

# Monetary Shocks and Bank Balance Sheets

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PRELIMINARY

June 2015

## Abstract

We propose a model to explain why banks' balance sheets are exposed to interest rate risk despite the existence of markets where that risk can be hedged. A rise in nominal interest rates raises the opportunity cost of holding currency; since bank liabilities are close substitutes of currency, demand for bank liabilities rises and banks earn higher spreads. If risk aversion is higher than 1, the optimal dynamic hedging strategy is to sustain capital losses when nominal interest rates rise and, conversely, capital gains when they fall. A traditional bank balance sheet with long duration nominal assets achieves that.

Keywords: Monetary shocks, bank deposits, interest rate risk

JEL codes: E41, E43, E44, E51

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# 1 Introduction

Monetary policy can have major redistributive effects. One of the channels of redistribution is through banking system. Typically, banks hold long duration nominal assets such as fixed-rate mortgages and therefore sustain capital losses (in mark-to-market terms) when nominal interest rates rise. One could conjecture that a maturity-mismatched balance sheet is inherent to the banking business and the resulting interest rate risk is an inevitable side effect. However, there exist deep and liquid markets for interest rate derivatives where banks could hedge against interest rate changes if they wanted. Furthermore, Begenau et al. (2013) show that, if anything, banks tend to use interest rate derivatives to increase rather than reduce their exposure to interest rate risk. Why, then, do they choose this exposure?

We argue that banks choose to bear interest rate risk as part of optimal dynamic hedging. We model a flexible price, complete markets, monetary economy, with three key ingredients. First, the economy consists of banks and households, who are identical except that banks can issue deposits which are close substitutes to currency, up to a leverage limit. Second, there are indeed monetary shocks which move nominal interest rates. Third, risk aversion is high, with a CRRA coefficient greater than 1. In this economy, banks optimally choose to be exposed to interest rate risk.

The mechanism works as follows. Because deposits provide liquidity services, banks earn the spread between the nominal interest rate on bonds and the lower interest rate on deposits. If nominal interest rates rise, the opportunity cost of holding currency rises so, given that currency and deposits are substitutes, demand for deposits rises. This drives up the spread between the nominal interest rate and the interest rate on deposits, increasing banks' return on wealth. This has both income and substitution effects. Because risk aversion is higher than 1, the income effect dominates and banks want to transfer wealth from states of the world with high-return-on-wealth to states of the world with low-return-on-wealth. They are willing to take capital losses when interest rates rise because spreads going forward will be high, and want to make gains when interest rates fall because spreads going forward will be low. Choosing a portfolio of long-duration nominal assets is a way to achieve this exposure and they do not want to undo it even if complete markets allow them to do so.

The fact that bank deposit rates move less than one-for-one with market interest rates has been observed before. Hannan and Berger (1991) and Driscoll and Judson (2013) attribute it to a form of price stickiness; Drechsler et al. (2014) attribute it to imperfect competition among bank branches. Nagel (2014) makes a related observation: the premium on other near-money assets (besides banks deposits) also co-moves with interest rates. He attributes

this, as we do, to the substitutability between money and other liquid assets. Krishnamurthy et al. (2015) document a negative correlation between the supply of publicly issued liquid assets and the supply of liquid bank liabilities, also consistent with their being substitutes. Relative to this literature, the contribution of our work is to derive the implications for equilibrium risk management in a model where the underlying risk is modeled explicitly. Landier et al. (2013) shows cross-sectional evidence that exposure to interest rate risk has consequences for bank lending.

## 2 The Model

**Preferences and technology.** Time is continuous. There is a fixed capital stock  $k$  which can be used to produce a flow of consumption goods with a linear technology  $y_t = ak$ . There are two types of agents in the economy: households and bankers, a continuum of each. Both have identical Epstein-Zin preferences with intertemporal elasticity of substitution equal to 1, risk aversion  $\gamma$  and discount rate  $\rho$ :

$$U_t = \mathbb{E}_t \left[ \int_t^\infty f(x_s, U_s) ds \right]$$

with

$$f(x, U) = \rho(1 - \gamma) U \left( \log(x) - \frac{1}{1 - \gamma} \log((1 - \gamma) U) \right)$$

$x$  is a Cobb-Douglas composite of consumption  $c$  and liquidity services from money holdings  $m$ :

$$x(c, m) = c^\beta m^{1-\beta} \tag{1}$$

Money itself is a CES composite of real currency holdings  $h$  and real bank deposits  $d$ , with elasticity of substitution  $\epsilon$ :<sup>1</sup>

$$m(h, d) = \left( \alpha^{\frac{1}{\epsilon}} h^{\frac{\epsilon-1}{\epsilon}} + (1 - \alpha)^{\frac{1}{\epsilon}} d^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} \tag{2}$$

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<sup>1</sup>Throughout, uppercase letters denote nominal variables and their corresponding lowercase letter are real variables. Hence  $h \equiv \frac{H}{p}$  and  $d \equiv \frac{D}{p}$  where  $p$  is the price of consumption goods in terms of currency, which we take as the numeraire.

**Currency supply.** The government issues a nominal amount of currency  $H$ . We take monetary policy as exogenously given by the following stochastic process

$$\frac{dH_t}{H_t} = \mu_{H,t}dt + \sigma_{H,t}dB_t$$

where  $B$  is a standard Brownian motion. The process  $B$  drives equilibrium dynamics. The government distributes or withdraws currency to and from agents through lump-sum transfers or taxes.

**Markets.** There are complete markets. We denote the real price of capital by  $q$ , the nominal interest rate by  $i$ , the real interest rate by  $r$ , and the price of risk by  $\pi$  (so an asset with exposure  $\sigma$  to the process  $B$  will pay an excess return  $\sigma\pi$ ). All these processes are contingent on the history of shocks  $B$ .

The total real wealth of private agents in the economy includes the value of the capital stock  $qk$ , the real value of outstanding currency  $h$  and the net present value of future government transfers and taxes, which we denote by  $g$ . Total wealth is denoted by  $\omega$ :

$$\omega = qk + h + g$$

Total household wealth is denoted by  $w$  and total bankers' wealth is denoted by  $n$ , so

$$n + w = \omega \tag{3}$$

Notice that with complete markets it is not necessary to specify who receives government transfers when the supply of currency changes: all those transfers are priced in and included in the definition of wealth. We denote by  $z \equiv \frac{n}{\omega}$  the share of the aggregate wealth that is owned by bankers.

The only difference between households and bankers is that bankers may issue deposits. These pay a nominal interest rate  $i^d$  and also enter the utility function according to equation (2).<sup>2</sup>

The amount of deposits bankers can issue is subject to a leverage limit. A banker whose

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<sup>2</sup>Note that even though deposit contracts are specified in nominal terms, nothing prevents a banker and a deposit holder from also trading securities such as interest rate or inflation swaps to choose any exposure to nominal variables.

individual wealth is  $n$  can issue deposits  $d^S$  up to

$$d^S \leq \phi n \tag{4}$$

where  $\phi$  is an exogenous parameter. Constraint (4) may be interpreted as either a regulatory constraint or a level of capitalization required for deposits to actually have the liquidity properties implied by (1).

**Monetary policy.** As is standard, monetary policy can be described in terms of the supply of currency or in terms of the nominal interest rate. We assume that the government chooses a path for  $H$  such that  $i$  follows the Cox et al. (1985) stochastic process:<sup>3</sup>

$$di_t = -\lambda(i_t - \bar{i}) dt + \sigma\sqrt{i_t}dB_t \tag{5}$$

Shocks to  $B$  are our representation of monetary shocks.

There is more than one stochastic process  $H$  that will result in (5). Let

$$\frac{dp_t}{p_t} = \mu_{p,t}dt + \sigma_{p,t}dB_t$$

be the stochastic process for the price level (which is endogenous). We assume that the government implements the unique process  $H$  such that in equilibrium (5) holds and  $\sigma_{p,t} = 0$ . Informally, this means that monetary shocks affect the rate of inflation  $\mu_p$  but the price level moves smoothly.

### 3 Equilibrium

**Households' problem.** Starting with some initial nominal wealth  $W_0$ , each household solves a standard portfolio problem:

$$\max_{W,x,c,h,d,\sigma_W} U(x)$$

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<sup>3</sup>This is a square root process. It is always nonnegative and if  $2\lambda\bar{i} \geq \sigma^2$  then it is strictly positive almost surely and has a stationary distribution.

subject to the budget constraint:

$$\frac{dW_t}{W_t} = \left( i_t + \sigma_{W,t}\pi_t - \hat{c}_t - \hat{h}_t i_t - \hat{d}_t (i_t - i_t^d) \right) dt + \sigma_{W,t} dB_t \quad (6)$$

$$W_t \geq 0$$

and equations (1) and(2). A hat denotes the variable is normalized by wealth, i.e.  $\hat{c} = \frac{pc}{W} = \frac{c}{w}$ . The household obtains a nominal return  $i_t$  on its wealth. It incurs an opportunity cost  $i_t$  on its holdings of currency. I also incurs an opportunity cost  $(i_t - i_t^d)$  on its holdings of deposits. Let  $s_t = i_t - i_t^d$  denote the spread between the deposit rate and the market interest rate. Furthermore, the household chooses its exposure  $\sigma_W$  to the monetary shock and obtains the risk premium  $\pi\sigma_W$  in return.

Constraint (6) can be rewritten in real terms as

$$\frac{dw_t}{w_t} = \left( r_t + \sigma_{w,t}\pi_t - \hat{c}_t - \hat{h}_t i_t - \hat{d}_t (i_t - i_t^d) \right) dt + \sigma_{w,t} dB_t \quad (7)$$

where  $r_t = i_t - \mu_{p,t}$  is the real interest rate.

**Bankers' problem.** Bankers are like households, except that they can issue deposits (denoted  $d^S$ ) up to the leverage limit and earn the spread  $s_t$  on these. The banker's problem, expressed in real terms, is:

$$\max_{n,x,c,h,d,d^S,\sigma_n} U(x)$$

subject to:

$$\frac{dn_t}{n_t} = \left( r_t + \sigma_{n,t}\pi_t - \hat{c}_t - \hat{h}_t i_t + \left( \hat{d}_t^S - \hat{d}_t \right) s_t \right) dt + \sigma_{n,t} dB_t \quad (8)$$

$$\hat{d}_t^S \leq \phi$$

$$n_t \geq 0$$

and equations (1) and(2).

**Equilibrium definition** Given an initial distribution of wealth between households and bankers  $z_0$  and an interest rate process  $i$ , a competitive equilibrium is

1. a process for the supply of currency  $H$
2. processes for prices  $p, i^d, q, g, r, \pi$
3. a plan for the household:  $w, x^h, c^h, m^h, h^h, d^h, \sigma_w$

4. a plan for the banker:  $n, x^b, c^b, m^b, h^b, d^b, d^S, \sigma_n$

such that

1. Households and bankers optimize taking prices as given and  $w_0 = (1 - z_0)(q_0k + h_0 + g_0)$  and  $n_0 = z_0(q_0k + h_0 + g_0)$
2. The goods, deposit and currency markets clear:

$$\begin{aligned} c_t^h + c_t^b &= ak \\ d_t^h + d_t^b &= d_t^S \\ h_t^h + h_t^b &= h_t \end{aligned}$$

3. Wealth holdings add up to total wealth:

$$w_t + n_t = q_tk + h_t + g_t$$

4. Capital and government transfers and nominal claims are priced by arbitrage:

$$q_t = \mathbb{E}_t^Q \left[ a \int_t^\infty \exp \left( - \int_t^s r_u du \right) ds \right] \quad (9)$$

$$g_t = \mathbb{E}_t^Q \left[ \int_t^\infty \exp \left( - \int_t^s r_u du \right) \frac{dH_s}{p_s} \right] \quad (10)$$

where  $Q$  is the equivalent martingale measure implied by  $r$  and  $\pi$ .

5. Monetary policy is consistent

$$\begin{aligned} i_t &= r_t + \mu_{p,t} \\ \sigma_{p,t} &= 0 \end{aligned}$$

## 4 Equilibrium Characterization

**Hamilton-Jacobi-Bellman equations and FOCs.** We study the banker's problem first. It can be separated into a static problem (choosing  $c, m, h$  and  $d$  given  $x$ ) and a dynamic problem (choosing  $x$  and  $\sigma_n$ ).

Consider the static problem first. Given the form of the aggregators (1) and (2), we immediately get that the minimized cost of one unit of liquidity services is given by  $\iota$ :

$$\iota(i, s) = (\alpha i^{1-\epsilon} + (1-\alpha) s^{1-\epsilon})^{\frac{1}{1-\epsilon}} \quad (11)$$

and the minimized cost of one unit of the aggregator  $x$  is given by  $\chi$ :

$$\chi(i, s) = \beta^{-\beta} \left( \frac{\iota}{1-\beta} \right)^{1-\beta} \quad (12)$$

and the static choices of  $c$ ,  $m$ ,  $h$  and  $d$  are given by:

$$\frac{c}{x} = \beta \chi \quad (13)$$

$$\frac{m}{x} = (1-\beta) \frac{\chi}{\iota} \quad (14)$$

$$\frac{h}{m} = \alpha \left( \frac{\iota}{i} \right)^\epsilon \quad (15)$$

$$\frac{d}{m} = (1-\alpha) \left( \frac{\iota}{s} \right)^\epsilon \quad (16)$$

Turn now to the dynamic problem. In equilibrium it will be the case that  $i^d < i$  so bankers' leverage constraint will always bind. This means that (8) reduces to

$$\frac{dn_t}{n_t} = \underbrace{(r_t + \sigma_{n,t}\pi_t - \chi(i_t, s_t) \hat{x}_t + \phi s_t)}_{\equiv \mu_{n,t}} dt + \sigma_{n,t} dB_t \quad (17)$$

Given the homotheticity of preferences and the linearity of budget constraints the problem of the banker has the value function:

$$V_t^b(n) = \frac{(\xi_t n)^{1-\gamma}}{1-\gamma}$$

$\xi_t$  captures the value of the banker's investment opportunities, i.e. his ability to convert units of wealth into units of lifetime utility, and follows the law of motion

$$\frac{d\xi_t}{\xi_t} = \mu_{\xi,t} dt + \sigma_{\xi,t} dB_t$$

where  $\mu_{\xi,t}$  and  $\sigma_{\xi,t}$  are equilibrium objects.

The associated Hamilton-Jacobi-Bellman equation is

$$0 = \max_{x, \sigma_n, \mu_n} f(x, V_t^b) + \mathbb{E}_t [dV_t^b]$$

Using Ito's lemma and simplifying, we obtain:

$$\begin{aligned} 0 &= \max_{\hat{x}, \sigma_n, \mu_n} \rho(1-\gamma) \frac{(\xi_t n_t)^{1-\gamma}}{1-\gamma} \left[ \log(\hat{x} n_t) - \frac{1}{1-\gamma} \log((\xi_t n_t)^{1-\gamma}) \right] \\ &\quad + \xi_t^{1-\gamma} n_t^{1-\gamma} \left( \mu_n + \mu_{\xi_t} - \frac{\gamma}{2} \sigma_n^2 - \frac{\gamma}{2} \sigma_{\xi_t}^2 + (1-\gamma) \sigma_{\xi_t} \sigma_n \right) \\ \text{s.t. } \mu_n &= r_t + \sigma_n \pi_t + \phi s_t - \hat{x} \chi_t \end{aligned}$$

The household's problem is similar. The only difference is that the term  $\phi s_t$  is absent from the budget constraint. The value function has the form

$$V_t^h(w) = \frac{(\zeta_t w)^{1-\gamma}}{1-\gamma}$$

where

$$\frac{d\zeta_t}{\zeta_t} = \mu_{\zeta, t} dt + \sigma_{\zeta, t} dB_t$$

and the HJB equation is

$$\begin{aligned} 0 &= \max_{\hat{x}, \sigma_w, \mu_w} \rho(1-\gamma) \frac{(\zeta_t w_t)^{1-\gamma}}{1-\gamma} \left[ \log(\hat{x} w_t) - \frac{1}{1-\gamma} \log((\zeta_t w_t)^{1-\gamma}) \right] \\ &\quad + \zeta_t^{1-\gamma} w_t^{1-\gamma} \left( \mu_w + \mu_{\zeta_t} - \frac{\gamma}{2} \sigma_w^2 - \frac{\gamma}{2} \sigma_{\zeta_t}^2 + (1-\gamma) \sigma_{\zeta_t} \sigma_w \right) \\ \text{s.t. } \mu_w &= r_t + \sigma_w \pi_t - \hat{x} \chi_t \end{aligned}$$

**Aggregate state variables.** We look for a recursive equilibrium taking the static optimization (choosing  $c$ ,  $m$ ,  $h$  and  $d$  given  $x$ ) as given. There are two state variables: the interest rate  $i$  (which is exogenous) and the bankers' share of aggregate wealth  $z$  (which is endogenous). Using the definition of  $z = \frac{n}{n+w}$ , we obtain a law of motion for  $z$  from Ito's

lemma and the budget constraints

$$\begin{aligned} \frac{dz_t}{z_t} = & \underbrace{\left( (1-z_t) \left( (\sigma_{n,t} - \sigma_{w,t})\pi_t + \phi s_t - (\hat{x}_t^b - \hat{x}_t^h)\chi_t + \sigma_{w,t}(\sigma_{w,t} - \sigma_{n,t}) \right) - \frac{z_t}{1-z_t} \sigma_{z,t}^2 \right)}_{\equiv \mu_{z,t}} dt \\ & + \underbrace{(1-z_t)(\sigma_{n,t} - \sigma_{w,t})}_{\equiv \sigma_{z,t}} dB_t \end{aligned} \quad (18)$$

Equilibrium objects are then functions of  $i$  and  $z$ , e.g.  $\xi_t = \xi(i_t, z_t)$ . We can use Ito's lemma to compute their laws of motion, e.g.

$$\begin{aligned} \mu_{\xi,t} &= \frac{\xi_i}{\xi} \lambda (\bar{i} - i_t) + \frac{\xi_z}{\xi} \mu_{z,t} z_t + \frac{1}{2} \left( \frac{\xi_{ii}}{\xi} i_t \sigma^2 + 2 \frac{\xi_{iz}}{\xi} \sqrt{i_t} \sigma \sigma_{z,t} z_t + \frac{\xi_{zz}}{\xi} \sigma_{z,t}^2 z_t^2 \right) \\ \sigma_{\xi,t} &= \frac{\xi_i}{\xi} \sigma \sqrt{i} + \frac{\xi_z}{\xi} \sigma_{z,t} z_t \end{aligned}$$

**Definition 1.** A recursive equilibrium is a set of functions of  $i$  and  $z$ : value functions  $\xi$  and  $\zeta$ , policy functions  $(\hat{x}^b, \sigma_n, \mu_n)$  and  $(\hat{x}^h, \sigma_w, \mu_w)$ ; prices  $q, g, h, r, \pi, i^d$ ; and functions  $\mu_z$  and  $\sigma_z$  that define a law of motion for  $z$ :  $dz_t = \mu_z z_t dt + \sigma_z z_t dB_t$  such that

1.  $\xi$  and  $\zeta$ , and the corresponding policy functions solve the HJB equations of bankers and households respectively.
2. Markets clear:

(a) for goods:

$$[\hat{x}^h(1-z) + \hat{x}^b z] (qk + h + g) \beta \chi = ak$$

(b) For deposits:

$$[\hat{x}^h(1-z) + \hat{x}^b z] (qk + h + g) (1-\alpha)(1-\beta) \frac{\chi}{l} \left(\frac{l}{s}\right)^\epsilon = \phi z (qk + h + g)$$

(c) For currency:

$$[\hat{x}^h(1-z) + \hat{x}^b z] (qk + h + g) \alpha (1-\beta) \frac{\chi}{l} \left(\frac{l}{i}\right)^\epsilon = h$$

3. Arbitrage pricing:

(a) For capital:

$$\frac{a}{q} + \mu_q - r = \pi \sigma_q$$

(b) For government transfers

$$(\mu_h + i - r)h + \mu_g - rg = (\sigma_h h + \sigma_g)\pi$$

4. The law of motion of  $z$  satisfies (18)

The goods market clearing condition is obtained by using (13),  $n = z(qk + h + g)$  and  $w = (1 - z)(qk + h + g)$ . The deposit market clearing condition is obtained similarly, using (14) and (16) and the fact that deposit supply is  $\phi n$ . The currency market clearing condition is obtained similarly, using (14) and (15). The arbitrage pricing conditions are just the differential form of (9) and (10).

**Total Wealth, spreads and currency holdings.** The first order conditions for  $\hat{x}$  in the banker and household problem are both given by:

$$\hat{x}_t = \frac{\rho}{\chi_t} \quad (19)$$

Since the intertemporal elasticity of substitution is 1, both the banker and the household spend their wealth at a constant rate  $\rho$  independent of prices.

Using (19) and the goods market clearing condition we can solve for total wealth:

$$\omega = \frac{ak}{\beta\rho} \quad (20)$$

Hence in this economy total wealth will be constant. This is because the Cobb-Douglas form of the  $x$  aggregator implies that consumption is a constant share of spending (the rest is liquidity services), the rate of spending out of wealth is constant and total consumption is constant and equal to  $ak$ .

Using (19), the deposit market clearing condition simplifies to:

$$\rho(1 - \alpha)(1 - \beta)\iota^{\epsilon-1}s^{-\epsilon} = \phi z \quad (21)$$

Solving (21) for  $s$  implicitly defines bank spreads  $s(i, z)$  as a function of  $i$  and  $z$ . Replacing (11) into (21) and using the implicit function theorem:

$$\frac{\partial s(i, z)}{\partial i} = -\frac{\alpha(1-\epsilon)i^{-\epsilon}}{(1-\alpha)s(i, z)^{-\epsilon} + \alpha i^{1-\epsilon}\epsilon s(i, z)^{-1}} \quad (22)$$

$$\frac{\partial s(i, z)}{\partial z} = -\frac{\phi(\alpha i^{1-\epsilon} + (1-\alpha)s(i, z)^{1-\epsilon})^2 s(i, z)^\epsilon}{\rho(1-\alpha)(1-\beta)[(1-\alpha)s(i, z)^{-\epsilon} + \alpha i^{1-\epsilon}\epsilon s(i, z)^{-1}]} \quad (23)$$

By equation (22), the spread is increasing in  $i$  as long as  $\epsilon > 1$ . If currency and deposits are close substitutes, an increase in  $i$ , which increases the opportunity cost of holding currency, increases the demand for deposits, so the spread must rise to clear the deposit market. By equation (23), the spread is always decreasing in  $z$ . If bankers have a larger fraction of total wealth, they can supply more deposits so the spread must fall to clear the deposit market.

Finally, using (19) and (20), the currency market clearing condition simplifies to:

$$\frac{ak}{\beta}\alpha(1-\beta)i^{\epsilon-1}i^{-\epsilon} = h \quad (24)$$

Having solved for  $s$ , (24) immediately defines currency holdings  $h(i, z)$  as a function of  $i$  and  $z$ .

**Aggregate risk sharing.** The first order conditions for the banker's choice of  $\sigma_n$  and the household's choice of  $\sigma_w$  are respectively:

$$\sigma_{n,t} = \frac{\pi_t}{\gamma} + \frac{1-\gamma}{\gamma}\sigma_{\xi,t} \quad (25)$$

$$\sigma_{w,t} = \frac{\pi_t}{\gamma} + \frac{1-\gamma}{\gamma}\sigma_{\zeta,t} \quad (26)$$

The first term in each of (25) and (26) relates exposure to  $B$  to the risk premium  $\pi$ ; this is the myopic motive for choosing risk exposure. The second term captures the dynamic hedging motive, which depends on an income and a substitution effect. If the agent is sufficiently risk averse ( $\gamma > 1$ ), then the income effect dominates. The agent will want to have more wealth when his investment opportunities (captured by  $\xi$  and  $\zeta$  respectively) are worse.

From (25) and (26) we obtain the following expression for  $\sigma_z$ :

$$\sigma_{z,t} = (1-z_t)\frac{1-\gamma}{\gamma}(\sigma_{\xi,t} - \sigma_{\zeta,t}) \quad (27)$$

The object  $\sigma_z$  measures how the bankers' share of wealth responds to the aggregate shock. The term  $\sigma_{\xi,t} - \sigma_{\zeta,t}$  in (27) captures the relative sensitivity of bankers' and households' investment opportunities to the aggregate shock. How this differential sensitivity feeds into changes in the wealth share depends on income and substitution effects. If agents are not very risk averse ( $\gamma < 1$ ), then the substitution effect dominates and in equilibrium they will shift aggregate wealth to bankers after aggregate shocks that improve their investment opportunities relative to households', i.e. when  $\frac{\xi}{\zeta}$  goes up. In contrast, if agents are highly risk averse ( $\gamma > 1$ ) they will shift aggregate wealth to bankers after shocks that worsen their investment opportunities relative to households', i.e.  $\frac{\xi}{\zeta}$  goes down. In the quantitative section we focus on this second, more empirically relevant, case.

We can use Ito's lemma to obtain an expression for  $\sigma_\xi - \sigma_\zeta$ :

$$\sigma_\xi - \sigma_\zeta = \left( \frac{\xi_z}{\xi} - \frac{\zeta_z}{\zeta} \right) \sigma_z z + \left( \frac{\xi_i}{\xi} - \frac{\zeta_i}{\zeta} \right) \sigma \sqrt{i} \quad (28)$$

Notice that  $\sigma_z$  enters the expression for  $\sigma_\xi - \sigma_\zeta$ : the response of relative investment opportunities to aggregate shocks depends in part on aggregate risk sharing decisions as captured by  $\sigma_z$ . This is because in equilibrium investment opportunities depend on the distribution of wealth  $z$ , so we must look for a fixed point. Replacing (28) into (27) and solving for  $\sigma_z$ :

$$\sigma_z = \frac{(1-z)^{\frac{1-\gamma}{\gamma}} \left( \frac{\xi_i}{\xi} - \frac{\zeta_i}{\zeta} \right)}{1 - z(1-z)^{\frac{1-\gamma}{\gamma}} \left( \frac{\xi_z}{\xi} - \frac{\zeta_z}{\zeta} \right)} \sigma \sqrt{i} \quad (29)$$

**Implementation.** With complete markets, there is more than one way to attain the exposure dictated by equations (25) and (26). We are interested in seeing whether one possible way to do this is for banks to have a “traditional” balance sheet: long-term nominal assets, deposits as the only liability and no derivatives. To be concrete, imagine that a banker with wealth  $n$  issues deposits  $\phi n$  in order to buy  $(1 + \phi)n$  worth of nominal zero-coupon bonds that mature in  $T$  years. It's easy to show that with the interest process (5), this balance sheet produces the following exposure:

$$\sigma_n = - (1 - e^{-\lambda T}) \frac{1 + \phi}{\lambda} \sigma \sqrt{i} < 0 \quad (30)$$

Expression (30) has a standard interpretation: assets will be more sensitive to changes in interest rates if they have longer maturity (high  $T$ ) or if interest changes are more persistent (low  $\lambda$ ). Furthermore, a more highly leverage bank will have greater exposure, other things

being equal. Notice that this implementation only works if the desired  $\sigma_n$  is negative, i.e. if bankers want to lose wealth when interest rates rise.

Conversely, if the banker wants exposure  $\sigma_n$ , inverting (30) tells us the maturity of the nominal assets he needs to hold:

$$T = -\frac{1}{\lambda} \log \left( 1 + \frac{\lambda}{1 + \phi} \frac{\sigma_n}{\sigma \sqrt{i}} \right) \quad (31)$$

## 5 Numerical Results

We solve for the recursive equilibrium by mapping it into a system of partial differential equations for the equilibrium objects. We solve them numerically using a finite difference scheme. In order to obtain a stationary wealth distribution we add tax on bankers' wealth at a rate  $\tau$  that is redistributed to households as a wealth subsidy. Appendix A explains the numerical procedure in detail.

**Parameter values.** We choose parameter values so that the model economy matches some key features of the US economy. Our choice of parameters is shown on Table 1. TO BE COMPLETED

Parameter	Value
$a$	1
$k$	1
$\gamma$	15
$i$	0.04
$\lambda$	0.1
$\sigma$	0.03
$\rho$	0.08
$\alpha$	0.95
$\beta$	0.75
$\phi$	8
$\tau$	0.19
$\epsilon$	8

Table 1: Parameter values

**Aggregate risk sharing.** Figure 1 shows aggregate risk sharing. The top panels show bankers' exposure to interest rate risk. If the nominal interest rate rises by 100 basis points,

bankers' net worth changes by  $\frac{\sigma_n}{\sigma_i}\%$ . It is always negative, so banks face large financial losses after an increase in nominal interest rates. This means it can be implemented with a “traditional” banking structure as explained above. Expression (31) gives us the maturity of nominal assets bankers' need to hold in order to implement the desired exposure to interest rate risk. This is shown in the lower panels of Figure 1.

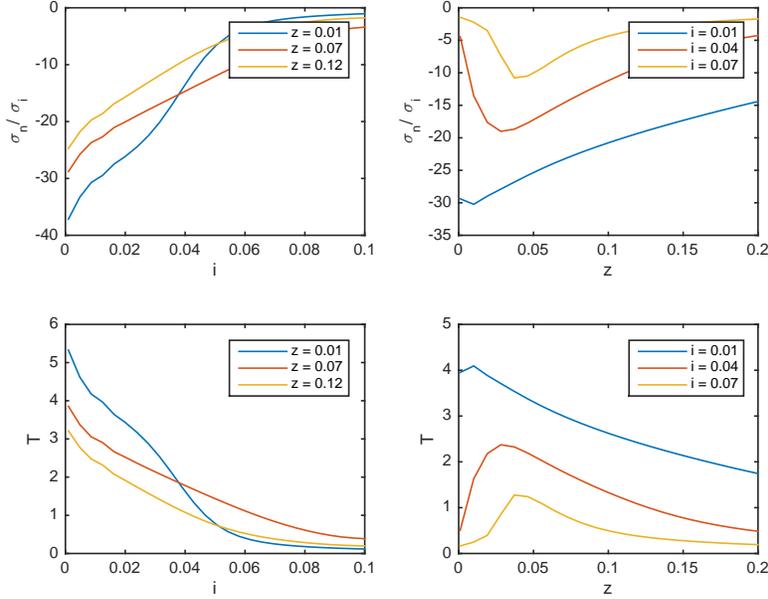


Figure 1: Aggregate risk sharing as captured by  $\frac{\sigma_n}{\sigma_i}$  (upper panels) and the implied maturity of nominal assets  $T$  (lower panels).

**Exploring the mechanism.** To understand the mechanism, it is useful to split it into several parts. First, an increase in the nominal interest rate  $i$  makes holding currency more costly for agents. Since currency and deposits are substitutes, this increases the demand for deposits, other things being equal. We can see this in the expression for total demand for deposits

$$\rho\omega(1-\alpha)(1-\beta)\left(\alpha i^{1-\epsilon} + (1-\alpha)s^{1-\epsilon}\right)^{-1}s^{-\epsilon}$$

which is increasing in  $i$ . Since the supply of deposits  $\phi z\omega$  is fixed, the spread on deposits  $s$  must go up to clear the market for deposits. Intuitively, banks are able to charge a higher spread for the liquidity services they provide when holding currency becomes more costly. In addition, if bankers' share of aggregate wealth  $z$  goes down, this reduces the supply of

deposits and also drives the spread  $s$  up. This is shown in the upper panels of Figure 2. Since banks earn the spread  $s$ , and households don't, bankers' relative investment opportunities are better when the interest rate  $i$  is high and their share of aggregate wealth  $z$  is low. This is captured by the ratio  $\frac{\xi}{\zeta}$ , shown in the lower panels of Figure 2. As a result, bankers' relative investment opportunities are better after a monetary shock that raises the interest rate. Since  $\gamma > 1$ , this means that the right hand side of (27) is negative: bankers' share of aggregate wealth  $z$  goes down after interest rates go up. Since bankers benefit from a monetary shock that raises interest rates (via higher spreads) it makes sense that they are willing to sustain financial losses relative to households in that state. These financial losses in turn reduce the supply of deposits and drive the spread  $s$  up, further improving bankers' investment opportunities and amplifying their incentives to take on interest rate risk ex-ante.

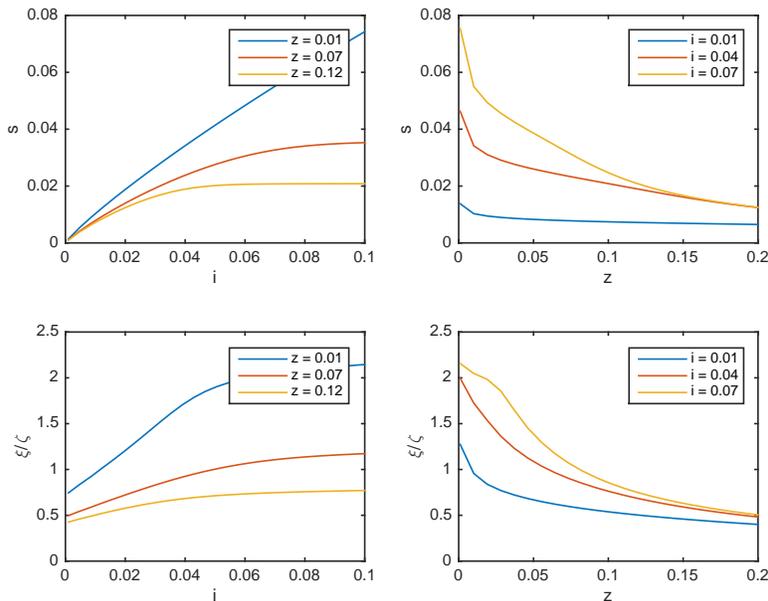


Figure 2: Spread on deposits  $s$  (upper panels) and bankers' relative investment opportunities captured by  $\frac{\xi}{\zeta}$  (lower panels).

**Dynamics.** Agents' endogenous exposure to interest rate risk leads to interesting equilibrium dynamics, shown in Figure 3. The upper panels show the drift of bankers' share of aggregate wealth  $z$ . The volatility  $\sigma_z$  is negative throughout, so after a positive shock  $dB$  that increases nominal interest rates,  $z$  goes down. The drift of  $z$  is positive for small  $z$  and negative for high  $z$ . The drift is also higher when the interest rate  $i$  is high. This dynamic

behavior is driven primarily by the spread  $s$ , which is higher when  $i$  is high and  $z$  low, and leads to a stationary distribution. This is shown in Figure

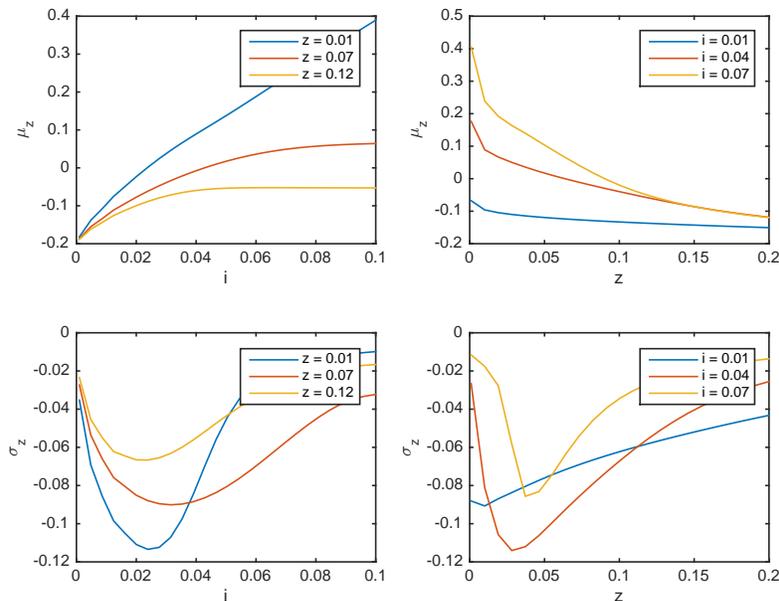


Figure 3: The drift of  $z$ ,  $\mu_z$  (upper panels) and its volatility  $\sigma_z$  (lower panels).

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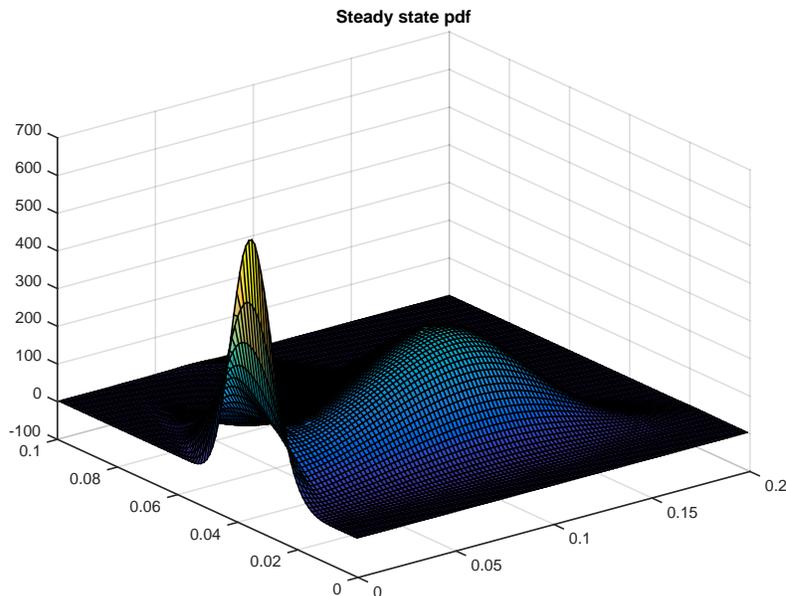


Figure 4: Stationary distribution over  $(i, z)$ .

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## Appendix A: Solution Method (NOT UP TO DATE)

**Retirement** In order to have a stationary distribution for  $z$  we assume that bankers retire randomly with Poisson intensity  $\theta$ . Upon retirement, they keep their wealth but lose their

ability to issue deposits, effectively becoming households. The HJB equation then becomes:<sup>4</sup>

$$\begin{aligned} \rho \frac{(\xi_t \tilde{n})^{1-\gamma}}{1-\gamma} &= \max_{x, \sigma_n, \mu_n} \frac{\hat{x}^{1-\gamma}}{1-\gamma} \tilde{n}^{1-\gamma} + \xi_t^{1-\gamma} \tilde{n}^{1-\gamma} \cdot \\ &\quad \left( \mu_n + \mu_{\xi,t} - \frac{\gamma}{2} \sigma_n^2 - \frac{\gamma}{2} \sigma_{\xi,t}^2 + (1-\gamma) \sigma_{\xi,t} \sigma_n + \theta \left( \frac{(\zeta_t \tilde{n})^{1-\gamma}}{1-\gamma} - \frac{(\xi_t \tilde{n})^{1-\gamma}}{1-\gamma} \right) \right) \\ \text{s.t.} \quad \mu_n &= r_t + \sigma_n \pi_t + \phi(i_t - i_t^d) - \hat{x} \chi_t \end{aligned}$$

Replacing the first order conditions (19) and (??) (which are unaffected), we obtain:

$$\frac{\rho + \theta}{1-\gamma} = \frac{\gamma}{1-\gamma} \xi_t^{\frac{\gamma-1}{\gamma}} \chi_t^{\frac{\gamma-1}{\gamma}} + r_t + \frac{\gamma}{2} \sigma_{n,t}^2 + \phi(i_t - i_t^d) + \mu_{\xi,t} - \frac{\gamma}{2} \sigma_{\xi,t}^2 + \theta \frac{\left(\frac{\zeta_t}{\xi_t}\right)^{1-\gamma}}{1-\gamma} \quad (32)$$

Similarly, replacing (??) and (26) in (??) we obtain the following HJB equation for households:

$$\frac{\rho}{1-\gamma} = \frac{\gamma}{1-\gamma} \zeta_t^{\frac{\gamma-1}{\gamma}} \chi_t^{\frac{\gamma-1}{\gamma}} + r_t + \frac{\gamma}{2} \sigma_w^2 + \mu_{\zeta,t} - \frac{\gamma}{2} \sigma_{\zeta,t}^2 \quad (33)$$

**Overview of the solution procedure** The solution method finds endogenous objects as functions of state variables. We'll divide the equilibrium objects into two groups. Denote the first group of variables by  $X = \{\xi(i, z), \zeta(i, z), q(i, z), h(i, z), g(i, z), i^d(i, z)\}$ . We'll express these as a system of differential equations and solve it backwards. Denote the second group of variables by  $Y = \{\hat{x}^b, \hat{x}^h, \sigma_z, \sigma_n, \sigma_w, \pi, r\}$ . These variables can be solved statically for every possible value of  $X$ .

**Solving for  $Y$  given  $X$**  Suppose we had found all the variables in  $X$  as functions of  $(i, z)$ . By Ito's Lemma it follows that the law of motion of any of these variables  $X$  is:

$$dX(i, z) = \mu_X(i, z) dt + \sigma_X(i, z) dB \quad (34)$$

---

<sup>4</sup>Introducing retirement implies that there is a distinction between the net worth of an individual banker and the collective net worth of all bankers, since the group of individuals who are bankers keeps shrinking. We retain the notation  $n$  to refer to the collective net worth and denote the net worth of an individual banker by  $\tilde{n}$ .

where the drift and volatility are

$$\begin{aligned}\mu_X(i, z) &= X_z(i, z) \mu_z(i, z) + X_i(i, z) \mu_i(i) \\ &\quad + \frac{1}{2} [X_{zz}(i, z) \sigma_z^2(i, z) z^2 + X_{ii}(i, z) \sigma_i^2(i) + 2X_{zi}(i, z) \sigma_i(i) z \sigma_z(i, z)] \\ \sigma_X &= X_z(i, z) \sigma_z(i, z) z + X_i(i, z) \sigma_i(i)\end{aligned}$$

or, in geometric form:

$$\frac{dX(i, z)}{X(i, z)} = \mu_X(i, z) dt + \sigma_X(i, z) dB \quad (35)$$

where the drift and volatility are

$$\begin{aligned}\mu_X(i, z) &= \frac{X_z(i, z)}{X(i, z)} \mu_z(i, z) + \frac{X_i(i, z)}{X(i, z)} \mu_i(i) \\ &\quad + \frac{1}{2} \left[ \frac{X_{zz}(i, z)}{X(i, z)} \sigma_z^2(i, z) z^2 + \frac{X_{ii}(i, z)}{X(i, z)} \sigma_i^2(i) + 2 \frac{X_{zi}(i, z)}{X(i, z)} \sigma_i(i) z \sigma_z(i, z) \right] \\ \sigma_X &= \frac{X_z(i, z)}{X(i, z)} \sigma_z(i, z) z + \frac{X_i(i, z)}{X(i, z)} \sigma_i(i)\end{aligned}$$

Hence if we know the functions  $X$  and their derivatives, we know their drifts and volatilities at every point of the state space. Numerically, we approximate the derivatives with finite-difference matrices such for any set of values of  $X$  on a grid, the values of the derivatives on the grid are:

$$\begin{aligned}X_i &\approx D_i X \\ X_z &\approx X D_z \\ X_{ii} &\approx D_{ii} X \\ X_{zz} &\approx X D_{zz} \\ X_{iz} &\approx D_i X D_z\end{aligned}$$

The variables in  $Y$  can be found as follows.  $\iota(i, z)$  and  $\chi(i, z)$  are immediate from (11) and (12).  $\hat{x}^b(i, z)$  and  $\hat{x}^h(i, z)$  follow from the first order conditions (19) and (??).  $\sigma_z(i, z)$  follows from (29).

By definition,

$$z = \frac{n}{qk + h + g}$$

which implies:<sup>5</sup>

$$\sigma_z = \sigma_n - \frac{qk\sigma_q + h\sigma_h + \sigma_g}{qk + h + g} \quad (36)$$

Knowing  $g$ ,  $q$ ,  $h$  and their volatilities, as well as  $\sigma_z$ ,  $\sigma_n(i, z)$  can be obtained from (36).

$\pi(i, z)$  can then be obtained from the FOC (??).  $\sigma_w(i, z)$  follows from the FOC (26).  $r(i, z)$  follows from (33).

**Solving for  $X$**  The remaining equilibrium conditions are:

$$[\hat{x}^h(1-z) + \hat{x}^b z] \beta \chi^\eta = a \frac{k}{qk + h + g} \quad (37)$$

$$[\hat{x}^h(1-z) + \hat{x}^b z] (1-\alpha)(1-\beta) \left(\frac{X}{\iota}\right)^\eta \left(\frac{\iota}{i-i^d}\right)^s = \phi z \quad (38)$$

$$[\hat{x}^h(1-z) + \hat{x}^b z] \alpha(1-\beta) \left(\frac{X}{\iota}\right)^\eta \left(\frac{\iota}{i}\right)^s = \frac{h}{qk + h + g} \quad (39)$$

$$\frac{\gamma}{1-\gamma} \xi^{\frac{\gamma-1}{\gamma}} \chi^{\frac{\gamma-1}{\gamma}} + r + \frac{\gamma}{2} \sigma_n^2 + \phi(i_t - i_t^d) + \mu_\xi - \frac{\gamma}{2} \sigma_\xi^2 + \theta \frac{\left(\frac{\zeta}{\xi}\right)^{1-\gamma}}{1-\gamma} = \frac{\rho + \theta}{1-\gamma} \quad (40)$$

$$\frac{a}{q} + \mu_q - r = \pi \sigma_q \quad (41)$$

$$(\mu_h + \mu_p) h + \mu_g - r g = (\sigma_h h + \sigma_g) \pi \quad (42)$$

Equation (37) is the market clearing condition for the goods market; (38) is a market clearing condition for the deposits market; (39) is a market clearing condition for the currency market; (40) is the banker's HJB equation; (41) is an arbitrage-pricing condition for capital and (42) is an arbitrage-pricing condition for government transfers.

We find the functions  $X$  by differentiating equations (37)-(42) with respect to time and finding  $X$  such that the time derivatives are equal to zero. Differentiating yields the following system of differential equations:

$$A \cdot \begin{pmatrix} \dot{\xi} \\ \dot{\zeta} \\ \dot{q} \\ \dot{h} \\ \dot{g} \\ \dot{i}^d \end{pmatrix} = B \quad (43)$$

---

<sup>5</sup> $g$  is expressed in absolute terms using (34) but  $n$ ,  $q$ ,  $h$  are expressed in geometric terms using (35)

where  $A$  is a  $6 \times 6$  matrix with entries:

$$\begin{aligned}
a_{11} &= \beta \chi^\eta z \frac{\gamma-1}{\gamma} \xi^{-\frac{1}{\gamma}} \chi^{-\frac{1}{\gamma}} \\
a_{12} &= \beta \chi^\eta (1-z) \frac{\gamma-1}{\gamma} \zeta^{-\frac{1}{\gamma}} \chi^{-\frac{1}{\gamma}} \\
a_{13} &= a \frac{k^2}{(qk+h+g)^2} \\
a_{14} &= a \frac{k}{(qk+h+g)^2} \\
a_{15} &= a \frac{k}{(qk+h+g)^2} \\
a_{16} &= \left[ - \left( z \xi^{\frac{\gamma-1}{\gamma}} + (1-z) \zeta^{\frac{\gamma-1}{\gamma}} \right) \frac{1}{\gamma} \chi^{-\frac{1+\gamma}{\gamma}} \beta \chi^\eta + [\hat{x}^h(1-z) + \hat{x}^b z] \beta \eta \chi^{\eta-1} \right] a_\chi
\end{aligned}$$

$$\begin{aligned}
a_{21} &= (1-\alpha)(1-\beta) \left( \frac{\chi}{\iota} \right)^\eta \left( \frac{\iota}{i-i^d} \right)^s z \frac{\gamma-1}{\gamma} \xi^{-\frac{1}{\gamma}} \chi^{-\frac{1}{\gamma}} \\
a_{22} &= (1-\alpha)(1-\beta) \left( \frac{\chi}{\iota} \right)^\eta \left( \frac{\iota}{i-i^d} \right)^s (1-z) \frac{\gamma-1}{\gamma} \zeta^{-\frac{1}{\gamma}} \chi^{-\frac{1}{\gamma}} \\
a_{23} &= 0 \\
a_{24} &= 0 \\
a_{25} &= 0 \\
a_{26} &= [\hat{x}^h(1-z) + \hat{x}^b z] (1-\alpha)(1-\beta) (i-i^d)^{-s} \iota^{s-\eta} \chi^\eta \left[ \eta \chi^{-1} a_\chi + (s-\eta) \iota^{-1} a_\iota + s (i-i^d)^{-1} \right] \\
&\quad - \left[ \zeta^{\frac{\gamma-1}{\gamma}} (1-z) + \xi^{\frac{\gamma-1}{\gamma}} z \right] \frac{1}{\gamma} \chi^{-\frac{1+\gamma}{\gamma}} (1-\alpha)(1-\beta) \left( \frac{\chi}{\iota} \right)^\eta \left( \frac{\iota}{i-i^d} \right)^s a_\chi
\end{aligned}$$

$$\begin{aligned}
a_{31} &= \alpha(1 - \beta) \left(\frac{\chi}{\iota}\right)^\eta \left(\frac{\iota}{i}\right)^s z \frac{\gamma - 1}{\gamma} \xi^{-\frac{1}{\gamma}} \chi^{-\frac{1}{\gamma}} \\
a_{32} &= \alpha(1 - \beta) \left(\frac{\chi}{\iota}\right)^\eta \left(\frac{\iota}{i}\right)^s (1 - z) \frac{\gamma - 1}{\gamma} \zeta^{-\frac{1}{\gamma}} \chi^{-\frac{1}{\gamma}} \\
a_{33} &= \frac{hk}{(qk + h + g)^2} \\
a_{34} &= -\frac{qk + g}{(qk + h + g)^2} \\
a_{35} &= \frac{h}{(qk + h + g)^2} \\
a_{36} &= [\hat{x}^h(1 - z) + \hat{x}^b z] (\alpha)(1 - \beta) \iota^{s-\eta} \chi^\eta i^{-s} [\eta \chi^{-1} a_\chi + (s - \eta) \iota^{-1} a_\iota] + \\
&\quad - \left[ \zeta^{\frac{\gamma-1}{\gamma}} (1 - z) + \xi^{\frac{\gamma-1}{\gamma}} z \right] \frac{1}{\gamma} \chi^{-\frac{1+\gamma}{\gamma}} (\alpha)(1 - \beta) \left(\frac{\chi}{\iota}\right)^\eta \left(\frac{\iota}{i}\right)^s a_\chi
\end{aligned}$$

$$a_{41} = -\frac{1}{\xi}$$

$$a_{42} = \frac{1}{\zeta}$$

$$a_{43} = 0$$

$$a_{44} = 0$$

$$a_{45} = 0$$

$$a_{46} = 0$$

$$a_{51} = 0$$

$$a_{52} = -\frac{1}{\zeta}$$

$$a_{53} = -\frac{1}{q}$$

$$a_{54} = 0$$

$$a_{55} = 0$$

$$a_{56} = 0$$

$$\begin{aligned}
a_{61} &= 0 \\
a_{62} &= -\frac{g+h}{\zeta} \\
a_{63} &= 0 \\
a_{64} &= -1 \\
a_{65} &= -1 \\
a_{66} &= 0
\end{aligned}$$

and  $B$  is a  $6 \times 1$  vector with entries

$$\begin{aligned}
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= \frac{\gamma}{1-\gamma} \xi^{\frac{\gamma-1}{\gamma}} \chi^{\frac{\gamma-1}{\gamma}} + \frac{\rho}{1-\gamma} - \frac{\gamma}{1-\gamma} \zeta^{\frac{\gamma-1}{\gamma}} \chi^{\frac{\gamma-1}{\gamma}} - \frac{\gamma}{2} \sigma_w^2 - \tilde{\mu}_\zeta + \frac{\gamma}{2} \sigma_\zeta^2 + \frac{\gamma}{2} \sigma_n^2 + \phi(i_t - i_t^d) + \tilde{\mu}_\xi \\
&\quad - \frac{\gamma}{2} \sigma_\xi^2 + \theta \frac{\left(\frac{\zeta}{\xi}\right)^{1-\gamma}}{1-\gamma} - \frac{\rho+\theta}{1-\gamma} \\
b_5 &= \frac{a}{q} + \tilde{\mu}_q - \frac{\rho}{1-\gamma} + \frac{\gamma}{1-\gamma} \zeta^{\frac{\gamma-1}{\gamma}} \chi^{\frac{\gamma-1}{\gamma}} + \frac{\gamma}{2} \sigma_w^2 + \tilde{\mu}_\zeta - \frac{\gamma}{2} \sigma_\zeta^2 - \pi \sigma_q \\
b_6 &= (\tilde{\mu}_h + i)h + \tilde{\mu}_g - \left( \frac{\rho}{1-\gamma} - \frac{\gamma}{1-\gamma} \zeta^{\frac{\gamma-1}{\gamma}} \chi^{\frac{\gamma-1}{\gamma}} - \frac{\gamma}{2} \sigma_w^2 - \tilde{\mu}_\zeta + \frac{\gamma}{2} \sigma_\zeta^2 \right) (g+h) - (\sigma_h h + \sigma_g) \pi
\end{aligned}$$

where for any variable  $X$ ,  $\tilde{\mu}_X$  is defined as

$$\tilde{\mu}_X \equiv \mu_X - \frac{\dot{X}}{X}$$

The algorithm for finding  $X$  is as follows.

1. Guess values for  $X$  at every point in the state space
2. Compute the derivatives with respect to  $i$  and  $z$  by a finite difference approximation
3. Compute  $Y$  at every point in the state space given the guess for  $X$ .
4. Compute  $\dot{X}$  at every point in the state space using (43)
5. Take a time-step backwards to define a new guess for  $X$

6. Repeat steps 1-5 until  $\dot{X} \approx 0$ .

The condition  $\dot{X} = 0$  is equivalent to saying that equilibrium conditions (37)-(42) hold.

**Finding the steady state** Once we solve for the equilibrium, this defines drifts and volatilities for the two state variables:  $\mu_i(i, z)$ ,  $\sigma_i(i, z)$ ,  $\mu_z(i, z)$ ,  $\sigma_z(i, z)$ . The density  $f(i, z)$  of the steady state distribution is the solution to the stationary Kolmogorov Forward Equation:

$$0 = -\frac{\partial}{\partial i} [\mu_i(i, z) f(i, z)] - \frac{\partial}{\partial z} [\mu_z(i, z) f(i, z)] \quad (44)$$

$$+ \frac{1}{2} \left( \frac{\partial^2}{\partial i^2} [\sigma_i(i, z)^2 f(i, z)] + \frac{\partial^2}{\partial z^2} [\sigma_z(i, z)^2 f(i, z)] + 2 \frac{\partial^2}{\partial i \partial z} [\sigma_i(i, z) \sigma_z(i, z) f(i, z)] \right)$$

We solve this equation by rewriting it in matrix form.<sup>6</sup> The first step is to discretize the state space into a grid of  $N_i \times N_z$  points and then convert it to a  $N_i N_z \times 1$  vector. Let  $vec(\cdot)$  be the operator that does this conversion. We then convert the differentiation matrices so that they are properly applied to vectors:

$$\begin{aligned} D_i^{vec} &\equiv I_{N_i} \otimes D_i \\ D_{ii}^{vec} &\equiv I_{N_i} \otimes D_{ii} \\ D_z^{vec} &\equiv M' (I_{N_z} \otimes D_z) M \\ D_{zz}^{vec} &\equiv M' (I_{N_z} \otimes D_{zz}) M \\ D_{iz}^{vec} &\equiv D_i^{vec} D_z^{vec} \end{aligned}$$

where  $\otimes$  denotes the Kreonecker product and  $M$  is the vectorized transpose matrix such that  $Mvec(A) = vec(A')$ .

Now rewrite (44):

$$-D_i^{vec} \cdot (diag(vec(\mu_i)) vec(f)) - D_z^{vec} (diag(vec(\mu_z)) vec(f))$$

$$+ \frac{1}{2} \left[ \begin{array}{l} D_{ii}^{vec} \cdot (diag(vec(\sigma_i^2)) vec(f)) + D_{zz}^{vec} (diag(vec(\sigma_z^2)) vec(f)) \\ + 2D_{iz}^{vec} (diag(vec(\sigma_i)) diag(vec(\sigma_z)) vec(f)) \end{array} \right] = 0$$

and therefore

$$Avec(f) = 0 \quad (45)$$

---

<sup>6</sup>See Achdou et al. (2014) for details on this procedure.

where

$$\begin{aligned}
A = & -D_i^{vec} \cdot diag(vec(\mu_i)) - D_z^{vec} \cdot diag(vec(\mu_z)) \\
& + \frac{1}{2} [D_{ii}^{vec} \cdot diag(vec(\sigma_i^2)) + D_{zz}^{vec} \cdot diag(vec(\sigma_z^2)) + 2D_{iz}^{vec} \cdot (diag(vec(\sigma_i)) diag(vec(\sigma_z)))]
\end{aligned}$$

Equation (45) defines an eigenvalue problem. We solve it by imposing the additional condition that  $f$  integrates to 1.