	125.6 (11.0)
5	3,988.2 (34.4)	4,386.8 (39.4)	3,883.4 (33.8)	+5	79.4 (8.9)
6	2,628.8 (29.6)	2,910.1 (33.5)	2,519.5 (28.4)	+6	51.7 (6.9)
7	1,732.7 (26.0)	1,919.1 (28.0)	1,654.6 (24.1)	+7	26.9 (5.1)
8	1,099.1 (23.3)	1,212.2 (26.8)	1,034.8 (22.1)	+8	17.0 (4.1)
9	761.9 (17.8)	817.1 (21.7)	716.7 (17.4)	+9	10.2 (3.1)
10	526.4 (15.4)	548.4 (19.4)	485.6 (15.1)	+10	4.7 (2.0)
11	348.0 (13.2)	353.2 (12.8)	316.3 (12.3)	+11	2.0 (1.1)
12	236.0 (10.9)	229.3 (10.5)	211.2 (10.4)		
Unassigned	5,613.4 (26.5)	5,426.7 (21.4)	5,613.4 (26.5)	Total:	1,487.5

Trade-off between efficiency and incentives

Preferences

$i_1 : s_2 - s_1 - s_3$

$i_2 : s_1 - s_2 - s_3$

$i_3 : s_1 - s_2 - s_3$

Priorities

$s_1 : i_1 - i_3 - i_2$

$s_2 : i_2 - i_1 - i_3$

$s_3 : i_2 - i_1 - i_3$

Trade-off between efficiency and incentives

Preferences
$i_1 : s_2 - s_1 - s_3$
$i_2 : s_1 - s_2 - s_3$
$i_3 : s_1 - s_2 - s_3$

Priorities
$s_1 : i_1 - i_3 - i_2$
$s_2 : i_2 - i_1 - i_3$
$s_3 : i_2 - i_1 - i_3$

DA:

i_1	i_2	i_3
s_1	s_2	s_3

Trade-off between efficiency and incentives

Preferences
$i_1 : s_2 - s_1 - s_3$
$i_2 : s_1 - s_2 - s_3$
$i_3 : s_1 - s_2 - s_3$

Priorities
$s_1: i_1 - i_3 - i_2$
$s_2 : i_2 - i_1 - i_3$
$s_3 : i_2 - i_1 - i_3$

DA:

i_1	i_2	i_3
s_1	s_2	s_3

Φ :

i_1	i_2	i_3
s_2	s_1	s_3

Trade-off between efficiency and incentives

Preferences	Priorities	Preferences
$i_1 : s_2 - s_1 - s_3$	$s_1: i_1 - i_3 - i_2$	$i_1 : s_2$
$i_2 : s_1 - s_2 - s_3$	$s_2 : i_2 - i_1 - i_3$	$i_2 : s_1 - s_2 - s_3$
$i_3 : s_1 - s_2 - s_3$	$s_3 : i_2 - i_1 - i_3$	$i_3 : s_1 - s_2 - s_3$

DA:

i_1	i_2	i_3
s_1	s_2	s_3

Φ :

i_1	i_2	i_3
s_2	s_1	s_3

Trade-off between efficiency and incentives

Preferences
$i_1 : s_2 - s_1 - s_3$
$i_2 : s_1 - s_2 - s_3$
$i_3 : s_1 - s_2 - s_3$

Priorities
$s_1 : i_1 - i_3 - i_2$
$s_2 : i_2 - i_1 - i_3$
$s_3 : i_2 - i_1 - i_3$

Preferences
$i_1 : s_2$
$i_2 : s_1 - s_2 - s_3$
$i_3 : s_1 - s_2 - s_3$

DA:

i_1	i_2	i_3
s_1	s_2	s_3

DA:

i_1	i_2	i_3
i_1	s_2	s_1

Φ :

i_1	i_2	i_3
s_2	s_1	s_3

Trade-off between efficiency and incentives

Preferences
$i_1 : s_2 - s_1 - s_3$
$i_2 : s_1 - s_2 - s_3$
$i_3 : s_1 - s_2 - s_3$

Priorities
$s_1 : i_1 - i_3 - i_2$
$s_2 : i_2 - i_1 - i_3$
$s_3 : i_2 - i_1 - i_3$

Preferences
$i_1 : s_2$
$i_2 : s_1 - s_2 - s_3$
$i_3 : s_1 - s_2 - s_3$

DA:

i_1	i_2	i_3
s_1	s_2	s_3

Φ :

i_1	i_2	i_3
s_2	s_1	s_3

DA:

i_1	i_2	i_3
i_1	s_2	s_1

Φ :

i_1	i_2	i_3
i_1	s_2	s_1

Trade-off between efficiency and incentives

Preferences
$i_1 : s_2 - s_1 - s_3$
$i_2 : s_1 - s_2 - s_3$
$i_3 : s_1 - s_2 - s_3$

Priorities
$s_1 : i_1 - i_3 - i_2$
$s_2 : i_2 - i_1 - i_3$
$s_3 : i_2 - i_1 - i_3$

Preferences
$i_1 : s_2$
$i_2 : s_1 - s_2 - s_3$
$i_3 : s_1 - s_2 - s_3$

DA:

i_1	i_2	i_3
s_1	s_2	s_3

DA:

i_1	i_2	i_3
i_1	s_2	s_1

Φ :

i_1	i_2	i_3
s_2	s_1	s_3

Φ :

i_1	i_2	i_3
i_1	s_2	s_1

Trade-off between efficiency and incentives

Theorem:(Abdulkadiroglu, Pathak and Roth 2009) Given any single tie breaking rule and an associated DA, there is no strategy-proof mechanism that does at least as well as the associated DA at every preference profile for every students.

That is, DA with single tie breaking lies on the Pareto frontier of strategy-proof stable mechanisms. Any inefficiency associated with it is the cost of providing straightforward incentives.

Extent of Inefficiency in NYC

TABLE 2—WELFARE CONSEQUENCES OF STABILITY FOR GRADE 8 APPLICANTS IN 2006–2007

Choice	Student-optimal stable matching (1)	Efficient matching (2)	Improvement from student-optimal stable matching	Number (3)	k	Count of students with k blocking pairs (4)
1	32,701.5 (58.4)	34,707.8 (50.5)	+1	1,819.7 (41.3)	1	22,287.5 (205.1)
2	14,382.6 (50.9)	14,511.4 (51.1)	+2	1,012.8 (26.4)	2	6,707.8 (117.9)
3	9,208.6 (46.0)	8,894.4 (41.2)	+3	592.0 (19.5)	3	2,991.0 (79.6)
4	5,999.8 (41.4)	5,582.1 (40.3)	+4	369.6 (16.0)	4	1,485.4 (56.5)
5	3,883.4 (33.8)	3,492.7 (31.4)	+5	212.5 (12.0)	5	716.6 (32.5)
6	2,519.5 (28.4)	2,222.9 (24.3)	+6	132.1 (9.1)	6	364.6 (22.9)
7	1,654.6 (24.1)	1,430.3 (22.4)	+7	77.0 (7.1)	7	183.1 (13.6)
8	1,034.8 (22.1)	860.5 (20.0)	+8	43.0 (5.6)	8	85.6 (10.9)
9	716.7 (17.4)	592.6 (16.0)	+9	26.3 (4.5)	9	44.7 (6.4)
10	485.6 (15.1)	395.6 (13.7)	+10	11.6 (2.8)	10	22.6 (4.9)
11	316.3 (12.3)	255.0 (10.8)	+11	4.8 (2.0)	11	9.9 (3.0)
12	211.2 (10.4)	169.2 (9.3)			12	3.2 (1.6)
Unassigned	5,613.4 (26.5)	5,613.4 (26.5)	Total:	4,296.6		34,898.8

Canonical Model

Rest of the talk focuses on the following canonical model with no indifference in school priorities (Abdulkadiroğlu and Sönmez 2003):

Primitives

1. a set of students $I = \{i_1, ..., i_n\}$,
2. a set of schools $S = \{s_1, ..., s_m\}$,
3. a capacity vector $q = (q_{s_1}, ..., q_{s_m})$,
4. a list of strict student preferences $P = (P_{i_1}, ..., P_{i_n})$, and
5. a list of strict school priorities $\pi = (\pi_{s_1}, ..., \pi_{s_m})$.

Gale's Top Trading Cycles

Adapted by Abdulkadiroğlu and Sonmez (2003) as follows:

- In the first step, all individuals and schools are available.

- Every available school points to its highest priority individual among all individuals. Every individual points to her most preferred school among all available schools. A cycle $c = \{s_k, i_k\}_{k=1,\ldots,K}$ is an ordered list of schools and individuals such that s_k points to i_k and i_k points to s_{k+1} for every k where $s_{K+1} = s_1$.

- For every cycle c, match each individual with the school she points to in that cycle and remove the individual and decrease the capacity of the school. If the capacity is zero, remove it along with individuals in the cycle from the problem.

- Repeat the algorithm in the next round until no more individuals are matched.

	Strategy-proof	Stable	Pareto Efficient
DA	YES	YES	NO
TTC	YES	NO	YES

- DA and TTC are not Pareto-comparable, i.e. although DA is not Pareto efficient, it may also produce an efficient matching different than the TTC matching
- How to choose between DA and TTC?

Outline

Policy debate: Boston

- 2003-05: Task Force recommended TTC:

  ...the Gale-Shapley algorithm [...] cuts down on the amount of choice afforded to families. The Top Trading Cycles algorithm also takes into account priorities while leaving some room for choice. [...] choice was very important to many families who attended community forums

- Recommendation was overturned following further deliberation, and TTC was faulted in the final school committee report:

  [TTC's] trading shifts the emphasis onto the priority and away from the goals BPS is trying to achieve by granting these priorities in the first place.

Outline

OneApp in New Orleans

- New Orleans Recovery School District was formed in 2003 to facilitate state control of schools in New Orleans; role expanded considerably following Hurricane Katrina in 2005

- In 2012, the RSD became the nation's first to integrating assignment between traditional public and charter schools; by 2014, the district became 100% charter

- Early on, officials decided that all RSD schools would only use two different priorities: sibling and walk-zone, and an even lottery number would break ties within applicants

- RSD environment closely resembles original school choice environment of Abdulkadiroğlu and Sönmez (2003)

INNER WORKINGS OF THE NEW CENTRAL ENROLLMENT SYSTEM

Students who want a spot at a Recovery School District school all filled out a common application this year and ranked their top eight choices. Using that information, the RSD will use a complex algorithm to match as many students as possible with their highest ranked school. Here's a simplified version of how it will work:

STEP 1

Students fill out a common application for a seat in one of the RSD's 67 schools, ranking their top eight choices by order of preference.

STEP 2

The RSD takes that data — this year from roughly 28,000 students — and uploads it into a central computer.

STEP 3

Every student is assigned a random lottery number. Schools play no role in assigning that lottery number or in ranking students. Students with a sibling at a particular school will move to the top of the list, followed by students living in that school's attendance zone.

STEP 4

The computer, using a complex mathematical formula, attempts to match as many students as possible to their top choice, followed by their second choices, and so on.

STEP 5

Students who don't get a spot at any of their top eight choices will be manually assigned, and every student will have a chance to appeal their placement.

Inside a computer at the RSD, School A offers its first open seat to Student No.1, who has the highest rank for that particular school among the 28,000 applicants based on that student's random lottery number, sibling preference and proximity.

SCENARIO A:

And Student No.1 has ranked School A as her top choice. In this scenario, student No.1 gets a seat at her top ranked school and available seats at School A decreases by one.

SCENARIO B:

But, say Student No.1 has ranked School B as her first choice of school.

The top ranked student for School B, however, is Student No.2 ...

... who in turn has selected as his top choice School C.

The top ranked student for School C is Student No.3.

Luckily, Student No.3 has selected School A, closing the loop and ensuring that all three students get their top choice.

Review

- Roth (1982): A matching free of justified envy need not be Pareto efficient; therefore there is no mechanism that is both Pareto efficient and without justified envy

- Kesten (2010, Prop 1): There is no Pareto efficient and strategy-proof mechanism that selects the Pareto efficient and justified-envy free matching when it exists

- Gale and Shapley (1962): A justified envy-free matching which is not Pareto dominated by any other justified-envy free matching exists (e.g., the student-optimal matching).

  ✓ If value elimination of justified envy first, then Pareto efficiency, this is an obvious choice
  ✓ If value Pareto efficiency first, and then elimination of justified envy, what is a good choice?

- A longstanding open issue: why choose TTC?

- Three students

$$i_1 : s_2 \succ s_1$$
$$i_2 : s_1 \succ s_2$$
$$i_3 : s_2 \succ s_1$$

- Two schools, where $q_{s_1} = 2$, priorities:

$$s_1 : i_1 \succ i_2 \succ i_3$$
$$s_2 : i_2 \succ i_3 \succ i_1$$

- TTC produces:

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_1 \end{pmatrix}$$

where i_2 trades into s_1, even though she directly qualifies

- Compare with justified envy-free and efficient

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_1 & s_2 \end{pmatrix}$$

Other strategy-proof and efficient mechanisms

- Serial Dictatorship

- Morrill (2015): **Clinch and Trade**

  ✓ TTC with counters allows i to trade her priority at other objects even when i is among the highest q_s priority students at s

  ✓ Since i is among the highest ranked, would receive s without such a trade; possible that this trade results needlessly creates envy

      - Assign a counter for each school q_s. For each student i, if i is one of the q_s highest ranked students and i most preferred school is s, then assign i to s (**clinch**). Reduce q_s by one, and remove school if capacity is zero. Update preferences and priority rankings. Iterate clinching until no student has one of the top q_s ranking at her most preferred choice

      - Then look for cycle as in TTC

- How to make comparisons between SP and PE mechanisms?

Problem-wise comparisons

- Related ideas in Maskin (2011), Kesten (2011), Pathak and Sönmez (2008) for different contexts

- φ **has less envy** than ψ at $\succ$, if for any P and pair (i, s), if (i, s) blocks $\varphi(P, \succ)$, then (i, s) blocks $\psi(P, \succ)$

    ✓ Not a count of blocking pairs, but subset relationship
    ✓ Implies that set of individuals and objects are in blocking pairs is at least as large

- **has strictly less envy**: φ has less envy than ψ, but ψ does not have less envy than φ

- φ **minimizes envy**: there is no ψ that has strictly less envy than φ

TTC has least envy

- **Theorem**. Suppose each school has one seat. Let φ be a Pareto efficient and strategy-proof mechanism. If φ has less envy than TTC at $\succ$, then $\varphi(\cdot, \succ) = TTC(\cdot, \succ)$

  ✓ **Corollary**: TTC minimizes envy in the class of Pareto efficient and strategy-proof mechanisms.

- This result leaves open the possibility that there are Pareto efficient and strategy-proof mechanisms that have less envy than TTC, but not strictly less envy than TTC

  ✓ For each s, define $f_s :\succ_s \to \succ_s$ to be an arbitrary transformation of priorities
  ✓ Consider a class of mechanisms $\varphi(\cdot, \succ) = TTC(\cdot, f(\succ))$. (run TTC with modified priorities)

    - Serial dictatorship would have for given $\succ$, $f_s(\succ_s) = \succ$ for any $\succ_s$ and s

  ▶ **Proposition 1**: Suppose $f_s(\succ_s) \neq \succ_s$ for some s. Then the mechanism $\varphi(\cdot, \succ) = TTC(\cdot, f(\succ))$ is not justified-envy minimal

Proof idea: By contradiction

- Strategy of argument closely related to Ma (1994)

- Suppose φ is PE and SP mechanism that has less envy than TTC given priority $\succ$

- Look at steps of TTC, and argue that at each step, φ cannot have less justified envy than TTC without contradicting either Pareto efficiency or strategy-proofness of φ

Proof Sketch

1
S2
⋮
S1

3
S1
⋮
S3

2
S3
⋮
S2

Suppose $\Phi_1(R) \neq TTC_1(R)$
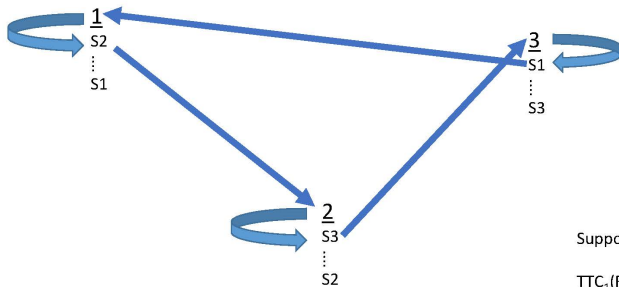
$TTC_1(R) = S2$ so $\Phi_1(R) \neq S2$

Suppose $\Phi_1(R) \neq TTC_1(R)$
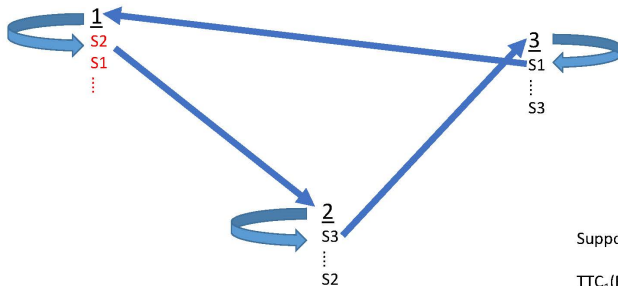
$TTC_1(R) = S2$ so $\Phi_1(R) \neq S2$

$TTC_1(R') = S2$
Φ is SP so $\Phi_1(R') \neq S2$ (single seats at schools)
But Φ has less envy than TTC so $\Phi_1(R') = S1$ must hold
Then $\Phi_3(R') \neq S1$
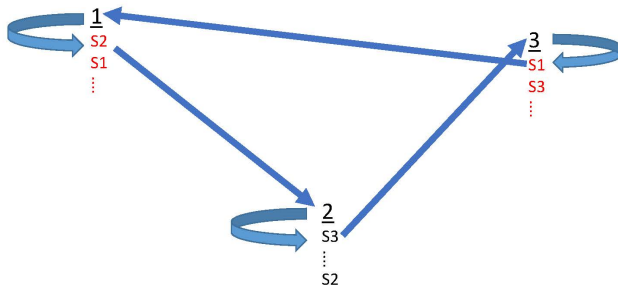
$TTC_1(R'') = S2$ and $TTC_3(R'') = S1$
Φ is SP so $\Phi_3(R'') \neq S1$ (single seats at schools)
But Φ has less envy than TTC so $\Phi_3(R'') = S3$ must hold
Then $\Phi_2(R'') \neq S3$
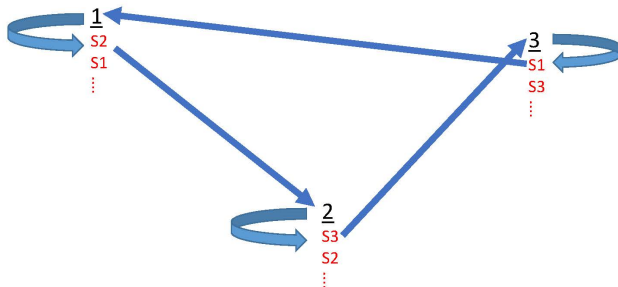
TTC$_1$(R''') = S2, TTC$_2$(R''') = S3 and TTC$_3$(R''') = S1
Φ is SP so Φ_2(R''') ≠ S3 (single seats at schools)
But Φ has less envy than TTC so Φ_2(R''') = S2 must hold

Then Φ assigns 1,2 and 3 to their second choices
Contradiction with PE of Φ
So Φ = TTC

Proof Sketch

TTC$_1$(R''') = S2, TTC$_2$(R''') = S3 and TTC$_3$(R''') = S1
Φ is SP so Φ$_2$(R''') ≠ S3 (single seats at schools)
But Φ has less envy than TTC so Φ$_2$(R''') = S2 must hold
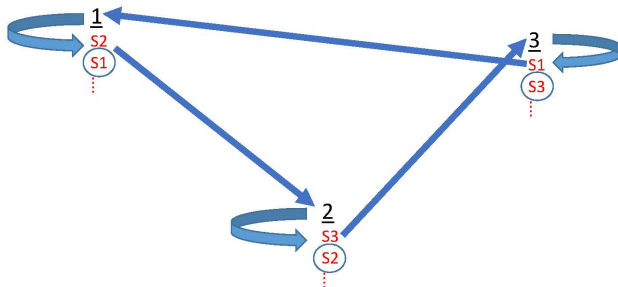
Then Φ assigns 1,2 and 3 to their second choices
Contradiction with PE of Φ
So Φ = TTC

RSD's Switch to DA

RSD switched to DA because of three main reasons

- Act 2: expanded Louisiana's Student Scholarships for Excellence Program statewide and integrated within OneApp; blocking pairs created by TTC would potentially be in violation of law

- TTC was not easy to explain to participants; esp why someone obtained a seat at a school

- RSD thought that DA would encourage more schools from competing Orleans Parish School Board to participate; many OPSB schools screen applicants

Lecture wrap up

- How to think about choosing a mechanism?
  - ▶ trade-offs among stability, efficiency and strategy-proofness
- Single vs multiple (i.e. school-specific) tie breaking in DA
- DA with various tie breaking rules vs SIC vs fully efficient matching
- TTC and minimizing envy with efficient matchings

	TTC-Counters (1)	TTC-Clinch and Trade (2)	Equitable TTC (3)	Serial Dictatorship (4)	Student-Proposing Deferred Acceptance (5)
A. Choice Assigned					
1	772	770	771	777	762
2	126	129	127	121	137
3	46	47	47	44	51
4	18	18	18	17	19
5+	11	11	11	8	10
Unassigned	222	221	222	228	217
Total	1196	1196	1196	1196	1196
B. Statistics on Blocking Pairs					
Students with justified envy (i)	158	157	159	213	0
Schools with justified envy (s)	7	7	7	12	0
Blocking pairs (i,s)	228	224	215	308	0
Instances of justified envy (i, (j,s))	1111	1086	1100	6546	0

Notes: Main transition grades are PK and grade 9 for 2012. TTC-counters defined in Abdulkadiroglu and Sonmez (2003). TTC-Clinch and Trade defined in Morrill (2013). Equitable-TTC defined in Hakimov and Kesten (2014). Instance of justified envy $(i,(j,s))$ means student i complains about student j's assignment at s. Blocking pair (i,s) means there exists at least one applicant j such that $(i,(j,s))$ is a blocking instance. Students with justified envy (i) means there exists a school s where (i,s) is a blocking pair. School with justified envy (s) means there is a school s such that there exists student i such that (i,s) is a blocking pair. The numbers represent averages over 100 different lottery draws for each grade, and then averaged over grades PK and 9. The standard deviation across lottery draws in column 1 for first choice assigned is 5.2, for unassigned is 5.2, for students with justified envy is 8.5, for blocking pairs is 20.2, for instances of justified envy is 110.7, and for schools with justified envy is 0.4. Standard deviations are similar for the other columns.

	TTC-Counters (1)	TTC-Clinch and Trade (2)	Serial Dictatorship (3)	Student-Proposing Deferred Acceptance (4)
A. Choice Assigned				
1	1240	1240	1236	1227
2	322	323	315	336
3	134	134	132	138
4	56	55	51	57
5+	39	39	34	40
Unassigned	102	101	124	96
Total	1893	1893	1893	1893
B. Statistics on Blocking				
Blocks defined by priority and lottery number				
students with justified envy (i)	389	368	280	0
blocking pairs (i,s)	538	506	369	0
instances of justified envy (i, (j,s))	1943	1752	3650	0
schools with justified envy (s)	30	29	44	0
Blocks defined by priority				
students with justified envy	129	126	280	0
blocking pairs (i,s)	160	156	369	0
instances of just envy (i, (j,s))	768	711	3650	0
schools with justified envy (s)	18	18	44	0

Table 2. Comparison of Mechanisms for Main Transition Grades (K1, K2, 6, and 9) in Boston

School Choice, II

Parag Pathak

MIT and NBER

Boston: Background

- 1970s: court-supervised forced busing
  - ✓ Boston in 1974: If school more than 50% nonwhite, then it was racially imbalanced
  - ✓ Judge Garrity's 14-year court supervision of Boston Public Schools, longest anywhere
- 1980-90s: introduce element of choice into busing plans; Avles and Willie establish geographic boundaries in Boston and controlled choice plan in 1989, explicitly using race; by 1999, race no longer a factor

- Today's debate: Rationing oversubscribed schools
  - ✓ E.g., zone geographies, proximity set-asides, sibling and family-link policies, special ed kids
  - ✓ Major fault-line of debate: pro-neighborhood vs. pro-choice

"The Soiling of Old Glory" by Stanley J. Forman
1977 Pulitzer Prize for Spot Photography

Student Assignment in Boston (up to 2005)

- Over 60,000 students from grades K-12 in almost 140 schools, divided into three zones: East, West, and North.

- Main new school entry points are K2, 6th and 9th grade: about 3,300 entering Kindergarten, 5,400 entering grade 6, and about 6,300 entering grade 9.

- In January, students asked to rank at least three schools in order of preference.

- For elementary and middle school, parents are asked to consider schools in their zone plus five schools open to all neighborhoods. High school admissions are citywide.

Boston Mechanism

✓ For each school a priority ordering is determined according to the following hierarchy:

1) First priority: sibling and walk zone
2) Second priority: sibling
3) Third priority: walk zone
4) Fourth priority: other students

Students in the same priority group are ordered based on an even lottery.

✓ Each student submits a preference ranking of the schools (with no constraint)

✓ The final phase is the student assignment based on preferences and priorities:

Round 1: In Round 1 only the first choices of the students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her first choice.

In general, at

Round k: Consider the remaining students. In Round k only the kth choices of these students are considered. For each school with still available seats, consider the students who have listed it as their k^{th} choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her k^{th} choice.

Cambridge mechanism: Advice for participants

Question: How would you participate if you were in this system?

"Strategies for Getting Your First Choice" (2007)

> *Because assignments are made randomly by a computer, the only way to strategize is to look at supply and demand for seats and try to predict human behavior. One way to do this is to obtain a list of estimated kindergarten seats.*
>
> *...*
>
> *Nowadays, parents who might prefer Graham & Parks sometimes don't choose it at all for fear of wasting their top choice.*

These types of anecdotes can be found in nearly every city using this type of mechanism

- **From the Media:**
  Consider the following quotation from St. Petersburg Times:

  > Make a realistic, informed selection on the school you list as your first choice. It's the cleanest shot you will get at a school, but if you aim too high you might miss.
  >
  > Here's why: If the random computer selection rejects your first choice, your chances of getting your second choice school are greatly diminished. That's because you then fall in line behind everyone who wanted your second choice school as their first choice. You can fall even farther back in line as you get bumped down to your third, fourth and fifth choices.

- **From the Education Literature**:

  Glenn [Public Interest 1991] states

  > As an example of how school selections change, analysis of
  > first-place preferences in Boston for sixth-grade enrollment in 1989
  > (the first year of controlled choice in Boston) and 1990 shows that
  > the number of relatively popular schools doubled in only the second
  > year of controlled choice. The strong lead of few schools was
  > reduced as others "tried harder."

Performance of Boston mechanism

Sample year, 2001-2002

	K2	6	9
1st choice	2,598	4,157	5,497
2nd choice	301	415	428
3rd choice	131	294	100
4th choice	61	61	42
5th choice	33	26	11
Unassigned	202	476	302

- Roughly 80% get their top choice, 8% get 2nd choice, ..., 5-9% unassigned
- Similar patterns across the years before 2005

- Nov 2003-June 2005: Ongoing discussion in Boston about school choice, one aspect was assignment mechanism

- July 2005: Boston School Committee voted to change their student assignment mechanism to the student-optimal stable mechanism

- Superintendent Thomas Payzant's Report to School Committee (5/11/2005): "A strategy-proof algorithm levels the playing field by diminishing the harm done to parents who do not strategize or do not strategize well."

- Formal investigation of this argument in Pathak and Sönmez (2008)

Burden of strategizing

- *Evidence from West Zone Parents Group*:

  Date: Fri, 28 Jan 2005
  Subject: Re: Philbrick School

  Have you gotten any sense if a lot of people are choosing Philbrick as a 1st choice? We really like Philbrick (love the K2 teacher) but are not in the walk zone. We are putting Manning 1st since we're in the walk zone and Philbrick 2nd but I'm getting very nervous that Philbrick has gotten so popular that it might only be a good #1 selection. We're also looking for a good safety for 4th place, perhaps Hale or Mendell.

Parent comments at public hearing

I find the current system of maximizing first choice to be insidious and destructive. I urge each school committee member to vote enthusiastically for this new algorithm proposal. [...] My wife and I take dozens of phone calls around choice time in Dorchester. We have to tell people that it hdoesn't make sense to choose our children's elementary school. And that is absurd. And the people who get that advice get very angry. [...] Because to get into the O'Hearn you need to be luckier than megabucks. So I have to say [to these parents], don't make your first choice your first choice. That's enraging. It is at the bottom of the anger that you [the School Committee] get from West Roxbury.

It angers the parents who figure it out because they are told not to make their first choice the first one. And it hurts those who don't figure it out because they choose a popular school and end up in the administrative assignment bin.

Information of Participants

Why might parents understand?

- BPS School Guide (2004, p3, quotes in original):

For a better chance of your "first choice" school... consider choosing less popular schools.

- *Advice from the West Zone Parent's Group*:
Introductory meeting minutes, 10/27/03

One school choice strategy is to find a school you like that is undersubscribed and put it as a top choice, OR, find a school that you like that is popular and put it as a first choice and find a school that is less popular for a "safe" second choice.

$\Rightarrow$ Evidence of sophisticated behavior among some players, unsophisticated behavior by others.

Model: Sincere and Sophisticated Students

Empirical and experimental work has documented existence of heterogenous levels of sophistication

$\mathcal{N}$: Sincere students

$\mathcal{M}$: Sophisticated students

For each $i \in \mathcal{N}$, restrict the strategy space to be a singleton, corresponding to truthful preference revelation.

Focus on the Nash equilibria of the preference revelation game induced by the Boston mechanism.

Modeling Choice

We assume that sincere parents are *truthful*.

- Natural default behavior.

- Chen & Sönmez (*JET* 2006): In laboratory about 20% of participants report their true preferences under the Boston mechanism.

- Hastings, Kane, Staiger (2009): In Charlotte, "we believe that the extent of strategic manipulation in the first year was limited and that parents were generally reporting their true preferences"

- Since truth-telling is a dominant strategy in Boston's new mechanism, this is the relevant case for comparative static analysis.

Sincere & Sophisticated Students

$\mathcal{N}$: Sincere students
$\mathcal{M}$: Sophisticated students

For each $i \in \mathcal{N}$, restrict the strategy space to be a singleton, corresponding to truthful preference revelation.

Focus on the Nash equilibria of the preference revelation game induced by the Boston mechanism.

Example: There are three schools a, b, c each of which has one seat, two strategic students i_1, i_2 and one sincere student i_3.
Utilities:

	a	b	c
u_{i1}	1	2	0
u_{i2}	0	2	1
u_{i3}	2	1	0

Priorities:

$$\pi_a : i_2 - i_1 - i_3$$
$$\pi_b : i_3 - i_2 - i_1$$
$$\pi_c : i_1 - i_3 - i_2$$

	abc	acb	bac	bca	cab	cba
abc	(0,0,1)	(0,0,1)	**(1,2,0)**	**(1,2,0)**	(1,1,1)	(1,1,1)
acb	(0,0,1)	(0,0,1)	**(1,2,0)**	**(1,2,0)**	(1,1,1)	(1,1,1)
bac	(2,0,0)	(2,0,0)	(0,2,2)	(0,2,2)	(2,1,2)	(2,1,2)
bca	(2,0,0)	(2,0,0)	(0,2,2)	(0,2,2)	(2,1,2)	(2,1,2)
cab	(0,0,1)	(0,0,1)	(0,2,2)	(0,2,2)	(0,2,2)	(0,2,2)
cba	(0,0,1)	(0,0,1)	(0,2,2)	(0,2,2)	(0,2,2)	(0,2,2)

There is only one Nash equilibrium outcome:

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ a & b & c \end{pmatrix}$$

Observations:

1. Truthful revelation is not a Nash equilibrium.

2. The sincere player i_3 is assigned a seat at school c and received a utility of 0 at all equilibria although she had the highest priority at school b where her utility is 1.

3. No reason to expect that the equilibrium outcome will be a stable matching. This is indeed the case here since (i_3, b) is a blocking pair.

Augmented Priorities

Given an economy (P, π) and a school s, partition the set of students I into m sets as follows

I_1: Sophisticated students and sincere students who rank s as their first choices under P,

I_2: sincere students who rank s as their second choices under P,

I_3: sincere students who rank s as their third choices under P,

$\vdots$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\vdots$

I_m: sincere students who rank s as their last choices under P.

Given an economy (P, π) and a school s, construct an **augmented priority ordering** $\tilde{\pi}_s$ as follows:

- each student in I_1 has higher priority than each student in I_2, each student in I_2 has higher priority than each student in I_3, ..., each student in I_{m-1} has higher priority than each student in I_m, and

- for any $k \leq m$, priority among students in I_k is based on π_s.

Define $\tilde{\pi} = (\tilde{\pi}_s)_{s \in S}$.

Let $(P, \tilde{\pi}_s)$ be the **augmented economy**.

Example (continued): Only student i_3 is sincere. So $\tilde{\pi}$ is constructed from π by pushing student i_3 to the end of the priority ordering at each school except his top choice a (where he has the lowest priority anyways):

$$\begin{aligned}
\pi_a : i_2 - i_1 - i_3 &\quad \Rightarrow \quad \tilde{\pi}_a : i_2 - i_1 - i_3 \\
\pi_b : i_3 - i_2 - i_1 &\quad \Rightarrow \quad \tilde{\pi}_b : i_2 - i_1 - i_3 \\
\pi_c : i_1 - i_3 - i_2 &\quad \Rightarrow \quad \tilde{\pi}_c : i_1 - i_2 - i_3
\end{aligned}$$

In this example the unique Nash equilibrium outcome μ of the preference revelation game induced by the Boston mechanism is the unique stable matching for the augmented economy $(P, \tilde{\pi})$.

Characterizing Nash Equilibrium Outcomes

Proposition 1: The set of Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism for economy (P, π) is equivalent to the set of stable matchings for augmented economy $(P, \tilde{\pi})$.

- Sincere students **lose their priority** to sophisticated students.

- Set of Nash equilibrium outcomes inherits the same properties as set of stable matchings for $(P, \tilde{\pi})$: Set of students who are single is the same in all equilibrium outcomes, set of occupied seats always the same, lattice structure, Pareto-dominant equilibrium allocation, etc.

Equilibrium Assignments of Sincere Students

Proposition 2: Fix an economy (P, π) and a sincere student $i \in \mathcal{N}$. Student i receives the same assignment at each Nash equilibrium outcome.

- Multiple stable matchings created by "conflict" between school priorities and student preferences

- Under augmented priorities, sincere students are never involved in conflicts
  - ▶ They have lower priority than sophisticated students
  - ▶ Among sincere students, a school gives higher priority to the sincere student who ranks it higher in her preferences
  - ▶ This is the reason why a sincere student receives the same assignment

- Useful result for comparative statics.

Comparative Statics: Becoming Sophisticated

- What happens when a parent joins WZPG?

- By Proposition 2, a sincere student always receives the same outcome in all Nash equilibria.

- When the student becomes sophisticated, her priority weakly improves at each school

- When a student's priority weakly improves, she receives a school that is at least as good under the Pareto dominant Nash equilibrium (although she could receive a worse outcome at other equilibria).

- This is called the **respecting improvements** property of Balinski and Sönmez (1999)

Truth-telling is a dominant strategy for sophisticated students in the student-optimal stable mechanism and the only strategy for sincere ones.

Example (continued): The outcome of SOSM is the following:

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ a & c & b \end{pmatrix}$$

- Sincere student i_3 improves and obtains a seat at school b.
- Sophisticated student i_1 receives a seat at school a under both mechanisms.
- Strategic student i_2 suffers a loss under SOSM and receives a seat at her second choice school c.

Comparing Mechanisms for Sincere Students

Is a sincere student always better off under the SOSM? No.

A sincere student can prefer the Boston mechanism since

- she gains priority at her second choice school over sincere students who rank it third or lower,
- she gains priority at her third choice school over sincere students who rank it fourth or lower, etc.

In a way an sincere student may luck out (at the expense of another sincere student) under the Boston mechanism!

Comparing Mechanisms for Sophisticated Students

Proposition 3: Fix an economy (P, π) and a sophisticated student $i \in \mathcal{M}$. The assignment of student i under the Pareto-dominant Nash equilibrium outcome of the Boston mechanism is at least as good as her assignment under the dominant strategy equilibrium outcome of the SOSM.

- Sophisticated players could be worse off in other Nash equilibrium outcomes of the Boston mechanism.

- Coordination at Pareto dominant Nash equilibrium may be difficult.

Resistance to Change by the WZPG

June 8th, 2005: Community testimony from WZPG leader

"There are obviously issues with the current system. If you get a low lottery number and don't strategize or don't do it well, then you are penalized. But this can be easily fixed. When you go to register after you show you are a resident, you go to table B and the person looks at your choices and lets you know if you are choosing a risky strategy or how to re-order it.

Don't change the algorithm, but give us more resources so that parents can make an informed choice."

Comparative Statics: Becoming Strategic

- What happens when a parent joins WZPG?

  Proposition 4: A sincere student weakly benefits from becoming sophisticated in the Pareto-dominant Nash equilibrium of the Boston game, whereas all other sophisticated students weakly suffer.

- When the student becomes strategic, her priority weakly improves at each school and she receives a school that is at least as good under the Pareto dominant Nash equilibrium (although she could receive a worse outcome at other equilibria).

# Does the Boston Mechanism have any virtues?

- When all applicants prefer the same school the most, say school X, the tie among everybody has to be broken

- If school X does not rank students, priorities cannot be used to break ties

- DA uses a lottery to break ties

- Assignment of X will be efficient ex-post, regardless of the realization of the lottery

- Assigning X to those who really value it very highly and does not have a better alternative is still important

- Yet the DA cannot differentiate among students based on preference intensities; Boston may be able to elicit cardinal information

- Abdulkadiroğlu, Che and Yasuda (2011): Present scenarios without priorities where Boston may dominate DA; Troyan (2012) results are not robust to introduction of (weak) priorities
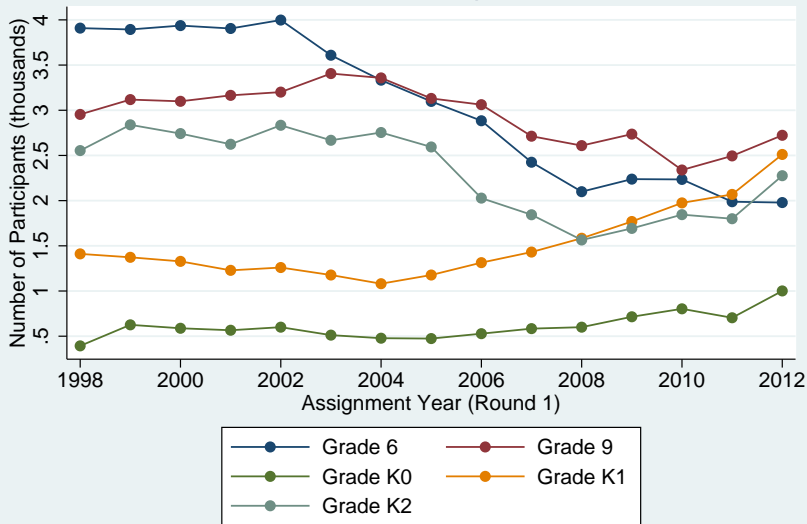   - Empirical work quantifies trade-offs

Number of Programs Available: Zone ALL
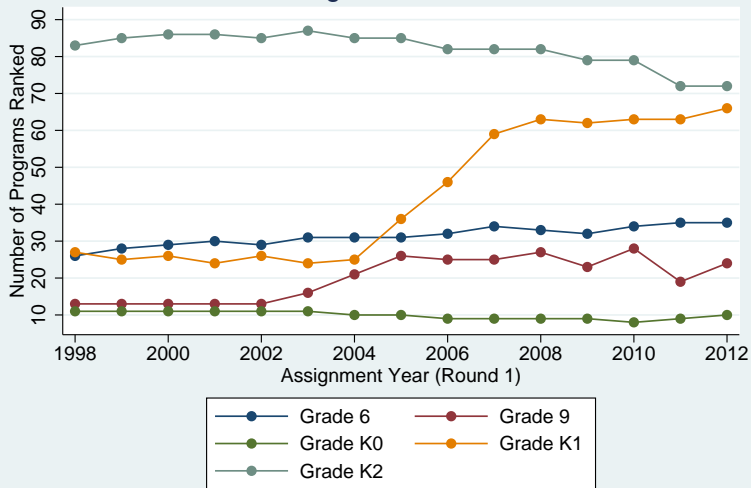
Total Number of Participants: Zone ALL

Grade 6, Grade 9, Grade K0, Grade K1, Grade K2

Length of Applicant Rank Order Lists: Zone ALL
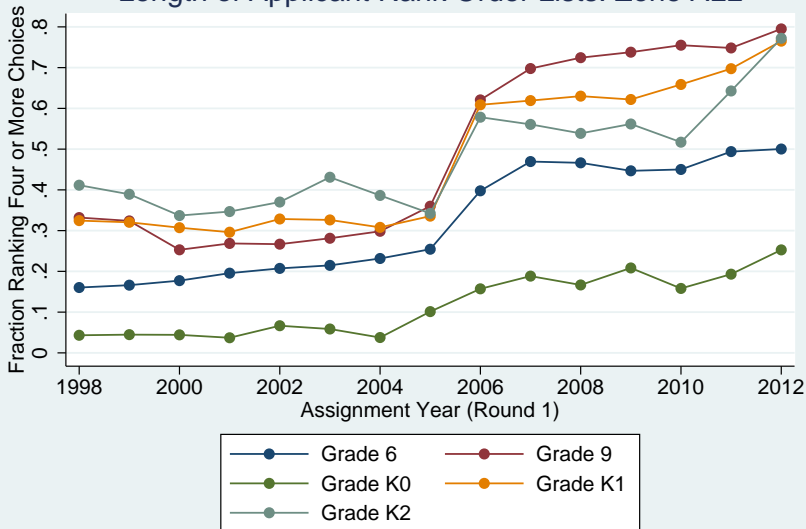
Fraction of Students Receiving Top Choice: Zone ALL
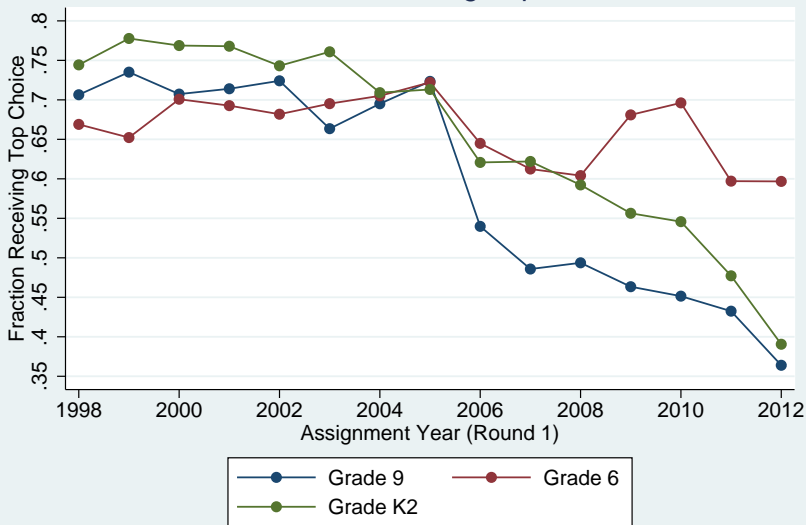
10

Condemnation of certain assignment mechanisms

- Chicago and England: mechanisms changed without direct consultation with economists
  - ✓ Public discussion resembles academic arguments made in Boston

- Like another prominent "natural experiment" in mechanism design: US Medical Match (NRMP)
  - ✓ Participants (<u>not</u> game-theorists) organized rules, mostly still in place since 1952
  - ✓ Seen as support for positive interpretation of *stability*

- For school assignment, the broader game-theoretic concept advanced is aversion to strategic manipulation or "gaming"

8th-graders' shot at elite high schools better

Poring over data about eighth-graders who applied to the city's elite college preps, Chicago Public Schools officials discovered an alarming pattern.

High-scoring kids were being rejected simply because of the order in which they listed their college prep preferences.

"I couldn't believe it," schools CEO Ron Huberman said. "It's terrible."

CPS officials said Wednesday they have decided to let any eighth-grader who applied to a college prep for fall 2010 admission re-rank their preferences to better conform with a new selection system.

Previously, some eighth-graders were listing the most competitive college preps as their top choice, forgoing their chances of getting into other schools that would have accepted them if they had ranked those schools higher, an official said.

Under the new policy, Huberman said, a computer will assign applicants to the highest-ranked school they quality for on their list.

"It's the fairest way to do it." Huberman told Sun-Times.

Chicago Public Schools

9 selective high schools

Applicants: Any current 8th grader in Chicago

Composite test score: entrance exam $+$ 7th grade scores

Up to Fall 2009, system worked as follows:

- Take admissions test
- Rank up to 4 schools

Chicago Selective Enrollment Mechanism

Round 1: Only the first choices of the students are considered. For each school, consider the students who have listed it first. Assign school seats to these students following their composite test score until either there are no seats left or there is no student left listing it as her first choice.

In general, for $k = 2, ..., 4$

Round k: For the remaining students, only the kth choices are considered. For each school with still available seats, consider the students who have listed it as their k^{th} choice. Assign the remaining school seats to these students following their composite test score until either there are no seats left or there is no student left listing it as her k^{th} choice.

- New Chicago mechanism (SD^4)

  ✓ Rank up to 4 schools

  ✓ Students ordered by composite score

  ✓ First student obtains her top choice, the second student obtains her top choice among remaining, and so on.

- Somewhat surprising midstream change, especially given that both mechanisms are manipulable...

Comparing Mechanisms

- Mechanism ψ is **manipulable** by player i at problem R if there exists a type R'_i such that

$$\psi(R'_i, R_{-i}) P_i \psi(R).$$

- Mechanism ψ is **at least as manipulable as** mechanism φ if for any problem where mechanism φ is manipulable, mechanism ψ is also manipulable.

- Mechanism ψ is **more manipulable than** mechanism φ if
  - ✓ ψ is at least as manipulable as φ,
  - ✓ there is at least one problem where ψ is manipulable though φ is not.

Equivalent definition: if truth-telling is a Nash equilibrium of the preference revelation game induced by mechanism ψ, it is also a Nash equilibrium of the game induced by mechanism φ (even though the converse does not hold).

Proposition. Suppose there are at least k schools and let $k > 1$. The old Chicago mechanism (CHI^k) is more manipulable than the truncated serial dictatorship Chicago adopted (SD^k) in Fall 2009.

- Outrage expressed in quotes from Chicago Sun-Times:

    "I couldn't believe it," schools CEO Ron Huberman said. "It's terrible."

  suggests that the old mechanism was quite undesirable.

- We'd like to compare it to a larger class of mechanisms:
    ◇ stable mechanisms?
    ◇ not satisfied by many school choice mechanisms, including Chicago's old one

A matching is **strongly unstable** if a student who ranks school s as his first choice loses a seat to a student who has a lower composite score.

A **weakly stable** matching is one that is not strongly unstable.

✓ old Chicago mechanism is weakly stable

✓ new mechanism is weakly stable

✓ variants of new mechanism where can rank more choices are weakly stable

Theorem. *The old Chicago mechanism* (CHI^k) *is at least as manipulable as any weakly stable mechanism.*

Chicago in 2010-11

- Based on the last two results, the new mechanism in Chicago is an improvement in terms of our criteria

- However, 2009 mechanism is not Pareto efficient

- Possible to have a completely non-manipulable mechanism by considering all choices...so why not?

- In 2010-11 school year, Chicago decided to consider 6 out of 9 choices, so the mechanism is still manipulable; this remains true this past spring

Constrained School Choice

- Consider more general environment where students may be ordered in different ways across school

- Vulnerability of school choice mechanisms to manipulation played a role in NYC's adaptation of a version of the student-optimal stable mechanism in NYC, where students can rank up to 12 choices

  > NYC DOE press release on change: "to reduce the amount of gaming families had to undertake to navigate a system with a shortage of good schools" (New York Times, 2003)

- Based on the strategy-proofness of the student-optimal stable mechanism, the following advice was given to students:

  > You must now rank your **12** choices according to your true preferences.

Next result formalizes the idea that the greater the number of choices students can make, the less vulnerable this mechanism is to manipulation:

Theorem: *Let $\ell > k > 0$ and suppose there are at least ℓ schools. The student-optimal stable mechanism where students can rank k schools is more manipulable than the student-optimal stable mechanism where students can rank ℓ schools.*

Corollary: The 2009 Chicago mechanism (SD^4) is more manipulable than the newly adopted 2010 Chicago mechanism (SD^6).

English context

- Forms of school choice for decades

- 2003 School Admissions Code
  - ⋄ "National Offer Day": coordinated admissions nationwide, under authority of Local Education Authority; 800,000 students given offer

- 2007 School Admissions Code
  - ⋄ Strengthened enforcement of admissions rules

Section 2.13: In setting oversubscription criteria the admission authorities for all maintained schools must not:
give priority to children according to the order of other schools named as preferences by their parents, including 'first preference first' arrangements.

- A **first preference first system** is any "oversubscription criterion that gives priority to children according to the order of other schools named as a preference by their parents, or only considers applications stated as a first preference" (School Admissions Code, 2007, Glossary, p. 118).

- Best-known first preference first system is **Boston mechanism** (pre-2005)

- Rationale given by Dept. for Ed & Skills (2007):

    "'first preference first' criterion made the system unnecessarily complex to parents who had to play an 'admissions game' with their children's future"

- Echoes themes from our 2003-05 Boston discussion, where policymakers said the new mechanism (BPS 2005):

    "adds transparency and clarity to the assignment process, by allowing for clear and straightforward advice to parents regarding how to rank schools."

First preference first (FPF) mechanism: definition

- A school is either a **first-preference-first school** or an **equal preference school**

- At each first-preference-first school, priorities modified:
  - ✓ any student who ranks school *s* as his first choice has higher priority than any student who ranks school *s* as his second choice,
  - ✓ any student who ranks school *s* as his second choice has higher priority than any student who ranks school *s* as his third choice,
  - ✓ ...

- Outcome determined by the student-proposing deferred acceptance algorithm

✓ FPF mechanism is a hybrid between Boston and the student-optimal stable mechanism

Ban of FPF Mechanism in 2007

2007 Admissions Code outlaws FPF at more than 150 Local Education Authorities (LEAs) across the country; continued through 2012

Some LEAs abandoned earlier:

- Pan London Admissions Authority adopted an 'equal preference' system in 2005 (=student-optimal stable mechanism)

  designed to "eliminate the need for tactical preferences and make the admissions system fairer"; it will "create a level playing field for school admissions"

- cf. June 2005 comments by Boston superintendent that new algorithm

  "levels the playing field by diminishing the harm done to parents who do not strategize or do not strategize well."

- In 2006, Coldron report: 101 LEAs used equal preference, 47 used first preference first, nearly all with constraints on rank order list length

Theorem: *Suppose there are more than k schools where $k > 1$. FPF^k is more manipulable than the student-optimal stable mechanism where students can rank k schools.*

◇ **Corollary**: *The old abandoned Chicago Selective Enrollment mechanism is more manipulable than the new 2009 mechanism.*

Finishing the Job on Student Assignment in Boston

Mayor Thomas Menino, State of the City Address in 2012

> "Something stands in the way of taking our [public school] system to the next level: a student assignment process that ships our kids to schools across our city."

> "Pick any street. A dozen children probably attend a dozen different schools. Parents might not know each other; children might not play together. They can't carpool, or study for the same tests."

> [...]

> "Boston will have a radically different school assignment process—one that puts priority on children attending schools closer to their homes."

- Mayor Menino and Supt. Johnson form 27-member External Advisory Committee (EAC) to help BPS develop a new plan in partnership with the community; over 70 public meetings since March 2012

An unexpected advocate for more neighborhood assignment...

The Boston Globe

TED LANDSMARK

It's time to end busing in Boston

The city's demographics have changed since busing began, says Ted Landsmark, who was the target of this angry demonstrator brandishing an American flag during the school busing riots on April 5, 1976. (©Stanley Forman)

The city's demographics have changed. In the 1970s, Boston was largely a "bicultural" city with a "white" majority and a significantly smaller "black" minority. Federal court cases indicated that financial and facility allocations clearly favored the white majority. The current student population is "majority minority."

We can no longer afford to spend millions a year to bus children across Boston to schools that are not demonstrably better than schools near their homes.

By Ted Landsmark
January 31, 2009

Neighborhoods vs. Choice

- Motivated by court challenges, Boston eliminates racial set-aside for assignments in 1999; city-wide debate about return to neighborhood schools

- **November 1999:** School Committee adopts choice plan which reduces use of walk zone priority from 100% to 50%

> *Fifty percent walk zone preference means that half of the seats at a given school are subject to walk zone preference. The remaining seats are open to students outside the walk zone.*

> *One hundred percent walk zone preference would limit choice and access for too many families to the schools they want their children to attend. On the other hand, the policy also should and does recognize the interests of families who want to choose a walk zone school.*
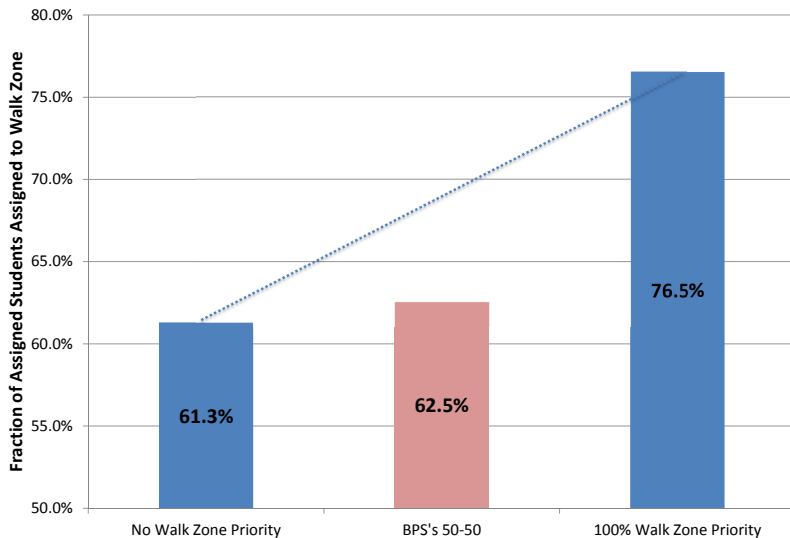
Role of Walk Zone Priority

- Boston's Walk zones: 1 mile radius around school, intersected with 867 geocodes

- In Boston, walk zone applicants are prioritized at half of the school seats and then have a second chance for each school

- Has the change from 100% to 50% walk zone shifted the balance too much to the detriment of neighborhood assignment?

- Strategy-proofness of DA allows us to consider two counterfactuals:
    1) How would the outcome change if walk zone priority was maintained for all seats? (100% Walk)
    2) On the other extreme, how would the outcome change if walk zone priority was abandoned altogether? (0% Walk)

Grade K1, 2009-2012

Motivating Puzzle

- Outcome under BPS 50-50 "compromise" is incredibly close to the outcome in the absence of any walk zone priority

- Since students only rank schools (and not halves), mechanism must decide how to "convert" student preferences over schools to student preferences over school-halves

- In BPS, walk-half seats are systematically ranked ahead of open-half; ranking between schools unchanged

- Decision was a detail left to BPS IT

  "A great deal of market design is going to be done by Java programmers..."
  Roth (2002)

For simplicity, this example assumes same number of walk-zone applicants and outside walk-zone applicants.

Scenario 1: All Slots are open (0% Walk-Zone Priority)

Best random tie-breaker

Worst random tie-breaker

Walk-zone Applicants

Walk-Zone Applicants matched

School Seats

Final Allocation:
Walk-Zone: 50%
Outside Walk-Zone: 50%

Outside Walk-Zone Applicants matched

Outside Walk-zone Applicants

Best random tie-breaker

Worst random tie-breaker

For simplicity, this example assumes same number of walk-zone applicants and outside walk-zone applicants.

Scenario 2: 50-50 slot split (50% Walk-Zone Priority – 50% Open Priority), Walk-half first – Open-half next, Same tie-breaker for both halves (Current BPS)
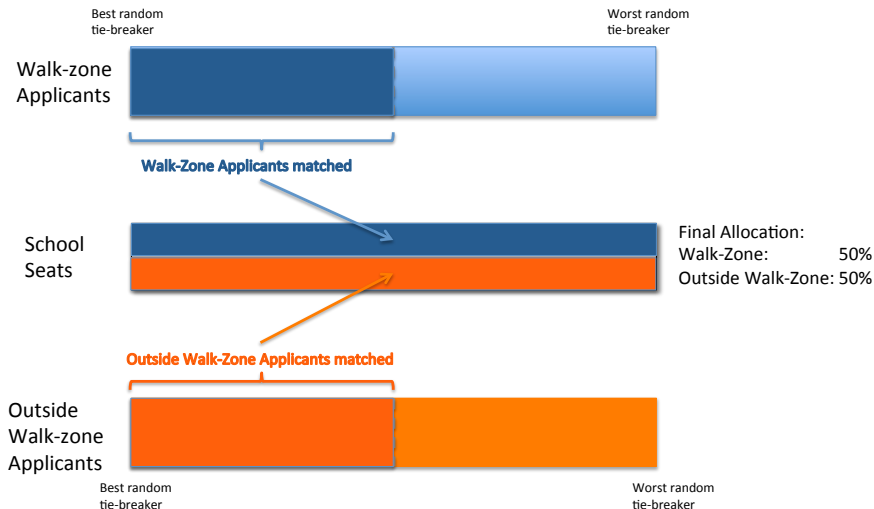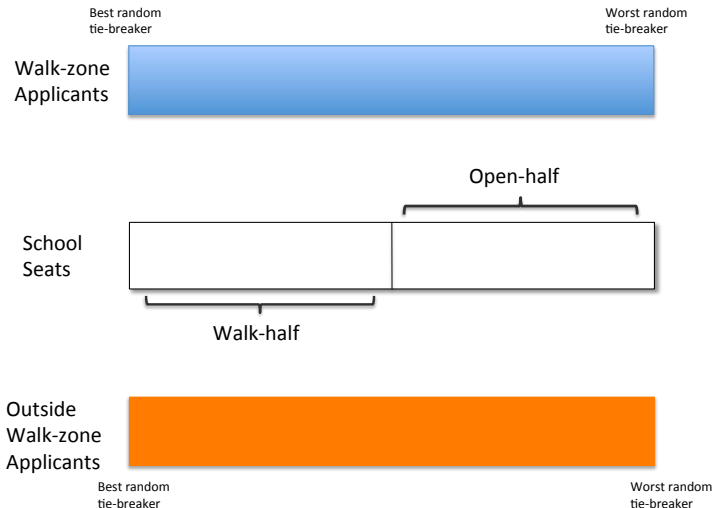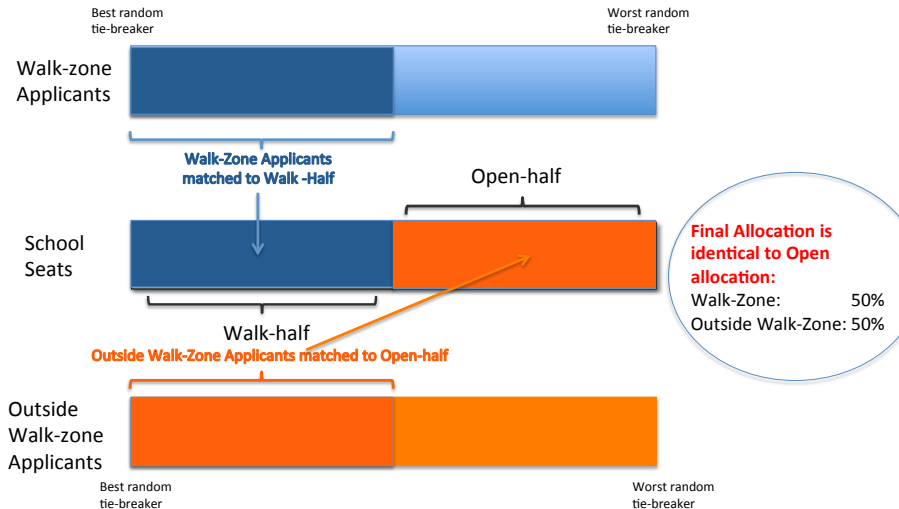
For simplicity, this example assumes same number of walk-zone applicants and outside walk-zone applicants.

Scenario 2: 50-50 slot split (50% Walk-Zone Priority – 50% Open Priority), Walk-half first – Open-half next, Same tie-breaker for both halves (Current BPS)

Best random tie-breaker — Worst random tie-breaker

Walk-zone Applicants

Walk-Zone Applicants matched to Walk-Half

Open-half

School Seats

Walk-half

Outside Walk-Zone Applicants matched to Open-half

Outside Walk-zone Applicants

Best random tie-breaker — Worst random tie-breaker

Final Allocation is identical to Open allocation:
Walk-Zone: 50%
Outside Walk-Zone: 50%

- **March 2013:** Supt. Johnson supports the EAC recommendation of Home-Based Plan A, but recommends against keeping the walk zone priority.

  "After reviewing that struggle and after viewing the final MIT and BC presentations on the way the walk zone priority actually works, it seems to me that **it would be unwise** to add a second priority to the Home-Based model by allowing the walk zone priority to be carried over.

  Leaving the walk zone priority to continue as it currently operates is not a good option. We know from research that it does not make a significant difference the way it is applied today: although people may have thought that it did, the walk zone priority does not in fact actually help students attend schools closer to home. The External Advisory Committee suggested taking this important issue up in two years, but I believe we are ready to take this step now. **We must ensure the Home-Based system works in an honest and transparent way from the very beginning.**"

# Organization of Kidney Markets

Nikhil Agarwal[1]

MIT and NBER

Motivation

- Transplantation is the best treatment for kidney failure
  - Improves quality and length of life
  - Each transplant is estimated to save Medicare hundreds of thousands of dollars

- Almost 100K patients are waiting on the kidney list
  - List has been growing, and thousands die while waiting

- Two sources of kidney transplants
  1. ∼12K receive a deceased donor transplant each year
  2. ∼6K receive a kidney from a living donor

- Potential for many more living donor transplants
  - ✓ Biological compatibility prevents many direct donations

Biological Compatibility

- Donation requires biological compatibility
  - ▸ Blood-type compatibility
  - ▸ Tissue-type compatibility
    - ✓ Common immune sensitivity measure: Panel Reactive Antibody (PRA)
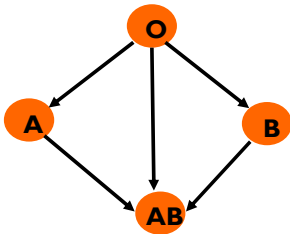
A Barrier to Organ Markets

- In principle, shortage of organs can be solved using monetary incentives
  [Becker and Elias, '07]

- Section 301, National Organ Transplant Act (NOTA), 42 U.S.C. 274e 1984:

    - "it shall be unlawful for any person to knowingly acquire, receive or otherwise transfer any human organ for valuable consideration for use in human transplantation"

- Near universal norm (exception: Republic of Iran)
    - Societies often constrain transactions [Roth, '07]
    - Concerns about inadequate protections against exploitation and coercion
      [see also Satz, '12; Sandel, '13]

Double Coincidence of Wants

- Chapter I [Jevons, 1876]:
  - ▶ "The first difficulty in barter is to find two persons whose disposable possessions mutually suit each other's wants. ... to allow of an act of barter, there must be a double coincidence, which will rarely happen."
  - ▶ "Sellers and purchasers can only be made to fit by the use of some commodity... which all are willing to receive... This common commodity is called a medium, of exchange..."

- Pairwise Kidney Exchange
  - ▶ Each living donor donates a kidney
  - ▶ In return, her intended recipient receives one
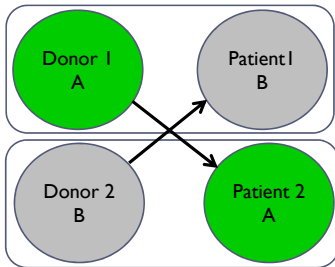
What about NOTA?

- Charlie W. Norwood Living Organ Donation Act (2007)

  - Section 301 of the National Organ Transplant Act (42 U.S.C. 274e) is ammended – (I) in subsection (a), by adding at the end the following:

  "The preceding sentence does not apply with respect to human organ paired donation"

- Pioneering work by Roth, Sonmez and Unver ('04)
  - ▶ Based on model for housing market by Shapley and Scarf ('74)
  - ▶ Gale's Top Trading Cycles to form cycles and chains
  - ✓ Show that individual patient-donor pairs have an incentive to enter the market
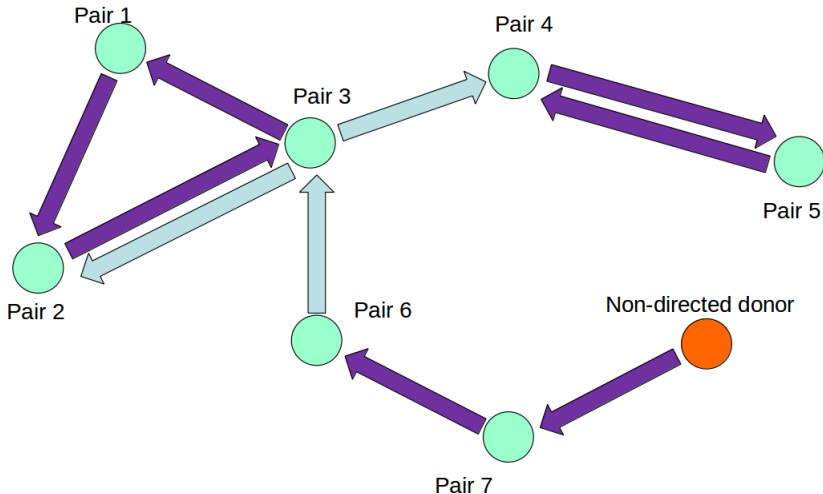
Constraints on Cycles Lengths

- Large cycles are logistically challenging
    - Most exchanges through 2-way and 3-way cycles
    - In addition, priorities need to be incorporated [Roth, Sonmez and Unver, '05]

- Non-simultaneous altruistic donor chains [Rees et.al. '09]
  - ▶ Vast majority of transplants in large exchanges
  - ▶ Typically four to five donors long, although long chains are possible and useful [Ashlagi et.al., '12]

Exchanges in Large Market

- Erdos-Renyi Model: Random graph $G(n, p)$ with n nodes and probability p of a non-directed edge
  - If $p > \dfrac{(1 + \varepsilon) \ln n}{n}$, then $G(n, p)$ is almost surely connected
  - ✓ Interpret p as probability of tissue-type compatibility

- Implications: Kidney exchange platforms exhibit natural scale economies
  - Tissue-type incompability can be overcome in large platforms
  - Blood-type incompatibility persists
    [see Roth, Sonmez and Unver '07; Unver '09; Ashlagi and Roth, '15]
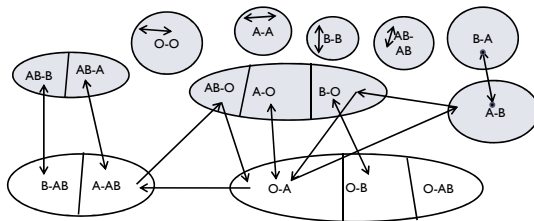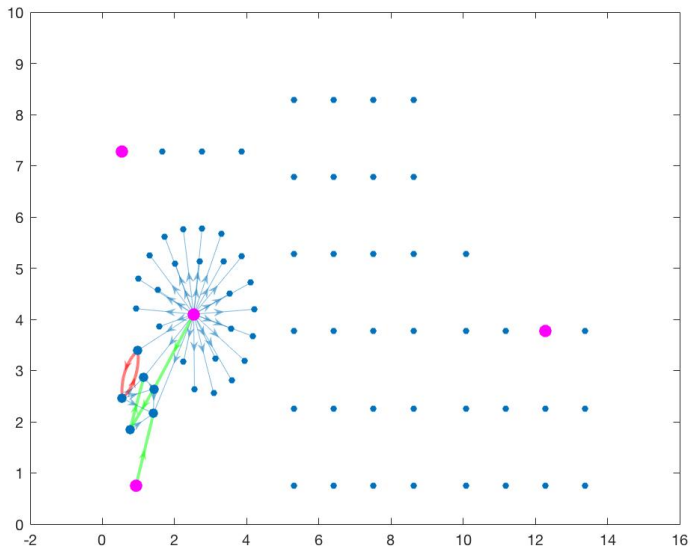
Hospital Free-Riding: Causes and Consequences

Center 1   Center 2
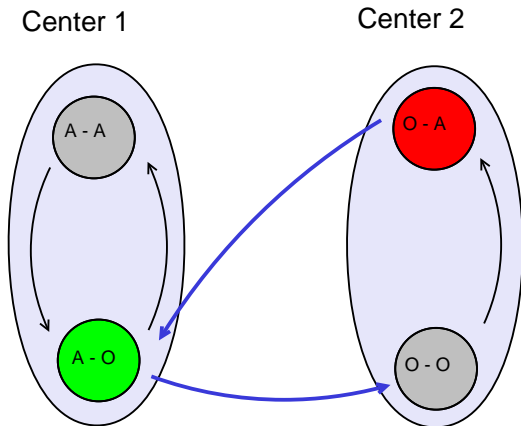
- Hospitals may have incentives to withold valuable types, resulting in thin markets
- Mechanisms can better align incentives

  [Roth, Sonmez and Unver, '07; Ashlagi and Roth, '14; Agarwal, Ashlagi, Azevedo, Featherstone, Karaduman, '17]

- Mike Rees (APD Director):

  "As you predicted, competing matches at home centers is becoming a real problem. Unless it is mandated, I'm not sure we will be able to create a national system. I think we need to model this concept to convince people of the value of playing together"

Problem I: A Fragmented and Inefficient Market

Size of Various Exchange Platforms
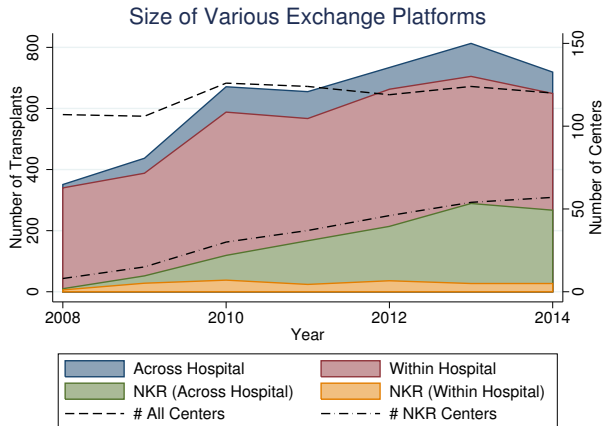
- Moreover, market is inefficient
  - ✓ Within Hospital exchanges use O donors used to transplant non-O patients

  [Source: Agarwal et.al., '17]

Problem II: A Selected Market

	Altruistic Donors		Pairs		Unpaired Patients	
N	164		1265		501	
	Mean	s.d.	Mean	s.d.	Mean	s.d.
Patient Blood Type						
A	---	---	23.8%	(0.43)	51.1%	(0.50)
B	---	---	15.0%	(0.36)	16.0%	(0.37)
AB	---	---	2.6%	(0.16)	19.0%	(0.39)
O	---	---	58.6%	(0.49)	14.0%	(0.35)
Donor Blood Type						
A	44.5%	(0.50)	44.8%	(0.50)	---	---
B	14.0%	(0.35)	18.5%	(0.39)	---	---
AB	3.7%	(0.19)	5.1%	(0.22)	---	---
O	37.8%	(0.49)	31.5%	(0.46)	---	---
Match Power						
Recipient/Pair	---	---	21.6%	(0.21)	43.0%	(0.39)
Donor	27.6%	(0.16)	25.4%	(0.16)	---	---
Panel Reactive Antibody (PRA)	---	---	48.8%	(0.41)	44.4%	(0.45)
Pair Type						
Overdemanded	---	---	13.8%	(0.35)	---	---
Underdemanded	---	---	42.2%	(0.49)	---	---

[Source: Agarwal et.al., '17]

Problem III: Financial Barriers/Agency Problems
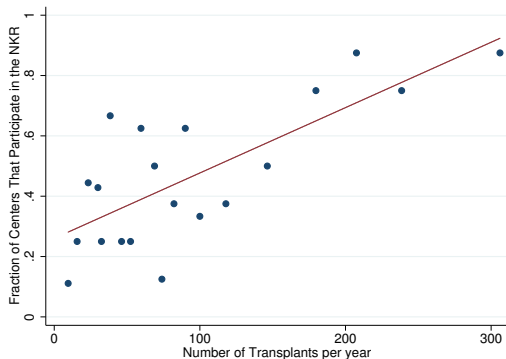
[Source: Agarwal et.al., '17]

- Small Hospitals Unlikely to Participate
  - ▸ Consistent with financial barriers of participating at large platforms
    [Rees et.al.,'12; Agarwal et.al., '17]

Value of a Thick Market

- Kidney exchange platform: An abstraction [Agarwal et.al., '17]
    - Inputs: Patients and donors, q
    - Outputs: Transplants, $f(q)$
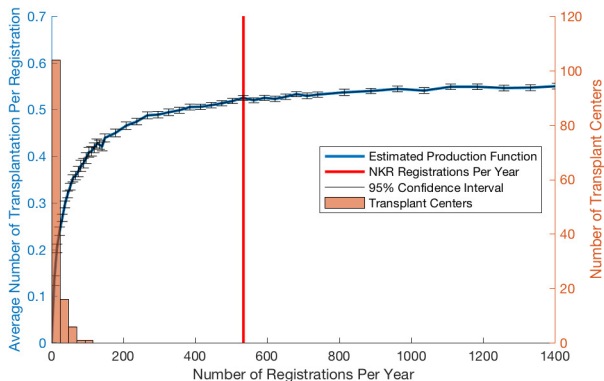
- ✓ Gains from co-ordinating on a few large platforms exceed 200 transplants per year

Batching/Market Timing

- Exchanges differ on how frequently they match
  - NKR matches everyday $\rightarrow$ market is effectively small
  - UNOS and APD match less frequently

✓ Frequent matching effectively makes the market thinner
  - Suppose pairs arrive in the order (a), (b), (c) and (d)

- Not easy to create thickness via batching in this market
  - ✓ Knowledge of whether a patient is critical is invaluable
    [Akbarpour, Li and Gharan, '17]
  - ✓ Must wait very long for significant gains ($\sim 3 - 6$ months)
    [Anderson, Ashlagi, Gamarnik, Kanoria, '15; Ashlagi, Burq, Jaillet, Manshadi, '13]
  - ✓ Waiting costs can be incorporated into the exchange design
    [Unver, '09; Lee and Yariv, '15]

A Points System

- Hospitals are key players advising patients and donors
  - Need incentives for participation [Ashlagi and Roth, '15]
  - Optimal incentives approximate marginal product, $p^* \approx \nabla f$ [Agarwal et.al., '17]

Private vs. Social Value of Submissions
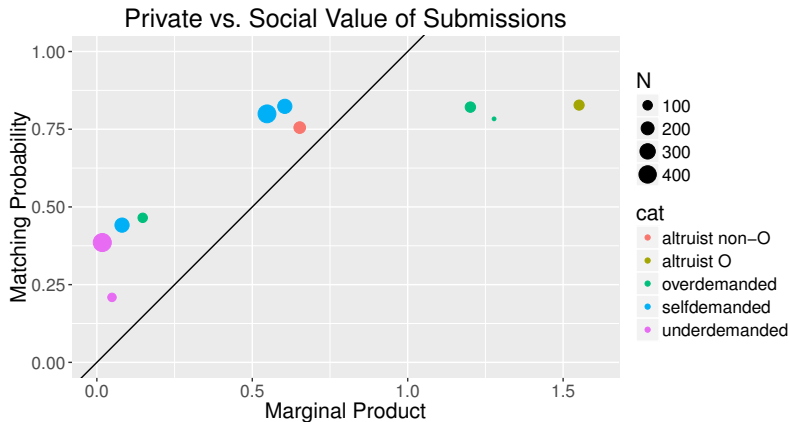
A Points System

- Proposed implementation as a "scrip system" or "frequent flyer" system
  [Ashlagi and Roth, '15; Agarwal et.al., '17]
  - ✓ Currently working with the APD on implementation
  - ✓ Many unsolved theoretical questions [see Mobius, '01; Hauser and Hopenhayn, '08; Abdulkadiroglu and Bagwell, '13]

	Match Probability	Marginal Product		Points per Transplantation
	Mean	Mean	Within Category Standard Deviation	
	Panel A: Altruistic Donors			
Non-O Donor	0.85	0.81	0.17	0.95
O Donor	0.93	1.88	0.23	2.02
	Panel B: Patient-Donor Pairs			
O Patient, Non-O Donor	0.28	0.04	0.14	-0.87
O Patient, O Donor, PRA >= 82%	0.35	0.08	0.13	-0.77
O Patient, O Donor, PRA < 82%	0.79	0.64	0.19	-0.20
Non-O Patient, O Donor, PRA >= 94%	0.32	0.12	0.15	-0.63
Non-O Patient, non-O Donor, PRA < 94%	0.84	0.62	0.23	-0.25
Non-O Patient, O Donor, PRA < 94%	0.82	1.32	0.33	0.61
	Panel C: Unpaired Patients			
Non-AB Patients	0.20	-0.01	0.12	-1.03
AB Patients	0.36	0.09	0.14	-0.74

Global Kidney Exchange

- Financial barriers commonly prevent transplantation in the developing world
- ✓ Global Kidney Exchange (GKE) overcomes these financial barriers
  - ▶ Bringing biologically compatible pairs to the US
  - ▶ Pay for transplants and post-transplant care
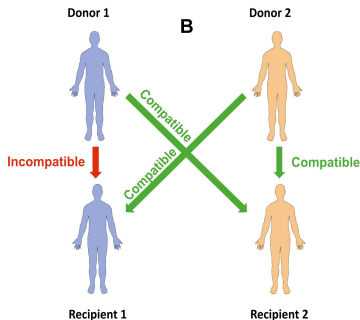
- GKE pair (pair 2) is not transplantable
  - ✓ Enables a transplant for pair 1
  - ✓ Saves US health-insurer $$ [Rees et.al., '17; Nikzad, Akbarpour, Rees, Roth, '17]
  - ✓ First successful chain involving a couple from the Phillippines

Outline

Other Directions

- Endowment bias has kept compatible pairs out of the market
  - ✓ Compatible pairs may be very valuable
  - ▶ Incentive schemes based on future transplants could encourage participation
    [see Sonmez, Unver and Yenmez, '17]

- Interactions between kidney exchange and deceased donor exchange
  - ✓ Mostly analyzed as a separate problem [Su and Zenios (several papers); Agarwal, Ashlagi, Rees, Somaini, Waldinger, '17]
  - ▶ Potential value from interactions between the two systems
  - ▶ Current efforts:
    - Kidney Donor Waiting List Exchange [Roth, Sonmez, Unver, Delmonico, Saidman, '06]
    - Chains initiated by deceased donors [Melcher, Roberts, Leichtman, Roth, Rees, '16]
  - ✓ Need to analyze willingness to accept a deceased donor instead

Conclusion

- Kidney exchange is a poster child of market design
  - ✓ Transparent, measurable improvements enabled by matching theory

- Close interaction between theory, practice, and empirics
  - ✓ Lessons from practice incorporated into research and vice-versa
  - ✓ Inolvement of surgeons, academics and policy-makers

- Design of the market has evolved in response to new challenges
  - ✓ Pushed economic analysis and practice on the field

Revealed Preference for Matching Markets

Nikhil Agarwal

MIT and NBER

Introduction

- Theory of matching markets
  - Comparison of mechanisms (Efficiency, Incentives, Fairness, Stability)

- Debates on the best forms of market organization
  - Organization of school choice systems
    - Immediate Acceptance vs. Deferred Acceptance
      [Pathak and Sonmez, 2008; Abdulkadiroglu, Che and Yasuda, 2011]
    - Centralized vs. Decentralized Systems
  - Effects of centralized systems on salaries (medical match)
    [Jung et.al. 2002; Bulow and Levin, 2006; Niederle and Roth, 2003]
  - ✓ Theory does not always yield unambiguous answers

- Effects of Policy Proposals
  - Impact of financial aid reform on access to college

- Outcomes are mediated through agent choices
  - Preferences are primitives in the theory
  - Practical designs are often based on reported preferences
  - Matches in decentralized/informal markets also mediated through preferences

✓ Formal mechanisms produce rich administrative data

Role of Using Choice Models

- Positive Analysis:
    - ▶ Quantifying preferences
        - ✓ What do parents value in a school or college?
        - ✓ What is the value of certain job amenities?
    - ▶ Effects of market interventions are intermediated through agent choices
        - ✓ Taxes, tuition subsidies, free tuition, quotas
        - ✓ Preference estimates facilitate General Equilibrium policy analysis

- Normative Analysis:
    - ▶ Welfare and distributional consequences

✓ Complementary to theory in evaluation of trade-offs
    - ▶ Magnitudes of effects identified in the theory
    - ▶ Analysis in cases where theory is intractible or ambiguous

Revealed Preference Approach

- Traditional revealed preference approach
  - ✓ Use data on consumer decisions to deduce most preferred option (given price)

- Matching Markets: Cannot choose your preferred option $\rightarrow$ must also be chosen
  - ▶ Cannot decide to enroll at MIT
  - ▶ Your partner needs to agree to marry you
  - ▶ Cannot show up at work at Google
  - ▶ Peer-to-peer platforms require mutual consent (eg. AirBnb)

Revealed Preference Approach

✓ Rules of the market determine the interpretation of the data
- ▶ Matched partner need not be preferred to others
- ▶ College application decisions consider chances of admission
- ▶ Agents need not submit a truthful ranking of options if incentives are not straightforward

✓ Organized marketplaces present a unique opportunity for analysis
- ▶ Administrative data on outcomes and/or submitted rankings
- ▶ Well understood rules of the game assist modeling choices

● NB: Related to significant body of work on labor and marriage markets with transferable utility, and search markets matching more broadly

[Abowd, Kramarz and Margolis, 1999; Burdett, Judd and/or Mortensen ('90s);
Postel-Vinay and Robin, 2002; Choo and Siow, 2006; Chiappori, Galichon, Salanie and
co-authors (several papers); Menzel, 2015; Fox and co-authors (several papers)]

Random Utility Preference Model

- (Indirect) Utility of agent i over possible options j:

$$\begin{aligned}
v_{ij} &= v(x_{ij}, \xi_j, \varepsilon_i; \beta) \\
&= x_{ij}\beta_i + \xi_j + \varepsilon_{ij}
\end{aligned}$$

- x_{ij} are observed in the data
- β_i are individual preference factors
- ξ_j is an unobserved option quality indicator
- ε_{ij} is an idiosyncratic taste

- Other details
  - Value of a reference (outside) option normalized to zero $v_{i0} = 0$
  - Scale of utility also requires a normalization

✓ Parametric assumptions on ε_{ij} and β_i are commonly made for estimation

Numeraire and Welfare

- Often interested in making welfare statements
    1. Within agent: Does agent *i* benefit? Do all agents benefit?
    2. Across agents: Who benefits the most? Does the average student benefit?

- Inter-personal comparisons are difficult if lump-sum transfers are prohibited
  [c.f. Kaldor-Hicks]

- Two approaches, depending on the setting
    - Standard: Setting involves monetary payments, e.g. tuition

    $$v_{ij} = v(x_{ij}, \xi_j, \varepsilon_i) - p_{ij}$$

      ✓ Increase in welfare coupled with a transfer is a pareto improvement

    - Non-Standard: Utility metric in terms of another variable, e.g. distance

    $$v_{ij} = v(x_{ij}, \xi_j, \varepsilon_i) - d_{ij}$$

      ✓ Willingness to travel metric
      ✓ Inter-personal comparisons and subgroup analysis based on chosen units

Discrete Choice Model

- Consumer's decision to purchase a product or pick from a set of alternatives

✓ Choice reveals the region in utility space

Estimation Approaches

- Several possible estimation methods:
  - ▶ Maximum likelihood   [McFadden, 1974; Train, 2004]
  - ▶ Bayesian MCMC methods   [Rossi, McCulloch and Allenby, 1996]
  - ▶ Maximum Score   [Manski, 1985]
  - ▶ Moment Inequality   [Ciliberto and Tamer, 2009; Pakes 2010; Chernozhukov, Hong and Tamer, 2007]
  - ▶ Method of Moments (with price endogeneity)   [Berry, 1994; Berry, Levinsohn and Pakes, 1995]

✓ Logit choice probabilities when $\varepsilon \sim$ EV I

$$\mathbb{P}(i \text{ chooses } j | X; \beta) = \frac{\exp(x_{ij}\beta)}{\sum_k \exp(x_{ik}\beta)}$$

  - ▶ Yields simple maximum likelihood methods

Rank-Ordered Data

- Several public school choice systems use incentive compatible mechanisms in which reported rank-order lists are used to assign students
  - ✓ Deferred Acceptance, Top Trading Cycles and Serial Dictatorship mechanisms
  - ▸ Good source of administrative data to learn about preferences

Rank-Ordered Data

✓ Multiple choices useful for individual specific (random) co-efficients β_i

$$v_{ij} = x_{ij}\beta_i + \varepsilon_{ij}$$

▸ When ε_{ij} has a GEV Type I distribution, we get the exploded logit form:

$$\mathbb{P}(i \text{ ranks } j > j'|X; \beta_i) = \frac{\exp(x_{ij}\beta_i)}{\sum_k \exp(x_{ik}\beta_i)} \frac{\exp(x_{ij'}\beta_i)}{\sum_{k \neq j} \exp(x_{ik}\beta_i)}$$

[Beggs, Cardell and Hausman (1981) and Berry, Levinsohn and Pakes (2004)]

- Several studies estimate value of various schools using data from organized school choice systems
  [Abdulkadiroglu, Agarwal and Pathak, 2015; Hastings, Kane and Staiger, 2009; Ajayi, 2015; amongst others]

	No student interactions (1)	School characteristics × Student characteristics		
			Models with random coefficients	
		Without random coefficients (2)	All choices (3)	Choice among eligible programs (4)
High math achievement				
Main effect	0.016 (0.016)	0.027 (0.014)	−0.029 (0.018)	−0.058 (0.039)
Baseline math		0.031 (0.001)	0.039 (0.001)	0.050 (0.001)
Percent subsidized lunch				
Main effect	−0.085 (0.007)	−0.057 (0.004)	−0.069 (0.009)	−0.113 (0.058)
Size of ninth grade (in 100s)				
Main effect	−0.164 (0.036)	−0.092 (0.032)	−0.113 (0.048)	−0.153 (0.178)
Percent white				
Main effect	−0.002 (0.014)	0.070 (0.012)	0.062 (0.016)	0.093 (0.062)
Asian		−0.054 (0.002)	−0.075 (0.003)	−0.100 (0.004)
Black		−0.084 (0.002)	−0.124 (0.002)	−0.189 (0.003)
Hispanic		−0.047 (0.002)	−0.084 (0.002)	−0.119 (0.003)
Standard deviation of ε	7.226 (0.010)	7.385 (0.011)	7.858 (0.013)	10.059 (0.022)
Standard deviation of ξ	3.519 (0.121)	2.954 (0.100)	3.676 (0.129)	5.151 (0.650)
Random coefficients (covariances)				
Size of ninth grade (in 100s)			1.584 (0.009)	1.837 (0.012)
Percent white			−0.006 (0.001)	−0.009 (0.001)
Percent subsidized lunch			−0.002 (0.000)	−0.002 (0.000)
High math achievement			−0.011 (0.001)	−0.015 (0.001)
Percent white			0.008 (0.000)	0.013 (0.000)
Percent subsidized lunch			−0.001 (0.000)	−0.002 (0.000)
High math achievement			0.005 (0.000)	0.007 (0.000)
Percent subsidized lunch			0.002 (0.000)	0.003 (0.000)
High math achievement			0.000 (0.000)	−0.001 (0.000)
High math achievement			0.016 (0.000)	0.022 (0.000)
		X	X	X
		X	X	X
	69,907	69,907	69,907	69,907
	542,666	542,666	542,666	542,666

Welfare Gains From Coordination

- In 2003, NYC replaced an uncoordinated mechanism with one based on DA
  - Uncoordinated mechanism did not automate offer processing
    - ✓ One-third were unassigned and placed administratively in a nearby school

Source: Abdulkadiroglu, Agarwal and Pathak (2017)

Gains Correlated with Administrative Assignment

✓ Students most likely to be administratively assigned gained the most!

Source: Abdulkadiroglu, Agarwal and Pathak (2017)

Alternative Mechanisms

✓ Student welfare modestly affected by further modifications of the algorithm

Source: Abdulkadiroglu, Agarwal and Pathak (2017)

Strategic Choices

- However, many mechanisms do not incentivize truthful reporting
  - ✓ Immediate Acceptance (Boston) Mechanism prioritizes students at higher ranked choices
  - ▶ Trade-off: Gaining priority at true second-choice or try for true first choice
  - ▶ Need to interpret choices in terms of preferences:

✓ Baseline case of optimal choices [see He 2012; Agarwal and Somaini, 2015; Calsamiglia et.al., 2016; Hwang, 2016; for extensions and applications]

Revealed Preferences

- ✓ Similar to simpler discrete choice setting!
- Several recent developments with varying behavioral assumptions

  [He, 2014; Agarwal and Somaini, 2015; Calsamiglu, Guell and Fu, 2015; Hwang, 2015]

| | Probability of Assignment as First Choice School | | Estimated Willingness to Travel (in miles) | | | |
| | | | Assumption: Truthful Behavior | | Assumption: Sophisticated Behavior | |
	Paid Lunch	Free Lunch	Paid Lunch	Free Lunch	Paid Lunch	Free Lunch
Graham Parks	22%	82%	1.29	0.40	1.9	-0.17
			[0.06]	[0.08]	[0.12]	[0.19]
Haggerty	45%	87%	1.39	0.72	1.49	0.61
			[0.07]	[0.11]	[0.10]	[0.15]
Baldwin	49%	89%	1.26	0.50	1.35	0.73
			[0.05]	[0.09]	[0.07]	[0.10]
Morse	54%	64%	0.66	0.70	0.84	1.00
			[0.07]	[0.08]	[0.08]	[0.10]
Amigos	73%	74%	-0.01	-0.38	0.02	-0.39
			[0.13]	[0.15]	[0.11]	[0.15]
Cambridgeport	51%	77%	0.77	0.18	0.68	0.39
			[0.06]	[0.08]	[0.08]	[0.10]
King Open	100%	100%	0.65	0.40	0.59	0.22
			[0.06]	[0.07]	[0.08]	[0.10]
Peabody	94%	95%	0.22	0.48	0.06	0.27
			[0.08]	[0.09]	[0.09]	[0.12]
Tobin	93%	72%	-0.49	0.64	-0.82	0.37
			[0.11]	[0.12]	[0.18]	[0.19]
Fletcher Maynard	100%	76%	-1.30	-0.05	-2.29	-0.19
			[0.14]	[0.10]	[0.24]	[0.14]
Kenn Long	100%	100%	-0.19	0.47	-0.36	0.17
			[0.09]	[0.07]	[0.14]	[0.11]
MLK	100%	100%	-0.66	0.08	-1.27	-0.32
			[0.10]	[0.09]	[0.16]	[0.13]
King Open Ola	100%	100%	-3.60	-4.13	-2.20	-2.69
			[0.35]	[0.39]	[0.32]	[0.44]
Outside Option	---	---	-2.08	-1.44	-0.50	-0.94
			[0.10]	[0.09]	[0.04]	[0.06]

Notes: Estimates from Agarwal and Somaini (2015). Utility Estimates relative to the mean value of inside options.

Comparison Between IA and DA

	All Students	Paid Lunch	Free Lunch
	Deferred Acceptance		
Percent Assigned to First Choice	67.8	58.4	86.4
Percent Assigned to Second Choice	15.8	18.7	10.0
Percent Assigned to Third Choice	5.2	7.1	1.5
	Immediate Acceptance		
Percent Assigned to First Choice	72.3	63.9	88.8
Percent Assigned to Second Choice	14.7	18.1	7.9
Percent Assigned to Third Choice	3.9	5.1	1.3
	Comparison		
Mean Utility DA - Cambridge	-0.078	-0.107	-0.021
Std. Utility DA - Cambridge	0.109	0.120	0.046
Percent DA > Cambridge	17.3	15.6	20.6
Percent DA ≈ Cambridge	31.2	28.0	37.5
Percent DA < Cambridge	51.5	56.4	41.9
Percent with Justified Envy	2.5	2.7	2.3

Empirical Findings and Limitations: School Choice

- Other Key findings
  - ▶ Significant preference heterogeneity for various schooling options based on socio-economic characteristics [Hastings, Kane and Staiger, 2009; Abdulkadiroglu, Agarwal and Pathak, 2015]
  - ▶ Biases in beliefs can diminish screening benefits of Immediate Acceptance [Agarwal and Somaini, 2015; Kapor, Neilson and Zimmerman, 2016]
  - ▶ Improving the organization of after-markets is promising [Narita, 2016]

- Some Limitations and avenues for future research
  - ▶ Models take strong stances on parent information [Hastings and Weinstein, 2008]
  - ▶ Peer effects are largely ignored
  - ▶ Limited evidence on monetary value for better schools
  - ▶ Limited evidence on effects of assignment on achievement outcomes

Estimating Preferences Using Stability

- So far...
    - ▸ Settings where individual's choices are observed
    - ✓ Interpreting choices may still be challenging without truthful behavior

- Often observe who matches with whom
    - ▸ College/school enrollment data
    - ▸ Employer-employee match dta

- Typical datasets include characteristics of both sides (eg. workers and firms)
    - ▸ Significant sorting in proxies for quality
    - ✓ Cannot choose your most preferred option $\rightarrow$ must also be chosen

- ✓ Two issues:
    1. Cannot directly use revealed preferences approaches developed earlier
    2. Need two-sided preference model

Stability

- Pairwise stable equilibrium for frictionless markets
  - ▸ IR: Each firm is assigned no more than its capacity
  - ▸ IC: No worker prefers a firm that prefers that worker to an assigned worker (at fixed salaries)

✓ Substantiating this assumption requires knowledge of market institutions
  - ▸ Medical matching market using stable matching algorithms [Agarwal (2015)]
  - ▸ Schools using test scores for admissions [Fack, Grenet and He (2015)]
  - ▸ College admission settings [Akyol and Krishna (2017); Bucarey (2017)]
  - ▸ Decentralized settings [Boyd et.al. (2013); Jiang (2016); Vissing (2017)]

One Side: School Choice or College Admissions

- Suppose we know preferences of schools for students
  - ▸ Student i has test score e_{ij} for school j

- Stability admits a cutoff representation [Azevedo and Leshno (2016)]
  - ▸ Each school has a cutoff

$$P_j = \min_{i \in \mu^{-1}(j)} e_{ij}$$

  - ▸ Students can enroll in any eligible school

$$S(e_i, P) = \{j \in J | P_j \le e_{ij}\}$$

  - ▸ Students enroll at their most preferred eligible school

$$\mu(i) = \arg \max_{j \in S(e_i, P)} u_{ij}$$

  - ✓ Cutoffs ensure that no school is over-subscribed

- Model has student specific choice sets that depend on scores e_{ij}
- ✓ Can construct choice sets using data on P and e_i

✓ Similar to simplest discrete choice setting!

- Less information because no information about unattainable options

Empirical Details

- Applicable to settings with rank-data
  - ▶ Ranks are hard to interpret if students know which schools are unattainable
  - ✓ Stability is more robust, especially in large markets
    [Fack, Grenet and He (2015); Artemov, Che and He (2017)]

- Choice probabilities are similar to standard discrete choice
  - ▶ In mixed-logit case:

$$\mathbb{P}(i \text{ enrolls in } j | X; \beta_i) = \frac{\exp(x_{ij}\beta_i)}{\sum_{k \in S(e_i, P)} \exp(x_{ik}\beta_i)}$$

  - ✓ Notice the denominator

- Extrapolation of preferences for lower performing students

Application: College Financial Aid Design in Chile

- Current Chilean government was elected on the promise of making college free by 2020
  - Active policy debates around the world, including the US

- Several major implications
  - Increase in financial aid across the board
  - Smallest change for current beneficiaries – low income students

✓ What should we expect to happen in 2020?
  - Who gains the most?
  - Are there any losers?

Outline of Analysis: Bucarey (2017)

- Chilean Higher Education
  - ► Most selective colleges use a DA based admission scheme
  - ✓ Known aggregate of test scores used for admission
  - ► Administrative data on student enrollment

- Regression discontinuity and Differences-in-differences analysis
  - ► Students barely eligible for financial aid are more likely to attend 4-year college
    [Solis (2017)]
  - ► Past expansions, although modest, resulted in crowding out of the poor

- Structural Model
  - ► Preferences for various college-majors
  - ► Willingness to pay estimated using regression discontinuity in financial aid

✓ Effects of free tuition?
  - ► Who would enroll where if capacities did not change?
  - ► How much expansion in capacities would offset any adverse effects?
  - ► Can we design a better policy?

	Type of Program			
	All	University	Centralized System	Vocational
A. Percentage Excess of Demand				
Common Price Coefficient Model	47%	58%	44%	28%
Income-heterogeneous Price Coefficient Model	35%	37%	25%	31%
B. Program Characteristics				
Share of Students Enrolled in Baseline	100%	64%	40%	34%
Number of Programs	2,370	2,361	1,287	-

✓ Cutoffs or capacity would have to change!

Effects of Free Tuition
Fixed Capacity

Admission Test Score (t)

Admitted | Applicants

New Admission Cutoff

Displaced Students

Income (I)

Rejected | Applicants

Effects of Free Tuition
Fixed Capacity

	All Programs		University		Centralized System	
	Baseline	Change with Free College	Baseline	Change with Free College	Baseline	Change with Free College
A. Common Price Coefficient Model						
Poorest 20%	0.57	-10%	0.27	-11%	0.15	-13%
2nd Income Quintile	0.78	0%	0.44	-11%	0.27	-20%
3rd Income Quintile	0.73	-5%	0.46	-6%	0.29	-11%
4th Income Quintile	0.69	8%	0.50	9%	0.32	11%
Richest 20%	0.69	17%	0.59	15%	0.38	17%
B. Income-heterogeneous Price Coefficient Model						
Poorest 20%	0.56	-7%	0.27	-12%	0.15	-13%
2nd Income Quintile	0.72	-2%	0.40	-8%	0.25	-12%
3rd Income Quintile	0.72	3%	0.46	-5%	0.29	-12%
4th Income Quintile	0.71	11%	0.53	6%	0.34	7%
Richest 20%	0.72	18%	0.62	11%	0.40	13%

Effects of Free Tuition
Welfare Effects

	Utility	Utility Net of Price	Sticker Tuition	Received Scholarship
			Change in average:	
A. Common Price Coefficient Model				
Family Income				
Poorest Quintile	-$3,396	-$1,180	-$567	$1,137
Second Quintile	-$4,586	-$1,454	-$243	$1,458
Third Quintile	-$2,994	-$1,109	-$524	$1,274
Fourth Quintile	-$1,247	-$776	$630	$2,736
Richest Quintile	-$96	-$490	$1,460	$3,484
Test Scores				
Lowest Quartile	-$8,533	-$2,485	-$2,184	$24
Top Quartile	$1,955	$178	$3,328	$4,515
B. Income-heterogeneous Price Coefficient Model				
Family Income				
Poorest Quintile	-$6,530	-$1,078	-$506	$1,271
Second Quintile	-$3,684	-$990	-$323	$1,379
Third Quintile	-$1,461	-$778	-$25	$1,629
Fourth Quintile	$404	-$572	$675	$3,070
Richest Quintile	$1,486	-$332	$1,204	$3,832
Test Scores				
Lowest Quartile	-$10,980	-$2,178	-$2,160	$34
Top Quartile	$5,480	$614	$2,509	$5,038

Estimating Preferences on Both Sides

- Key problem: Final matches depend on two sets of preferences

- Build intuition using simple model with no preference heterogeneity

$$
\begin{aligned}
u_{ij} &= z_j\beta + w_j + \xi_j \\
h_i &= x_i\alpha + \varepsilon_i
\end{aligned}
$$

  ▶ Perfect assortative matching on u and h

  1. Information in sorting patterns
  2. Usefulness of many-to-one matching structure
     [see Diamond and Agarwal, 2017 for formal analysis]
  ✓ Wage endogeneity can be dealt with using a control function approach

- Related work and alternative approaches
  ▶ Econometric issues [Diamond and Agarwal (2017) and Menzel (2015)]
  ▶ Transferable Utility (flexible salary/transfers) Case [Choo and Siow, 2006; Fox, 2010; Chiappori, Galichon and Salanie, and co-authors (several papers)]
  ▶ Imperfectly Transferable Utility [Galichon, Kominers and Weber, 2015]

Limitation of Sorting Patterns

Resident	Program Characteristic	
Characteristic	Large	Small
High	30%	20%
Low	20%	30%

- Assume that residents from "High" NIH funded schools tend to be better

- Cannot learn about preferences on both sides from sorting patterns alone
  - ► Consistent a strong preference for large hospitals +
    moderate association between high NIH funding and resident skill

  - ► Cannot distinguish from the reverse

  - ► Degree of sorting on observables increases with both α and β
    - Large β and small α vs. large α and small β

Usefulness of Data from Many-to-One Matching

- Data from many-to-one matching provides additional identifying information
  - Do residents matched at the same program have similar characteristics?

- Two residents matched at the same program must be similarly qualified
  - Otherwise, the program can replace the lower quality resident or the higher quality resident can find a better match

- Residents at a program have similar values of x if it strongly predicts human capital $\Rightarrow$ small within-program variation
  - If x is important, then programs pick residents with most desirable x
  - Low preference for x results in programs picking residents with varying x

- Provides crucial information that is not available in one-to-one matching
  - Combine with sorting patterns to learn about preferences on both sides

Information in Many-to-One Matching

- Fraction of variation within-program decreases with importance of resident characteristic
  - Programs are more segregated by degree type than gender
  - Consistent with degree but not gender being associated with skills

	Fraction of Variation Within Program-Year
Log NIH Fund (MD)	77.8%
Median MCAT (MD)	72.1%
Osteopathic/DO Degree	85.2%
Foreign Degree	57.2%
Allopathic/MD Degree	64.8%
Female	96.4%

Estimates for Job Characteristics

Select Variables	Full Heterogeneity (1)	Geographic Heterogeneity (2)	Geo. Het. w/ Instrument (3)
Case Mix Index (1 sd.)	$4,792	$2,320	$6,088
Random Coeff. (sigma)	$4,503		
Log NIH Fund (Major) (1 sd.)	$491	$6,499	$4,402
Random Coeff. (sigma)	$5,498		
Log Beds (1 sd.)	$6,900	$3,528	$8,837
Random Coeff. (sigma)	$11,107		
Log NIH Fund (Minor) (1 sd.)	$4,993	$5,560	$7,620
Medical School State	$9,820	$2,302	$4,529
Birth State	$6,342	$1,320	$2,451
Rural Birth x Rural Program	$1,189	$109	$233

Centralization and Salary Setting

- Centralized mechanism in the medical market has been criticized as a mechanism for wage suppression [Jung et.al., 2002]
  - ▸ Medical residents make approximately $40,000 less than substitute labor
  - ✓ Lawsuit argument based on a perfect competition benchmark

- Debate on the effects of this market reform on wage suppression
  - ▸ Presence of a match does not seem to be associated with lower wages [Niederle and Roth, 2003]
  - ▸ Theoretical results on effects of centralization on wages are ambiguous [Bulow and Levin, 2006; Kojima, 2007]

- Medical labor markets are characterized by imperfect competition
  - ▸ Sources: Accreditation restrictions on programs, fixed costs of operating and heterogeneity in quality
  - ▸ Conservative estimates suggest that these sources result in wage depression (implicit tuition) of at least $23,000 relative to MPL net training costs [Agarwal, 2015]
  - ✓ These sources of imperfect competition are not directly related to the match

✓ Centralized designs with ordered contracts allows for salary flexibility [Kelso and Crawford, 1982; Niederle, 2007; Crawford, 2008]

Empirical Findings and Conclusions

- Approaches that use related equilibrium models on the final outcomes have been used to quantify
  - ▸ Match surplus in the market for venture capital   [Sorenson, 2007]
  - ▸ Factors that determine the marital surplus   [Choo and Siow, 2006; Chiappori, Salanie and Wiess, 2016, and others]
  - ▸ Efficiency of various market mechanisms   [Bajari and Fox, 2013; Jiang, 2016]
  - ▸ Value of mergers in the Industrial Organizations literature   [Akkus, Cookson and Hortacsu, 2015]
  - ▸ Determinants of public school teacher matching   [Boyd et.al., 2013]
  - ▸ Effects of market power on drilling leases   [Vissing (2016)]

- Limitations and avenues for future research
  - ▸ Restrictive assumptions on preference distribution
  - ▸ Richer matching function at the cost of incorporating market frictions   [see Sorkin, 2016, for an exception]

Summary and Future Directions

- Avenues for future research
  - What are the effects of matching systems on education/health outcomes?
  - Comparison with decentralized markets

✓ Features commonly arise in the allocation of rival and non-excludable goods
  - Heterogeneity and capacity constraints
  - Limitations on the price mechanism

- A variety of other settings
  - Course allocation mechanisms
  - College applications in non-centralized schemes
  - Allocation of medical care via wait-lists
  - Public housing and organ allocation   [Agarwal et.al., 2018; Waldinger, 2017]

✓ Key: Empirical approach depends on
  1. Rules of the market
  2. Available data
  3. Nature of the agent's decision problem

Matching in Richer Domains

Parag Pathak[1]

MIT and NBER

Kelso & Crawford (1982)

- $i \in I$ workers, $f \in F$ firms

- Each firm can hire as many workers as it wants, while each worker can get at most one job

- Workers care for their jobs and salaries, but not for co-workers

$$u^i(f, s_i^f) \quad : \quad \text{Utility of worker } i \text{ matched at firm } f \text{ at wage } s_i^f$$
$$\text{Strictly increasing and continuous in } s_i^f$$

$$y^f(C^f) \quad : \quad \text{Gross product of firm } f \text{ which hires set of workers } C^f$$

$$\pi^f(C^f, s^f) \quad \equiv \quad y^f(C^f) - \sum_{i \in C^f} s_i^f, \quad \text{Net profits of firm } f$$

Firm's Problem

Suppose firm f face a vector of salaries $s^f \equiv (s_i^f)_{i \in I}$

$D^f(s^f)$: Set of solutions to firm's profit maximization problem

$$\max_C \pi^f(C, s^f)$$

$D^f(s^f)$ is the set of optimal groups of workers for firm f given salary vector s^f

Gross Substitutes

Workers are **gross substitutes** for a firm f if, for any pair of salary vectors $s^f, \tilde{s}^f$, with $\tilde{s}^f \geq s^f$,

$$C^f \in D^f(s^f) \implies \exists \tilde{C}^f \in D^f(\tilde{s}^f) \text{ s.t. } j \in \tilde{C}^f \text{ for all } j \text{ with } \tilde{s}_j^f = s_j^f$$

That is, workers are **gross substitutes** for a firm if the demand for a worker j does not go down when the salary of another worker i goes up.

Recall from GE theory that gross substitutes of Marshallian demand implies uniqueness of competitive equilibrium

Allocations

- **Matching**, $\mu : I \to F \cup \emptyset$, function that assigns each worker to at most one job

- **Allocation**: a matching μ together with a salary schedule $\left(s_i^{\mu(i)}\right)_{i \in I}$

  If $\mu(i) = \emptyset$, then $s_i^{\mu(i)} = 0$

- Allocation is **individually rational** if no worker or firm instead prefers to remain unmatched

The Strict Core and Core

- Individually rational allocation $\big(\mu; (s_i^{\mu(i)})_{i \in I}\big)$ is in the strict core if there is no firm+group of workers combination (g, J) and salary vector p^g such that
  1. $u^j(g, p_j^g) \geq u^j\big(\mu(j), s_j^{\mu(j)}\big)$ for all $j \in J$, and
  2. $\pi^g(J, p^g) \geq \pi^g\big(\mu^{-1}(g), s^g\big)$

  where at least one of the inequalities is strict.

- Individually rational allocation $\big(\mu; (s_i^{\mu(i)})_{i \in I}\big)$ is in the core if there is no firm+group of workers combination (g, J) and salary vector p^g such that
  1. $u^j(g, p_j^g) > u^j\big(\mu(j), s_j^{\mu(j)}\big)$ for all $j \in J$, and
  2. $\pi^g(J, p^g) > \pi^g\big(\mu^{-1}(g), s^g\big)$.

- Clearly strict core is a subset of the core. The two are equivalent when salaries are continuous.

Kelso-Crawford 'Auction'

Step 0: Set all salaries to 0

Step 1: Firms make offers to their most preferred set of workers. No offer can be withdrawn

Step 2: Each workers evaluates his offers. He tentatively holds the best acceptable offer (if any), and rejects the rest

Step 3: If no offer is rejected, terminate the procedure. Otherwise, for each rejected offer a firm made, increase the salary of the rejecting worker by one unit. This defines the revised salaries.

Step 4: Return to Step 1

Kelso-Crawford 'Auction'

- Gross substitutes condition assures that the "No offer can be withdrawn" condition (Step 1) does not hinder firms' ability to optimize in Step 1 no matter how salaries evolve

- Also observe that nothing changes in Step 2 if workers evaluate "all offers to date" rather than merely the existing offers

- As the algorithm progresses
  a. the set of feasible offers shrink for firms,
  b. the set of available offers grow for the workers

  Therefore the optimal choices of firms and workers evolve in opposite directions as the algorithm proceeds

  Like the deferred acceptance algorithm

Kelso-Crawford 'Auction'

- **Theorem**: Assume workers are gross substitutes for each firm. Then
  - a. the Kelso-Crawford auction terminates,
  - b. it produces an allocation in the core, and
  - c. its outcome is weakly preferred by every firm to any other allocation in the core.

- Alternatively we can consider a worker proposing version of the above auction where workers start offers at the highest feasible salary

- Modern treatments have been based on tools from lattice theory, largely thanks to Hatfield and Milgrom (2005)

- Standard reference is Topkis (1998): Supermodularity and Complementarity

Matching with Contracts

$$
\begin{aligned}
D &= \text{Set of doctors} \\
H &= \text{Set of Hospitals} \\
X &= \text{Finite set of contracts}
\end{aligned}
$$

Each contract (except the null contract) is associated with one doctor and one hospital.

Examples:

- Gale & Shapley (1962) College Admissions: $X = D \times H$
- Kelso & Crawford (1982) Labor Market: $X = D \times H \times$ Salaries

Hatfield and Milgrom (2005)

- Hatfield and Milgrom (2005): use tools from lattice theory to generalize Gale-Shapley algorithm
- Preferences expressed in terms of choice sets: $C_h(X)$: choice of hospital h from set of contracts X; $R_h(X) \equiv X \backslash C_h(X)$: contracts rejected by hospital h
- Gale-Shapley expressed as an isotone operator defined via "opportunity sets": e.g., the set of doctors not yet rejected by a hospital
- Two key ideas:
  - **Substitutes**:

  $$X' \subseteq X'' \Rightarrow R_h(X') \subseteq R_h(X'')$$

  Addition of a contract to a choice set never induces a hospital to take a contract it previously rejected
  - **Law of Aggregate Demand**:

  $$X' \subseteq X'' \Rightarrow |C_h(X')| \leq |C_h(X'')|$$

- Under substitutes, we obtain key results on lattice structure, existence, side-optimality, opposite interests as before
- Under Substitutes and Law of Aggregate Demand, Hatfield and Milgrom show that generalized Gale-Shapley algorithm is strategy-proof; and rural hospitals theorem

Junior Officer Retention in the Army

- There are two main programs the U.S. Army relies on to recruit officers:
  - United States Military Academy (USMA)
  - Reserve Officer Training Corps (ROTC)

- Graduates of USMA and ROTC enter active duty for an initial period of obligatory service upon completing their programs.

- The Active Duty Service Obligation (ADSO) is
  - 5 years for USMA graduates,
  - 4 years for ROTC scholarship graduates, and
  - 3 years for ROTC non-scholarship graduates.

Army's Difficulty Retaining Junior Officers

- Upon completion of this obligation, an officer may apply for voluntary separation or continue on active duty

- The low retention rate of these junior officers has been a major issue for the U.S. Army since the late 1980s.

- In the last few years, the Army has responded to this challenge with unprecedented retention incentives, including **branch-for-service** incentives programs offered by both USMA and ROTC (Wardynski, Lyle, and Colarusso 2010)

Army Branches

- During the fall semester of their senior year, USMA and ROTC cadets "compete" for a slot from the following 16 branches:

 Adjutant General's Corps

 Air Defense Artillery

 Armor

 Aviation

 Chemical Corps

 Corps of Engineers

 Field Artillery

 Finance Corps

 Infantry

 Medical Service Corps

 Military Intelligence

 Military Police Corps

 Ordnance Corps

 Quartermaster Corps

 Signal Corps

 Transportation Corps

- Career advancement possibilities vary widely across different branches

Cadet-Branching Prior to 2006

- Long tradition of assigning branches to cadets based on their preferences and their merit ranking

- Merit ranking is known as the **order-of-merit list** (OML) in the military and is based on a weighted average of academic performance, physical fitness test scores, and military performance

- Until 2006, the **simple serial dictatorship** based on OML was used by the USMA and ROTC

Cadet-Branching Reform in 2006

- In 2006, both programs changed their mechanisms in response to historically low retention rates of their graduates

- **Idea:** Since branch choice is essential for most cadets, why not allow them to bid an additional period of obligatory service for their desired branches?

- Fraction of slots up for bidding is
  - 25% for USMA
  - 50% for ROTC

- New matching process is referred as the **branch-for-service program** for both USMA and ROTC

Cadet-Branch Matching Problem

A **cadet-branch matching problem** consists of
1) a finite set of cadets $I = \{i_1, i_2, \ldots, i_n\}$,
2) a finite set of branches $B = \{b_1, b_2, \ldots, b_m\}$,
3) a vector of branch capacities $q = (q_b)_{b \in B}$,
4) a set of "terms" or "prices" $T = \{t_1, \ldots, t_k\} \in \mathbb{R}_+^k$
   where t_1 is the cheapest, $\ldots$, and t_k is the most expensive term,
5) a list of cadet preferences $P = (P_i)_{i \in I}$ over $(B \times T) \cup \{\emptyset\}$, and
6) a list of base priority rankings $\pi = (\pi_b)_{b \in B}$.

- $\pi_b : I \to \{1, \ldots, n\}$: The function that represents the base priority ranking of cadets for branch b
- $\pi_b(i) < \pi_b(j)$ means that cadet i has higher claims to a slot at branch b than cadet j, other things being equal

Primitives

- Cadet **preferences** over branch-price pairs:
  - $\succ_i$:   Cadet preferences over branches alone
  - $\mathcal{P}$:   Set of all preferences over $(B \times T) \cup \{\emptyset\}$
  - $\mathcal{Q}$:   Set of all preferences over B

- **Contract** $x = (i, b, t) \in I \times B \times T$: a cadet i, a branch b, and the terms of their match

  $X \equiv I \times B \times T$: set of all contracts

- **Allocation** $X' \subset X$ is a set of contracts such that each cadet appears in at most one contract and no branch appears in more contracts than its capacity.

$\mathcal{X}$:	Set of all allocations
$X'(i) = (b, t)$:	Assignment of cadet i under allocation X'
$X'(i) = \emptyset$:	Cadet i remains unmatched under X'

Fairness

- For a given problem, an allocation X' is **fair** if

$$\forall i, j \in I, \qquad \underbrace{X'(j)}_{=(b,t)} P_i\, X'(i) \Rightarrow \pi_b(j) < \pi_b(i).$$

  That is, a higher-priority cadet can never envy the <u>assignment</u> of a lower-priority cadet under a fair allocation.

- Possible for a higher-priority cadet to envy the branch assigned to a lower-priority cadet under a fair allocation only if the latter pays the price of a longer term

Mechanisms

- A **mechanism** is a **strategy space** S_i for each cadet i along with an **outcome function**

$$\varphi : \prod_{i \in I} S_i \to \mathcal{X}$$

  that selects an allocation for each strategy vector $(s_1, s_2, \ldots, s_n) \in \prod_{i \in I} S_i$.

- A **direct mechanism** is a mechanism where the strategy space is simply the set of preferences $\mathcal{P}$ for each cadet i.

- A direct mechanism is **fair** if it always selects a fair allocation.

The USMA Mechanism

- All cadets receive an assignment under the USMA mechanism (unassigned manually placed).

    $\mathcal{P}$: Set of preferences over $B \times T$

- Since 2006, $T = \{t_1, t_2\}$.

    t_1: **Base price**
    t_2: **Increased price**

- Denote any contract with increased price t_2 as a **branch-of-choice contract**

Strategy Space under the USMA Mechanism

- Each cadet is asked to choose
  1) a ranking of branches alone
  2) a number of branches (possibly none) for which the cadet is asked to sign a branch-of-choice contract.

  Hence $S_i = \mathcal{Q} \times 2^B$ for each cadet i.

- Let $(\succ_i', B_i)$ be the strategy choice of cadet i under the USMA mechanism for a given problem

  Interpretation of B_i:
  - For each branch $b \in B_i$, cadet i is willing to pay the increased price t_2 in exchange for favorable treatment for the last 25 percent of slots
  - Cadet i will need to pay the increased price <u>only if</u> he receives one of the last 25 percent of the slots for which he is favored.

Strategy Space under the USMA Mechanism

- For each branch b,
    - priority for the top 75 percent of slots determined by the order-of-merit list $\pi_b = \pi^{OML}$
    - cadets who sign a branch-of-choice contract for branch b receive favorable treatment for the last 25 percent of slots

- Priority for the last 25 percent of slots is based on the following **adjusted priority ranking** π_b^+:

For any $i, j \in I$,
- if $b \in B_i$ and $b \notin B_j$, then $\pi_b^+(i) < \pi_b^+(j)$,
- if $b \in B_i$ and $b \in B_j$, then $\pi_b^+(i) < \pi_b^+(j) \Leftrightarrow \pi_b(i) < \pi_(j)$,
- if $b \notin B_i$ and $b \notin B_j$, then $\pi_b^+(i) < \pi_b^+(j) \Leftrightarrow \pi_b(i) < \pi_b(j)$.

Outcome Function of USMA Mechanism

For a given strategy profile $(\succ_i', B_i)_{i \in I}$, the USMA mechanism determines the final outcome with the following USMA algorithm:

Step 1: Each cadet i "applies" to his top-choice under $\succ_i'$.

* Each branch b *holds* the top $0.75q_b$ candidates based on π_b.
* Among the *remaining applicants* it *holds* the top $0.25q_b$ candidates based on the adjusted priorities π_b^+.

Any remaining applicants are rejected.

In general, at

Step k: Each cadet i who is rejected at Step (k-1) "applies" to his next-choice under $\succ_i'$.

* Each branch b reviews the new applicants along with those held from Step (k-1), and *holds* the top $0.75q_b$ based on π_b.
* For the remaining slots, branch b considers all remaining applicants and *holds* the top $0.25q_b$ of them based on the adjusted priorities π_b^+.

Any remaining applicants are rejected.

Outcome Function under the USMA Mechanism

Algorithm terminates when no applicant is rejected. All tentative assignments are finalized at that point.

For any branch b,

- any cadet who is assigned one of the top 75 percent of slots is charged the base price t_1,
- any cadet who is assigned one of the last 25 percent of slots is charged
    - the increased price t_2 if he has signed a branch-of-choice contract for branch b, and
    - the base price t_1 if he has not signed a branch-of-choice contract for branch b.

$\psi^{WP}(s):$ Outcome of USMA mechanism under $s = (\succ'_i, B_i)_{i \in I}$

Preliminary Observations

Let λ be the fraction of slots where branch-of-choice contracts are favored

- When $\lambda = 0$:
  - USMA mechanism reduces to the simple serial dictatorship induced by the order-of-merit list

- When $\lambda > 0$:
  - Truthful preference revelation be suboptimal under the USMA mechanism and the optimal choice of branch-of-choice contracts is not obvious

  - Issue: Mechanism tries to "infer" cadet preferences over branch-price pairs from their submitted preferences over branches alone and signed branch-of-choice contracts

    Strategy-space provided by the USMA mechanism is not rich enough to reasonably represent cadet preferences.

Representation of Priorities via Choice Sets

- Branches have priorities over cadet-price pairs, and these **priorities induce choice sets**

- Hence, the cadet-branch matching problem is a special case of **matching with contracts**

- In general, the choice set of branch b from a set of contacts X' depends on the policy on who has higher claims for slots in branch b. So let's represent the current **USMA priorities,** or any other priorities by adequate construction of choice sets.

- For a given priority structure for branch b,

$C_b(X')$:	Set of contracts chosen from $X' \subseteq X$
$R_b(X') \equiv X' \setminus C_b(X')$:	**Rejected set**

Phase 0: Remove all contracts that involve another branch b' and add them all to rejected set $R_b(X')$.

(Hence each contract that survives Phase 0 involves branch b.)

Phase 1: For the first $0.75q_b$ potential elements of $C_b(X')$, choose the contracts with highest-OML cadets one at a time. When two contracts of the same cadet are available, choose the contract with the base price t_1 and reject the other one.

Continue until either all contracts are considered or $0.75q_b$ elements are chosen for $C_b(X')$.

If the former happens, terminate the procedure.
If the latter happens proceed with Phase 2.1.

Phase 2.1: For the last $0.25q_b$ potential elements of $C_b(X')$, give priority to contracts with increased price t_2. In this phase only consider branch-of-choice contracts and among them include in $C_b(X')$ the contracts with highest-OML cadets. If any cadet covered in Phase 2.1 has two contracts in X' reject the contract with the base price t_1. Continue until either all branch-of-choice contracts are considered in X' or $C_b(X')$ fills all q_b elements. For the latter case, reject all remaining contracts, and terminate the procedure. For the former case, terminate the procedure if all contracts in X' are considered and proceed with Phase 2.2 otherwise.

Phase 2.2: By construction, all remaining contracts in X' have the base price t_1. Include in $C_b(X')$ the contracts with highest-priority cadets one at a time until either all contracts in X' are considered or $C_b(X')$ fills all q_b elements. Reject any remaining contracts, placing them in $R_b(X')$.

- Allocation X' is **stable** if
  1) no cadet or branch is imposed an **unacceptable** contract, and
  2) there exists no cadet i, branch b, and contract $x = (i, b, t) \in X \setminus X'$ s.t.

$$(b, t) \, P_i \, X'(i) \quad \text{and} \quad x \in C_b(X' \cup \{x\}).$$

- Motivation:
  - ✓ If the first condition fails, then the outcome is not **individually rational**
  - ✓ If the second requirement fails, then there exists an unselected contract (i, b, t) where cadet i prefers pair (b, t) to his assignment and also contract x has sufficiently high priority to be selected by branch b.

Two cadets i_1, i_2, one branch b with two slots and $\lambda = 0.5$

Cadet i_1 has higher priority than cadet i_2

$$(b_1, t_1)P_{i_1}(b_1, t_0) \quad \text{and} \quad (b_1, t_0)P_{i_2}(b_1, t_1)$$

In allocation

$$X' = \{(i_1, b_1, t_0), (i_2, b_1, t_1)\},$$

the higher-priority cadet i_1 envies i_2, so X' not fair.

But its stable because branch b gives priority to contract (i_1, b_1, t_0) over contract (i_1, b_1, t_1) for the first slot

Irrelevance of Rejected Contracts

- Three properties of choice sets (branch priorities in our context) are central in matching with contracts.

- Priorities satisfy the **irrelevance of rejected contracts** for branch b if $\forall X' \subset X, \forall x \in X \setminus X'$,

$$x \notin C_b(X' \cup \{x\}) \implies C_b(X') = C_b(X' \cup \{x\}).$$

That is, the removal of rejected contracts have no effect on the choice set under the IRC condition.

Lemma: *USMA priorities satisfy IRC.*

- Aygun and Sönmez (2013) show that IRC is implicitly assumed throughout the analysis of Hatfield and Milgrom (2005)

Law of Aggregate Demand

- Priorities satisfy the **law of aggregate demand (LAD)** for branch b if

$$X' \subset X'' \Rightarrow |C_b(X')| \leq |C_b(X'')|$$

That is, the size of the choice set never shrinks as the set of contracts grows under the LAD condition.

Lemma: *USMA priorities satisfy the LAD.*

Substitutes Condition

- Elements of X are **substitutes** for branch b if

$$\forall X' \subset X'' \subseteq X, \qquad R_b(X') \subseteq R_b(X'').$$

- Matching with contracts: Substitutes + IRC $\Rightarrow$ existence of a stable allocation

- Elements of X are **unilateral substitutes** for branch b if, whenever a contract $x = (i, b, t)$ is rejected from a smaller set X' even though x is the only contract in X' that includes cadet i, contract x is also rejected from a larger set X'' that includes X'.

Contracts not Substitutes

$I = \{i_1, i_2\}$ in OML order and $B = \{b_1\}$ with $q_{b_1} = 2$, $\lambda = 0.5$

$$X' = \{(i_2, b_1, t_0), (i_2, b_1, t_1)\}$$
$$X'' = \{(i_1, b_1, t_0), (i_2, b_1, t_0), (i_2, b_1, t_1)\}$$

- $C_{b_1}(X') = \{(i_2, b_1, t_0)\}$ and $R_{b_1}(X') = \{(i_2, b_1, t_1)\}$
- $C_{b_1}(X'') = \{(i_1, b_1, t_0), (i_2, b_1, t_1)\}$ and $R_{b_1}(X'') = \{(i_2, b_1, t_0)\}$

Contract (i_2, b_1, t_1) is rejected from X', but not from $X'' \supset X'$

When two contracts include the same cadet, $(i, b, t_0), (i, b, t_1)$ contract (i, b, t_0) might be rejected at the expense of (i, b, t_1) from a larger set X'' while the contract choice is reversed for a subset X' of X''

Only reason a contract can be rejected from a smaller set despite being chosen from a larger one, so we satisfy unilateral substitutes

Cumulative Offer Algorithm and COSM

Cadet-optimal stable mechanism (COSM)

- Strategy space of each cadet is $\mathcal{P}$ under the COSM, and hence it is a direct mechanism.

- Fix branch priorities (and hence the choices sets). Given a profile $P \in \mathcal{P}$, the following **cumulative offer algorithm (COA)** can be used to find the outcome of COSM.

  Step 1: Start the offer process with the highest OML cadet $\pi(1) = i(1)$. Cadet $i(1)$ offers his first-choice contract $x_1 = (i(1), b(1), t)$ to branch $b(1)$ that is involved in this contract. Branch $b(1)$ holds the contract if $x_1 \in C_{b(1)}(\{x_1\})$ and rejects it otherwise. Let $A_{b(1)}(1) = \{x_1\}$ and $A_b(1) = \emptyset$ for all $b \in B \setminus \{b(1)\}$.

Cumulative Offer Algorithm and COSM

In general, at

Step k: Let $i(k)$ be the highest OML cadet for whom no contract is currently held by any branch. Cadet $i(k)$ offers his most-preferred unrejected contract to branch $b(k)$. Branch $b(k)$ holds the contract if $x_k \in C_{b(k)}(A_{b(k)}(k-1) \cup \{x_k\})$ and rejects it otherwise. Let $A_{b(k)}(k) = A_{b(k)}(k-1) \cup \{x_k\}$ and $A_b(k) = A_b(k-1)$ for all $b \in B \setminus \{b(k-1)\}$.

The algorithm terminates when each cadet either has an offer that is on hold or has no remaining acceptable contracts. Since there are a finite number of contracts, the algorithm terminates after a finite number T of steps. All contracts held at this final Step T are finalized and the final allocation is $\bigcup_{b \in B} C_b(A_T)$.

COSM and Unilateral Substitutes

Theorem: *Suppose that the priorities satisfy the unilateral substitutes condition and the IRC. Then the COA produces a stable allocation that is weakly preferred by any cadet to any stable allocation. If in addition the priorities satisfy the LoAD, then the COSM is also strategy-proof.*

Improving the USMA Mechanism

φ^{USMA}: COSM induced by USMA priorities

- COSM induced by USMA priorities fixes all previously mentioned deficiencies of the USMA mechanism.

  Proposition. *The outcome of φ^{USMA} is stable under USMA priorities and it is weakly preferred by any cadet to any stable allocation. Moreover φ^{USMA} is strategy-proof and fair.*

- Note that stability does not imply fairness here since we haven't assumed that a cadet prefers (b, t_0) to (b, t_1)

Matching with Contracts

- Literature pursuing generalizations of Gale-Shapley is very active

- Relationship between models
  - Echenique (AER 2012); Hatfield and Kojima (JET 2010); Hatfield-Kominers (2015)

- Several papers consider matching in even richer domains – relaxing two-sidedness
  - Ostrovsky (2008); Hatfield-Kominers-Westkamp (2016)

- Other directions include applied papers
  - Sönmez (2013)

Research Design Meets Market Design: Using Centralized Assignment for Impact Evaluation

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Outline

Road Map

- How to utilize centralized assignment data for impact evaluation?
- Selection Bias
- Centralized assignment and solving the selection bias problem
- Large market approximation to assignment probabilities under DA
- Propensity score based research design
- Evaluating Denver's charter school sector

- How can centralized assignment be used for impact evaluation?
- The impact on educational outcomes of
  - Early childhood education
  - School size
  - Teacher training
  - School resources
  - Peer effects
  - Charter schools
  - Exam schools
- Portfolio planning
  - System evaluation beyond simple summary stats
- Major challenge: Selection Bias

Data

= student

QUESTION:
What is the causal impact of
attending BLUE on test scores?

Data

10 means student has a test score of 10

Selection Bias

$$\frac{10+10}{2}=10$$

$$\frac{10+0+0+10+10}{5}=6$$

Average effect:

10 $-$ 6 $= 4$

Selection Bias

$$\frac{10+10}{2}=10$$

$$\frac{10+10}{2}=10$$

10

0

0

10

10

10

10

GREEN:
Highly motivated students

Average effect (if we could observe Green):

10 — 10 = 0

Gale and Shapley Go to School

- Parents in large urban districts increasingly have the opportunity to choose schools from a menu of options; they express strict **preferences** by ranking schools on this menu

- Schools establish coarse **priorities** over applicants, determined by neighborhood, sibling enrollment, and poverty status
  - ✓ Boston, Chicago, Denver, New Orleans, Newark, NYC, and Washington DC convert preferences and priorities into school assignments using variants of Gale and Shapley's **Deferred Acceptance** (DA)
  - ✓ Ties for applicants with the same priority are often broken by randomly assigned lottery numbers

- Can these lotteries be utilized for impact evaluation?
- A growing literature uses quasi-experiments embedded in centralized matches for research

Outline

Randomized Experiments

Randomization in Deferred Acceptance Assignment

Group 1 Students

Preferences	Priority Ranking
School A	4
School B	5

Group 2 Students

Preferences	Priority Ranking
School C	2
School A	2

Group 3 Students

Preferences	Priority Ranking
School C	2
School B	5

- Group 2 and 3 will compete at C, same priority, lottery will determine assignment at C
- Group 2 students that lose lottery at C are considered at A and compete against Group 1
- Group 2 students have higher priority at A than Group 1, despite that they might have worse lottery numbers, so the are all assigned as long as there is capacity
- Remaining seats at A are lotteried among Group 1 students
- Group 1 students that lose at A and Group 3 students that lose at C compete at school B
- They both have same priority, lottery determines assignment

Random assignment
- among 2 and 3 at C
- among 2 at A
- among 1 and 3 at B

Randomized Experiments in Universal Enrollment

Wrong Math – Selection Bias

Equal Treatment of Equals

- DA and other assignment mechanisms satisfy the **equal treatment of equals** property: applicants with the same preferences and priorities (or "type") have the same probability distribution over assignments

- Embedded in DA, therefore, is a stratified randomized trial

- Conditional on preferences and priorities schools are assigned randomly

- Perform impact evaluation among students with same preferences and priorities?

- Few students share the same preferences and priorities $\Rightarrow$ eliminates data

Wrong Math – Selection Bias

Correct Math

75 seats

10   5

20

25 of each type

rest of the students

10   0

20

Average Gain for Greens:

$$\frac{(25\times10)}{25} - \frac{(25\times10)}{25} = 0$$

Average Gain for Purples:

$$\frac{(25\times20)}{25} - \frac{(75\times20)}{75} = 0$$

Average Gain for Reds:

$$\frac{(25\times5)}{25} - \frac{(25\times0)}{25} = 5$$

Average effect:

$$\frac{(50\times0)+(100\times0)+(50\times5)}{200} = \text{plus } 1.25$$

50 of

50 of

100 of

Universal Enrollment

Correct Assignment Probabilities → Correct Math

Identifying randomization in assignment is not trivial

Group 1 Students

Preferences	Priority Ranking
School A	4
School B	5

Group 2 Students

Preferences	Priority Ranking
School C	2
School A	4

Group 3 Students

Preferences	Priority Ranking
School C	2
School B	5

- Group 2 students that lose lottery at C are considered at A and compete against Group 1
- Group 2 students have the same priority at A as Group 1
- Group 1 will compete against 2 at A who have already lost lottery at C
- The set of Group 1 losers at A changes, so does Group 3's competition at B
- There is also another random assignment among 1 and 2 at A

A Formula to Identify <u>Randomization</u> and <u>Odds of Assignment</u> at <u>All Choices</u>

- Student choice lists
- School priority lists
- Random numbers
- Assignment

Group students with equal assignment odds together

Correct Assignment Probabilities → Correct Math

Correct Assignment Probabilities → Correct Math

Correct Assignment Probabilities → Correct Math

75 seats

10   5

25 of each type

20

50 of

50 of

100 of

Universal Enrollment

rest of the students

10   0

20

Average effect for Green and Red : 2.5

Average effect for Purple : 0

Average effect for the population:

$$\frac{(50\times2.5)+(50\times2.5)+(100\times0)}{200} = 1.25$$

Selection bias eliminated!

Ad hoc RD/MD

- Previous use of centralized assignment schemes for research:
  - Random tie-breakers include Abdulkadiroğlu, Angrist, Dynarski, Kane, Pathak (2011); Abdulkadiroğlu, Hu, Pathak (2013); Bloom, Unterman (2014); Deming (2011); Deming, Hastings, Kane, Staiger (2013); Hastings, Kane, and Staiger (2009), among others
  - Regression-discontinuity tie-breakers include Abdulkadiroğlu, Angrist, Pathak (2013); Ajayi (2013); Dobbie and Fryer (2013); Jackson (2010); Lucas and Mbiti (2013); Pop-Eleches-Urqiuola (2013)

- This work uses *first-choice* and *qualification instruments* that discard much of the causality-revealing power of quasi-randomized DA

- The **DA propensity score** harnesses <u>all</u> random variation in school offers
  - The DA score retains students and schools (for research) that methods like first-choice discard
  - Furthermore, the DA score reveals why we do or don't see random assignment at one school or another

Outline

Defining the Problem

Market Design: DA Details

- Each student applies to his or her most preferred school.

  - Each school ranks its initial applicants (those who've ranked it first) by priority then by lottery number within priority groups, *tentatively* admitting the highest-ranked applicants up to its capacity
  - Other applicants are rejected

- Each rejected student applies to his or her next most preferred school.

  - Each school ranks these new applicants *together with applicants tentatively admitted in the previous round*, first by priority and then by lottery number
  - From this pool, schools again tentatively admit those it's ranked highest, up to capacity, rejecting the rest

- DA terminates when there are no new applications or each applicant has exhausted the schools he or she has ranked

  - Some students may remain unassigned

Research Design: Ignorable Assignments

- Let $D_i(s)$ indicate whether student i is offered a seat at school s

  - Applicants are characterized by prefs and priorities, their type, θ
  - Type affects assignment and is correlated w/outcomes, hence a powerful source of omitted variables bias (OVB)

- DA induces a stratified RCT

  - Let W_i be any r.v. independent of lottery numbers

  $$Pr[D_i(s) = 1 | W_i, \theta_i = \theta] = Pr[D_i(s) = 1 | \theta_i = \theta] \qquad (1)$$

  - W_i includes potential outcomes and student characteristics like sibling and free lunch status
  - Conditioning on type therefore eliminates any OVB in comparisons by offer status

- But full type conditioning is **impractical**: it eliminates many students and schools from statistical analyses

  - Denver's 5,000 charter applicants include 4,300 types

Propensity Score

We condition instead on the **propensity score**, the probability of assignment to school s for a given type:

$$p_s(\theta) = Pr[D_i(s) = 1 | \theta_i = \theta]$$

Theorem (Rosenbaum & Rubin 1983)

Conditional independence property (1) *implies that for any W_i that is independent of lottery numbers,*

$$P[D_i(s) = 1 | W_i, p_s(\theta_i)] = P[D_i(s) = 1 | p_s(\theta_i)] = p_s(\theta_i)$$

- Why is this useful?
  - The score is much coarser than θ: many types share a score
  - The score identifies the maximal set of applicants for whom we have a randomized school-assignment experiment
  - The score reveals the experimental design embedded in DA: we know (and will show) its structure

Example 1: The Score Pools Types

- Five students $\{1,2,3,4,5\}$; three schools $\{a,b,c\}$, each with one seat
  - student preferences

$$1 : a \succ b$$
$$2 : a \succ b$$
$$3 : a$$
$$4 : c \succ a$$
$$5 : c$$

  - school priorities
    - 2 has priority at b
    - 5 has priority at c

- Types are unique, ruling out research with full-type conditioning
- The score pools: DA assigns students 1, 2, 3, and 4 to school a each with probability 0.25
  - 5 beats 4 at c by virtue of priority; this leaves 1, 2, 3, and 4 all applying to a in the second round and no one advantaged there

Example 2: Further Pooling in Large Markets

- Four students $\{1, 2, 3, 4\}$; three schools $\{a, b, c\}$, each with one seat and no priorities
  - student preferences

$$
\begin{array}{l}
1 : c \\
2 : c \succ b \succ a \\
3 : b \succ a \\
4 : a
\end{array}
$$

  Types are again unique

- There are $4! = 24$ possible assignments. Enumerating these, we find
  - $p_a(1) = 0$, since 1 doesn't rank a
  - $p_a(2) = 2/24 = 1/12$
  - $p_a(3) = 1/24$
  - $p_a(4) = 1 - p_a(1) - p_a(2) = 21/24$
- No pooling

- The probability of assignment to a is determined by
  - Failure to be seated at schools ranked more highly than a
  - Success in the competition for a conditional on this failure

- Type 2 is seated at a when:
  - Schools he's ranked ahead of a (schools b and c) are filled by others
  - He also beats type 4 in competition for a seat at a
    - This happens for two realizations of the form $(s, t, 2, 4)$ for $s, t = 1, 3$

- Type 3 is seated at a when:
  - Schools he's ranked ahead of a (school b) are filled by another and he beats type 4 at a
    - This happens only when the lottery order is $(1, 2, 3, 4)$

The Large-Market P-Score

- An $n-scaled$ version of Example 2:
    - n each of types 1-4 apply to 3 schools, each with n seats
    - Enumeration with large n is a chore, but repeating lottery draws reveals a common score for types 2 and 3 for n more than a few:

Probability of Assignment to School a

Outline

Deriving the DA Score

Score Computation

- $p_s(\theta)$ is generated by a permutation distribution, a relative frequency generated by all possible lottery realizations
- That is 26,000! lotteries for DPS... I'll get back to you...
  - The LLN tells us it's enough to sample lotteries. But since covariates are discrete, the resulting empirical $\hat{p}_s(\theta)$ has as many points of support as does θ (cf. HIR 2003)
  - Sim scores are a black box; sample-based $\hat{p}_s(\theta)$ must be smoothed
- Is there a formula?
  - Except for special cases, $p_s(\theta)$ as no closed form
- Our large market continuum model provides the formula we need
  - The DA score for a continuum market approximates the score as a function of a few easily-computed sufficient statistics
  - The DA score is automatically coarse: no simulation, smoothing or rounding required
  - The DA score reveals the nature of the stratified trial embedded in DA: which schools have random assignment and why

Outline

DA Formalities

- *I* students with preferences $\succ_i$ and priorities for school s given by $\rho_{is} \in \{1, ..., K, \infty\}$
- Student i's type is $\theta_i = (\succ_i, \rho_i)$, where ρ_i is the vector of i's ρ_{is}
- $s = 1, ..., S$ schools, with capacity vector $\mathbf{q} = (q_1, ..., q_S)$
  - In the continuum (large market), $I = [0, 1]$ and q_s is the proportion of I that can be seated at s

- Student i's *lottery number*, r_i, is i.i.d. uniform $[0, 1]$
- Student i's *rank* at school s is

$$\pi_{is} = \rho_{is} + r_i$$

- DA assignment is determined by a *vector of cutoffs*, c_s: applicants to s with $\pi_{is} \leq c_s$ and $\pi_{i\tilde{s}} > c_{\tilde{s}}$ $\forall \tilde{s}$ they prefer to s, are seated at s
  - Lottery numbers matter for assignment to s only in the *marginal priority group*

Illustrating Cutoffs and Marginal Priorities

Rank	Priority	Lottery No.	Offer	
1.13	1	.13	1	
1.99	1	.99	1	
2.05	2	.05	1	
2.35	2	.35	1	c_s=2.35
2.57	2	.57	0	
2.61	2	.61	0	
3.12	3	.12	0	
3.32	3	.32	0	

- Marginal priority, denoted ρ_s, is the integer part of c_s; here, $\rho_s = 2$
- The *lottery cutoff, denoted* τ_s, is the decimal part of c_s; here, $\tau_s = .35$

Assignment Outcomes: Partitioning Types

- Let Θ_s denote the set of types who rank s
  - $B_{\theta s}$ denotes the set of schools that type θ prefers to s
- This set is partitioned by:
  - Θ_s^n, defined by $\rho_{\theta s} > \rho_s$
    - These **never-seated** applicants have worse than marginal priority at s
    - *No one* in this group is seated at s
  - Θ_s^a, defined by $\rho_{\theta s} < \rho_s$
    - These **always-seated** applicants clear marginal priority at s
    - *Everyone* in this group is seated at s when not seated at a school in $B_{\theta s}$
  - Θ_s^c, defined by $\rho_{\theta s} = \rho_s$
    - These **conditionally seated** applicants have marginal priority at s
    - Members of this group are seated at s when not seated at a school in $B_{\theta s}$ *and* they clear the lottery cutoff at s

Assignment Risk: Most Informative Disqualification

- Define

$$\text{MID}_{\theta s} = \begin{cases} 0 & \text{if } \rho_{\theta\tilde{s}} > \rho_{\tilde{s}} \text{ for all } \tilde{s} \in B_{\theta s} \\ 1 & \text{if } \rho_{\theta\tilde{s}} < \rho_{\tilde{s}} \text{ for some } \tilde{s} \in B_{\theta s} \\ \max\{\tau_{\tilde{s}} \mid \rho_{\theta\tilde{s}} = \rho_{\tilde{s}}, \tilde{s} \in B_{\theta s}\} & \text{if } \rho_{\theta\tilde{s}} \geq \rho_{\tilde{s}} \text{ for all } \tilde{s} \in B_{\theta s} \end{cases}$$

- MID tells us how the lottery number distribution for applicants to s is truncated by qualification at more preferred schools
  - MID is 0 when priority status is worse-than-marginal at all higher ranked schools (no truncation)
  - MID is 1 if $B_{\theta s}$ includes a school where θ is seated with certainty (complete truncation)
  - For those who are marginal or worse at all schools they prefer to s, and marginal somewhere, MID is the most forgiving cutoff in the set of schools at which they're marginal
    - Applicants who clear $\max\{\tau_{\tilde{s}} \mid \rho_{\theta\tilde{s}} = \rho_{\tilde{s}}, \tilde{s} \in B_{\theta s}\}$ are seated in $B_{\theta s}$, and so not at risk for a seat at s

Outline

The DA Propensity Score

Theorem

In a continuum economy, $\Pr[D_i(s) = 1 | \theta_i = \theta] = \varphi_s(\theta) \equiv$

$$\begin{cases} 0 & \text{if } \theta \in \Theta_s^n \\ (1 - MID_{\theta s}) & \text{if } \theta \in \Theta_s^a \\ (1 - MID_{\theta s}) \times \max\left\{0, \dfrac{\tau_s - MID_{\theta s}}{1 - MID_{\theta s}}\right\} & \text{if } \theta \in \Theta_s^c \end{cases}$$

where we set $\varphi_s(\theta) = 0$ *when* $MID_{\theta s} = 1$ *and* $\theta \in \Theta_s^c$

- $MID_{\theta s}$, τ_s, and Θ are population quantities, fixed in the continuum
  - Our second theorem shows that the sample analog of $\varphi_s(\theta)$ converges uniformly to the finite market score as market size grows

Example 2 in the Continuum

- 4 types determined by student preferences:

$$1 : c$$
$$2 : c \succ b \succ a$$
$$3 : b \succ a$$
$$4 : a$$

- $\tau_c = 0.5$ and $\tau_b = 0.75$
- $B_{2a} = \{b, c\}$; $B_{3a} = \{b\}$
- Hence, $\mathsf{MID}_{2a} = \mathsf{MID}_{3a} = \tau_b = 0.75$
  - Types 2 and 3 have the same large market score at school a

- It remains to compute $\tau_a = 5/6$. We then have

$$\varphi_a(\theta) = \max\{0, 5/6 - 0.75\} = \frac{1}{12}$$

Outline

DA Econometrics

- Estimating the DA score described by Theorem 1
  - **Formula**: assign students by priority status to Θ_s^n, Θ_s^a, or Θ_s^c as realized in the match; plug empirical τ_s and $\text{MID}_{\theta s}$ into $\varphi_s(\theta)$
  - **Frequency**: use empirical offer rates in cells with constant $\varphi_s(\theta)$
    - This is looks more like an estimated score of the sort discussed by Abadie and Imbens (2012)

- Simulating the finite-market score
  - We do this using 1,000,000 lottery draws, running DA for each, and computing the empirical assignment rate for each type
    - This converges (as # of draws increase) to the finite-market score
    - No dimension reduction in our data: 1148 score values for 1523 applicants with non-degenerate charter risk. The simulated score requires smoothing
  - By contrast with simulation, Theorem 1 reveals *why* we have random assignment at one school or another

Outline

MILE HIGH CHARTER EFFECTS

Empirical Strategy

- DPS has a large charter sector, part of the SchoolChoice match.
- Impact evaluation for the charter sector
    - An *any-charter* offer dummy, D_i, is the sum of all individual charter offers (our instrument)
    - The *any-charter p-score* (our key control) is the sum of the scores for each charter that type θ ranks
    - C_i indicates any-charter enrollment (our "endogenous" variable)

- 2SLS First and Second stages

$$C_i = \sum_x \gamma(x) d_i(x) + \delta D_i + v_i$$
$$Y_i = \sum_x \alpha(x) d_i(x) + \beta C_i + \varepsilon_i$$

    - $d_i(x)$: dummies for propensity score values (cells), indexed by x
    - $\gamma(x)$ and $\alpha(x)$: associated "score effects"

IV Foundations

- Conditional independence:

$$\{Y_{1i}(d), Y_{0i}(d), C_{1i}, C_{0i}\} \perp\!\!\!\perp D_i | \theta_i; \quad d = 0, 1$$

  where $Y_{ji}(d)$ is potential outcome for applicant i in sector j when the offer instrument, D_i, equals d (CI seems likely given (1))
- Exclusion:

$$Y_{ji}(1) = Y_{ji}(0) \equiv Y_{ji}; \quad j = 0, 1$$

  - Holds if we assume away within-sector differences in potential outcomes; mitigated by a finer-grained parametrization of school sector effects
  - Violated if school quality changes for never-takers in Round 2

- Monotonicity: $C_{1i} \geq C_{0i}$
- By the propensity score and LATE theorems, we have

$$\frac{E[Y_i | D_i = 1, p_D(\theta_i)] - E[Y_i | D_i = 0, p_D(\theta_i)]}{E[C_i | D_i = 1, p_D(\theta_i)] - E[C_i | D_i = 0, p_D(\theta_i)]} = E[Y_{1i} - Y_{0i} | p_D(\theta_i), C_{1i} > C_{0i}]$$

  - 2SLS and a semiparametric estimand marginalize this

Table 1: DPS charter schools

School	Total applicants	Applicants offered seats	Propensity score in (0,1)			
			DA score (frequency)	DA score (formula)	Simulated score	Simulated score (first choice)
	(1)	(2)	(3)	(4)	(5)	(6)
Elementary and middle schools						
Cesar Chavez Academy Denver	62	9	7	9	8	3
Denver Language School	4	0	0	0	0	0
DSST: Cole	281	129	31	40	44	0
DSST: College View	299	130	47	67	68	0
DSST: Green Valley Ranch	1014	146	324	344	357	291
DSST: Stapleton	849	156	180	189	221	137
Girls Athletic Leadership School	221	86	18	40	48	0
Highline Academy Charter School	159	26	69	78	84	50
KIPP Montbello College Prep	211	39	36	48	55	20
KIPP Sunshine Peak Academy	389	83	41	42	44	36
Odyssey Charter Elementary	215	6	20	21	22	14
Omar D. Blair Charter School	385	114	135	141	182	99
Pioneer Charter School	25	5	0	2	2	0
SIMS Fayola International Academy Denver	86	37	7	18	20	0
SOAR at Green Valley Ranch	85	9	41	42	43	37
SOAR Oakland	40	4	0	9	7	2
STRIVE Prep - Federal	621	138	170	172	175	131
STRIVE Prep - GVR	324	112	104	116	118	0
STRIVE Prep - Highland	263	112	2	21	18	0
STRIVE Prep - Lake	320	126	18	26	26	0
STRIVE Prep - Montbello	188	37	16	31	35	0
STRIVE Prep - Westwood	535	141	235	238	239	141
Venture Prep	100	50	12	17	17	0
Wyatt Edison Charter Elementary	48	4	0	3	2	0
High schools						
DSST: Green Valley Ranch	806	173	290	343	330	263
DSST: Stapleton	522	27	116	117	139	96
Southwest Early College	265	76	34	47	55	0
Venture Prep	140	39	28	42	45	0
KIPP Denver Collegiate High School	268	60	29	37	40	24
SIMS Fayola International Academy Denver	71	15	6	22	22	0
STRIVE Prep - SMART	383	160	175	175	175	175

Notes: This table describes DPS charter applications. Column 1 reports the number of applicants ranking each school. Columns 3-6 count applicants with propensity score values strictly between zero and one according to different score computation methods. Column 6 shows the subset of applicants from column 5 who rank each school as their first choice.

Sample Size Gains: Non Charter Schools

Score Anatomy at STRIVE Prep

Table 2: DA Score anatomy

| Campus | Eligible applicants (1) | Capacity (2) | Offers (3) | DA Score = 0 | | | DA Score in (0,1) | | DA Score = 1 |
				Θ_s^n $0\leq\text{MID}\leq1$ (4)	Θ_s^c $\text{MID}\geq\tau_s$ (5)	Θ_s^a $\text{MID}=1$ (6)	Θ_s^c $\text{MID}<\tau_s$ (7)	Θ_s^a $0<\text{MID}<1$ (8)	Θ_s^a $\text{MID}=0$ (9)
GVR	324	147	112	0	0	159	0	116	49
Lake	274	147	126	0	0	132	0	26	116
Highland	244	147	112	0	0	121	0	21	102
Montbello	188	147	37	0	0	128	0	31	29
Federal	574	138	138	78	284	3	171	1	37
Westwood	494	141	141	53	181	4	238	0	18

Notes: This table shows how formula scores are determined for STRIVE school seats in grade 6 (all 6th grade seats at these schools are assigned in a single bucket; ineligible applicants, who have a score of zero, are omitted). Column 3 records offers made to these applicants. Columns 4-6 show the number of applicants in partitions with a score of zero. Columns 7 and 8 show the number of applicants subject to random assignment. Column 9 shows the number of applicants with certain offers.

Every STRIVE campus has random assignment, though many are undersubscribed and only two have have first-choice applicants at risk

Demographic Characteristics

Table 3: DPS student characteristics

| | Denver students (1) | SchoolChoice applicants (2) | Charter applicants (3) | Propensity score in (0,1) | | | |
| | | | | DA score (frequency) | | Simulated score | |
				Charter applicants (4)	Charter students (5)	Charter applicants (6)	Charter students (7)
Origin school is charter	0.133	0.080	0.130	0.259	0.371	0.230	0.357
Female	0.495	0.502	0.518	0.488	0.496	0.506	0.511
Race							
Hispanic	0.594	0.593	0.633	0.667	0.713	0.636	0.711
Black	0.141	0.143	0.169	0.181	0.161	0.192	0.168
White	0.192	0.187	0.124	0.084	0.062	0.098	0.059
Asian	0.034	0.034	0.032	0.032	0.039	0.033	0.037
Gifted	0.171	0.213	0.192	0.159	0.152	0.165	0.149
Bilingual	0.039	0.026	0.033	0.038	0.042	0.032	0.037
Subsidized lunch	0.753	0.756	0.797	0.813	0.818	0.800	0.823
Limited English proficient	0.285	0.290	0.324	0.343	0.378	0.337	0.380
Special education	0.119	0.114	0.085	0.079	0.068	0.083	0.070
Baseline scores							
Math	0.000	0.015	0.021	0.037	0.089	0.037	0.062
Reading	0.000	0.016	0.005	-0.011	0.007	0.008	-0.002
Writing	0.000	0.010	0.006	-0.001	0.039	0.016	0.035
N	40,143	10,898	4,964	1,436	828	1,523	781

Notes: This table decribes the population of Denver 3rd-9th graders in 2011-2012, the baseline and application year. Statistics in column 1 are for charter and non-charter students. Column 2 describes the subset that submitted an application to the SchoolChoice system for a seat in grades 4-10 at another DPS school in 2012-2013. Column 3 reports values for applicants ranking any charter school. Columns 4-7 show statistics for charter applicants with propensity score values strictly between zero and one. Test scores are standardized to the population in column 1.

Balance: Traditional Tests

Table 5a: Statistical tests for balance in application covariates

					Propensity score controls			
		DA score (frequency)			Simulated score			
			Nonparametric			Nonparametric		
	No controls	Linear control	Rounded (hundredths)	Saturated	Linear control	Rounded (hundredths)	Rounded (ten thousandths)	Full applicant type controls
Application variable	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Number of schools ranked	-0.341***	0.097	0.059	0.028	0.014	0.001	-0.061	-0.015
	(0.046)	(0.103)	(0.095)	(0.094)	(0.102)	(0.095)	(0.125)	(0.042)
Number of charter schools ranked	0.476***	0.143***	0.100**	0.074	0.020	-0.017	0.009	0.007
	(0.024)	(0.052)	(0.047)	(0.047)	(0.048)	(0.043)	(0.061)	(0.010)
First school ranked is charter	0.612***	0.012	0.002	-0.001	-0.030	-0.042*	0.012	0.000
	(0.011)	(0.025)	(0.022)	(0.020)	(0.027)	(0.022)	(0.027)	(0.000)
N	4,964	1,436	1,289	1,247	1,523	1,290	681	301
Risk set points of support		88	40	47	1,148	51	126	61
Robust F-test for joint significance	1190	2.70	1.70	1.09	0.49	1.26	0.31	0.34
p-value	0.000	0.044	0.165	0.352	0.688	0.287	0.817	0.710

Notes: This table reports coefficients from regressions of the application variables in each row on a dummy for charter offers. The sample includes applicants for 2012-13 charter seats in grades 4-10 who were enrolled in Denver at baseline. Columns 1-7 are from regressions like those used to construct expected balance in Table 4, except that the tests reported here use realized DA offers, with test statistics and standard errors computed in the usual way. Column 8 reports the balance test generated by a regression with saturated controls for applicant type (that is, unique combinations of applicant preferences over school programs and school priorities in those programs). Robust standard errors are reported in parentheses. P-values for robust joint significance tests are estimated by stacking outcomes and clustering at the student level.
*significant at 10%; **significant at 5%; ***significant at 1%

Imbalance too small to detect under saturated DA score control

2SLS and Semiparametric Alternatives

Table 6: Comparison of 2SLS and semiparametric estimates of charter effects

	Frequency (saturated)		Formula (saturated)		Simulation (hundredths)	
	2SLS	Semiparametric	2SLS	Semiparametric	2SLS	Semiparametric
	(1)	(2)	(3)	(4)	(5)	(6)
Math	0.496***	0.443***	0.524***	0.486***	0.543***	0.474**
	(0.076)	(0.105)	(0.071)	(0.105)	(0.075)	(0.212)
	{0.071}		{0.076}		{0.079}	
Reading	0.127*	0.106	0.120*	0.118	0.106	0.127
	(0.065)	(0.107)	(0.073)	(0.115)	(0.069)	(0.173)
	{0.065}		{0.069}		{0.071}	
Writing	0.325***	0.326***	0.356***	0.364***	0.324***	0.305**
	(0.079)	(0.102)	(0.082)	(0.113)	(0.079)	(0.145)
	{0.077}		{0.080}		{0.080}	
N	1,102	1,093	1,083	1,081	1,137	1,137

Notes: This table compares 2SLS and semiparametric estimates of charter attendance effects on the 2012-13 TCAP scores of Denver 4th-10th graders. The instrument is an any-charter offer dummy. The semiparametric estimator is described in Section 3.5. In addition to score variables, 2SLS estimates include controls for grade tested, gender, origin school charter status, race, gifted status, bilingual status, subsidized price lunch eligibility, special education, limited English proficient status, and baseline test scores. Semiparametric models use these same variables as controls when computing the score weighting function. Standard errors in parentheses are from a Bayesian bootstrap. Conventional robust standard errors for 2SLS estimates are reported in braces.
*significant at 10%; **significant at 5%; ***significant at 1%

The semiparametric scheme uses a score-weighted Abadie (2003)-style estimand; 2SLS estimates are close, but more precise

Semiparametric Details

- Abadie (2003) shows the LATE framework implies

$$E[Y_{0i}|C_{1i} > C_{0i}] = \frac{1}{Pr(C_{1i} > C_{0i})} E\left[\frac{C_i Y_i (D_i - p_D(\theta_i))}{(1 - p_D(\theta_i)) p_D(\theta_i)}\right]$$

$$E[Y_{1i}|C_{1i} > C_{0i}] = \frac{1}{Pr(C_{1i} > C_{0i})} E\left[\frac{(1 - C_i) Y_i ((1 - D_i) - (1 - p_D(\theta_i)))}{(1 - p_D(\theta_i)) p_D(\theta_i)}\right].$$

- Subtracting and rearranging, we have

$$E[Y_{1i} - Y_{0i}|C_{1i} > C_{0i}] = \frac{1}{Pr(C_{1i} > C_{0i})} E\left[\frac{Y_i (D_i - p_D(\theta))}{(1 - p_D(\theta_i)) p_D(\theta_i)}\right] \quad (2)$$

- The first stage in this case is constructed using

$$P[C_{1i} > C_{0i}] = E\left[\frac{C_i (D_i - p_D(\theta_i))}{(1 - p_D(\theta_i)) p_D(\theta_i)}\right] \quad (3)$$

- The estimator used here is the sample analog of the rightmost term in (2) divided by the sample analog of (3)
  - Non-score covs are added to a logit model for $E[D_i|X_i, p_D(\theta_i)]$ and the fitted values used for $p_D(\theta_i)$)

2SLS

Table 7: Comparison of 2SLS and OLS estimates of charter attendance effects

		2SLS estimates					
		DA score					
		Frequency (saturated)	Formula (saturated)	Simulated score rounded (hundredths)	No score controls	OLS	OLS with score controls
		(1)	(2)	(3)	(4)	(5)	(6)
First stage		0.410***	0.389***	0.377***	0.683***		
		(0.031)	(0.032)	(0.032)	(0.012)		
Math		0.496***	0.524***	0.543***	0.306***	0.304***	0.386***
		(0.071)	(0.076)	(0.079)	(0.021)	(0.015)	(0.034)
Reading		0.127**	0.120*	0.106	0.093***	0.103***	0.093***
		(0.065)	(0.069)	(0.071)	(0.020)	(0.014)	(0.029)
Writing		0.325***	0.356***	0.324***	0.183***	0.180***	0.202***
		(0.077)	(0.080)	(0.080)	(0.023)	(0.015)	(0.036)
	N	1,102	1,083	1,137	4,317	4,317	1,102

Notes: This table compares 2SLS and OLS estimates of charter attendance effects using the same sample and instruments as for Table 6. The OLS estimates in column 6 are from a model that includes saturated control for frequency estimates of the DA score. In addition to score variables, all models include controls for grade tested, gender, origin school charter status, race, gifted status, bilingual status, subsidized price lunch eligibility, special education, limited English proficient status, and baseline test scores. Robust standard errors are reported in parentheses.
*significant at 10%; **significant at 5%; ***significant at 1%

Outline

Multi Sector Models

Outline

Compared to What?

- 2SLS estimates so far contrast charter outcomes with mix of traditional public schools and schools from other non-charter sectors

- To characterize these states, let W_i denote the sector in which i enrolls; $1(W_i = j)$ indicates enrollment in sector j

- Define *potential* sector enrollment variables, W_{1i} and W_{0i}, indexed against charter offers, D_i

$$W_i = W_{0i} + (W_{1i} - W_{0i})D_i$$

In the population of charter-offer compliers, $W_{1i} = charter$ for all i

- We're interested in $E[1(W_{0i} = j)|C_{1i} > C_{0i}]$, the sector dsn for charter-offer compliers in the scenario where they aren't offered a charter seat ("counterfactual destinies")
- These are identified by a simple 2SLS estimand and reported in column 5 of Table 9

Table 9: Enrollment destinies for charter applicants

	All charter applicants		Charter applicants with DA score (frequency) in (0,1)			
			All applicants		Compliers	
	No charter offer	Charter offer	Non-offered mean	First stage + col 3	No charter offer	Charter offer
	(1)	(2)	(3)	(4)	(5)	(6)
Enrolled in a study charter	0.147	0.865	0.347	0.757	--	1.000
… in a traditional public	0.405	0.081	0.257	0.054	0.497	--
… in an innovation school	0.234	0.023	0.241	0.107	0.328	--
… in a magnet school	0.192	0.021	0.116	0.078	0.094	--
… in an alternative school	0.009	0.005	0.018	0.006	0.030	--
… in a contract school	0.012	0.004	0.018	-0.001	0.047	--
… in a non-study charter	0.001	0.000	0.002	0.000	0.005	--
N	2,555	1,833	498	1,102	--	--

Notes: This table describes school enrollment outcomes for charter applicants in the sample used to construct the estimates reported in Table 7. Columns 1-2 show enrollment by sector for all applicants without and with a charter offer. The remaining columns look only at those with a DA (frequency) score strictly between zero and one. Column 4 adds the non-offered mean in column 3 to the first stage estimate of the effect of charter offers on charter enrollment. School sectors are classified by grade. Innovation schools design and implement innovative practices to improve student outcomes. Magnet schools serve students with particular styles of learning. Alternative schools serve students struggling with academics, behavior, attendance, or other factors that may prevent them from succeeding in a traditional school environment; the latter offer faster pathways toward high school graduation, such as GED preparation and technical education. There is a single contract school, Escuela Tlatelolco, a private school contracted to serve DPS students, and a single non-study charter that closed in May 2013. Complier means in columns 5 and 6 were estimated using the 2SLS procedures described by Abadie(2002), with the same propensity score and covariate controls as were used to construct the estimates in Table 7.

Outline

Multi-Sector Models

- In addition to being of intrinsic interest, introducing innovation sector effects turns the non-charter counterfactual into one of mostly traditional publics
- Within charter sector heterogeneity: CMO vs. non-CMO
- Innovation and CMO/non-CMO charter FX are jointly identified by 2SLS system w/three endogenous vars, C_i^1, C_i^2, and C_i^3

$$Y_i = \sum_x [\alpha_1(x)d_i^1(x) + \alpha_2(x)d_i^2(x) + \alpha_3(x)d_i^3(x)] + \beta_1 C_i^1 + \beta_2 C_i^2 + \beta_3 C_i^3 + \varepsilon_i$$

$$C_i^1 = \sum_x [\gamma_{11}(x)d_i^1(x) + \gamma_{12}(x)d_i^2(x) + \gamma_{13}(x)d_i^3(x)] + \delta_{11} D_i^1 + \delta_{12} D_i^2 + \delta_{13} D_i^3 + \nu_i$$

$$C_i^2 = \sum_x [\gamma_{21}(x)d_i^1(x) + \gamma_{22}(x)d_i^2(x) + \gamma_{23}(x)d_i^3(x)] + \delta_{21} D_i^1 + \delta_{22} D_i^2 + \delta_{23} D_i^3 + \eta_i$$

$$C_i^3 = \sum_x [\gamma_{31}(x)d_i^1(x) + \gamma_{32}(x)d_i^2(x) + \gamma_{33}(x)d_i^3(x)] + \delta_{31} D_i^1 + \delta_{32} D_i^2 + \delta_{33} D_i^3 + \upsilon_i$$

where $d_i^j(x)$ saturate the (estimated) propensity scores for each offer IV, $p_j(\theta_i) = E[D_i^j | \theta_i]$ for $j = 1, 2, 3$

- We also try joint score control instead of additive

Table 11A: DPS charter and innovation school attendance effects for academic years 2013 and 2014

| | DA score (frequency) controls (saturated) | | | | |
| | | | | Charters and innovation | |
	3 CMOs only (1)	Other Charters only (2)	Innovation only (3)	Additive score controls (4)	Joint score controls (5)
3 CMOs First Stage	0.490***	--	--	0.490***	0.468***
	(0.024)	--	--	(0.027)	(0.028)
Other Charters First Stage	--	0.344***	--	0.333***	0.351***
	--	(0.057)	--	(0.056)	(0.056)
Innovation First Stage	--	--	0.378***	0.354***	0.359***
	--	--	(0.033)	(0.033)	(0.034)
			A. Math		
3 CMOs	0.440***	--	--	0.450***	0.432***
	(0.046)	--	--	(0.057)	(0.063)
Other Charters	--	-0.095	--	-0.028	-0.044
	--	(0.157)	--	(0.155)	(0.148)
Innovation	--	--	-0.159*	0.061	0.099
	--	--	(0.088)	(0.094)	(0.102)
N	2,012	405	960	2,922	2,404

Table 11B: DPS charter and innovation school attendance effects for academic years 2013 and 2014 (cont'd)

		DA score (frequency) controls (saturated)			
				Charters and innovation	
	3 CMOs only (1)	Other Charters only (2)	Innovation only (3)	Additive score controls (4)	Joint score controls (5)
B. Reading					
3 CMOs	0.207***	--	--	0.135**	0.128*
	(0.046)	--	--	(0.064)	(0.070)
Other Charters	--	-0.236	--	-0.228	-0.112
	--	(0.180)	--	(0.185)	(0.181)
Innovation	--	--	-0.144	-0.094	-0.092
	--	--	(0.098)	(0.110)	(0.117)
C. Writing					
3 CMOs	0.294***	--	--	0.348***	0.320***
	(0.052)	--	--	(0.068)	(0.075)
Other Charters	--	-0.021	--	0.002	0.018
	--	(0.157)	--	(0.164)	(0.170)
Innovation	--	--	-0.067	0.145	0.061
	--	--	(0.098)	(0.111)	(0.114)
N	2,012	405	960	2,922	2,404

Comparison to Other IVs

Outline

Comps: Qualification and First-Choice IV Strategies

- First-choice strategy
  - $R(\theta_i)$ identifies the charter i ranks *first*, along with his priority there
  - The first-choice instrument, D_i^f, indicates qualification (lottery number below cutoff) at this school
  - First-choice 2SLS:

$$Y_i = \sum_x \alpha(x) d_i(x) + \beta\, C_i + \varepsilon_i$$
$$C_i = \sum_x \gamma(x) d_i(x) + \delta D_i^f + \nu_i$$

    where dummies $d_i(x) = 1[R(\theta_i) = x]$

- Qualification instruments
  - The qualification 2SLS model swaps as follows:
    - $R(\theta_i)$ identifies *all* charters i ranks and i's priority at each
    - D_i^q indicates i's qualification at *any* charter he or she has ranked

Alternative IV Results

Table 8: Other IV strategies

	Charter attendance effect			Sample size increase for equivalent gain (col 2 vs col 1)	Sample size increase for equivalent gain (col 3 vs col 1)
	Offer instrument with DA score (frequency) controls (saturated)	First choice charter offer with risk set controls	Qualification instrument with risk set controls		
	(1)	(2)	(3)	(4)	(5)
	A. First stage estimates				
First stage for charter offers	1.000	0.774***	0.476***		
	--	(0.026)	(0.024)		
First stage for charter enrollment	0.410***	0.323***	0.178***		
	(0.031)	(0.035)	(0.027)		
	B. 2SLS estimates				
Math	0.496***	0.596***	0.409***		
	(0.071)	(0.102)	(0.149)	2.0	4.4
Reading	0.127**	0.227**	0.229		
	(0.065)	(0.102)	(0.144)	2.5	4.9
Writing	0.325***	0.333***	0.505***		
	(0.077)	(0.119)	(0.162)	2.4	4.5
N (students)	1,102	1,125	1,969		
N (schools)	30	15	24		

Unpacking first choice advantage

- First choice estimates on math and reading are noticeably larger than estimates from DA
  - May reflect an advantage for those awarded a seat at their first choice school
  - Since DA offer instrument covers 2X more charter schools, estimate may be more representative of an overall charter effect

- To investigate further, we use multi-sector model to estimate models with first choice charter vs. other charter

- Also of independent interest, given longstanding practice of evaluating matches based on fraction who obtain top choice

Version 4: DPS first-choice and other-choice charter school attendance effects for academic years 2012-2014

	DA score (frequency) controls (saturated)			
			First chcice charter and other charters	
	First choice is charter only	Other choices are charters only	Additive score controls	Joint score controls
	(1)	(2)	(3)	(4)
First choice charter First Stage				
First-choice offer	0.625***	--	0.603***	0.600***
	(0.022)	--	(0.023)	(0.024)
Other charters offer	--	--	-0.049*	-0.051**
	--	--	(0.025)	(0.025)
Other choice charter First Stage				
First-choice offer	--	--	-0.142***	-0.149***
	--	--	(0.017)	(0.017)
Other charters offer	--	0.544***	0.464***	0.457***
	--	(0.034)	(0.035)	(0.035)
	A. Math			
First choice charter	0.287***	--	0.386***	0.381***
	(0.036)	--	(0.048)	(0.049)
Other choice charter	--	-0.041	0.227***	0.207***
	--	(0.063)	(0.075)	(0.078)

Outline

Conclusion

- Centralized assignment produces data with quasi-experimental variation in student assignment to schools
- Such data offers unprecedented opportunities for program evaluation
  - Informed parental choice
  - Informed school portfolio planning
- Probability of assignment to schools can be estimated as a function of a few simple statistics
- Research design via propensity score matching