Techniques of Empirical Econometrics

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Overview

A huge literature estimates dynamic causal effects to various shocks in various ways:

- Monetary policy shocks
  Romer and Romer (1989), Christiano, Eichenbaum, and Evans (1999), Kuttner (2001), Christiano, Eichenbaum, and Evans (2005), Gertler and Karadi (2015), Antolín-Díaz, Petrella, and Rubio-Ramírez (2021), Bauer and Swanson (2022), ...
- Tax shocks
  Romer and Romer (2010), Blanchard and Perotti (2002), Mountford and Uhlig (2009), Mertens and Ravn (2013), Mertens and Ravn (2014), Mertens and Montiel Olea (2018), Lewis (2021), ...
- Government spending shocks
  Ramey and Shapiro (1998), Blanchard and Perotti (2002), Ramey (2011), Mountford and Uhlig (2009), Lewis (2021) ...
- General aggregate demand or supply shocks
  Blanchard and Quah (1989), Angeletos, Collard, and Dellas (2020), Shapiro and Watson (1988)...
- Technology shocks
  Galí (1999), Fisher (2006), Beaudry and Portier (2006)
- Oil shocks, credit shocks, uncertainty shocks, etc.

See e.g. Ramey (2016), Kilian and Lütkepohl (2017) for recent overviews.

These impulse response estimates inform policy and guide macroeconomic theory

IRFs (often) also quantify contributions of shocks to fluctuations in macro aggregates

Forecast Error Variance (FEV) Decomposition

The share of the FEV for $z_{i,t}$ at horizon h explained by $\epsilon_{j,t}$ is

$$\Omega_h = \frac{\sum_{n=0}^{h}(m_n^j(i))^2}{\sum_{l=1}^{N_z}\sum_{n=0}^{h}(m_n^l(i))^2}$$

where $m_h^j(i)$ is the i-th element in M_h^j

Historical Counterfactuals

Let $\epsilon_{j,t}^* = \epsilon_{j,t}$ but $\epsilon_{-j,t}^* = 0$ for all t, then

$$B(L)z_t^* = \mathcal{D}\epsilon_t^* = \mathcal{D}_j\epsilon_{j,t}$$

provides the counterfactual history of z_t^* with all $\epsilon_{-j,t}$ set to zero.

4. Other Uses of Impulse Responses

4.1 Estimating Theoretical Models with Impulse Response Matching

4.2 Estimating Single Structural Equations with Impulse Responses

4.3 Counterfactuals Under Alternative Policy Rules

4.4 Evaluating Optimality of Policy

Impulse Response Matching

Recall the SMA(∞) representation of the solution of a theoretical model

$$z_t = \left(\mathcal{D} + \mathcal{A}(\mathcal{I} - \mathcal{G}L)^{-1}\mathcal{F}L\right)\epsilon_t$$

where the coefficients of the state space respresentation $\{\mathcal{G}, \mathcal{F}, \mathcal{A}, \mathcal{D}\}$ are specific functions of deep structural parameters $\theta \in \Theta$

The 'constrained' SMA(∞) is $z_t = \mathcal{M}(\theta, L)\epsilon_t = \sum_{i=0}^{q} \mathcal{M}_i(\theta)\epsilon_{t-i}$

The 'unconstrained' SMA(∞) is $z_t = M(L)\epsilon_t = \sum_{i=0}^{q} M_i\epsilon_{t-i}$ (in population)

Impulse Response Matching Conditions

- $\gamma(\theta)$: $m \times 1$ theoretical IRF coefficients from the $\mathcal{M}_i(\theta)$'s
- γ_0: $m \times 1$ unconstrained coefficients from the M_i's corresponding to $\gamma(\theta)$
- $\hat{\gamma}_T \rightarrow \gamma_0$ for $T \rightarrow \infty$ (consistent IRF estimator)
- $\gamma_0 = \gamma(\theta)$, $\gamma(\theta)$ invertible (identification)

Impulse Response Matching

Impulse Response Matching Estimator Rotemberg and Woodford (1997)

An impulse response matching estimator of θ is

$$\hat{\theta} = \underset{\theta \in \Theta}{\text{argmin}} \; (\hat{\gamma}_T - \gamma(\theta))' W (\hat{\gamma}_T - \gamma(\theta))$$

where W is a p.s.d weighting matrix.

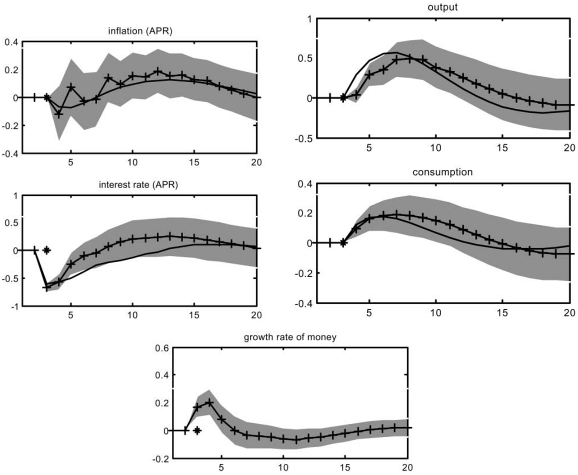
The estimator $\hat{\theta}$ minimizes the distance between the theoretical and empirical impulse responses, and $\hat{\theta} \to \theta$ for $T \to \infty$.

Let $\hat{\Sigma}$ be a consistent estimator of the covariance matrix of the IRFs $\hat{\gamma}_T$

Common choices for W are

- $W = \hat{\Sigma}^{-1}$, the optimal weighting matrix
- $W = diag(\hat{\Sigma}^{-1})$, only the diagonal elements $\hat{\Sigma}^{-1}$

Example: Christiano, Eichenbaum, and Evans (2005) Monetary Impulse Response Matching



Indirect Inference with Impulse Responses

There is not always a direct match between theoretical and empirical impulse responses (e.g. lag truncation, nonfundamentalness, violation of identification restrictions)

Suppose $\hat{\gamma}_T$ is from an approximating SVAR, $z_t = \sum_{i=1}^{p} Bz_{t-i} + u_t$ and some impact matrix D_j.

We can simulate the model and obtain N artificial samples of length T, and for each sample n obtain $\hat{\gamma}_T^n(\theta)$, the IRF estimate from the approximating SVAR

IRF Indirect Inference Conditions

- $\hat{\gamma}_T^n(\theta) \rightarrow \gamma(\theta)$ for $T \rightarrow \infty$
- $\hat{\gamma}_T \rightarrow \gamma_0$ for $T \rightarrow \infty$
- $\gamma_0 = \gamma(\theta)$, $\gamma(\theta)$ invertible (identification)

$\gamma(\theta)$ are auxiliary parameters rather than true theoretical impulse responses

Indirect Inference with Impulse Responses

Indirect Inference Estimator Gourieroux, Monfort, and Renault (1993)

An indirect inference estimator of θ is

$$\hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmin}} \left(\hat{\gamma}_T - \frac{1}{N}\sum_{n}^{N} \hat{\gamma}_T^n(\theta)\right)' W \left(\hat{\gamma}_T - \frac{1}{N}\sum_{n}^{N} \hat{\gamma}_T^n(\theta)\right)$$

where W is a p.s.d. weighting matrix.

The estimator $\hat{\theta}$ minimizes the distance between the theoretical and empirical auxiliary parameters, and $\hat{\theta} \to \theta$ for $T \to \infty$.

IRF Matching/indirect inference are similar to GMM and SMM, but with structural impulse response coefficients instead of moments.

Example: Mertens and Ravn (2011)

Impulse responses to unanticipated and pre-announced changes in income tax based on direct measures of Romer and Romer (2010)

Medium-scale RBC model
Tax experiments potentially informative about a range of important parameters:

Consumption dynamics:

- CES utility consumption parameter σ
- Degree of habit persistence μ
- Fraction of hand-to-mouth agents $1 - \zeta$

Investment dynamics:

- Investment adjustment cost in capital and durables ϕ_k and ϕ_v,

Labor supply and capacity utilization:

- Inverse Frisch labor supply elasticity κ
- Capacity utilization elasticity ψ

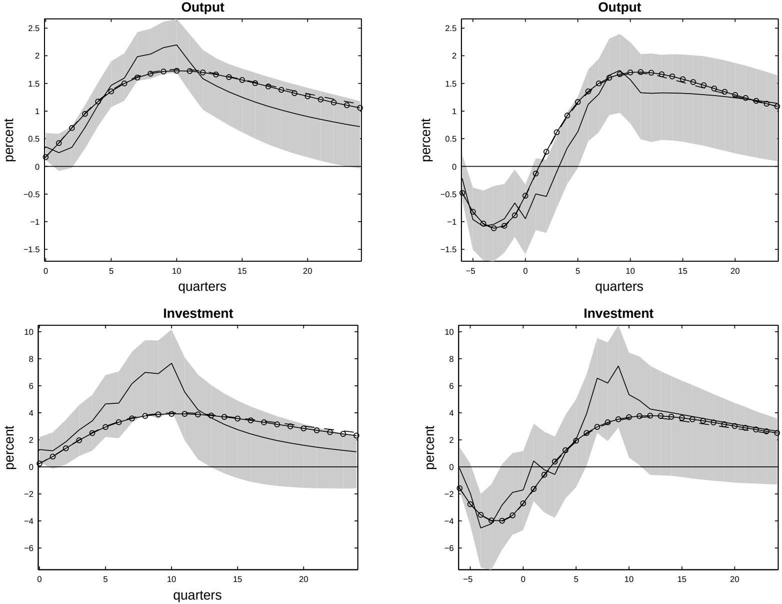
Fiscal policy rules:

- Elasticity of spending to tax revenues, π_G

Example: Mertens and Ravn (2011)

(a) Unanticipated Tax Cut

(b) Anticipated Tax Cut



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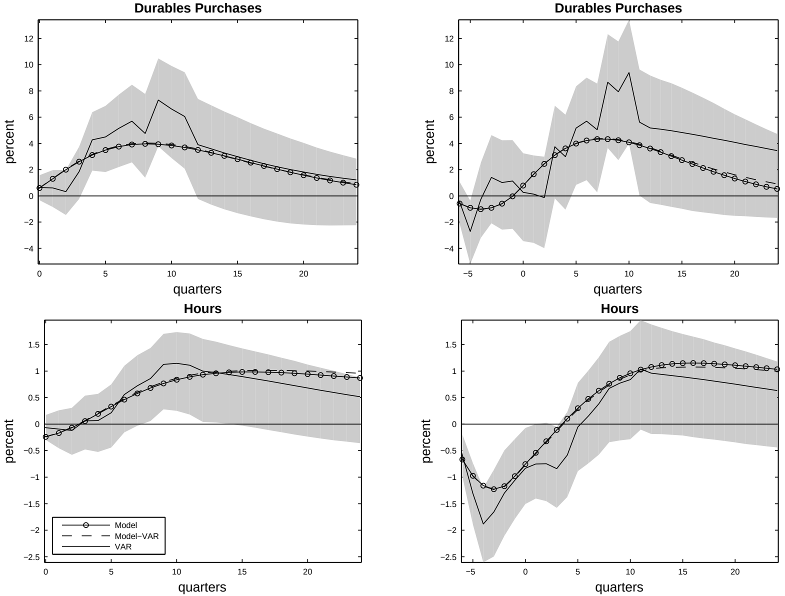

Example: Mertens and Ravn (2011)

Model	σ	μ	κ	ϕ_k	ϕ_v	ψ_k	$\rho_{n,1}$	$\rho_{n,2}$	$\rho_{k,1}$	$\rho_{k,2}$	π_G	ς	Obj.
(1) Benchmark	3.762	0.880	0.976	8.488	7.795	0.619	1.483	-0.484	1.707	-0.729	–	–	78.77
	(0.198)	(0.008)	(0.116)	(0.355)	(0.448)	(0.060)	(0.032)	(0.032)	(0.015)	(0.015)			
(2) No Durables	3.058	0.747	0.125	5.966	–	0.611	0.999*	0*	1.654	-0.684	–	–	95.28
	(0.145)	(0.020)	(0.036)	(0.184)		(0.044)			(0.011)	(0.012)			
(3) No Habits	7.183	–	1.103	10.786	7.995	0.626	1.564	-0.565	1.724	-0.743	–	–	113.22
	(0.201)		(0.130)	(0.445)	(0.430)	(0.044)	(0.026)	(0.026)	(0.011)	(0.011)			
(4) Endogenous G	1.829	0.926	0.477	6.032	7.265	0.459	1.102	-0.103	1.671	-0.698	0.221	–	68.15
	(0.176)	(0.006)	(0.058)	(0.274)	(0.448)	(0.0530)	(0.071)	(0.071)	(0.016)	(0.016)	(0.026)		
(5) Fixed Capital Tax	2.651	0.926	1.334	3.010	1.866	0.011	1.597	-0.598	–	–	–	–	145.01
	(0.134)	(0.007)	(0.116)	(0.252)	(0.184)	(0.007)	(0.018)	(0.018)					
(6) Fixed Labor Tax	0.411	0.906	0*	2.179	3.201	4.377	–	–	1.392	-0.406	–	–	110.46
	(0.021)	(0.006)		(0.097)	(0.158)	(0.084)			(0.009)	(0.008)			
(7) Rule-Of-Thumb	3.328	0.917	0.287	6.752	6.712	0.512	1.388	-0.389	1.707	-0.728	–	0.848	66.74
Households	(0.185)	(0.007)	(0.071)	(0.317)	(0.392)	(0.049)	(0.026)	(0.026)	(0.013)	(0.013)		(0.006)	

Standard errors are given in parentheses.

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Estimating Single Structural Equations with Impulse Responses

IRF matching/indirect inference require fully specified theoretical models

Sometimes we only want to estimate the parameters of a single structural equation without specifying a complete model

Turns out this is simple using **regressions in impulse response space**

- Regressions with impulses from distributed lag specifications — Barnichon and Mesters (2020)
- Regressions with impulses from VARs, LPs, ... — Lewis and Mertens (2022)

General Estimation Problem

$$y_t = \beta' Y_t + \upsilon_t \ ,$$

y_t: scalar outcome variable

Y_t: $N_Y \times 1$ endogenous variables

β: $N_Y \times 1$ structural parameters of interest

General Estimation Problem

$$y_t \quad = \quad \beta' Y_t + \upsilon_t \ ,$$

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Example: Hybrid NK Phillips curve

$$\pi_t = \gamma_b \pi_{t-1} + \gamma_f E_t \pi_{t+1} + \lambda gap_t + \upsilon_t$$

$y_t = \pi_t$

$Y_t = [\pi_{t-1}, E_t \pi_{t+1}, gap_t]'$

$\beta = [\gamma_b, \gamma_f, \lambda]'$

Well-known endogeneity problems (simultaneity, measurement error).

Common to use lags as instrumental variables (e.g. $\pi_{t-2}, gap_{t-1}, \ldots$) Galí and Gertler (1999)

Estimation Problem

$$y_t \quad = \quad \beta' Y_t + \upsilon_t \ ,$$

Using lagged endogenous variable z_{t-h} as instrument requires $E[\upsilon_t z_{t-h}] = 0$.

Strong assumption in macro applications if h is small.

Choosing large h weakens identification.

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Example: Hybrid NK Phillips curve

$$\pi_t = \gamma_b \pi_{t-1} + \gamma_f E_t \pi_{t+1} + \lambda gap_t + \upsilon_t$$

Allowing persistence in υ_t seems important empirically, e.g. Smets and Wouters (2007).

Lags of π_t, gap_t or other macro variables are not valid instruments in general.

Barnichon and Mesters (2020) propose lags of available direct measures of shocks m_t as instruments.

If uncorrelated with υ_t, these shocks are valid instruments.

IV with Direct Shock Measures

$$y_t = \beta' Y_t + \upsilon_t \ ,$$

Let m_t be a (scalar) direct shock measure (e.g. monetary policy shock)

Consider a distributed lag (DL) $m_t, m_{t-1}, ..., m_{t-H+1}$ as instrumental variables

IV Identification Conditions

$$E[m_{t-h} Y_t] \neq 0 \ \textit{for} \ h = 0, \ldots, H-1 \ , \ H \geq N_Y \qquad \text{(relevance)}$$

$$E[m_{t-h} \upsilon_t] = 0 \ \textit{for} \ h = 0, \ldots, H-1 \qquad \text{(exogeneity)}$$

Define $X_t = [m_t, ..., m_{t-H+1}]'$.

The data is demeaned: $\frac{1}{T}\sum y_t = 0$, $\frac{1}{T}\sum Y_t = 0$, $\frac{1}{T}\sum X_t = 0$

Let $y: \ T \times 1$, $Y: T \times N_Y$, $X: \ T \times H$

The 2SLS-DL estimator is $\hat{\beta}_{2SLS} = (Y' P_X Y)^{-1} Y' P_X y$ with $P_X = X(XX')^{-1} X'$

IV with Direct Shock Measures

$$Y'P_X Y/T = \underbrace{((X'X/T)^{-\frac{1}{2}}X'Y/T)'}_{\hat{\Theta}_Y^{DL\prime}}\underbrace{((X'X/T)^{-\frac{1}{2}}X'Y/T)}_{\hat{\Theta}_Y^{DL}}$$

$$Y'P_X y/T = \underbrace{((X'X/T)^{-\frac{1}{2}}X'Y/T)'}_{\hat{\Theta}_Y^{DL\prime}}\underbrace{((X'X/T)^{-\frac{1}{2}}X'y/T)}_{\hat{\Theta}_y^{DL}}$$

$\hat{\Theta}_Y^{DL}$: $H \times N_Y$ OLS estimator in regression of Y_t on $(X'X/T)^{-\frac{1}{2}}X_t$, i.e. impulse response coefficients of Y_t to a one-std innovation in m_t

$\hat{\Theta}_y^{DL}$: $H \times 1$ OLS estimator in regression of y_t on $(X'X/T)^{-\frac{1}{2}}X_t$, i.e. impulse response coefficients of y_t to a one-std innovation in m_t

2SLS-DL is a Regression in Impulse Response Space Barnichon and Mesters (2020)

The 2SLS estimator with $m_t, m_{t-1}, ..., m_{t-H+1}$ as instrumental variables is the OLS coefficient in a regression in impulse response space

$$\hat{\beta}_{2SLS} = (\hat{\Theta}_Y^{DL\prime}\hat{\Theta}_Y^{DL})\hat{\Theta}_Y^{DL\prime}\hat{\Theta}_y^{DL}$$

IV with Direct Shock Measures

The SMA(∞) representations of y_t and Y_t imply that

$$\upsilon_t = \sum_j \mu_{j,0}\epsilon_{j,t} + \sum_j \mu_{j,1}\epsilon_{j,t-1} + \dots$$

Restated Exogeneity Requirements for 2SLS-DL

For $h = 0, \dots, H-1$:

$$\mu_{j,l} E[m_t \epsilon_{j,t+h-l}] = 0, \;\; l = h; \forall j \qquad \text{(Contemporaneous Exogeneity)}$$
$$\mu_{j,l} E[m_t \epsilon_{j,t+h-l}] = 0, \;\; l = h+1, \dots, \infty; \forall j \qquad \text{(Lag Exogeneity)}$$
$$\mu_{j,l} E[m_t \epsilon_{j,t+h-l}] = 0, \;\; l = 0, \dots, h-1; \forall j \qquad \text{(Lead Exogeneity)}$$

m_t must be exogenous with respect to all non-excluded (i.e. $\mu_{j,l} \neq 0$) past, present and future shocks

We could add $z_{t-1}, z_{t-2}, \dots$ as controls to span the history of non-excluded shocks and get rid of lag exogeneity

Unfortunately, $z_{t-1}, z_{t-2}, \dots$ likely also spans the excluded shocks that correlate with m_t, so this generally weakens identification

From 2SLS-DL to SP-IV

- **2SLS-DL**: identifies structural parameters by regressing impulse responses from DLs with direct measures of shocks
- **SP-IV:** identifies structural parameters by regressing impulse responses from VARs or LPs using any identification scheme from Section 4

SP-IV: System Projections on Instrumental Variables (SP-IV) Lewis and Mertens (2022)

System Projections with Instrumental Variables (SP-IV)

Let $y_t^{\perp}(h)$, $Y_t^{\perp}(h)$ denote $h+1$-step ahead forecast errors conditional on $Z_{t-1} = [z'_{t-1}\ z'_{t-2}\ ...]'$.

If $y_t = \beta' Y_t + \upsilon_t$, then

$$y_t^{\perp}(h) \quad = \quad \beta' Y_t^{\perp}(h) + \upsilon_t^{\perp}(h)$$

Let $m_t^{\perp}$ denote one-step ahead forecast error conditional on $Z_{t-1} = [z'_{t-1}\ z'_{t-2}\ ...]'$.

H:# of horizons, $N_m = dim(m_t^{\perp})$.

$HN_m \geq N_Y$ **SP-IV identifying moments**: $E[\upsilon_t^{\perp}(h) m_t^{\perp}] = 0$ for $h = 0, \ldots, H-1$.

Without conditioning on Z_{t-1} and under stationarity, identical to

HN_m **2SLS-DL identifying moments**: $E[\upsilon_t m_{t-h}] = 0$ for $h = 0, \ldots, H-1$

The GMM problem

Consider forecasting models that are linear in Z_{t-1}, e.g. VARs or LPs

Let $y^{\perp}_{H,t}$ and $Y^{\perp}_{H,t}$ stack the forecast errors in y_t and Y_t.

$$E[\upsilon^{\perp}_{H,t}(\beta) \otimes m^{\perp}_t] = 0 \ , \ HN_m \ \textbf{identifying conditions}$$

$$E\left[\left[y^{\perp\prime}_{H,t}(\zeta), Y^{\perp\prime}_{H,t}(\zeta), m^{\perp\prime}_t(\zeta)\right]' \otimes Z_{t-1}\right] = 0 \ , \ \textbf{forecasting moments}$$

where $\upsilon^{\perp}_{H,t}(b) = y^{\perp}_{H,t} - (b' \otimes \mathcal{I}_H)Y^{\perp}_{H,t}$,
$y^{\perp}_{H,t}(d)$, $Y^{\perp}_{H,t}(d)$, $m^{\perp}_t(d)$ are functions of d with true value ζ.

Given a p.s.d. weighting matrix and mild assumptions, the GMM problem is separable in b and d.

Two-step procedure: (1) forecasting step, (2) structural estimation step

SP-IV with Local Projections

Using weights $\mathcal{I}_H \otimes E[m_t^{\perp} m_t^{\perp\prime}]^{-1}$, GMM problem is equivalent to minimizing $\text{Tr}(u_H^{\perp} P_{m^{\perp}} u_H^{\perp\prime})$ where

$$y_H^{\perp} = (\beta' \otimes \mathcal{I}_H) Y_H^{\perp} + u_H^{\perp}$$

Closed form solution a **restricted system 2SLS estimator**:

$$\hat{\beta} = \left(R'(Y_H^{\perp} P_{m^{\perp}} Y_H^{\perp\prime} \otimes \mathcal{I}_H) R \right)^{-1} R' \,\text{vec}(y_H^{\perp} P_{m^{\perp}} Y_H^{\perp\prime})$$

where $R = \mathcal{I}_{N_Y} \otimes \text{vec}(\mathcal{I}_H)$

SP-IV is a Regression in Impulse Response Space Lewis and Mertens (2022)

$$\hat{\beta} = (\hat{\mathbf{\Theta}}_Y' \hat{\mathbf{\Theta}}_Y)^{-1} \hat{\mathbf{\Theta}}_Y' \hat{\mathbf{\Theta}}_y$$

$\hat{\mathbf{\Theta}}_y$ $(HN_m \times 1)$ and $\hat{\mathbf{\Theta}}_Y$ $(HN_m \times N_Y)$ contain the LP estimates of the impulse responses to the standardized shocks $m_t^{\perp}$

Similarly, SP-IV VAR is a regression with SVAR impulse responses

Implementation in Phillips Curve Example

$$\pi_t = \gamma_b \pi_{t-1} + \gamma_f E_t \pi_{t+1} + \lambda gap_t + \upsilon_t$$

Let m_t, for example, denote a measure of monetary policy shocks

1. Estimate $\widehat{IRF}$ of π_t and gap_t to m_t, e.g. using VAR or LP that conditions on Z_{t-1}
2. Construct $\widehat{IRF}_Y$ using IRF_{gap} and lead and lag of IRF_π
3. Regress $\widehat{IRF}_\pi$ on $\widehat{IRF}_Y \to \hat{\gamma}_b, \hat{\gamma}_f, \hat{\lambda}$

Any of the identification schemes from Section 4 (proxies, recursive, ...) are OK as long as $E[\upsilon^\perp_{t+h} m^\perp_t] = 0$ holds for the identified shock $m^\perp_t$

Can stack IRFs to different (standardized) shocks $N_m \geq 1$

Exogeneity does not require IRFs are individual dynamic causal effects, can also just be a rotation to shocks that are exogenous

No need to use all horizons in practice

What are the Advantages of SP-IV over 2SLS-DL?

- More Identification Options using IRFs from LPs or VARs
- Weaker Exogeneity Conditions
- Potential Efficiency Gains
- Stronger Identification

Weaker Exogeneity Conditions

The SMA(∞) representations of y_t and Y_t imply that

$$\upsilon_t = \sum_j \mu_{j,0}\epsilon_{j,t} + \sum_j \mu_{j,1}\epsilon_{j,t-1} + \dots$$

Exogeneity Requirements for SP-IV

Assume Z_{t-1} spans the history of non-excluded shocks, for $h = 0, \dots, H-1$:

$$\mu_{j,l} E[m_t \epsilon_{j,t+h-l}] = 0,\ \ l = h; \forall j \qquad \text{(Contemporaneous Exogeneity)}$$
$$\mu_{j,l} E[m_t \epsilon_{j,t+h-l}] = 0,\ \ l = 0, \dots, h-1; \forall j \qquad \text{(Lead Exogeneity)}$$

For Z_{t-1} that spans the history non-excluded shocks, SP-IV does not require lag exogeneity .

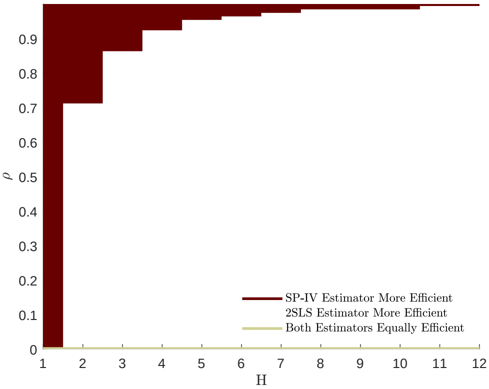
Note, weaker assumption than ϵ_t is fundamental for z_t (partial fundamentalness)

Efficiency Gains

SP-IV asymptotically more efficient if $Var(\upsilon^{\perp}_{H,t})$ is 'small' relative to $Var(\upsilon_t)$.

More likely if υ_t is persistent (predictable) and H is not too large.

AR(1) example: $\upsilon_t = \rho\upsilon_{t-1} + \upsilon_t$



Stronger Identification

Weak instrument problems are common.

Lead to small sample bias and incorrect inference.

Conditioning on Z_{t-1} removes predictable variation on Y_t and can improve the signal-to-noise ratio of m_t.

SP-IV performs better in small samples.

Including Z_{t-1} as exogenous regressors in 2SLS instead weakens identification, as Z_{t-1} is likely to span lags of m_t.

Inference for SP-IV

Inference under strong identification is standard:

$$\sqrt{T}(\hat{\beta} - \beta) \xrightarrow{d} N(0, V_\beta)$$

$$\hat{V}_\beta = \left(R'(Y_H^\perp P_{m\perp} Y_H^{\perp\prime} \otimes \mathcal{I}_H)R\right)^{-1} R' \left(Y_H^\perp P_{m\perp} Y_H^{\perp\prime} \otimes \hat{\Sigma}_{u_H^\perp}\right) R \left(R'(Y_H^\perp P_{m\perp} Y_H^{\perp\prime} \otimes \mathcal{I}_H)R\right)^{-1}$$

Inference for SP-IV

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$$\begin{aligned}
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\end{aligned}$$

Bias-based **first-stage test for weak instruments** based on the test statistic

$$\text{mineval}\{\hat{\Omega}^{-\frac{1}{2}} R'(Y_H^{\perp} P_{m^{\perp}} Y_H^{\perp\prime} \otimes \mathcal{I}_H)R\hat{\Omega}^{-\frac{1}{2}}\}$$

where $\hat{\Omega} = R'(\hat{\Sigma}_{v_H^{\perp}} \otimes \mathcal{I}_H)R$ and $\hat{\Sigma}_{v_H^{\perp}}$ is the $HK \times HK$ variance of the first-stage residuals. When $H = 1$, same as Stock-Yogo (2005). Critical values for $H > 1$ are non-standard, application-specific and obtained numerically.

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$$\sqrt{T}(\hat{\beta} - \beta) \quad \xrightarrow{d} \quad N(0, V_{\beta})$$

$$\hat{V}_{\beta} \quad = \quad \left(R'(Y_H^{\perp} P_{m^{\perp}} Y_H^{\perp\prime} \otimes \mathcal{I}_H)R\right)^{-1} R' \left(Y_H^{\perp} P_{m^{\perp}} Y_H^{\perp\prime} \otimes \hat{\Sigma}_{u_H^{\perp}}\right) R \left(R'(Y_H^{\perp} P_{m^{\perp}} Y_H^{\perp\prime} \otimes \mathcal{I}_H)R\right)^{-1}$$

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Identification robust inference can be based on

- AR-statistic (also Stock and Wright, 2000, S-statistic for GMM)
- KLM-statistic (Kleibergen, 2005, KLM-statistic for GMM)

Simulation Evidence

Phillips curve in Smets Wouters (2007)

$$\begin{aligned}\pi_t &= \gamma_b \pi_{t-1} + \gamma_f E_t \pi_{t+1} + \lambda gap_t + \upsilon_t \\ u_t &= \rho_u u_{t-1} + \epsilon_t^p - \mu_p \epsilon_{t-1}^p \ , \ \mid \rho_u \mid < 1\end{aligned}$$

- $N_m = 1$: monetary policy shock
- Controls Z_{t-1}: four lags of seven endogenous variables
- Fully exogenous, or lag endogenous (RR shock on inflation lags)
- $T = 200, 500, 5000$
- $H = 8, 20$

Lag Endogenous Instrument

Mean, $N_m = 1$, $T = 5000$

$H = 8$		Mean	
Estimator	γ_b	γ_f	λ
True	0.15	0.85	0.05
OLS	0.48	0.48	0.00
2SLS	0.26	0.58	-0.09
2SLS-C	0.41	0.18	1.46
SP-IV LP	0.26	0.60	-0.08
SP-IV LP-C	0.16	0.84	0.05
SP-IV VAR	0.12	0.83	0.09

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True Shocks as Instruments

Mean parameter estimates, $N_m = 1$

$H = 8$	$T = 250$			$T = 500$			$T = 5000$		
Estimator	γ_b	γ_f	λ	γ_b	γ_f	λ	γ_b	γ_f	λ
True	0.15	0.85	0.05	0.15	0.85	0.05	0.15	0.85	0.05
OLS	0.47	0.47	0.00	0.48	0.48	0.00	0.48	0.48	0.00
2SLS	0.27	0.51	0.01	0.23	0.60	0.01	0.17	0.83	0.04
2SLS-C	-0.08	0.33	0.14	-0.04	0.32	0.32	1.07	0.58	1.09
SP-IV LP	0.26	0.50	0.01	0.23	0.60	0.01	0.17	0.83	0.04
SP-IV LP-C	0.29	0.64	0.04	0.24	0.74	0.05	0.16	0.84	0.05
SP-IV VAR	0.23	0.81	0.03	0.18	0.84	0.05	0.12	0.83	0.09

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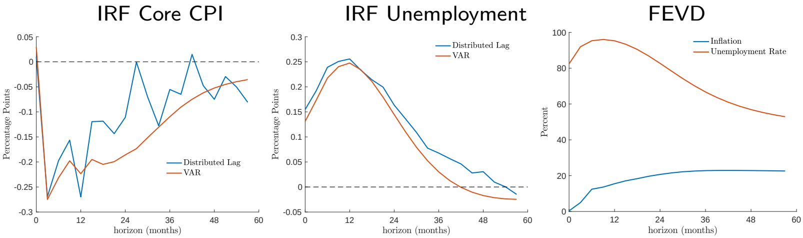
Application: Inflation-Activity Disconnect

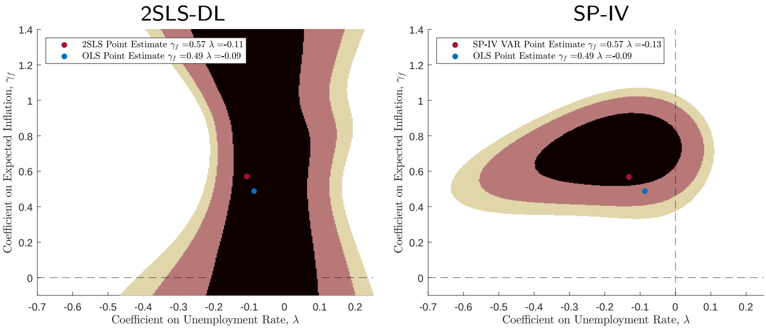
Based on the muted response of inflation to a Main Business Cycle Shock, Angeletos, Collard, and Dellas (2020) conclude inflation is disconnected from the business cycle

2SLS-DL and SP-IV allow a formal econometric investigation of claims about relationships across impulse responses



Parameter Estimates

$$\pi_t^{1q} = (1-\gamma_f)\pi_{t-3}^{1y} + \gamma_f \pi_{t+12}^{1y} + \lambda U_t + \upsilon_t \ ,$$



68%-90%-95% KLM Confidence sets

Counterfactuals Under Alternative Policy Rules

We can learn a lot from dynamic causal effects of e.g. monetary policy shocks

But ultimately, systematic monetary policy is much more important

Changes in systematic policy change expectations and therefore the dynamic causal effects to all shocks

However, we can still study the effects of changes in systematic policy using semi-structural evidence:

McKay and Wolf (2022)

Observed Economy

Suppose the actual economy follows the model

$$p_t = \alpha y_t + \sum_{n=0}^{\infty} \epsilon_{t-n}^{p,n} \qquad (E_1)$$

$$y_t = \mathcal{H}_y(\theta,\alpha) E_t[y_{t+1}] + \mathcal{H}_p(\theta,\alpha) p_t + \mathcal{H}_{\epsilon^y}(\theta,\alpha)\epsilon_t^y \qquad (E_2)$$

p_t is a scalar policy tool, e.g. the funds rate

$\epsilon_t^p = [\epsilon_t^{p,0}\ \epsilon_t^{p,1}\ \epsilon_t^{p,2}\ ...]$ is an infinite-dimensional vector of i.i.d 'policy news shocks'

y_t is $N_Y \times 1$ vector of macro variables, e.g. inflation, output, etc.

ϵ_t^y is $N_{\epsilon^y} \times 1$ vector of i.i.d 'non-policy shocks'

E_1 is a policy feedback rule, e.g. a Taylor rule

E_2 contains the structural equations, e.g. consumption Euler, Phillips curve, etc ...

$\{\mathcal{H}_y, \mathcal{H}_p, \mathcal{H}_{\epsilon^y}\}$ generally depend on the 'deep' structural parameters θ and α

Observed Economy

$$p_t = \alpha y_t + \sum_{n=0}^{\infty} \epsilon_{t-n}^{p,n} \qquad (E_1)$$

$$y_t = \mathcal{H}_y(\theta,\alpha) E_t[y_{t+1}] + \mathcal{H}_p(\theta,\alpha) p_t + \mathcal{H}_{\epsilon^y}(\theta,\alpha)\epsilon_t^y \qquad (E_2)$$

The observed data $z_t = [p_t \ y_t']'$ is generated by $\{\epsilon_i^y\}_{i=-\infty}^T$ and $\{\epsilon_i^p\}_{i=-\infty}^T$

Assumption: Uniqueness

There is unique solution satisfying (E_1)-(E_2), in SMA(∞) form

$$\begin{aligned} p_t &= M_{py}(L,\theta,\alpha)\epsilon_t^y + M_{pp}(L,\theta,\alpha)\epsilon_t^p \\ y_t &= M_{yy}(L,\theta,\alpha)\epsilon_t^y + M_{yp}(L,\theta,\alpha)\epsilon_t^p \end{aligned}$$

Assumption: Econometrician Information

- $\{\mathcal{H}_y(\theta,\alpha), \mathcal{H}_p(\theta,\alpha), \mathcal{H}_{\epsilon^y}(\theta,\alpha)\}$ is unknown
- $M_{yp}(L,\theta,\alpha)$ and $M_{pp}(L,\theta,\alpha)$, are known

The economic structure is unknown, but the dynamic causal effects of policy shocks on y_t and p_t are known.

Example: New Keynesian Model

$$
\begin{aligned}
R_t &= \phi_\pi \pi_t + \sum_{n=0}^{\infty} \epsilon_{t-n}^{p,n} \\
E_t \Delta gap_{t+1} &= R_t - E_t \pi_{t+1} - s_t^d \\
\pi_t &= \kappa gap_t + \beta E_t \pi_{t+1} + s_t^s
\end{aligned}
$$

$$
\underbrace{R_t}_{p_t} = \underbrace{\begin{bmatrix} 0 & \phi_\pi \end{bmatrix}}_{\alpha} \underbrace{\begin{bmatrix} gap_t \\ \pi_t \end{bmatrix}}_{y_t} + \sum_{n=0}^{\infty} \epsilon_{t-n}^{p,n}
$$

$$
\underbrace{\begin{bmatrix} gap_t \\ \pi_t \end{bmatrix}}_{y_t} = \underbrace{\begin{bmatrix} 1 & 1 \\ \kappa & \beta + \kappa \end{bmatrix}}_{\mathcal{H}_y} \underbrace{E_t \begin{bmatrix} gap_{t+1} \\ \pi_{t+1} \end{bmatrix}}_{E_t y_{t+1}} + \underbrace{\begin{bmatrix} -1 \\ -\kappa \end{bmatrix}}_{\mathcal{H}_p} \underbrace{R_t}_{p_t} + \underbrace{\begin{bmatrix} s_t^s \\ s_t^d \end{bmatrix}}_{\epsilon_t^y}
$$

with $\theta = [\kappa \ \beta]'$ and $\kappa > 0$, $\phi_\pi > 1$, $0 \leq \beta < 1$

Counterfactual Economy †

Consider the same economy, but with a different policy rule (and no policy shocks)

$$p_t^\dagger = \alpha^\dagger y_t \tag{$E_1^\dagger$}$$

$$y_t^\dagger = \mathcal{H}_y(\theta, \alpha^\dagger) E_t[y_{t+1}^\dagger] + \mathcal{H}_p(\theta, \alpha^\dagger) p_t + \mathcal{H}_{\epsilon^y}(\theta, \alpha^\dagger)\epsilon_t^y \tag{$E_2^\dagger$}$$

Assumption: Uniqueness †

There is unique solution satisfying ($E_1^\dagger$)-($E_2^\dagger$), in SMA(∞) form

$$p_t^\dagger = M_{py}(L, \theta, \alpha^\dagger)\epsilon_t^y$$
$$y_t^\dagger = M_{yy}(L, \theta, \alpha^\dagger)\epsilon_t^y$$

Same sequence of non-policy shocks $\{\epsilon_i^y\}_{i=-\infty}^T$ as in the actual economy

In general, $M_{yy}(L, \theta, \alpha^\dagger) \neq M_{yy}(L, \theta, \alpha)$

Entire impulse-propagation system changes when $\alpha \to \alpha^\dagger$

Counterfactual Economy $*$

Consider again the actual economy, but change sequence of policy shocks $\{\epsilon_{t-i}^{p*}\}_{i=-\infty}^{T}$

$$
\begin{aligned}
p_t^* &= M_{py}(L,\theta,\alpha)\epsilon_t^y + M_{pp}(L,\theta,\alpha)\epsilon_t^{p*} \\
y_t^* &= M_{yy}(L,\theta,\alpha)\epsilon_t^y + M_{yp}(L,\theta,\alpha)\epsilon_t^{p*}
\end{aligned}
$$

Same sequence of non-policy shocks $\{\epsilon_i^y\}_{i=-\infty}^{T}$ as in the actual economy

The stochastic processes p_t^* and y^* are solutions to (E_1)-(E_2)

All we are doing is changing the policy shock sequence to generate counterfactual realizations of p_t^* and y_t^*

A single policy shock per period suffices to ensure that $p_t^* = \alpha^\dagger y_t^*$ always holds

However, it is generally not be the case that $E_t[p_{t+h}^*] = \alpha^\dagger E_t[y_{t+h}^*]$.

Lucas' critique

Example: New Keynesian Model

New Keynesian model, but set $s_t^d = 0$, $\epsilon_t^{p,n} = 0$ for all $n > 0$, and $s_t^s = \rho s_{t-1}^s + \epsilon_t^s$

The solution is

$$\begin{bmatrix} R_t \\ gap_t \\ \pi_t \end{bmatrix} = \frac{1}{1+\phi_\pi \kappa} \begin{bmatrix} 1 & \phi_\pi(1-\rho) \\ -1 & -(\phi_\pi - \rho) \\ -\kappa & 1-\rho \end{bmatrix} \begin{bmatrix} \epsilon_t^{p,0} \\ \Delta^{-1}(1-\rho L)^{-1}\epsilon_t^s \end{bmatrix} \tag{1}$$

where $\Delta = det\left(\mathcal{I} - \rho \mathcal{C}^{-1}\right)$, $\mathcal{C}^{-1} = \frac{1}{1+\phi_\pi \kappa} \begin{bmatrix} 1 & 1-\beta\phi_\pi \\ \kappa & \beta+\kappa \end{bmatrix}$

The impulse response of inflation is

$$E_{t-1}[\pi_{t+h} \mid \epsilon_t^s = 1] - E_{t-1}[\pi_t] = \frac{(1-\rho)\rho^h \Delta^{-1}}{1+\phi_\pi \kappa}$$

In the counterfactual economy, the impulse response of inflation is

$$E_{t-1}[\pi_{t+h}^\dagger \mid \epsilon_t^s = 1] - E_{t-1}[\pi_t^\dagger] = \frac{(1-\rho)\rho^h (\Delta^\dagger)^{-1}}{1+\phi_\pi^\dagger \kappa}$$

where $\Delta^\dagger = det\left(\mathcal{I} - \rho (\mathcal{C}^\dagger)^{-1}\right)$, $(\mathcal{C}^\dagger)^{-1} = \frac{1}{1+\phi_\pi^\dagger \kappa} \begin{bmatrix} 1 & 1-\beta\phi_\pi^\dagger \\ \kappa & \beta+\kappa \end{bmatrix}$

Example: New Keynesian Model

Suppose for every h we choose the monetary policy shock ν_h such that

$$E_{t-1}[R^*_{t+h} \mid \epsilon^s_t = 1] - E_{t-1}[R_t] = \phi^\dagger_\pi \left(E_{t-1}[\pi^*_{t+h} \mid \epsilon^s_t = 1] - E_{t-1}[\pi_t] \right)$$

such that the counterfactual Taylor rule holds ex post at al horizons

The resulting impulse response of inflation is

$$E_{t-1}[\pi^*_{t+h} \mid \epsilon^s_t = 1, \nu_h] - E_{t-1}[\pi_t] = \frac{(1-\rho)\rho^h(\Delta)^{-1}}{1 + \phi^\dagger_\pi \kappa}$$

Generally not the same as in the counterfactual economy since $\Delta \neq \Delta^\dagger$ unless $\rho = 0$

The shocks that enforce the new rule at each horizon are not anticipated in advance.

Counterfactuals With Impulse Responses

Key Assumptions for Policy Counterfactuals with Impulse Responses

CFA1 $\{\mathcal{H}_y, \mathcal{H}_p, \mathcal{H}_{\epsilon^y}\}$ do not depend on α

$$\mathcal{H}_y(\theta,\alpha) = \mathcal{H}_y(\theta) \ , \ \mathcal{H}_y(\theta,\alpha) = \mathcal{H}_y(\theta) \ , \ \mathcal{H}_{\epsilon^y}(\theta,\alpha) = \mathcal{H}_y(\theta)$$

CFA2 There exists a sequence $\{\epsilon^{p*}_{t-i}\}^{T}_{i=-\infty}$ such that for all t and $h \geq 0$

$$E_t[p^*_{t+h}] = \alpha^{\dagger} E_t[y^*_{t+h}]$$

Policy Counterfactuals with Impulse Responses McKay and Wolf (2022)

Under **Uniqueness**, **Uniqueness** †, **CFA1** and **CFA2**,

$$p^{\dagger}_t = M_{py}(L,\theta,\alpha)\epsilon^y_t + M_{pp}(L,\theta,\alpha)\epsilon^{p*}_t = p^*_t \ , \ E_t[p^{\dagger}_{t+h}] = E_t[p^*_{t+h}]$$

$$y^{\dagger}_t = M_{yy}(L,\theta,\alpha)\epsilon^y_t + M_{yp}(L,\theta,\alpha)\epsilon^{p*}_t = y^*_t \ , \ E_t[y^{\dagger}_{t+h}] = E_t[y^*_{t+h}]$$

All conditional expectations in the counterfactual economy † can be replicated exactly by a (unique) sequence of policy shocks $\{\epsilon^{p*}_{t-i}\}^{T}_{i=-\infty}$ under the decision rules in the observed economy

Counterfactuals With Impulse Responses

Intuition:
Policy following the $\alpha^\dagger$-rule is exactly equivalent to deviating from the α-rule in a way that (1) perfectly mimicks the $\alpha^\dagger$-rule and (2) is known perfectly in advance by all private agents

The underlying model equations in $(E_2) - (E_2^\dagger)$ can arbitrarily more complicated as long as **CFA2**, and the other assumptions continue to hold, see McKay and Wolf (2022)

Throughout linearity is required

The methodology allows counterfactuals conditional on an identified non-policy shock

For counterfactuals with unconditional data, a fundamentalness assumption is also required.

Example: New Keynesian Model

Key Assumptions for Policy Counterfactuals with Impulse Responses

CFA1 $\{\mathcal{H}_y, \mathcal{H}_p, \mathcal{H}_{\epsilon^y}\}$ do not depend on α

$$\mathcal{H}_y(\theta,\alpha) = \mathcal{H}_y(\theta)\ ,\ \mathcal{H}_y(\theta,\alpha) = \mathcal{H}_y(\theta)\ ,\ \mathcal{H}_{\epsilon^y}(\theta,\alpha) = \mathcal{H}_y(\theta)$$

$$\underbrace{R_t}_{p_t} = \underbrace{\begin{bmatrix} 0 & \phi_\pi \end{bmatrix}}_{\alpha} \underbrace{\begin{bmatrix} gap_t \\ \pi_t \end{bmatrix}}_{y_t} + \sum_{n=0}^{\infty} \epsilon_{t-n}^{p,n}$$

$$\underbrace{\begin{bmatrix} gap_t \\ \pi_t \end{bmatrix}}_{y_t} = \underbrace{\begin{bmatrix} 1 & 1 \\ \kappa & \beta+\kappa \end{bmatrix}}_{\mathcal{H}_y} \underbrace{E_t \begin{bmatrix} gap_{t+1} \\ \pi_{t+1} \end{bmatrix}}_{E_t y_{t+1}} + \underbrace{\begin{bmatrix} -1 \\ -\kappa \end{bmatrix}}_{\mathcal{H}_p} \underbrace{R_t}_{p_t} + \underbrace{\begin{bmatrix} s_t^s \\ s_t^d \end{bmatrix}}_{\epsilon_t^y}$$

with $\theta = [\kappa\ \beta]'$ and $\kappa > 0$, $\phi_\pi > 1$, $0 \leq \beta < 1$

$\mathcal{H}_y$ and $\mathcal{H}_p$ indeed do not depend on α, CFA2 is satisfied

Counterfactuals With Impulse Responses

The solution to the NK model with news shocks is

$$\begin{bmatrix} gap_t \\ \pi_t \end{bmatrix} = \frac{1}{1+\phi_\pi \kappa} \sum_{m=0}^{\infty} \mathcal{C}^{-m} \begin{bmatrix} -1 \\ -\kappa \end{bmatrix} \sum_{n=m}^{\infty} \epsilon_{t+m-n}^{p,n}$$
$$+ \frac{1}{1+\phi_\pi \kappa} \begin{bmatrix} -(\phi_\pi - \rho) \\ 1-\rho \end{bmatrix} \Delta^{-1} (1-\rho L)^{-1} \epsilon_t^s$$

$$E_{t-1}\left[\begin{bmatrix} gap_{t+h}^* \\ \pi_{t+h}^* \end{bmatrix} | \epsilon_t^s = 1, \nu_0\right] - E_{t-1} \begin{bmatrix} gap_{t+h} \\ \pi_{t+h} \end{bmatrix}$$
$$= \frac{1}{1+\phi_\pi \kappa} \sum_{m=h}^{\infty} \mathcal{C}^{h-m} \begin{bmatrix} -1 \\ -\kappa \end{bmatrix} \nu_0(m) + \frac{1}{1+\phi_\pi \kappa} \begin{bmatrix} -(\phi_\pi - \rho) \\ 1-\rho \end{bmatrix} \Delta^{-1} \rho^h$$

$$E_{t-1}[R_{t+h}^* \mid \epsilon_t^s = 1, \nu_0] - E_{t-1}[R_t] = \phi_\pi \left(E_{t-1}[\pi_{t+h}^* \mid \epsilon_t^s = 1, \nu_0] - E_{t-1}[\pi_t] \right) + \nu_0(h)$$

Choose the news shocks $\nu_0(h)$ such that for all $h = 0, ..., N_p$

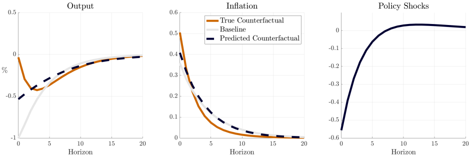
$$\nu_0(h) = (\phi_\pi^\dagger - \phi_\pi) \left(E_{t-1}[\pi_{t+h}^* \mid \epsilon_t^s = 1, \nu_0] - E_{t-1}[\pi_t] \right)$$

For $N_p \to \infty$, the resulting impulse response of inflation is

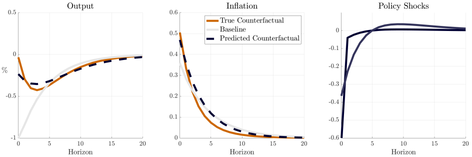
$$E_{t-1}[\pi_{t+h}^* \mid \epsilon_t^s = 1, \nu_0] - E_{t-1}[\pi_t] = \frac{(1-\rho)\rho^h (\Delta^\dagger)^{-1}}{1+\phi_\pi^\dagger \kappa}$$

McKay and Wolf (2022) HANK Theoretical Example

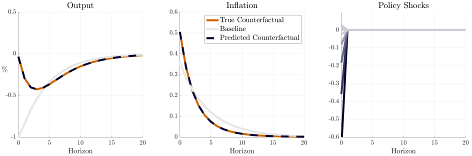
Rule switch from $R_t = \phi_\pi \pi_t + \sum_{n=0}^{\infty} \epsilon_{t-n}^{p,n}$ to $R_t = \phi_R R_{t-1} + (1-\phi_R)(\phi_\pi \pi_t + \phi_x gap_t)$



2 Shocks: Match 1-Period-Ahead Expectations



Full Date-0 Shocks: Match All Expectations

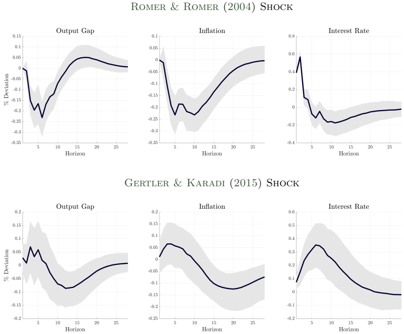


McKay and Wolf (2022) Monetary Policy Counterfactuals

In reality, we do not have a (rotation) of complete set of policy news shocks

Reasonable approximations of future policy paths with linear combinations of different identified monetary policy shocks

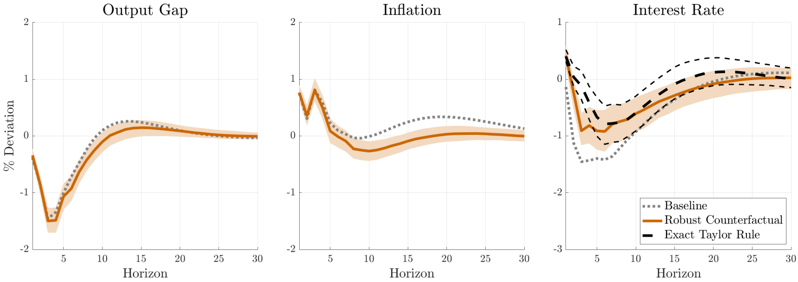
McKay and Wolf (2022) identify monetary policy shocks using Romer and Romer (2004) and Gertler and Karadi (2015) as m_t in an internal instrument SVAR



McKay and Wolf (2022) Monetary Policy Counterfactuals

Response to a technology news shock from Ben Zeev and Khan (2015) under counterfactual monetary policy rules

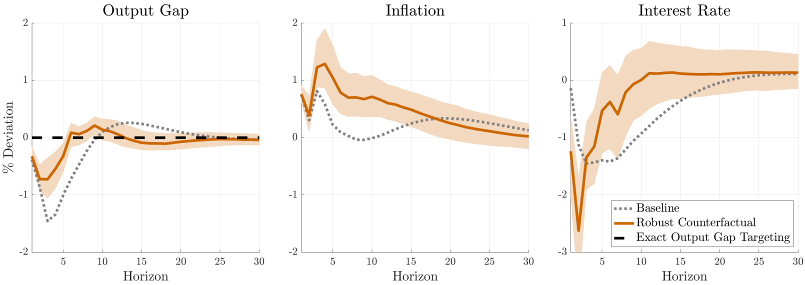
Approximate simple Taylor rule $R_t = 0.5R_{t-1} + (1 - 0.5) \times (1.5\pi_t + gap_t)$



McKay and Wolf (2022) Monetary Policy Counterfactuals

Response to a technology news shock from Ben Zeev and Khan (2015) under counterfactual monetary policy rules

Approximate output gap targeting rule



4. Other Uses of Impulse Responses

4.1 Estimating Theoretical Models with Impulse Response Matching

4.2 Estimating Single Structural Equations with Impulse Responses

4.3 Counterfactuals Under Alternative Policy Rules

4.4 Evaluating Optimality of Policy

Evaluating Optimality of Policy

We can approximate outcomes under counterfactual systematic policies with impulse responses to multiple policy shocks

Given a policy loss function, we can also approximate the optimal policy

Barnichon and Mesters (2022), McKay and Wolf (2022)

Optimal Policy

Consider again

$$y_t = \mathcal{H}_y E_t[y_{t+1}] + \mathcal{H}_p p_t + \mathcal{H}_{\epsilon^y} \epsilon_t^y \qquad (E_2)$$

where p_t is a scalar policy tool.

The policy loss function is

$$\mathcal{L}_t = \frac{1}{2} y_t' y_t$$

Optimal policy requires

$$\frac{\partial \mathcal{L}_t}{\partial p_t} = \mathcal{H}_p' y_t = 0$$

In the New Keynesian model $y_t = [gap_t \ \pi_t]'$

$$\mathcal{H}_p' y_t = \begin{bmatrix} -1 & -\kappa \end{bmatrix} \begin{bmatrix} gap_t \\ \pi_t \end{bmatrix} = 0 \Rightarrow gap_t = -\kappa \pi_t$$

Optimal Policy

Let the policy rule in the observed economy be

$$p_t = \alpha y_t + \sum_{n=0}^{\infty} \epsilon_{t-n}^{p,n} \tag{E_1}$$

$\epsilon_t^p = [\epsilon_t^{p,0}\ \epsilon_t^{p,1}\ \epsilon_t^{p,2}\ ...]$ is an infinite-dimensional vector of i.i.d 'policy news shocks'

Assume there is a unique solution in SMA(∞) form:

$$\begin{aligned} p_t &= M_{py}(L)\epsilon_t^y + M_{pp}(L)\epsilon_t^p \\ y_t &= M_{yy}(L)\epsilon_t^y + M_{yp}(L)\epsilon_t^p \end{aligned}$$

where $M_{ab}(L) = \sum_{h=0}^{\infty} M_{ab,h} L^h$

Consider

$$\frac{\partial \mathcal{L}_t}{\partial \epsilon_t^p} = M'_{yp,0} y_t = M_{pp,0} \mathcal{H}'_p y_t$$

If the policy rule is optimal, then $\mathcal{H}'_p y_t = 0$ such that $M'_{yp,0} y_t = 0$

Optimal Policy

If the impact impulse response coefficients $M_{yp,0}$ are known, than a feasible test of policy optimality is based on the condition $M'_{yp,0} y_t = 0$

Intuition: If policy is optimal, there should be no deviations from the policy rule that lead to a lower loss

In the New Keynesian model $M'_{yp,0} = \begin{bmatrix} \frac{-1}{1+\kappa\phi_\pi} & \frac{-\kappa}{1+\kappa\phi_\pi} \end{bmatrix}$

$$M'_{yp,0} y_t = \begin{bmatrix} \frac{-1}{1+\kappa\phi_\pi} & \frac{-\kappa}{1+\kappa\phi_\pi} \end{bmatrix} \begin{bmatrix} gap_t \\ \pi_t \end{bmatrix} = 0 \Rightarrow gap_t = -\kappa\pi_t$$

We can test deviations from optimal policy and even calculate policy improvements using empirical estimates of impulse responses to policy shocks

Optimal Policy

Let $\mathbf{Y}_t = [y_t' \ y_{t+1}' \ ...]'$ stack the current and all future values of y_t containing the arguments in the policy loss function

Policy loss function

$\mathcal{L}_t = E_t[\mathbf{Y}_t' W \mathbf{Y}_t]$ where p.s.d W contains policy weights

Uniqueness

There is a unique solution to the model generating the observed y_t, in SMA(∞) form

$$y_t = M_{yy}(L)\epsilon_t^y + M_{yp}(L)\epsilon_t^p$$

Optimal Policy Perturbations

Let $\mathbf{P}_t = [p_t' \; p_{t+1}' \; ...]'$ stack the current and all future values of p_t

Let $\mathbf{P}_t^e = E_t \mathbf{P}_t$ denote the observed expected policy path at time t

Let $\mathbf{P}_t^{e,\dagger}$ denote a proposed alternative expected policy path at time t

Let $\mathbf{P}_t^{e,opt}$ denote the expected policy path under optimal policy, i.e. minimizing $\mathcal{L}_t$

Uniqueness Conditions

- **Uniqueness Under Optimal Policy**: The optimal policy $\mathbf{P}_t^{e,opt}$ is unique
- **Uniqueness** $\dagger$: There is a unique solution to the model for $y_t^\dagger$ under $\mathbf{P}_t^{e,\dagger}$

Condition for Optimal Policy Barnichon and Mesters (2022)

$$\mathbf{P}_t^{e,\dagger} = \mathbf{P}_t^{e,opt} \Leftrightarrow \nabla \mathcal{L}_t|_{\mathbf{P}_t^{e,opt}} = \mathbf{M}^{\dagger\prime} W E_t \mathbf{Y}_t^\dagger = 0$$

where $\mathbf{M}^\dagger$ contains the dynamic causal effects on $\mathbf{Y}_t^\dagger$ under the proposed alternative policy

Optimal Policy Perturbations

Optimal Policy Perturbation Barnichon and Mesters (2022)

The optimal policy perturbation δ_t such that $\mathbf{P}_t^{e,\dagger} + \delta_t = \mathbf{P}_t^{e,opt}$ is given by

$$\delta_t = -(\mathbf{M}^{\dagger\prime} W \mathbf{M}^{\dagger})^{-1} \mathbf{M}^{\dagger\prime} W E_t \mathbf{Y}_t^{\dagger} = 0$$

Projection coefficient in the weighted projection of $E_t \mathbf{Y}_t^{\dagger}$ on -$\mathbf{M}^{\dagger}$

Policy deviations should not be able to reduce the sum of squared projection residuals (i.e. the policy objective)

Barnichon and Mesters (2022) check 'policy mistakes' δ_t at any time t in observed data with the following:

- Policymakers 'outlook', $E_t \mathbf{Y}_t$
- $\mathbf{M}$, the dynamic causal effect of policy shocks from the observed data
- Knowledge of the policy weights $\mathbf{W}$

Barnichon and Mesters (2022) Application to Monetary Policy

$E_t\mathbf{Y}_t$ and $\mathbf{M}$ are population objects, in practice both are sampled with error such that δ_t is a random variable

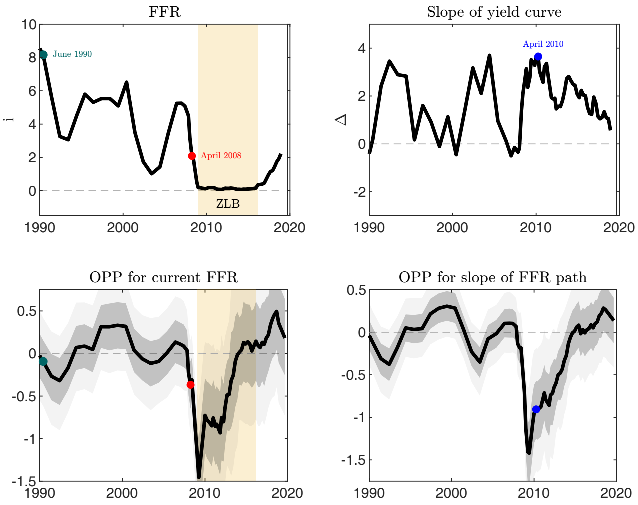
The test of policy optimality becomes a statistical test of the null that $\delta_t = 0$.

$E_t\mathbf{Y}_t$ is measured by median FOMC projection in the Summary of Economic Projections

$\mathbf{M}$ in theory requires a (rotation) of a full sequence of news shock. In practice, Barnichon and Mesters (2022) use impulse response to high frequency shocks around FOMC announcement in the FFR target and 10 year Treasury yield

The baseline policy loss function is $\mathcal{L}_t = ||\mathbf{\Pi}_t||^2 + ||\mathbf{U}_t||^2$ where $\mathbf{\Pi}_t$ and $\mathbf{U}_t$ stack the vector of inflation gaps and unemployment gaps from t to $t + H$

Barnichon and Mesters (2022) Application to Monetary Policy



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