# Why Have Interest Rates Fallen Far Below the Return on Capital

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December 27, 2017

**Abstract**

Risk-free rates have been falling since the 1980s while the return on capital has not. We analyze these trends in a calibrated OLG model designed to encompass many of the "usual suspects" cited in the debate on secular stagnation. Declining labor force and productivity growth imply a limited decline in real interest rates and deleveraging cannot account for the joint decline in the risk free rate and increase in the risk premium. If we allow for a change in the (perceived) risk to productivity growth to fit the data, we find that the decline in the risk-free rate requires an increase in the borrowing capacity of the indebted agents in the model, consistent with the increase in the sum of public and private debt since the crisis but at odds with a deleveraging-based explanation put forth in Eggertsson and Krugman (2012).

*We thank without implicating Marianne Andries, Adrien Auclert, Jean Barthelemy, Gilbert Cette, Nicolas Coeurdacier, Stephane Dupraz, Gauti Eggertsson, Emmanuel Farhi, François Geerolf, François Gourio, Christoph Grosse-Steffen, Stéphane Guibaud, Eric Mengus, Patrick Pintus, Franck Portier, Athanasios Orphanidès, Marc-Olivier Strauss-Kahn, Amir Sufi, Harald Uhlig and Ivan Werning, conference participants at the SED 2016, and seminar participants at the Federal Reserve Bank of Chicago, the Banque de France. The opinions herein do not necessarily represent those of the Banque de France, the Federal Reserve Bank of Chicago, or the Federal Reserve System.

# 1 Introduction

The "Global Financial Crisis" began ten years ago[1] and within two years short-term nominal interest rates were driven to near-zero levels in the large advanced economies (U.S., Euro-area, UK, Japan), staying there since. The Fed only recently "lifted off," the UK hasn't yet considered when to do so, and policy rates in Japan and the Euro area are still below zero. With low and (relatively) steady inflation, real rates have been negative for a while, and not just short-term rates but also rates at the 5-year and 10-year horizon (Hamilton et al., 2016).

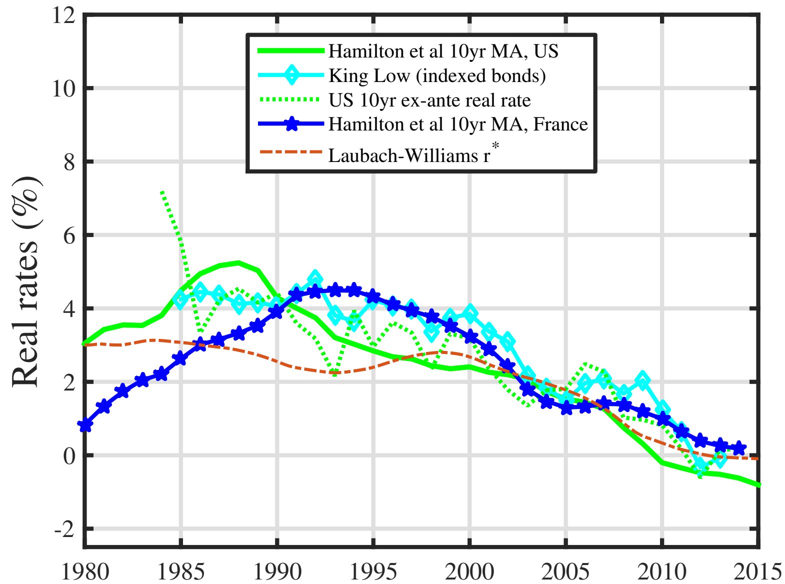


Figure 1: Real return on government bonds.

Much of the macroeconomic research responding to the financial crisis has taken place within the DSGE paradigm. Understanding the reasons for reaching the lower bound (the reason for low interest rate) was less urgent than understanding the proper responses to the situation. Also, the methodology relies on some approximation around a steady state, whether linear or nonlinear (Fernandez-Villaverde et al., 2012; Gust et al., 2012). Hence the low (real) interest rates are modeled as the result of an exogenous shock, for example to the discount rate or to a borrowing constraint (Eggertsson and Woodford, 2003; Eggertsson and Krugman, 2012), inducing deviations from a steady state whose dynamics (modified as needed by policy) are the core prediction of the model.

After a decade of low interest rates, the shock paradigm becomes less attractive because of the strains it places on the assumption of independent Gaussian shocks (Aruoba et al., 2013). At the same time observers are focusing increasingly "secular stagnation" hypothesis (Summers, 2014): low interest rates may not be temporary deviations but a

[1]The beginning of the crisis is commonly dated to the closure of two Paribas funds in August 2007.

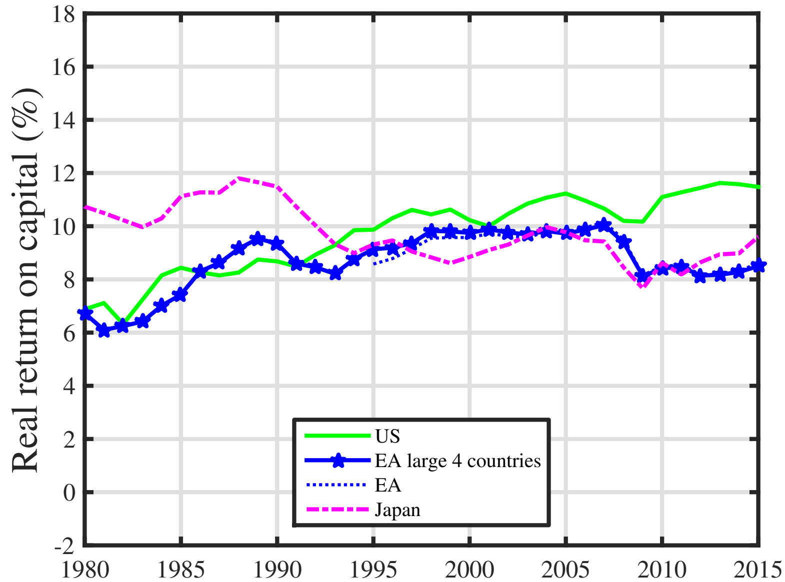


Figure 2: Return on capital.

now-permanent state of affairs. Are low rates the "new normal"? If so, why, and what can be done about it?

These are important policy questions (Fischer, 2016a,b). Three recent policy-oriented publications (Teulings and Baldwin, 2014; Bean et al., 2015; Gourinchas et al., 2016) have recently collected the possible explanations for such a permanent decline in interest rates. These include aging pressure on savings, income inequality, a slower pace of productivity, deleveraging, a collapse in the relative price of investment, a shortage of safe assets and an increase in the perception or risks. But so far there has been little quantitative evaluation of the competing explanations (but see Rachel and Smith, 2015) and little of it model-based. Yet answering the question requires a model. Our question is simple: can we account for current low interest rates in a model that encompasses the most likely factors?

To answer it we develop a framework that combines Coeurdacier et al. (2015) and Eggertsson and Mehrotra (2014) to encompass most of the current explanations for low interest rates. Importantly, we extend the model to include risk. There are two reasons for this. One is to make contact with the literature on the shortage of safe assets (Caballero et al., 2008; Caballero and Farhi, 2014) because safe assets only make sense in a context with risk. Second, we want to address an important fact on which we elaborate in the next section, namely the divergence between rates on (government) bonds, which have fallen, and the return on capital, which has not.

Using a single framework that encompasses the broad range of proposed explanations is like placing all the "usual suspects" in the same lineup. It comes at a cost if we want the model to remain tractable: there are only three generations and only one source of risk.

## 2 Related literature

Our paper relates to three literatures that investigate secular stagnation, the shortage of safe assets and long-term risk.

Several empirical papers document the role of demographics in explaining low interest rates. Ferrero et al. (2017) who find an effect of 0.5% in the last decade from a dynamic VAR on a panel of OECD countries and Busetti and Caivano (2017) who estimate such effects for eight advanced economies at low frequencies since the 1980s. Aksoy et al. (2016), however, in a broader analysis of demographic trends and the macro-economy, do not find significant effects on interest rates. Del Negro et al. (2017) who decompose changes in the US natural rate with either a DSGE or an identified VAR estimate that the slowdown of productivity can account for as much as 60bp in the decline of the natural rate since the mid 1990s, but attribute more of the decline (around 1%) to an increase in the convenience yield of safe assets (Treasury bonds). Favero et al. (2016) explore the role of demographic factors in an affine term structure model of interest rates.

A growing number of papers use OLG structures to assess how the aging of the baby-boomers explains either an increase in desired savings or the decelerating productivity or both. Eggertsson and Mehrotra (2014) provide a qualitative assessment in a closed economy set-up. Our results concur with the results of Gagnon et al. (2016), who use a rich OLG structure and find that aging can account for only as much at 1.2 pc decline in real interest rates. This accord with Carvalho et al. (2016), who find a 1.5% effect since 1990 in a simpler model with workers and retirees. We discuss Coeurdacier et al. (2015) below.

In our simulations, the contribution of ageing and the slowdown of productivity are consistent with these findings. They explain that the risk free rate has declined by 1 to 1.5% since 1990, i.e., much less than the declined observed in real rates. More importantly, these papers all consider a single asset class as a vehicle for savings. They do not account for the fact that, in the data, the return on capital, which equates the real interest rate in OLG models with production, has not declined. Instead we give households the choice either to own the capital used in production or to lend to the next generation. Hence we can use our model to replicate the evolution of the interest rate and the risk premia paid to own a capital stock of which the return is risky. Eggertsson et al. (2017) also differentiate the return on capital from the real interest rate. Their gap is due to a mark-up while ours reflect risk premia. They find that the fall in mortality, fertility and productivity since 1970 each explain nearly 2% drop in the real interest rate, a total effect of nearly -6 %, which is compensated by a 2% increase due to the rise in public debt. The much larger effects this paper finds for demographic factors and productivity is largely due to the choice of 1970 as their starting point, as a major part of the decline in these factors take place between 1970 and 1990, a period when real rates actually increased.

Our paper is also related to the literature on safe assets and their "shortage." In a seminal paper, Caballero et al. (2008) associated global imbalances to a growing demand of economic agents in emerging economies for "safe assets" that are typically issued by the US and other large OECD countries. Coeurdacier et al. (2015) use an OLG structure to estimate the effects of opening capital flows to China where severe credit constraints push down the equilibrium interest rate. As a result, the "world" interest rate can decline substantially. Coeurdacier et al. (2015) estimate that the equilibrium interest rate could by as much as 6%, however in a set up where the level of the steady state risk free interest rate is not consistent with the data. More recently Caballero et al. (2016) stress how the shortage of safe assets can slow economic growth, a force that would in turn push the risk free interest rate further down. Their model is purely stylised/qualitative. It cannot be used to quantify the role of each of the forces that influence the equilibrium interest rate. Caballero et al. (2017) introduce an accounting approach to jointly explain the decline in the risk free rate, the stability of the return of capital and the decline in labor share. Importantly, they show that even with a set of parameters that maximise the effects of increasing mark-ups on the gap between the interest rate and the return on capital, a large share of the increase in the risk premia remains unexplained. Hall (2016) models the decline of the risk free rate as resulting from a change in the composition of savers, with an increase in the weight of risk adverse savers in the economy. Our contribution with respect to this literature is that we offer a comprehensive and quantitative analysis of the role of risks in an OLG model where the other forces of secular stagnation can also have a role.

Third, we relate to the asset pricing literature on long term risk. Bansal and Yaron (2004) show that Epstein-Zin-Weil preferences combined with persistent growth rate of consumption and small uncertainty on its fluctuation can explain both a low risk free interest rate and a high risk premium. However, somewhat surprisingly, this finance literature has not investigated whether long term risks have changed over time. Our contribution is twofold: to put Epstein-Zin-Weil preferences in an OLG model and use such a model to compute the low frequency changes in long term risk that are consistent with the data.

Finally our paper contributes to the small literature that investigate whether inequality can explain the level of interest rates. Auclert and Rognlie (2016) present a new-Keynesian model with wage rigidities and agents facing uninsurable idiosyncratic risks. Calibrating to the present, their model shows that a rise in inequality similar to that observed in the US since the 1980s would induce a further drop of 0.90% in interest rates, but in the context of a permanently binding ZLB. We allow inequality to impact savings and interest rates in our OLG model through a bequest motive. We show that it plays no role in the increase of the risk premia.

## 3 Stylized facts

First, real interest rates have declined steadily over the last 2 decades (Figure 1). This downward trend is observed across OECD countries, for short-term and long-term interest rates as well as estimates of the natural rate of interest, and whatever the approach taken to approximate inflation expectations to define ex ante real interest rates (King and Low, 2014; Hamilton et al., 2016; Rachel and Smith, 2015; Laubach and Williams, 2016; Holston et al., 2016; Fries et al., 2016; Fischer, 2016a,b). Since the 1970s correspond to a period of financial repression with limited openness of euro area financial markets, we focus this paper on understanding the decline in real rates since 1980. Following Summers (2014) many economists have associated this decline to secular stagnation. The lower growth rates of per capita GDP and the demographic transition imply lower investment and higher savings so much so that the return to savings should decline.

Second, the return to capital as measured from national accounts has remained flat. Gomme et al. (2011) build the return to productive capital as the net operating surplus, which is equal to value added minus depreciation and payments to labor, divided by the capital stock. Gomme et al. (2015) and Caballero et al. (2017) stress that, in the US, return to productive capital has no trend. It fluctuates with the cycle around 10 to 11 percent before tax and around 7 percent after tax. In Figure 2 we report similar indicators of the return on productive capital for the Euro area and the US from the AMECO database. Again, we see no downward trend in this measure of the return on investment. [2] Altogether, we observe both a downward trend in real interest rates and stable return to productive capital.

## 4 The Model

Many of the factors cited in Bean et al. (2015), Rachel and Smith (2015) and Gourinchas et al. (2016) can be embedded in the OLG model we present here, which nests Eggertsson and Mehrotra (2014) and Coeurdacier et al. (2015), and adds risk. The determination of the interest in those models comes down to the Euler equation of savers, within which the constraints faced by borrower agents and other determinants enter through market-clearing. In the presence of risk, the savers also face a portfolio choice.

[2] The comparability of the data across countries is somewhat limited. In some countries, such as Italy and Germany, the income of unincorporated businesses which includes labor income of self employed are included (Garnier et al., 2015). In addition, and unlike the measure developed by Gomme et al. (2011, 2015), AMECO stock of capital includes dwellings and the flow of income to capital does not include rents. Caballero et al. (2017) who adjust their estimates for intangible intellectual property product introduced by the BEA since 2013 again find no evidence of a downward trend. However, there is little reason why either of these characteristics would modify the trend of the return to productive capital.

## 4.1 Description

In each discrete time period $t$ a generation is born that lives 3 periods $y$, $m$, $o$. The size of the generation born at $t$ is $N_t = g_{L,t}N_{t-1}$. Preferences are of the Epstein and Zin (1989) - Weil (1990) form:

$$V_t^y = \left( {c_t^y}^{1-\rho} + \beta \left( E_t V_{t+1}^m \right)^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\rho}} \tag{1}$$

$$V_{t+1}^m = \left( {c_{t+1}^m}^{1-\rho} + \beta \left( E_{t+1} V_{t+2}^o \right)^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\rho}} \tag{2}$$

$$V_{t+2}^o = {c_{t+2}^o}^{1-\gamma} \tag{3}$$

with $\beta$ the discount factor, $\gamma \geq 0$ the coefficient of relative risk aversion, $1/\rho \geq 1$ the intertemporal elasticity of substitution. The only source of risk is an aggregate productivity shock.

The factors of production are capital, which depreciates at a rate $\delta$, and labor, supplied inelastically by young and middle-aged agents. The labor productivity of a member of generation $t$ is $e_t^y$ when young and $e_t^m = 1$ when middle-aged. The aggregate productivity of labor over time is $A_t = g_{A,t}A_{t-1}$ and is stochastic. The neo-classical constant-returns production function combines capital (with share $\alpha$) and labor (with share $1-\alpha$) to produce output, one unit of which can become either one unit of consumption or $1/p_t^k$ units of investment; the relative price of investment goods is exogenous and follows $p_t^k = g_{I,t}p_{t-1}^k$. Markets are competitive and prices are flexible. Labor earns a wage $w_t$ while capital earns a return $r_t^k$.

Agents can purchase investment goods, and can also borrow from and lend to each other at a gross rate $R_{t+1}$, but they cannot owe more (principal and interest) than a fraction $\theta_t$ of next period's labor income. We will focus on situations in which the young borrow from middle-aged, and the middle-aged lend to the young and invest in physical capital by buying the depreciated stock in the hands of the old and purchasing investment goods.

The following equations summarize the above. Agents of generation $t$ choose $\{c_t^y$, $c_{t+1}^m$, $c_{t+2}^o$, $k_{t+2}^m$, $b_{t+1}^y$, $b_{t+2}^m\}$ to maximize (1–3) subject to three budget constraints and one borrowing constraint:

$$c_t^y = b_{t+1}^y + w_t e_t^y \tag{4}$$

$$c_{t+1}^m - b_{t+2}^m + p_{t+1}^k k_{t+2}^m = w_{t+1} - R_{t+1} b_{t+1}^y \tag{5}$$

$$c_{t+2}^o = (p_{t+2}^k(1-\delta) + r_{t+2}^k) k_{t+2}^m - R_{t+2} b_{t+2}^m \tag{6}$$

$$b_{t+1}^y \leq \theta_t E_t (w_{t+1}/R_{t+1}). \tag{7}$$

On the production side, the production function combines the current capital stock $N_{t-2}k_t^m$ (held by the old of generation $t-2$ but chosen at $t-1$ when they were middle-aged) and the labor supply of current young and middle-aged $e_t^y N_t + N_{t-1}$ to produce

$$Y_t = (N_{t-2} k_t^m)^\alpha \left[A_t (e_t^y N_t + N_{t-1})\right]^{1-\alpha}$$

which yields the wage rate and capital rental rate

$$w_t = (1-\alpha) A_t^{1-\alpha} k_t^\alpha \tag{8}$$

$$r_t^k = \alpha A_t^{1-\alpha} k_t^{\alpha-1} \tag{9}$$

both written in terms of the capital/labor ratio $k_t$ defined as

$$k_t \equiv \frac{N_{t-2} k_t^m}{e_t^y N_t + N_{t-1}} = \frac{k_t^m}{g_{L,t-1}(1 + e_t^y g_{L,t})}. \tag{10}$$

The final condition imposes clearing of the bond market at time $t$:

$$g_{L,t} b_{t+1}^y + b_{t+1}^m = 0. \tag{11}$$

## 4.2 Equilibrium conditions

The solution proceeds as follows. Following Giovannini and Weil (1989) we first express the middle-aged agent's first-order conditions in terms of a total return on their portfolio, and derive the demand for the two available assets, capital and loans to the young, as well as relation between the two returns. We then use market clearing: the demand for capital must equal the aggregate stock of capital, while the young's borrowing constraint, expressed in terms of their wages, determines the supply of the other asset. This allows us to derive the law of motion for the capital stock. We assume here that $\delta = 1$; the general case is treated in the appendix.

The problem of the middle-aged of generation $t-1$ is

$$\max_{c_t^m, c_{t+1}^o} \left( (c_t^m)^{1-\rho} + \beta E_t [{c_{t+1}^o}^{1-\gamma}]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\rho}}$$

subject to

$$c_t^m + p_t^k k_{t+1}^m - b_{t+1}^m = w_t - R_t b_t^y \tag{12}$$

$$c_{t+1}^o = R_{t+1}^k p_t^k k_{t+1}^m - R_{t+1} b_{t+1}^m \tag{13}$$

which leads to the first-order conditions

$$(c_t^m)^{-\rho} = \beta \left[E_t (c_{t+1}^o)^{1-\gamma}\right]^{\frac{\gamma-\rho}{1-\gamma}} E_t \left[(c_{t+1}^o)^{-\gamma} R_{t+1}^k\right] \tag{14}$$

$$(c_t^m)^{-\rho} = \beta \left[E_t (c_{t+1}^o)^{1-\gamma}\right]^{\frac{\gamma-\rho}{1-\gamma}} E_t \left[(c_{t+1}^o)^{-\gamma}\right] R_{t+1}. \tag{15}$$

To see it as a portfolio problem, express the budget constraints (12)–(13) in terms of income $Y_t \equiv w_t - \theta_{t-1} E_{t-1} w_t$ and total savings $W_t$ invested in capital $p_{t+1}^k k_{t+1}^m$ with return $R_{t+1}^k \equiv r_{t+1}^k / p_t^k$ and loans $-b_{t+1}^m$ with return $R_{t+1}$. Letting $\alpha_t$ be the portfolio weight on capital, the total return on the middle-aged agent's portfolio is $R^m_{t+1} = \alpha_t R^k_t + (1-\alpha_t)R_{t+1}$ and the budget constraints become

$$W_t = Y_t - c^m_t$$
$$c^o_{t+1} = R^m_{t+1} W_t.$$

The two first-order conditions (14)–(15), in which we substitute $c^o_{t+1} = R^m_{t+1}W_t$, determine the portfolio allocation between bonds and capital: $\alpha_t$ must be such that

$$E_t({R^m_{t+1}}^{-\gamma})R_{t+1} = E_t\left({R^m_{t+1}}^{-\gamma}R^k_{t+1}\right)$$

In equilibrium the return on capital must satisfy

$$R^k_{t+1} \equiv \frac{r^k_{t+1}}{p^k_t} = \frac{\alpha A^{1-\alpha}_{t+1}}{p^k_t} k^{\alpha-1}_{t+1}, \tag{16}$$

which implies that the risk-free rate and the return on capital are linked by

$$R^k_{t+1} = \frac{\tilde{a}_{t+1}}{\xi_t} R_{t+1} \tag{17}$$

where we have defined

$$\xi_t \equiv \frac{E_t({R^m_{t+1}}^{-\gamma}\tilde{a}_{t+1})}{E_t {R^m_{t+1}}^{-\gamma}}$$
$$\tilde{a}_{t+1} \equiv \frac{A^{1-\alpha}_{t+1}}{\mathbb{E}_t A^{1-\alpha}_{t+1}}.$$

The variable $\tilde{a}_{t+1}$ is a transformation of the exogenous shock $a_{t+1}$.

In addition, the first-order condition (15) yields the saving decision

$$Y_t = \left(1 + (\beta\phi_t R^{1-\rho}_{t+1})^{-\frac{1}{\rho}}\right) W_t \tag{18}$$

where we have defined

$$\phi_t \equiv \left[\mathbb{E}_t((\frac{R^m_{t+1}}{R_{t+1}})^{1-\gamma})\right]^{(\gamma-\rho)/(1-\gamma)} \mathbb{E}_t((\frac{R^m_{t+1}}{R_{t+1}})^{-\gamma}).$$

We now bring in the market-clearing conditions to express $Y_t$ and $W_t$ in (18) in terms of the aggregate capital stock $k_t$. First, using the fact that the middle-aged were credit-constrained in their youth, we express their income $Y_t$ as:

$$\begin{aligned} Y_t &= w_t - \theta_{t-1}E_{t-1}w_t \\ &= (1-\alpha)(\tilde{a}_t - \theta_{t-1})\mathbb{E}_{t-1}A^{1-\alpha}_t k^\alpha_t. \end{aligned} \tag{19}$$

Next, their portfolio choices must equal the supply of the two assets. In (11) the supply of bonds is given by (7) at equality, and the supply of capital is given by (10), which leads to:

$$W_t = p^k_t k^m_{t+1} - b^m_{t+1} = g_{L,t} v_t \frac{p^k_t k_{t+1}}{\alpha \xi_t} \tag{20}$$

where we have defined

$$v_t \equiv \alpha(1 + e^y g_{L,t+1})\xi_t + (1-\alpha)\theta_t.$$

Similarly we can express $R^m_{t+1}W_t$ as:

$$R^m_{t+1}W_t = R^k_{t+1}p^k_t k^m_{t+1} - R_{t+1}b^m_{t+1} = g_{L,t}u_{t+1}\mathbb{E}_t A^{1-\alpha}_{t+1}k^{\alpha}_{t+1} \tag{21}$$

where we have defined

$$u_{t+1} \equiv \alpha(1 + e^y g_{L,t+1})\tilde{a}_{t+1} + (1-\alpha)\theta_t.$$

Taking the ratio of (20) and (21) gives

$$R^m_{t+1} = \frac{u_{t+1}}{v_t}R_{t+1} \tag{22}$$

with $u_t$ and $v_{t+1}$ only function of exogenous variables. Replacing (22) in the definitions of $\xi_t$ and $\phi_t$ gives:

$$\xi_t = \frac{\mathbb{E}_t(u_{t+1}{}^{-\gamma}\tilde{a}_{t+1})}{\mathbb{E}_t(u_{t+1}{}^{-\gamma})} \tag{23}$$

$$\phi_t = \left[\mathbb{E}_t u_{t+1}{}^{1-\gamma}\right]^{(\gamma-\rho)/(1-\gamma)} \mathbb{E}_t u_{t+1}{}^{-\gamma} v_t{}^{\rho} \tag{24}$$

which are also functions of (moments of) the exogenous shock $\tilde{a}_{t+1}$.

We can now rewrite the middle-aged agent's savings decision (18) as a law of motion for capital by replacing income expressed as (19) and savings expressed as (20):

$$(1-\alpha)(1-\frac{\theta_{t-1}}{\tilde{a}_t})\alpha\frac{A_t{}^{1-\alpha}}{p^k_t}k^{\alpha}_t = \left(1 + (\beta\phi_t)^{-1/\rho}R^{1-1/\rho}_{t+1}\right) g_{L,t}\frac{v_t}{\xi_t}k_{t+1}. \tag{25}$$

with the left-hand side consisting entirely of variables pre-determined at $t$.

The expression involves both $k$ and $R$ but (17) allows us to express $k_{t+1}$ in terms of $R_{t+1}$ and vice-versa, so that (25) can be written in terms of the risk-free interest rate $R_{t+1}$ or, equivalently, in terms of the capital stock.

## 4.3 Discussion

The law of motion (25), which is the core of the model and the basis for our simulations, is the middle-aged agent's optimal choice of saving (18), but with market-clearing imposed on the quantities: the left-hand side represents the savers' income, while the right-hand side is (the inverse of) the saving rate multiplying savings. To develop more intuition we first examine the form it takes in a deterministic steady state, then examine the role of risk.

### Deterministic steady state

When we set $\tilde{a}_t = 1$ to shut down the only source of uncertainty the terms involving risk simplify: $\xi_t = 1$ and from (24) $\phi_t = 1$, and $R^m_t = R^k_t = R_t$ (no risk premium).

In steady state (16) implies that $(A_t/k_t)^{\alpha}/p^k_t$ is constant, hence the growth rate of capital must be $g_k = g_A/g_I{}^{1/(1-\alpha)}$, as it would be in an infinitely-lived representative agent model. Capital grows at the same rate as labor productivity; the trend in the price of investment goods acts in this respect like an additional form of technological change ($g_I < 1$ leading to growth in the capital stock).

From (19) it also follows that, in steady state, the income of the middle-aged $Y_t$ (and, by (18), their consumption as well as aggregate consumption) grows at the rate $g_I g_k$, that is, the growth rate of capital priced as investment goods. The only determinants of these steady state rates are the technological parameters $g_A$ and $g_I$. The other parameters affect $R$ and the allocation across generations.

The equation determining the steady state interest rate can be expressed as

$$g_A g_I^{-\frac{1}{1-\alpha}} = (1+\beta^{-\frac{1}{\rho}} R^{1-\frac{1}{\rho}})^{-1} \left[\frac{1-\alpha}{\alpha g_L} \frac{R}{g_I}\right] \frac{\alpha(1-\theta)}{\alpha(1+e^y g_L)+(1-\alpha)\theta}$$

(see Theorem 1 in Coeurdacier et al. (2015)).

The structure of the equation remains a modified Euler equation. On the left-hand side the term $g_A g_I^{-\frac{1}{1-\alpha}}$ is the steady state rate of growth of capital, which depends only on productivity growth (including the effect of the price in investment goods). This growth rate is unaffected by the various other features of the model. On the right-hand side are three terms. The first is the saving rate. The second term in square brackets represents the "pure" OLG component, specifically the fact that those who save do so out of labor income only; capital income is used by the old to finance their consumption. The last term captures the effect of the borrowing constraint: this can be seen by setting $e^y = 0$ and $\theta = 0$, which deprives the young of income and prevents them from borrowing, effectively eliminating them. Then that last term reduces to 1, and the model is isomorphic to a two-period overlapping generations model with no borrowing constraint.

### Risky steady state

We assume that uncertainty on the productivity can be modelled as an i.i.d process.

**Assumption 1** (Distribution of the productivity shock)**.** *Assume that the productivity shock is i.i.d, with mean* 1*.*

To account for the impact of risk while retaining tractability, we appeal to the concept of risky steady state (Juillard, 2011; Coeurdacier et al., 2011). Instead of setting $\tilde{a}_t = \tilde{a}_{t+1} = 1$ as in the deterministic steady state, we set $\tilde{a}_t$ to its mean of 1 but maintain $\tilde{a}_{t+1}$ as a stochastic variable, assuming that it is lognormal with variance $\sigma_{t+1}$. In effect, we assume that in every period the current realization of the shock is at its mean but agents take into account the risk in the next period.

The following result describes the behavior of the risky steady-state.

**Proposition 1.** *The risky steady-state satisfies the following equation*

$$g_A g_I^{-\frac{1}{1-\alpha}} = (1+(\phi\beta)^{-\frac{1}{\rho}} R^{1-\frac{1}{\rho}})^{-1} \left[\frac{1-\alpha}{\alpha g_L} \frac{R}{g_I}\right] \frac{\alpha(1-\theta)}{\alpha(1+e^y g_L)\xi+(1-\alpha)\theta} \tag{26}$$

*This equation admits a unique solution.*

Compared to the deterministic case, the presence of risk adds two channels, captured by the terms $\phi$ and $\xi$, functions of the exogenous factors only. [3].

The first channel $\xi$ relates to the risk premium and its impact on the portfolio choice of the agent. This can be seen in two ways. First, from (17) the risk premium $R^k/R$ is $\tilde{a}_{t+1}/\xi_t$. Second, the share of the agent's savings invested in capital (the risky asset) is $\alpha(1+e^y g_{L,t+1})\xi_t/v_t$, which is proportional to the term $\xi_t/v_t = \xi_t/(\alpha(1+e^y g_L)\xi_t+(1-\alpha)\theta$.

The following result describes the properties of the risky steady-state, when $\tilde{a}$ follows a log-normal law.

**Proposition 2.** *1. The risk-premium $\xi^{-1}$ admits the following asymptotic expansion*

$$\xi^{-1}(\theta, g_L, \sigma) = 1 + \gamma \frac{\alpha(1+e^y g_L)}{(1-\alpha)\theta + \alpha(1+e^y g_L)}\sigma^2 + o(\sigma^4)$$

*2. The distortion $\phi$ admits the following asymptotic expansion*

$$\phi_t = 1 + \frac{1}{2}\gamma(1-\rho)\frac{\alpha^2(1+e^y g_{L,t+1})^2}{(\alpha(1+e^y g_{L,t+1}) + (1-\alpha)\theta_t)^2}\sigma^2 + o(\sigma^4).$$

*3. The portfolio-share of middle-aged allocated to capital is given by*

$$\frac{\alpha(1+e^y g_{L,t+1})\xi_t}{\alpha(1+e^y g_{L,t+1})\xi_t + (1-\alpha)\theta_t}$$

The risk premium obviously increases with risk aversion $\gamma$, but also with a tightening of the borrowing constraint (lower $\theta$) or a fall in population growth $g_L$, both of which reduce the supply of bonds. These effects, however, are second-order only: there are no first-order terms for $\theta$ or $g_L$.

The second channel, $\phi_t$, acts like a distortion to the discount rate, and is familiar from the literature on recursive preferences.

There are two things to note. One is that the sign of the sensitivity of $\phi_t$ to risk depends on whether the intertemporal elasticity $\rho$ is high or low relative to 1. The discussion in Weil (1990, 38) applies here: a high IES ($\rho < 1$) means that the income effect is small relative to the substitution effect, and the "effective" discount factor $\phi\beta$ rises with risk: the agent behaves as if she were more patient, and a higher interest rate $R$ is required in equilibrium. The IES determines the sign, but the magnitude of the effect is determined by the risk aversion $\gamma$.

The second point is that the relative strength of the two channels (the ratio of $\partial\phi/\partial_{\sigma^2}$ to $\partial\xi/\partial_{\sigma^2}$) is $(1-\rho)\alpha(1+e^y g_L)/2(\alpha(1+e^y g_L)+(1-\alpha)\theta)$ which, for low $\theta$, is close to $(1-\rho)/2$. For a IES close to 1, the effect of risk through the precautionary channel will be much smaller (in our calibration of $\rho = 0.8$, one order of magnitude smaller) than through the portfolio channel. For log utility (IES=1) there is only the portfolio channel.

---

[3]When $\delta \neq 1$ they are also functions of the endogenous rates of return (see the appendix).

When $\rho = 1$ we find the following first-order approximation for $R$ and $R^k$ around $[g_I, g_A, g_L, \theta, \sigma^2] = [1, 1, 1, 0, 0]$:

$$\ln(R) = \bar{r} + \frac{1+2e^y}{1+e^y}\ln(g_L) + \ln(g_A) - \frac{\alpha}{1-\alpha}\ln(g_I) + \frac{1+\alpha e^y}{\alpha(1+e^y)}\theta - \gamma\sigma^2$$
$$\ln(R^k) = \bar{r} + \frac{1+2e^y}{1+e^y}\ln(g_L) + \ln(g_A) - \frac{\alpha}{1-\alpha}\ln(g_I) + \frac{1+\alpha e^y}{\alpha(1+e^y)}\theta$$

with

$$\bar{r} = \ln\left[\frac{\alpha(1+e^y)(1+\beta)}{(1-\alpha)\beta}\right]$$

Thus, even for $\rho = 1$ there is room for risk to affect interest rates, through the portfolio channel. The return to capital, however, does not depend on risk. Increasing risk raises the risk premium and compresses the risk-free rate, which is (roughly) what we see in the data. Indeed, risk is the only one of our "suspects" that affects only the risk-free rate.

# 5 A Quantitative Evaluation

In this section we use the model to match the data on the risk free real interest rate and the return of capital since 1970.

## 5.1 Calibration of the model

The spirit of the exercise is as follows.

We distinguish between (a) the structural parameters $\beta$, $\gamma$, $\rho$ $\alpha$, $\delta$, $e^y$ characterizing preferences and technology, held fixed throughout and calibrated in a standard way, and (b) the factors $g_I$, $g_A$, $g_L$ that vary over time but are readily observable (Table 1).

This leaves us with $\theta$ (the borrowing constraint) and $\sigma$ (the amount of risk) which we approach flexibly because we see them as less easily observable. We proceed in several steps. First, we fix both $\theta = 0.07$ and $\sigma = 0.09$ for the US and the EA to see how much the observable factors can explain.[4] Then we keep one fixed and compute a time series for the other in order to match the path of the risk-free rate $R$. Finally let both vary and we back out time series of $\theta_t$ and $\sigma_t$ that will result in sequences of risky steady states $R$ and $R^k$ matching the observations. Ultimately, of course, we will need to confront these time series to data in order to assess the model's success or failure at accounting for the decline in interest rates.

Our model periods last 10 years. In the figures that follow, each year $N$ on the $x$-axis corresponds to the average 10-year lagging average (years $N-9$ to $N$), both for data and simulations.[5]. Our reasoning is that deciding when our 10-year periods start and end is

[4]These calibrations are chosen roughly in terms of the estimated values on the whole period, the values have an incidence on the level of the interest rates, but they impact marginally their evolution

[5]At the beginning of the sample, we compute the average on the available data, which starts in 1960, except for the productivity for the EA which starts in 1970

| | | |
|---|---|---|
| *Parameters* | | |
| $T$ | length of period (years) | 10 |
| $\beta$ | discount factor | $0.98^T$ |
| $\alpha$ | capital share | 0.28 |
| $\gamma$ | risk aversion | 100 |
| $\rho$ | inverse of IES | 0.8 |
| $\delta$ | capital depreciation rate | $0.1 * T$ |
| $e^y$ | relative productivity of young | 0.3 |
| *Factors* | | |
| $g_{L,t}$ | growth rate of population 20-64 | US, EA (France), China, Japan: OECD |
| $g_{I,t}$ | investment price growth | DiCecio (2009) |
| $g_{A,t}$ | productivity growth | US: Fernald (2012), Euro: NAWM model |
| $R_t$ | real interest rate | US: Hamilton et al. (2016), France |
| $R_t^k$ | return on capital | US, EA: our calculations à la Gomme et al. (2015) |
| $\tilde{a}_t$ | productivity shock | $\ln(\tilde{a})$ is a i.i.d. $N(-\sigma^2/2, \sigma^2)$ |
| *Free parameters* | | |
| $\theta$ | borrowing constraint on young | |
| $\sigma^2$ | variance of $\tilde{a}_t$ | |

Table 1: Model calibration and data sources.

somewhat arbitrary. Presenting the data and simulations in this manner avoids making that decision, as long as the reader keeps in mind that we are not representing annual time series, but sequences of $\{t, t+10\}$ pairs.

## 5.2 Results

### The impact of observable factors

To measure this impact we fix $\theta$ and $\sigma$ and analyze the model-based interest rates, when we use as inputs the growth rate of productivity, the change in demography and in relative investment price that we observe in the data (Figures 4 and 5).These factors are represented in Figure 3. [6]. Combined, these inputs reduce both the risk free rate and the return on capital by about 0.7 percent from 1990 to 2014 with no effect on the risk premia for the US, by 2.3 percent for the EA on the same period. These estimates are comparable to the ones of Gagnon et al. (2016) and Carvalho et al. (2016). The former estimate that demography account for a decline of the US equilibrium real rate by 1.25 percent from 1980 to 2015. And the latter estimate a decline by 1.5 percent between 1990 and 2014.

### The borrowing constraint

To measure the explanatory power of the borrowing constraint, we fix $\sigma$ and compute the parameter $\theta$ which is consistent with the risk-free rate, and the observable inputs. The implied borrowing constraints and the model-based return on capital, and risk-premia are represented in Figures 6, for the US and 7, for the EA. This exercise shows that for both

[6] The risk premia in the euro area appears very high in the 1970s again because at that period, financial repression implies very low real interest rates. For the euro area, we believe that the data are more meaningful for the post 1985 period as capital controls are progressively removed in Europe

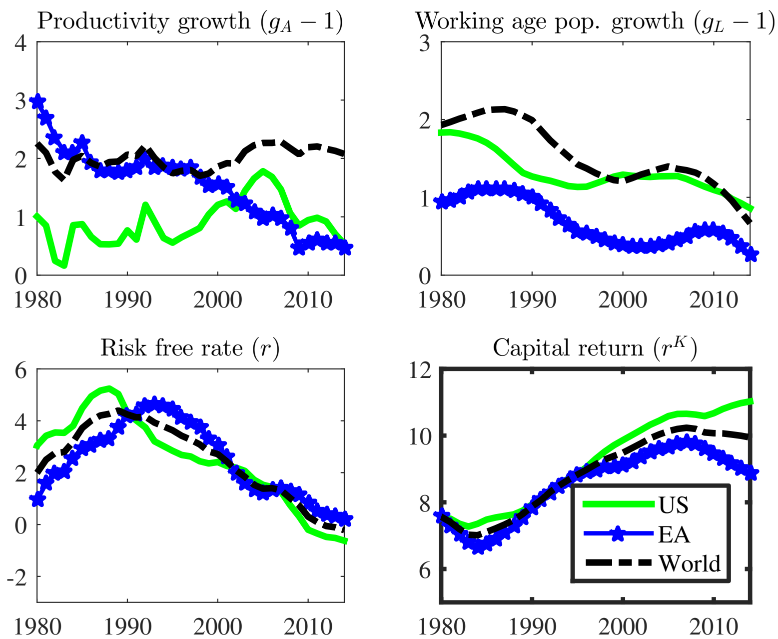


Figure 3: The inputs

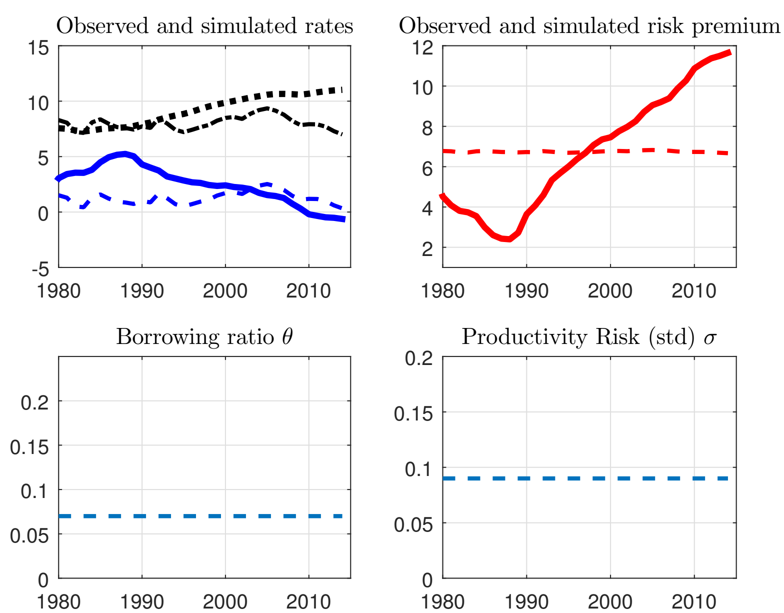


Figure 4: Impact of observable factors, in the US.

areas, the decline in the risk-free rate requires a tightening in the borrowing constraint, from 0.15 in 1990 to 0.05 in the US (from 0.12 to 0.08 in the EA). This evolution of the borrowing constraint hardly replicates the increase in the risk-premium in both areas, 1.1% in the US between 1990 and 2014 (0.2% in the EA on the same period).

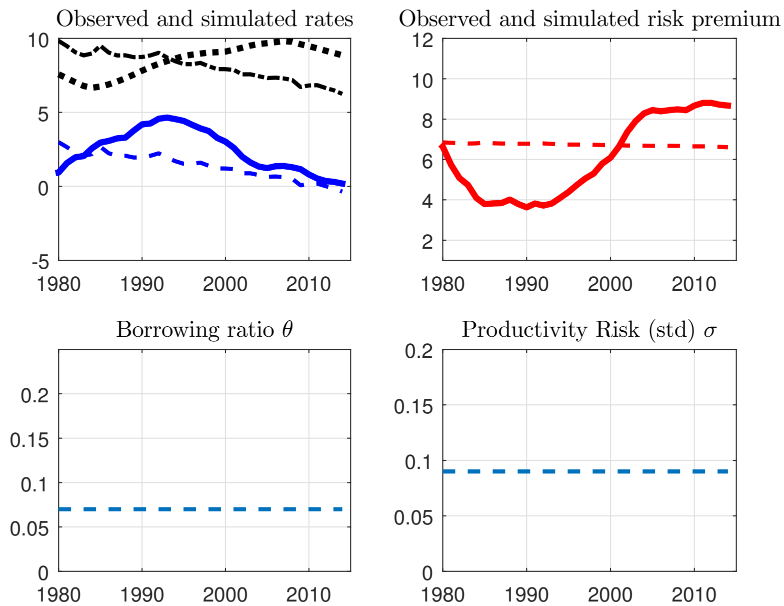


Figure 5: Impact of observable factors, in the EA.

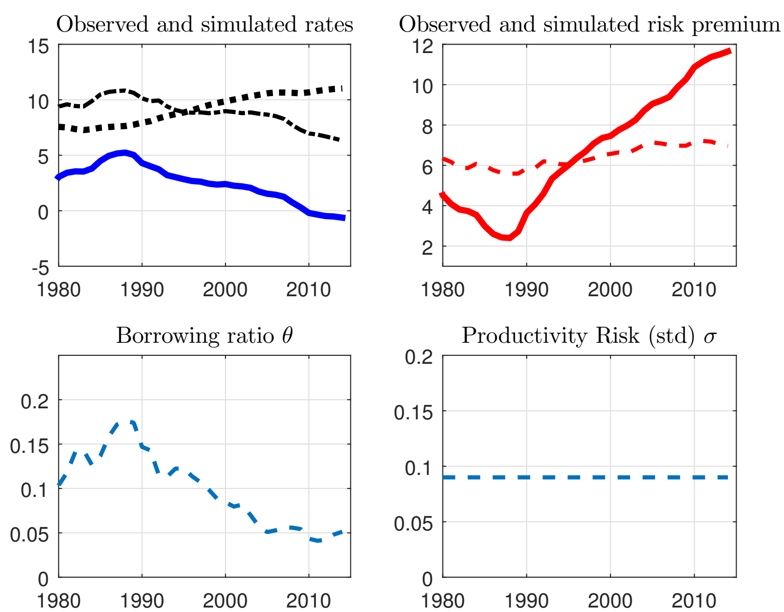


Figure 6: Impact of the borrowing constraint, in the US.

### Risk

We now consider the borrowing constraint as fixed over time, and assess the evolution of the variability in productivity that is required to reproduce the decline in the risk-free rate. The implied change in variability and the model-based return on capital are represented in Figures 8, for the US and 9, for the EA. The variance of productivity has to increase from about 0.04 per cent in 1990 to 0.1 today, for the US (from 0.06 to 0.09 for the EA). For both areas, this evolution since 1990 replicates quite well the evolution in the return on capital, and risk-premium.

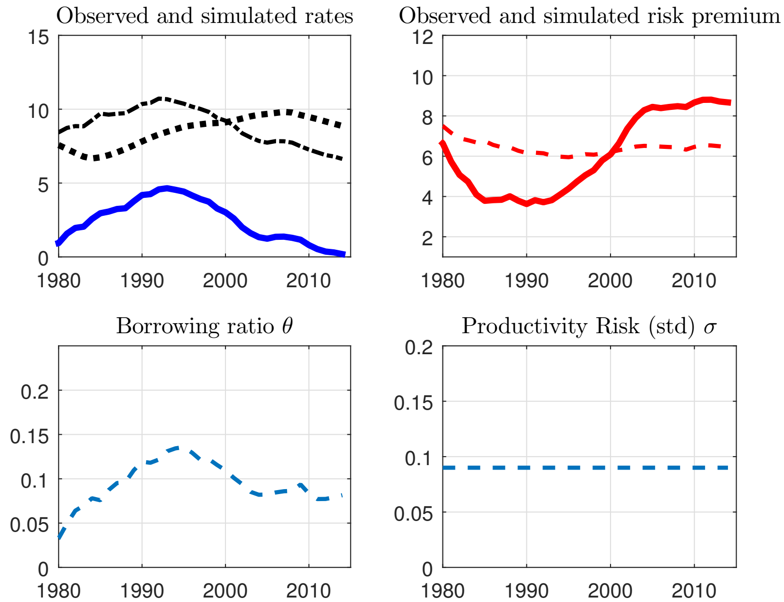


Figure 7: Impact of the borrowing constraint, in the EA.

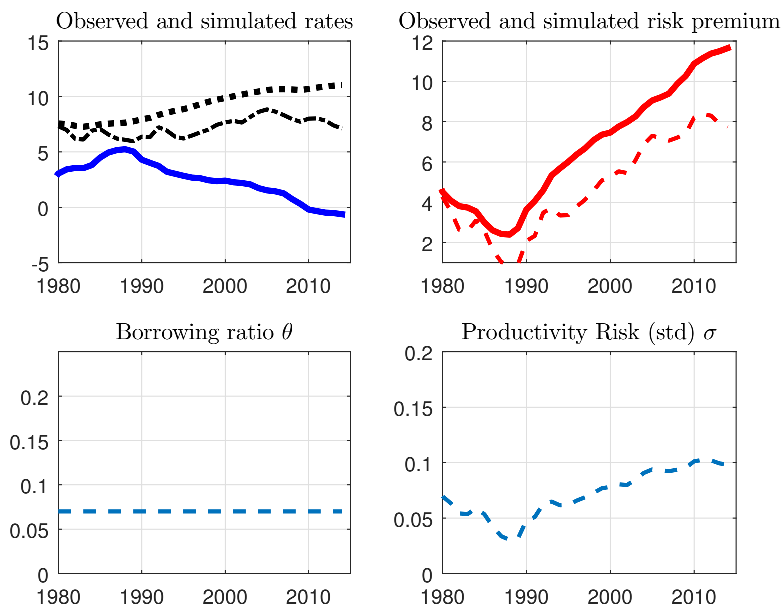


Figure 8: Impact of risk, in the US.

### Risk and the borrowing constraint

In Figures 10, for the US and 11, for the EA, we let both $\theta$ and $\sigma$ change over time so that we can replicate perfectly both the risk free rate and the return on capital. In particular, the trend decline in the risk free rate since 1990 is due exclusively to an increase in the risk of productivity, from 0.06 to 0.14 in the US (0.06 to 0.11 in the EA). Moreover, the evolution of the risk-free rate and the return on capital is consistent with a non decreasing pattern of the borrowing constraint. This shows that, according to the model, the drop in real interest rate need not reflect deleveraging headwinds. What evidence do we have that

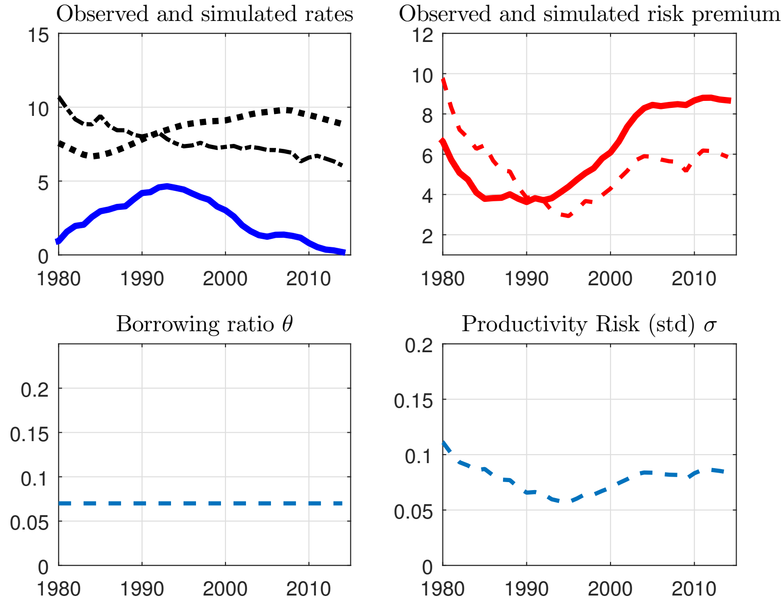


Figure 9: Impact of risk, in the EA.

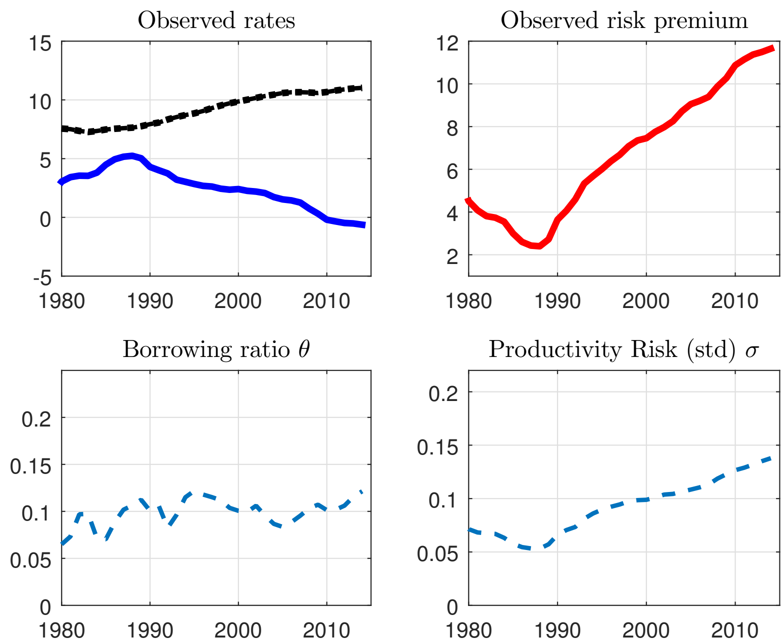


Figure 10: Impact of risk and the borrowing constraint, in the US.

uncertainty as effectively increased over the last 25 years? Baker et al. (2016) indicates that there may be an upward trend of economic uncertainty from the 1985 to 2012 and their a clear acceleration of political uncertainty over the last fifteen years. In particular the so called "great moderation" period, usually dated from 1985 to 2007 does not correspond to a decline in uncertainty as measured by Baker et al. (2016). Altogether, that our simulation point to an increase in perceived risk as suggested by the steady increase in the risk premia from 1990 to 2016 is not inconsistent with these other measures that show uncertainty

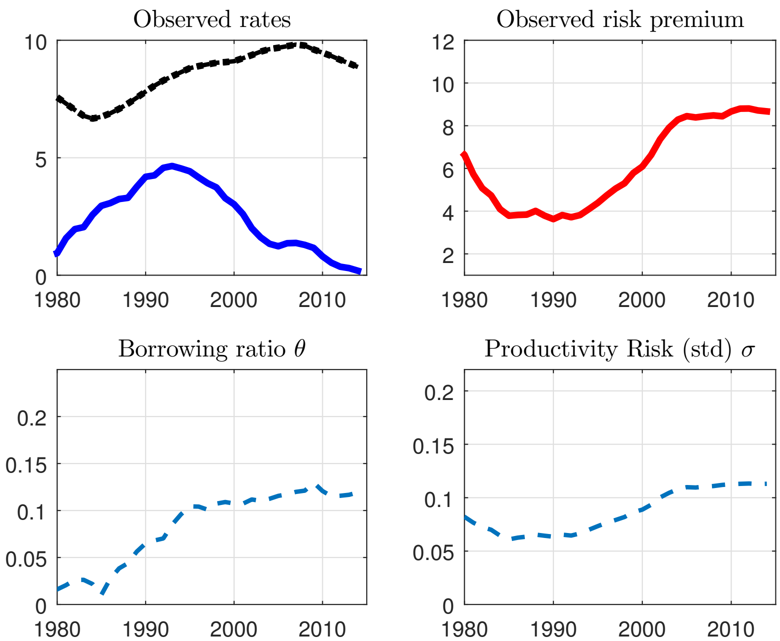


Figure 11: Impact of risk and the borrowing constraint, in the EA.

trending up, at least from 1990 to 2012.

### The evolution of the borrowing constraint ?

As shown by Buttiglione et al. (2014) there has been hardly any overall deleveraging since the crisis. Private debt in advanced economies adds up to 178 per cent of GDP in 2016, i.e. the same level than in 2010, while public debt increased from 75 to 87 per cent of GDP over the same period. Deleveraging of the private sector has been very large in Spain and in the United Kingdom, but it increased in France and Canada. And public debt increased in all G7 countries but Germany.

We thus use the model to infer the borrowing constraint consistent with the evolution of debt and the risk-free rate in both areas, depicted in Figures 12, for the US and 13, for the EA. In this exercise, we compute the borrowing constraint and the level of risk implied by the model to replicate the ratio debt/income[7] and the risk-free rate. This shows that the pattern of the borrowing constraint consistent with the evolution of the debt implies an increase in variability of productivity completely in line with what we observe in the US to replicate the risk-premium, a bit smaller in the EA.

## 5.3 A global perspective

A fair criticism of our calculations is that, by focusing on the US and the Euro area, we neglect the phenomenon described as "savings glut" or "global imbalances" of the 2000s,

[7]We consider that a proxy of this ratio is total debt over T times GDP.

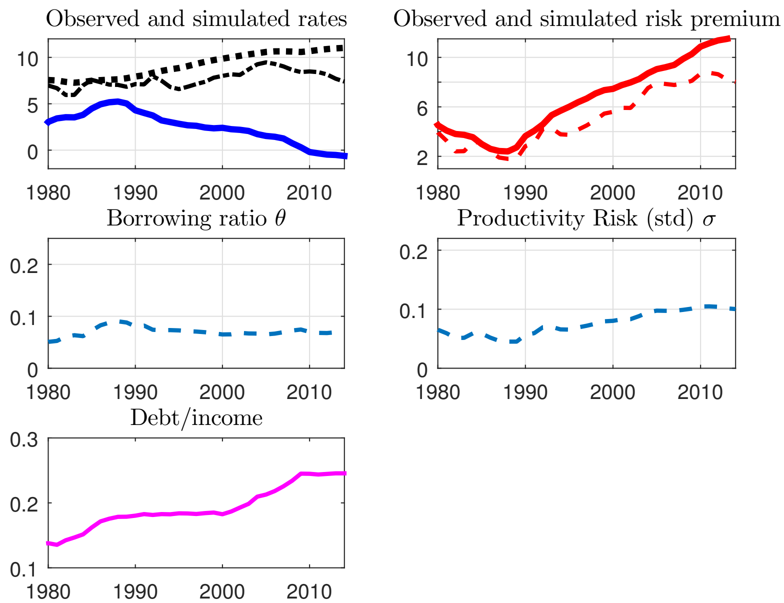


Figure 12: Borrowing constraint and risk, in the US.

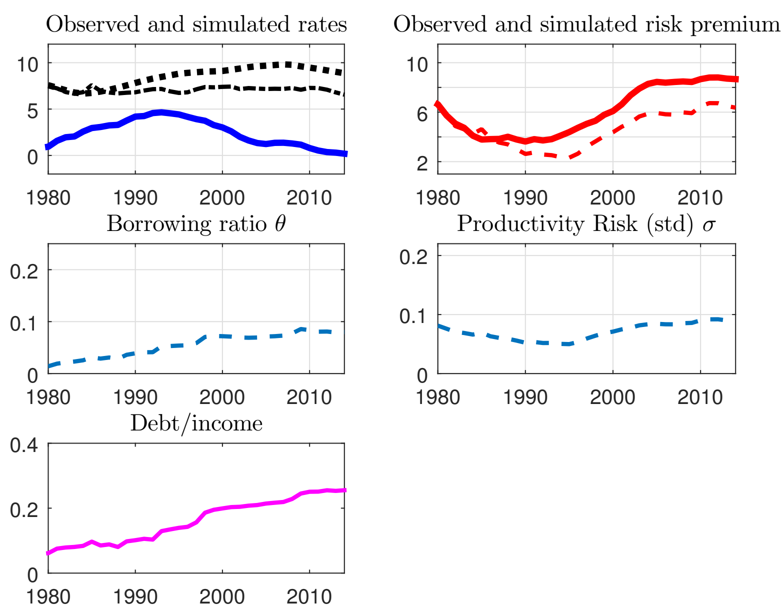


Figure 13: Borrowing constraint and risk, in the EA.

namely the increase in savings from emerging economies. We repeat our calculations for the world by adding Japan and China. We consider that a representation of the world is described by the aggregation of EA, US, China and Japan. The "world" population aged 20 to 64 is the ones of these four economies, investment price evolves as the American one, whole productivity is described as an aggregate of four productivities weighted by GDP. The target rates, both for return on capital and for the risk-free rate, are an average of the US and Euro area, considering that world capital markets are integrated. We think that our calculations are most likely to shed light on the period since 1990, once financial repression mechanisms have dropped out of the picture for most economies.

The results at the world level are broadly similar to the ones we found for the US and the EA: risk is the main factor that can account for the behavior of the risk premium since 1990. We note that, in the simulation that lets $\theta$ vary, in the early years the parameter $\theta$ is constrained at zero (which is why we cannot match the return on capital perfectly. The picture is nevertheless similar: we see $\theta$ rising since the mid-1980s, suggestive of the global savings glut. the borrowing ratio stops rising in the late 1990s and barely falls after that: deleveraging does not seem to be at play since the financial crisis.

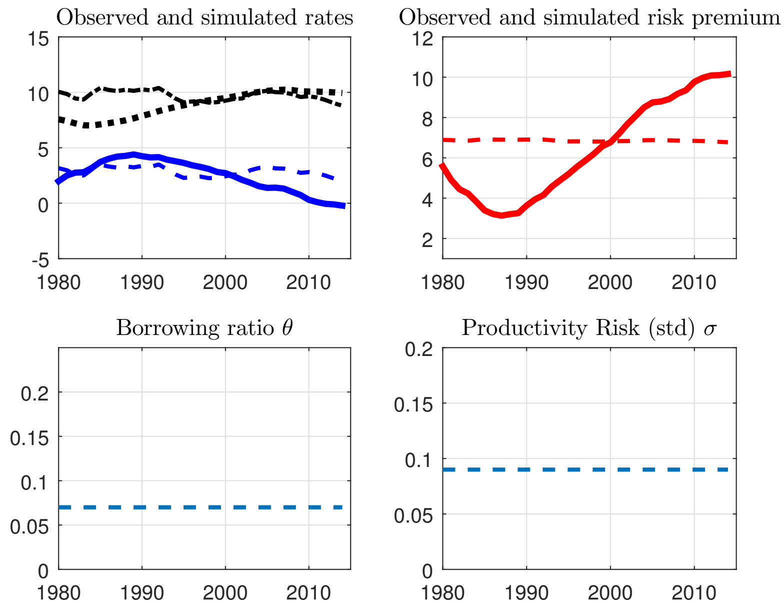


Figure 14: Impact of observable factors, world.

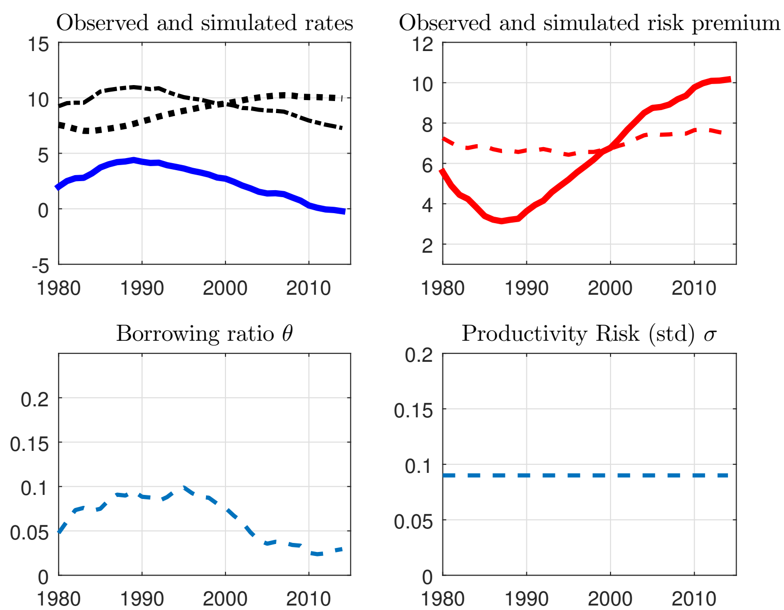


Figure 15: Impact of the borrowing constraint, world.

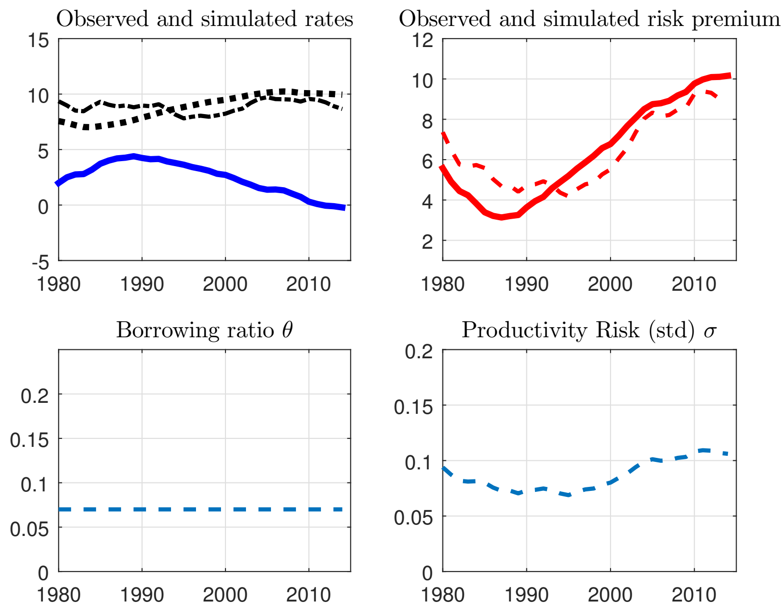


Figure 16: Impact of risk, world

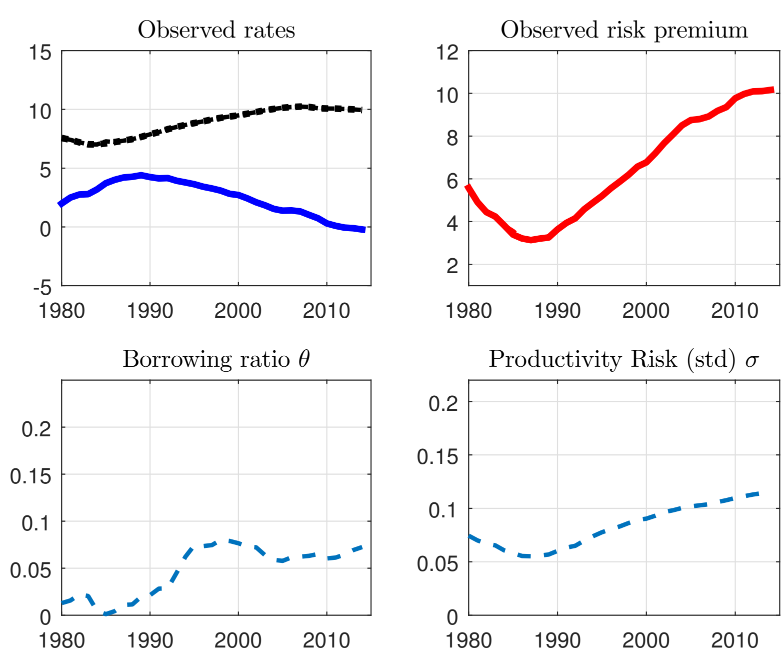


Figure 17: Impact of risk and the borrowing constraint, world.

## 5.4 Extensions

### Longevity

We introduce longevity as a survival probability. Specifically, we replace equation (2) with

$$V_{t+1}^m = \left( {c_{t+1}^m}^{1-\rho} + \beta\lambda_{t+1} \left( E_{t+1} V_{t+2}^o \right)^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\rho}} \tag{27}$$

The law of motion (25) becomes:

$$(1-\alpha)(1-\frac{\theta_{t-1}}{\tilde{a}_t})\alpha \frac{{A_t}^{1-\alpha}}{p_t^k} k_t^\alpha = \left(1 + (\beta\lambda_t\phi_t)^{-1/\rho} R_{t+1}^{1-1/\rho}\right) g_{L,t} \frac{v_t}{\xi_t} k_{t+1}$$

The data is taken from Bell and Miller (2005). We compute $\lambda_t$ as the probability of surviving from age 60 to age 70 at different points in time. This probability rises steadily through the sample from about 0.8 to about 0.9. In our model this is equivalent to a time-varying (and rising) discount factor.

The quantitative impact is to lower both the risk-free rate and the risky rate, by nearly identical amounts: 65bp for the risk-free rate and 69bp for the risky rate. The risk premium shrinks slightly.

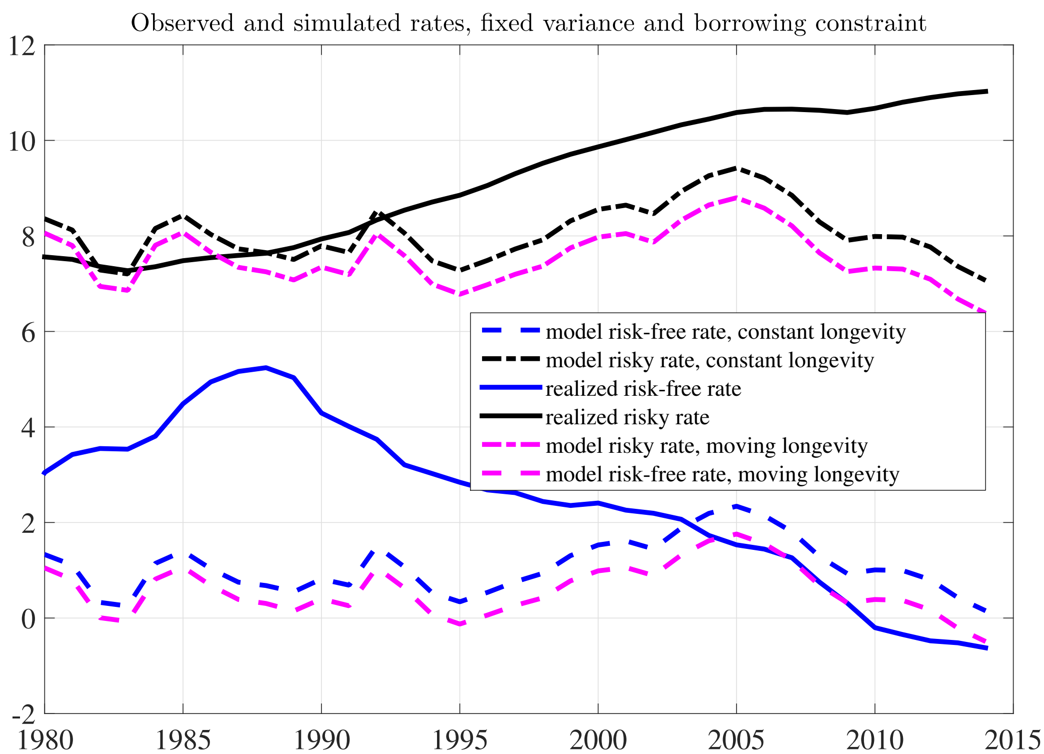


Figure 18: Effect of longevity, in the model

### Labor Share

We can take into account the decline in the labor share (Karabarbounis and Neiman, 2014; Jones and Philippon, 2016; Barka, 2017). The corresponding pattern for the capital share is a rise from around 33% in the early 1970s to 38% by 2014 (Koh et al., 2016).

In our model an increase in the capital share pushes up the risk-free rate, and increases slightly the risk-premium. Indeed, equation (40) can be rewritten as

$$(R^{-1} + \beta^{-\frac{1}{\rho}} R^{-\frac{1}{\rho}})^{-1} = g_L g_A g_I^{-\frac{\alpha}{1-\alpha}} \frac{\theta + \frac{\alpha}{1-\alpha}(1 + e^y g_L)}{1-\theta}$$

on the left hand, the function is increasing in $R$. Since $g_I < 1$, the right hand side is an increasing function in $\alpha$.

Thus, interest rate is an increasing function of the capital share $\alpha$. If we compare to our baseline calibration for which $\alpha$ is the sample average, the interest rate is lowered at the beginning of the sample by 0.6% but raised at the end by 1.1%. The risk premium is raised 20bp at the beginning and lowered 23bp at the end.

This factor, therefore, does not help in accounting for the patterns in the data.

### Inequality

Our final extension is to allow for changes in inequality.

We model inequality by introducing heterogeneity in middle-age productivity: agents of type $I$ have productivity $e_I$, and changes in inequality arise from mean-preserving changes in the distribution $\{e_I\}_I$. There is no additional uncertainty: young agents know which productivity they will have in the next period, and their borrowing constraint reflects their known productivity.

#### Introducing bequests

In our model consumption and savings in the second period are linear in lifetime wealth, hence mean-preserving spreads in inequality will have no aggregate effects. To break this linearity we introduce a bequest motive in the last period (De Nardi, 2004; Benhabib et al., 2011). The utility from leaving a bequest $B$ is assumed to be of the form $hB^{1-\varepsilon}$ with $h$ measuring the strength of the bequest motive. The elasticity $\varepsilon$, however, must be different from $\rho$.

This can be seen in a simple two-period planning problem, with an added bequest motive $h(B)$:

$$\max_{c_1, c_2, B} u(c_1) + \beta \left[ u(c_2) + h(B) \right]$$

subject to

$$\begin{aligned} c_1 + b &= Y_1 \\ c_2 + B &= Rb + Y_2 \end{aligned}$$

where $Y_i$ denotes income in each period. First-period consumption $c_1$ is the solution to

$$c_1 + \frac{1}{R} u'^{-1} \left( \frac{u'(c_1)}{\beta R} \right) + \frac{1}{R} h'^{-1} \left( \frac{u'(c_1)}{\beta R} \right) = Y_1 + \frac{Y_2}{R}$$

and will be linear in lifetime wealth unless $h'^{-1}(u'(c_1)/\beta R)$ is nonlinear in $c_1$. With $u(c) = c^{1-\rho}$ and $h(B) = hB^{1-\epsilon}$ this will hold if $\rho \neq \epsilon$. Moreover it can be shown that in equilibrium[8] a mean-preserving increase in inequality will lower $R$ iff $\rho > \varepsilon$.

With recursive preferences the corresponding equation for $c1$ is

$$c_1(1 + \beta^{\frac{1}{\rho}} R^{\frac{1}{\rho}-1} + (H\beta)^{\frac{1}{\omega}} R^{\frac{1}{\omega}-1} c_1{}^{\frac{\rho}{\omega}-1}) = Y_1 + \frac{Y_2}{R}.$$

with $\omega = 1-(1-\epsilon)\frac{1-\rho}{1-\gamma}$. A mean-preserving increase in inequality will lower $R$ iff $\frac{1-\rho}{1-\gamma}(\epsilon - \gamma) < 0$, so the comparison is between risk aversion and elasticity of bequests, but the sign of the effect of inequalities depends on whether $\rho \lesseqgtr 1$. In our calibration ($\rho < 1$ and $\gamma > 1$) a mean-preserving increase in inequality will lower $R$ iff $\epsilon > \gamma$. Since we want to "give inequality a chance," we choose $\epsilon$ accordingly.

**Bequests in our model**

For a steady-state with bequests to be well-defined, we need to make an adjustment to the formulation of preferences, as follows. Preferences are then represented as

$$\begin{aligned}
V_t^y &= \left( c_t^{y\,1-\rho} + \beta \left( E_t V_{t+1}^m \right)^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\rho}} \\
V_{t+1}^m &= \left( c_{t+1}^{m\;\;1-\rho} + \beta \left( E_{t+1} V_{t+2}^o \right)^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\rho}} \\
V_{t+2}^o &= \left( c_{t+2}^{o\;\;1-\rho} + \beta (E_{t+1} Y_{t+2})^{1-\rho} \left( h \left( \frac{B_{t+2}}{E_{t+1} Y_{t+2}} \right)^{1-\varepsilon} \right)^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\rho}}
\end{aligned}$$

Preliminary investigations with simple formulations of inequality, namely a binary distribution of productivities in middle age, reveal negligible impacts of inequality on the risk premium. At this stage it does not appear that differing propensities to save, in of themselves, will be of much help to explain the rising risk premium.

## 6 Conclusion

Risk-free rates have been falling since the 1980s while the return on capital has not. We analyze these trends in a calibrated overlapping generations model designed to encompass many of the "usual suspects" cited in the debate on secular stagnation. Declining labor force and productivity growth imply a limited decline in real interest rates and deleveraging cannot account for the joint decline in the risk free rate and increase in the risk premium. If we allow for a change in the (perceived) risk to productivity growth to fit the data, we find that the decline in the risk-free rate requires an increase in the borrowing capacity of the indebted agents in the model, consistent with the increase in the sum of public and private debt since the crisis but at odds with a deleveraging-based explanation put forth in Eggertsson and Krugman (2012).

[8] We close the simple two-period model with borrowers who have no bequest motives.

## Appendix

### General case ($\delta \neq 1$)

It will be convenient to denote the return to investment expressed in terms of consumption as $z_{t+1}$:

$$z_{t+1} \equiv \frac{p_t^k}{p_{t+1}^k} R_{t+1}^k - 1 + \delta = \alpha \frac{A_{t+1}^{1-\alpha}}{p_{t+1}^k} k_{t+1}^{\alpha-1}$$

Equation (17) becomes:

$$\begin{aligned} E_t({R_{t+1}^m}^{-\gamma}) R_{t+1} = E_t\left({R_{t+1}^m}^{-\gamma} R_{t+1}^k\right) \\ R_{t+1} &= \xi_t \alpha \frac{E_t A_{t+1}^{1-\alpha} k_{t+1}^{\alpha}}{p_t^k k_{t+1}} + (1-\delta)\frac{p_{t+1}^k}{p_t^k} &(28) \\ &= \xi_t \frac{\alpha}{1-\alpha} \frac{E_t w_{t+1}}{p_t^k k_{t+1}} + (1-\delta)\frac{p_{t+1}^k}{p_t^k} &(29) \end{aligned}$$

where we have defined

$$\begin{aligned} \xi_t &\equiv \frac{E_t({R_{t+1}^m}^{-\gamma} \tilde{a}_{t+1})}{E_t {R_{t+1}^m}^{-\gamma}} \\ \tilde{a}_{t+1} &\equiv \frac{A_{t+1}^{1-\alpha}}{\mathbb{E}_t A_{t+1}^{1-\alpha}}. \end{aligned}$$

Note that the risk-free rate and the return on capital are now linked as follows:

$$R_{t+1}^k = \frac{\tilde{a}_{t+1}}{\xi_t} R_{t+1} + (1 - \frac{\tilde{a}_{t+1}}{\xi_t}) \frac{p_{t+1}^k}{p_t^k}(1-\delta) \tag{30}$$

The auxiliary variables $v_t$ and $u_{t+1}$ take the more general form

$$\begin{aligned} v_t &\equiv \alpha(1 + e^y g_{L,t+1})\xi_t + (1-\alpha)\theta_t(1 - \frac{p_{t+1}^k}{p_t^k}\frac{1-\delta}{R_{t+1}}) \\ &= \alpha(1 + e^y g_{L,t+1})\xi_t + (1-\alpha)\theta_t \frac{\tilde{z}_{t+1}}{\tilde{z}_{t+1} + 1 - \delta}. \\ u_{t+1} &\equiv \alpha(1 + e^y g_{L,t+1})(\tilde{a}_{t+1} + p_{t+1}^k(1-\delta)\frac{k_{t+1}^{1-\alpha}}{\alpha \mathbb{E}_t A_{t+1}^{1-\alpha}}) + (1-\alpha)\theta_t \\ &= \alpha(1 + e^y g_{L,t+1})(\tilde{a}_{t+1} + \frac{\xi_t}{z_{t+1}}(1-\delta)) + (1-\alpha)\theta_t \end{aligned}$$

and $R_{t+1}^m$ becomes

$$\begin{aligned} R_{t+1}^m &= \frac{u_{t+1}}{v_t}\frac{\alpha\xi_t}{1-\alpha}\frac{(1-\alpha)\mathbb{E}_t A_{t+1}^{1-\alpha} k_{t+1}^{\alpha}}{p_t^k k_{t+1}} \\ &= \frac{u_{t+1}}{v_t}\frac{\alpha\xi_t}{1-\alpha}\frac{\mathbb{E}_t w_{t+1}}{p_t^k k_{t+1}} \\ &= \frac{u_{t+1}}{v_t}\left(R_{t+1} - \frac{p_{t+1}^k}{p_t^k}(1-\delta)\right) = \frac{u_{t+1}}{v_t}\frac{p_{t+1}^k}{p_t^k}\frac{\xi_t}{\tilde{a}_{t+1}} z_{t+1}. &(31) \end{aligned}$$

Replacing (31) in the definitions of $\xi_t$ and $\phi_t$ gives:

$$\xi_t = \frac{\mathbb{E}_t({u_{t+1}}^{-\gamma}\tilde{a}_{t+1})}{\mathbb{E}_t({u_{t+1}}^{-\gamma})} \tag{32}$$

$$\begin{aligned} \phi_t &= \left[\mathbb{E}_t {u_{t+1}}^{1-\gamma}\right]^{(\gamma-\rho)/(1-\gamma)} \mathbb{E}_t {u_{t+1}}^{-\gamma} {v_t}^{\rho} \left(1 - \frac{p_{t+1}^k}{p_t^k}\frac{1-\delta}{R_{t+1}}\right)^{-\rho} &(33) \\ &= \left[\mathbb{E}_t {u_{t+1}}^{1-\gamma}\right]^{(\gamma-\rho)/(1-\gamma)} \mathbb{E}_t {u_{t+1}}^{-\gamma}\left(u_{t+1} + \alpha(1+e^y g_{L,t+1})(\xi_t - \tilde{a}_{t+1})\right)^{-\rho} &(34) \end{aligned}$$

The general form of the law of motion is

$$(1-\alpha)(1-\frac{\theta_{t-1}}{\tilde{a}_t})z_t k_t = \left(1+(\beta\phi_t)^{-1/\rho}R_{t+1}^{1-1/\rho}\right)g_{L,t}\frac{v_t}{\xi_t}k_{t+1}. \tag{35}$$

with the left-hand side consisting entirely of variables pre-determined at $t$.

To compute the risky steady state, we first express the law of motion (35) in terms of $R_{t+1}$, we proceed as follows. First, we replace $z_t$ with $\tilde{z}_t\tilde{a}_t/\xi_{t-1}$ to write (35) as:

$$(1-\alpha)(\tilde{a}_t-\theta_{t-1})\tilde{z}_t = \left(1+(\beta\phi_t)^{-1/\rho}R_{t+1}^{1-1/\rho}\right)g_{L,t}v_t\frac{\xi_{t-1}}{\xi_t}\frac{k_{t+1}}{k_t}.$$

Then we use (28) to rewrite $k_{t+1}/k_t$:

$$\begin{aligned}\frac{R_{t+1}-(1-\delta)g_{I,t+1}}{R_t-(1-\delta)g_{I,t}} &= \frac{\xi_{t+1}}{\xi_t}g_{I,t+1}\frac{\mathbb{E}_t A_{t+1}^{1-\alpha}}{\mathbb{E}_{t-1}A_t^{1-\alpha}}\left(\frac{k_{t+1}}{k_t}\right)^{\alpha-1}\\ &= g_{I,t+1}\frac{\xi_{t+1}}{\xi_t}g_{A,t+1}^{1-\alpha}\left(\frac{k_{t+1}}{k_t}\right)^{\alpha-1}\\ \frac{k_{t+1}}{k_t} &= g_{I,t+1}^{1/(1-\alpha)}\left(\frac{\xi_t}{\xi_{t-1}}\right)^{1/(1-\alpha)} g_{A,t+1}\left(\frac{R_t-(1-\delta)g_{I,t}}{R_{t+1}-(1-\delta)g_{I,t+1}}\right)^{1/(1-\alpha)}\end{aligned}$$

so that

$$\begin{aligned}(1-\alpha)(\tilde{a}_t-\theta_{t-1})\left[\frac{R_t-(1-\delta)g_{I,t}}{g_{I,t}}\right] &= \left[1+(\beta\phi_t)^{-1/\rho}R_{t+1}^{1-1/\rho}\right]g_{L,t}\\ \times\left[\alpha(1+e^y g_{L,t+1})\xi_t+(1-\alpha)\theta_t(1-g_{I,t+1}\frac{1-\delta}{R_{t+1}})\right]&(g_{I,t+1})^{1/(1-\alpha)}\left(\frac{\xi_t}{\xi_{t-1}}\right)^{\alpha/(1-\alpha)}\end{aligned} \tag{36}$$

$$\times\, g_{A,t+1}\left(\frac{R_t-(1-\delta)g_{I,t}}{R_{t+1}-(1-\delta)g_{I,t+1}}\right)^{1/(1-\alpha)}. \tag{37}$$

This form of the law of motion involves additional variables $\xi_t$ and $\phi_t$ satisfying (32– 33) with

$$u_{t+1} = \alpha(1+e^y g_{L,t+1})\left(\tilde{a}_{t+1}+\frac{g_{I,t+1}(1-\delta)\xi_t}{[R_{t+1}-(1-\delta)g_{I,t+1}]}\right)+(1-\alpha)\theta_t. \tag{38}$$

Equation (37), along with (32), (33), and (38), define $(R_{t+1},\xi_t)$ implicitly as a recursive function of $(R_t,\xi_{t-1})$ and $\tilde{a}_t$:

$$\begin{bmatrix} R_{t+1}\\ \xi_t\end{bmatrix} = f(\left(\begin{bmatrix} R_t\\ \xi_{t-1}\end{bmatrix},\tilde{a}_t,\tilde{a}_{t+1}\right). \tag{39}$$

Note that the arguments of (39) include the realization of $\tilde{a}_t$ (known at $t$) and the (conditional) probability distribution of $\tilde{a}_{t+1}$, which is the only source of uncertainty in the model.

We define the risky steady-state as satisfying the relation

$$\begin{bmatrix}\bar{R}\\ \bar{\xi}\end{bmatrix} = f\left(\begin{bmatrix}\bar{R}\\ \bar{\xi}\end{bmatrix},1,\tilde{a}_{t+1}\right)$$

which leads to

$$\begin{aligned}(1-\alpha)(1-\theta)(\bar{R}/g_I-1+\delta) &= (1+\beta^{-1/\rho}R^{1-1/\rho})g_L g_A g_I^{-1/(1-\alpha)}\\ &\times\left[\alpha(1+e^y g_L)\bar{\xi}+(1-\alpha)\theta\left(1-(1-\delta)\frac{g_I}{\bar{R}}\right)\right]\\ \bar{\xi} &= \xi(\bar{R},\bar{\xi}).\end{aligned}$$

The equation determining the deterministic steady state interest rate can be expressed as

$$\frac{g_A}{g_I^{\frac{1}{1-\alpha}}} = (1+\beta^{-\frac{1}{\rho}}R^{1-\frac{1}{\rho}})^{-1} \quad \left[\frac{1-\alpha}{\alpha g_L}\left(\frac{R}{g_I}-1+\delta\right)\right] \frac{\alpha(1-\theta)}{\alpha(1+e^y g_L)+(1-\alpha)\theta(1-g_I\frac{1-\delta}{R})} \tag{40}$$

## Proof of Proposition 1

We give the details of the proof of Proposition 1. The proof is in two steps. First, we establish the dynamic equation of $R_t$, to obtain the equation at the steady-state. Then, we study the properties of $R$ as a function of the inputs.

### Dynamic relation

Starting from equation (25), the dynamic relation is rewritten as

$$(1-\alpha)(1-\frac{\theta_{t-1}}{\tilde{a}_t})\alpha\frac{A_t^{1-\alpha}}{p_t^k}k_t^\alpha = \left(1+(\beta\phi_t)^{-1/\rho}R_{t+1}^{1-1/\rho}\right)g_{L,t}\frac{v_t}{\xi_t}k_{t+1}$$

This leads to the following law of motion for $R_t$

$$(1-\alpha)R_t(\tilde{a}_t-\theta_{t-1}) = \left(1+(\beta\phi_t)^{-1/\rho}R_{t+1}^{1-1/\rho}\right)g_{L,t}g_{I,t}^{-\frac{\alpha}{1-\alpha}}g_{A,t}\left(\frac{\xi_t}{\xi_{t-1}}\right)^{\frac{\alpha}{1-\alpha}}v_t\left(\frac{R_{t+1}}{R_t}\right)^{-\frac{1}{1-\alpha}} \tag{41}$$

### Equation at the steady-state, for $\delta = 1$

We introduce

$$M(p) = \int [\alpha(1+e^y g_L)\tilde{a}+(1-\alpha)\theta]^{-p}\, d\tilde{a}$$

We compute

$$\alpha(1+e^y g_L)\xi+(1-\alpha)\theta = \frac{M(\gamma-1)}{M(\gamma)}$$

$$\phi = M(\gamma-1)^{\rho+(\gamma-\rho)/(1-\gamma)}M(\gamma)^{1-\rho} = \left(M(\gamma-1)^{\gamma/(\gamma-1)}M(\gamma)^{-1}\right)^{\rho-1}$$

$R$ is solution of the equation

$$\frac{R}{1+\beta^{-1/\rho}\left[M(\gamma-1)^{\gamma/(\gamma-1)}M(\gamma)R\right]^{1-1/\rho}} = \frac{g_L g_A g_I^{-\alpha/(1-\alpha}}{(1-\alpha)(1-\theta)}\frac{M(\gamma-1)}{M(\gamma)}$$

The proof mimics the proof of Theorem 1 in Coeurdacier et al. (2015), we consider the function

$$h(R) = \frac{R}{1+\beta^{-1/\rho}\left[M(\gamma-1)^{\gamma/(\gamma-1)}M(\gamma)R\right]^{1-1/\rho}}$$

$h$ is strictly increasing, for $\rho \leq 1$, it defines a bijection from $[0,+\infty)$ to $[0,+\infty)$, thus there exists a unique $R$ such that

$$h(R) = \frac{g_L g_A g_I^{-\alpha/(1-\alpha}}{(1-\alpha)(1-\theta)}\frac{M(\gamma-1)}{M(\gamma)}$$

The directions of variation of $R$ with $g_A$, $g_I$ and $\beta$ are obvious.

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