Debt Accumulation, Reserve Management and Sovereign Debt Cost in a World with Liquidity Crises*
PRELIMINARY VERSION

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Abstract

The accumulation of large amount of sovereign reserves has fuelled an intense debate on the associated costs. In a world with liquidity crises and strategic default, we model a contracting game between international lenders and a country, which delivers the country's optimal portfolio choice and the cost of sovereign debt: at equilibrium, the sovereign chooses the amount of debt and decides on the allocation of its resources between liquid and safe reserves, and an illiquid and risky production project. In line with recent empirical evidence, we find that the country holds positive amounts of sovereign debt and reserves. Indeed, we show that debt and reserves can be strategic complements: the accumulation of a larger amount of reserves can insure the country against the increase in the likelihood of a liquidity crisis induced by a larger amount of debt. Finally, we show that there might be cases in which the accumulation of sovereign reserves works independently from a precautionary motive.

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1 Introduction

Reinhart and Rogoff (2011) document that after the Great Depression and consequent banking system crisis, a wave of sovereign debt defaults arose during the 1930s. To establish this domino effect as a stylized fact in contemporary history a piece is missing: the failure of major financial institutions in the recent crisis to trigger a series of sovereign debt crises in the years ahead. This threat calls for a reflection on the management of sovereign liquidity and could become a test for the strategy, mainly employed by emerging economies after the experience of the late '90s Asian crisis, of accumulating large amounts of sovereign reserves as buffer stock to face liquidity shocks.

In early 2000 this accumulation seemed to be justified by the self-insurance motive and to be broadly in line with rules of thumb like the "Guidotti-Greenspan rule";[1] however, the recent increase in the resources devoted to reserves has made it urgent to understand the costs associated with this build-up. Rodrik (2006) measures the opportunity cost of accumulating large amounts of liquid assets (in particular US treasury bonds) as the spread between external borrowing costs and reserves' returns. Rodrik (2006) compares this "self-insurance premium" with the expected cost of a financial crisis in terms of reduced output and concludes that in the last decade the amount of reserves accumulated by some countries has grown too much. Moreover, Rodrik (2006) documents that, in recent years, emerging economies have not reduced their exposure to short-term debt, while accumulating large amounts of foreign liquid assets.

The goal of this work is analyze sovereign joint decision to issue sovereign debt and accumulate reserves. We develop a model of optimal portfolio choice where country's resources are determined by a contracting game with international lenders, and characterize the equilibrium level of debt, debt price, sovereign reserves and expected output in a world where the sovereign is subject to liquidity and productive shocks and can always default on debt if it finds optimal to do so. Sovereign debt and reserves are the country choice variables. Instead, the premium on debt borne by the country is set by international lenders. The country's expected output results from the share of total borrowed resources that is not devoted to reserves. We find that it is optimal for the country to hold positive amounts of both debt and reserves, and that the incentive to accumulate reservesgoes beyond the precautionary motive, it is part of the investment decision of the country. In some circumstances, the country can accumulate sovereign reserves independently from the need to employ them in the event of a liquidity crisis.

The strategic interaction between the country and international lenders is shaped by two financial frictions: the lenders cannot inject resources in the event of a liquidity shock and the country cannot commit not to default on debt.[2] These frictions introduce asset incompleteness into our model, and this is necessary to disentangle the relationship between the sovereign decision to default and the cost of sovereign debt. Indeed, if assets are not contingent, risk-neutral competitive lenders incorporate the probability of default in the premium set on the debt contract.

The optimal choice of reserves taken by the country arises from the equilibrium between two forces: on the one hand, reserves are liquid assets that (i) can be injected in the event of a liquidity shock and therefore help the sovereign to avoid default (precautionary motive) and (ii) cannot be seized by the lender in case of default, on the other hand, reserves distract resources from a more productive but illiquid project. The country also takes into account the impact of reserves on the cost of debt; there are two conflicting effects at work: the positive one is that, by allowing the country to avoid default in the event of liquidity crises, reserves raise the probability that lenders

[1] To be consistent with this rule, reserves should be equivalent to the country's short-term external debt.

[2] These assumptions are consistent with several papers in the literature and in particular with Holmström and Tirole (1996, 1998), Caballero and Krishnamurty (2002), Caballero and Panageas (2005) and Lorenzoni (2007).

will be repaid by the sovereign; the negative effect is that reserves distract resources from the production activity.

The novelty in our approach is that we do not assess whether the level of reserves is optimal in terms of its opportunity cost *per se*. We take a more general perspective, considering that when the country decides the optimal resource allocation it takes into account not only the opportunity cost of holding reserves in terms of the expected return of an alternative (illiquid) project, but also the effects of its choices on the price set by the lenders. In this way, we can address a number of policy relevant questions regarding the impact of reserves on the cost of sovereign debt and on expected output.

The country optimal choice of debt raises an additional trade-off: by borrowing a larger amount of debt the country has more resources to invest into the illiquid project (production activity) and accumulate more liquid assets (reserves). However, a larger level of debt increases the likelihood of a liquidity shock. This is because, as a country becomes more indebted, concerns about sovereign debt sustainability increase. Moreover, if the country borrows a larger amount of resources, also the likelihood that the country will find it optimal to strategically default on the productivity shock increases. Indeed, as the level of debt becomes bigger, it is more likely that the outcome of the production activity falls short of the amount that has to be repaid to international lenders after the productivity shock occurs.

Summarizing, we study the country's choices on the accumulation of debt and the allocation of its resources in a set-up that accounts for the impact of country's willingness-to-repay on lenders' optimal decision on debt cost. We find that the optimal level of both debt and reserves can be positive at equilibrium. The intuition is that reserves and debt can be strategic complements: since reserves can be employed in the event of a liquidity crisis, the country might decide to increase both reserves and debt so to, at the same time, increase its available resources (due to larger debt) and limit the consequences of a liquidity shock (thanks to larger reserves). However, the accumulation of a positive level of reserves can be independent from the precautionary motive. In our setting the sovereign solves the investment game by comparing the opportunity cost of reserves with respect to the production activity: provided the former is low enough, then hoarding reserves is rational. To understand to what extent the precautionary motive prevails, we perform an analysis of the role that reserves have in providing an insurance device against liquidity shocks and study whether self-insurance is preferable to a strategy that does not feature the injection of the necessary liquidity following a shock.

We contribute to the literature that studies the sovereign countries' optimal decision to accumulate international reserves and issue debt. Alfaro and Kanczuk (2009) employ a setup in which the sovereign keeps reserves while losing part of the output in the event of strategic default. In their model, to smooth consumption the country can either accumulate reserves or reduce external debt. The main result they obtain is that the sovereign optimal policy features nil international reserves, because the country prefers to smooth consumption by lowering debt exposure. They model sovereign liquidity crises in the form of contagion shocks (an abrupt variation in the interest rate borne by the sovereign) or sudden stops; however, in their setup reserves do not have a particular role in avoiding these crises.

We also contribute to the literature that studies optimal contractual arrangements in the presence of commitment problems and non-contingent contracts, as in Arellano (2008). In analogy to Arellano (2008), we show that default arises at equilibrium after an adverse shock occurs, consistently with the received empirical evidence. However, while Arellano (2008) studies the relationship between default risk and output, consumption and foreign debt, we look at the interaction between default risk, sovereign debt and reserves, cost of debt and output.

In the following, we present the model, discuss the equilibrium analysis and illustrate the features of model's equilibrium by undertaking numerical simulations.

2 The Model

Consider a sovereign country that needs to borrow D from international lenders. At stage 0, the lenders set the discount factor on the resources lent to the country and the country decides on the allocation of the same resources. At stage 1, a liquidity shock may take place: if a shock occurs the country has to decide whether to default at stage 2; in the absence of the shock the game proceeds to stage 3. At stage 3, the productivity shock takes place and at stage 4 the sovereign can again choose whether to default on debt. Figures 1 and 2 illustrate, respectively, the timing of the game and the game-tree. We solve the model by backward induction and the equilibrium concept we employ is the Sub-game Perfect Nash Equilibrium (SPNE).

In this Section, we present in detail how we model each relevant node and the main ingredients of the game.

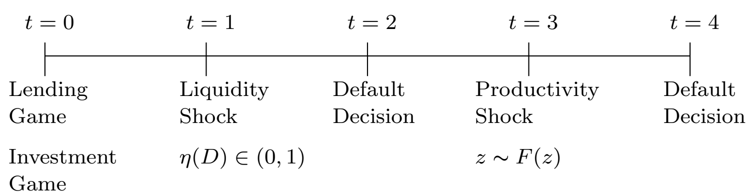

Figure 1: Timeline

[FIGURES 1 AND 2 ABOUT HERE]

2.1 The Lending Game

Our economy is populated by a continuum of atomistic and risk-neutral lenders (indexed by $i \in I$), from which the country can borrow. The sovereign borrows D from a subset of mass 1 of them. The total mass of lenders is large, ensuring that perfect competition prevails and lenders do not extract any rent; moreover, the lenders have unlimited access to fund at the risk-less interest (that we assume to be nil).

The lending game can be viewed as a general (common agency) contracting game between a principal (the sovereign) and multiple agents (the lenders):[3] as in Bolton and Jeanne (2007), lenders participate in a bidding game following the sovereign's announcement of a fund raising goal of D. Lenders move first by each making a bid simultaneously. The sovereign then decides which bids to accept.

The lenders' utility is equal to the value of the repayment D discounted by the probability that the sovereign repays in full. At the bidding stage of the game each lender i makes an offer on the rate of return, $r(i)$, that is required to break even in expectation. A lender i solves a problem of

[3] See, for example, Bernheim and Whinston (1986a,b), Hart and Tirole (1990).

the following sort:

$$\begin{aligned} D &= D(1+r(i))Prob\{\text{The Country is Solvent}\} \iff \\ 1/(1+r(i)) &= Prob\{\text{The Country is Solvent}\} \iff \\ \delta(i) &\equiv Prob\{\text{The Country is Solvent}\}. \end{aligned}$$

Consequently, in the model the contract specifies the discount factor $\delta(i) = 1/(1 + r(i))$ that a lender i asks in exchange for a loan D. A Nash equilibrium of the lending game is defined by a set of bids $(\delta(i))_{i\in I}$ such that, for all i, bid $\delta(i)$ maximizes lender i's utility taking all the other bids $\delta(j)$, with $j \neq i$, as given. Clearly, at equilibrium the sovereign squeezes all the surplus from the lending relationship and (randomly) selects among a set of identical bids, $\delta(j) = \delta(i) = \delta$, so we can focus on a representative sovereign-lender pair.

2.2 The Investment Game

The country is risk-neutral. It decides the amount of debt to borrow from international lenders (D) and how to invest these funds in public expenditure, g, and/or reserves, R. The sovereign feasibility constraint is given by

$$\delta D = g + R \iff g = \delta D - R. \tag{1}$$

This transformation implies that, once D and δ are determined, the country's decision on the value of R pins down the resources invested in g.

When deciding resource allocation, the country maximizes its expected utility, denoted by $E(U(.))$: this is determined by the amount of liquid resources gathered and by the expected value of output. In the model, reserves (R) yield the risk-less interest rate and act as a storage and liquid technology that can be carried over from the stage in which they are accumulated to the final stage of the game.

The production activity is the illiquid technology: it requires public expenditure (g) as sole input and is subject to a productivity shock, z. Output materializes at stage four, after the country has decided on the allocation of its resources and uncertainty over the shocks occurs. More specifically, the output is generated by a production function, $Y(z, g)$, that is linearly affected by the productivity shock and that, using (1), can be rewritten as:

$$Y(z, g) = zY(g) = zY(\delta, D, R).$$

The shock z is such that $z \sim F(z)$, where $F(z)$, the cumulative distribution function, is twice differentiable and continuous and $f(z)$, the probability distribution function, is given by $dF(z)/dz = f(z)$. We assume that z follows a continuous uniform distribution with positive density in $[1-c, 1+c]$.

Finally, the function $Y(\delta, D, R)$ is twice differentiable, with $Y'(.) \geq 0$, $Y''(.) < 0$, it is increasing in δ, decreasing in R and satisfies standard Inada Conditions ($\lim_{g\to 0} Y(.) = 0$, $\lim_{g\to 0} Y'(.) = \infty$ and $\lim_{g\to +\infty} Y'(.) = 0$).

2.3 The Liquidity Shock

We assume that the country incurs in the risk of a liquidity shock at an intermediate stage, before the outcome of the production activity.

Following Chang and Velasco (2000) we assume that, with probability $\eta(D) \in (0,1)$, the illiquid project needs a further infusion of capital ϵ at stage two in order to be completed. More specifically, $\lim_{D \to 0} \eta(D) = 0$, $\lim_{D \to \bar{D}} \eta(D) = 1$, and $\eta(.)', \eta(.)'' > 0$ for all $D \in (0, \bar{D})$.

Two things must be remarked. First, the likelihood of the liquidity shock increases in the amount of debt borrowed by the sovereign. This captures a negative relationship between the exposure to external debt and the likelihood of a liquidity crisis. Second, if a liquidity shock occurs the country can only use the accumulated reserves to inject ϵ, while it cannot dismantle the capital invested in the production process. Conversely, if it decides not to tackle the shock with the infusion of capital, the country defaults on the process and retains reserves.

2.4 The Default Decisions

There are two stages in the model at which the country may choose to default. The first is after the realization of the liquidity shock at stage two. The second is after the realization of the productivity shock when output realizes. In this way, we introduce the two main frictions of the model: the first is that the lender commits not to inject the resources needed by the country if a liquidity shock occurs. The second is that the country cannot undertake not to default when the realized output is lower than the face value of debt.[4]

In analogy to what is typically assumed in the literature (e.g. Bolton and Jeanne, 2007), the cost of default for the country consists in losing the entire output.[5] However, as in Alfaro and Kanczuk (2009), in the event of default the country keeps the reserves that it has accumulated.

For simplicity, we distinguish between two plans (that we denote $\mathcal{F}$ and $\mathcal{N}$) depending on the country's choice at stage two to inject the needed liquidity should the shock occur. Although our main focus is not on country's choice to default strategically on debt, we find it a useful distinction since it allows us to easily characterize all possible branches of the game-tree and study whether the sovereign expected utility from defaulting is greater that the one from continuation. Notice, however, that the crucial decision at time zero is the country's choice on D and the allocation of its resources on R.

3 Solution of the Model

In order to better disentangle the role of reserves, we first solve the model assuming that reserves cannot be accumulated: in this benchmark, the country decides only on the accumulation of D and can never cover the liquidity shock. Then, we allow the country to decide also on the amount of resources (δD) to allocate on R.

3.1 Benchmark without Reserves

If the country is hit by the shock in $t = 1$, then it is not able to repay ϵ and is already in default at $t = 1$. Conversely, if the country is not hit by the shock, then it defaults at the final stage if:

$$0 \geq zY(\delta, D) - D \iff z \leq \bar{z}(\delta, D) \equiv \frac{D}{Y(\delta, D)}.$$

$\bar{z}$ represents the threshold level of the productive shock: if the realized productivity is higher, then the country repays its creditors and keeps the residual, otherwise the country defaults on debt and

[4]Or, equivalently, that the lender cannot act as residual claimant if realized output is lower than the face value of debt.

[5]See Alfaro and Kanczuk (2005) for a discussion on output losses in the event of default.

loses the entire output. This introduces a problem of limited liability on the country side, which influences the outcome of the investment game.

At $t = 0$ country's resources are entirely invested in g. In turn the lender sets the optimal value of δ as to break-even in expectation:

$$\delta D = D(1 - \eta(D)) \int_{\bar{z}(\delta,D)}^{\infty} dF(z) \iff \delta = (1 - \eta(D))[1 - F(\bar{z}(\delta, D))]. \tag{2}$$

In the absence of reserves, the lender takes into account that the break-even is attained only if the liquidity shock does not take place, that is with probability $(1 - \eta(D))$.

LEMMA 1. *At the unique equilibrium, the country accumulates $D^* \in (0, \bar{D})$ and the lenders set $\delta^* \in (0, 1)$, with*

$$\frac{\delta^*}{D} < 0.$$

Proof. See Appendix A.

Lemma 1 yields the existence and uniqueness of both the equilibrium discount factor set by the lenders (δ^*) and the value of debt accumulated by the country (D^*). The equilibrium value of debt in Lemma 1 results from the trade-off between the increase in the expected marginal return of the production activity induced by an increase of D and the negative impact caused by the increase of D on the probability that a liquidity shock occurs ($\eta(.)$).

We show that the relationship between the exposure to external debt and the discount factor borne by the country is negative, that is, as D increases, the equilibrium value of δ becomes smaller. Intuitively, if the country is more indebted with the foreign lender, repayment concerns induce the lender to reduce the value of the discount factor.

The deal signed at the initial stage between the lender and the sovereign can be implemented by a debt contract in which the country receives $\delta^* D^*$ at $t = 0$ behind the promise to repay D^* at $t = 4$. Hence, the lender earns $(D^* - \delta^* D^*)$ (or, equivalently, $D^* r^*/(1 + r^*)$, with $r^*/(1 + r^*) = 1 - \delta^*$) provided the country repays in full, zero otherwise.

3.2 Framework with Reserves

In what follows, we allow the country to accumulate reserves at the investment game stage.

If the liquidity shock does not occur at $t = 1$, then the model proceeds as in the framework without reserves and at $t = 4$ the country defaults after the realization of the productivity shock if the following condition holds:

$$zY(\delta, D, R) - D + R - \epsilon \leq R - \epsilon \iff z \leq \bar{z}(\delta, D, R) \equiv \frac{D}{Y(\delta, D, R)}.$$

Conversely, if the liquidity shock occurs at $t = 1$, we distinguish between two cases, depending on whether the country decides to face the liquidity shock (plan $\mathcal{F}$) or not (plan $\mathcal{N}$).

Plan $\mathcal{F}$

At $t = 4$ the country defaults after the realization of the productivity shock if the following condition holds:

$$zY(\delta, D, R) - D + R - \epsilon \leq R - \epsilon \iff z \leq \bar{z}(\delta, D, R) \equiv \frac{D}{Y(\delta, D, R)}.$$

At $t = 2$, the country would employ its reserves to go on with the project. Clearly, a necessary condition for the country to do so is that it accumulates enough liquidity. In other words, the outcome of the investment stage must be such that $R \geq \epsilon$ (*resource constraint*).

Moreover, we analyze whether the decision to inject liquidity after the shock is sub-game perfect. Given the choice on D and R taken by the country and the premium (δ) offered at $t = 0$ by the lender, it must be that the sovereign has incentive to continue with the liquid project instead of defaulting strategically at $t = 2$, otherwise the same plan would not be time-consistent. The *no-default constraint* that has to hold follows:

$$\begin{aligned} R - \epsilon + \int_{\bar{z}(\delta,D,R)}^{\infty} [zY(\delta, D, R) - D] dF(z) &\geq R \iff \\ \int_{\bar{z}(\delta,D,R)}^{\infty} [zY(\delta, D, R) - D] dF(z) &\geq \epsilon. \end{aligned}$$

At $t = 0$, the lender solves the following zero-profit condition:

$$\delta D = D \int_{\bar{z}(\delta,D,R)}^{\infty} dF(z) \iff \delta = 1 - F(\bar{z}(\delta, D, R)). \tag{3}$$

The formulation of the lender's problem in this case takes into account that the sovereign does not default after the liquidity shock occurs.

At $t = 0$, the country decides on the accumulation of D and whether to invest δD in R. To do so, it solves the following problem:

$$\max_{D,R} E(U_F(\delta, D, R)) = R + \int_{\bar{z}(\delta,D,R)}^{\infty} [zY(\delta, D, R) - D] dF(z) - \eta(D)\epsilon. \tag{4}$$

The first term in $E(U_F(\delta, D, R))$ reflects the fact that the country is certain about R, because reserves are not lost in the event of default. The second term in (4) is the expected value of the production project net of D: this second term is positive by construction, because the expected value of output is truncated downwards by $\bar{z}$. The third term in (4) corresponds to the expected value of the liquidity shock.

The maximization problem in (4) is solved under the aforementioned constraints:

$$\int_{\bar{z}(\delta,D,R)}^{\infty} [zY(\delta, D, R) - D] dF(z) \geq \epsilon, \tag{5}$$

$$R \geq \epsilon, \tag{6}$$

which make sure that the implementation of plan $\mathcal{F}$ is sub-game perfect.

Plan $\mathcal{N}$

At $t = 2$, if the sovereign chooses to default after the liquidity shock, it loses output but keeps the accumulated liquidity (R).

In this case, the lender anticipates that the country defaults on the liquidity shock in $t = 2$. Thus, at $t = 0$, the lender sets δ as to break-even in expectation:

$$\delta D = D(1 - \eta(D)) \int_{\bar{z}(\delta,D,R)}^{\infty} dF(z) \iff \delta = (1 - \eta(D))[1 - F(\bar{z}(\delta, D, R))]. \tag{7}$$

At $t = 0$, the country's choice on D and R is obtained by solving the following problem:

$$\max_{D,R\in[0,\delta D]} E(U_N(\delta, D, R)) = R + (1 - \eta(D)) \int_{\bar{z}(\delta,D,R)}^{\infty} [zY(\delta, D, R) - D]dF(z). \tag{8}$$

The decision to default on the liquidity shock implies two things: the first is that the sovereign obtains the net expected payoff from the production project only if exempted from the liquidity shock (with probability $1 - \eta(.)$); the second is that the sovereign does not need to inject ϵ to complete the illiquid project.

3.3 Equilibrium Definition

The equilibrium is defined by the vector $\{\delta, D, R\}$ such that the country's behavior is optimal and the lender breaks even in expectation. Moreover, with liquidity shock coverage (that is, when plan $\mathcal{F}$ is chosen), the country's actions must be sub-game perfect, insofar as both the no-default constraint and the resource constraint must be satisfied.

DEFINITION 1. Denote by $\mathcal{F} \equiv \{\delta_F^*, D_F^*, R_F^*\}$ the vector that characterizes the plan in which the country decides to face the liquidity shock. At $\mathcal{F} \equiv \{\delta_F^*, D_F^*, R_F^*\}$, the resource constraint is satisfied

$$R_F^* \geq \epsilon,$$

and the no-default constraint is satisfied

$$\int_{\bar{z}(\delta_F^*,D_F^*,R_F^*)}^{\infty} [zY(\delta_F^*, D_F^*, R_F^*) - D]dF(z) \geq \epsilon.$$

Denote by $\mathcal{N} \equiv \{\delta_N^*, D_N^*, R_N^*\}$ the vector that characterizes the plan in which the country decides not to face the liquidity shock.

The SPNE of the game is given by plan $\mathcal{F}$ if $E(U(\delta_F^*, D_F^*, R_F^*)) \geq E(U(\delta_N^*, D_N^*, R_N^*))$ and the relevant constraints are satisfied, otherwise plan $\mathcal{N}$ is chosen at the equilibrium.

In the following, we first determine $\{\delta_N^*, D_N^*, R_N^*\}$ and $\{\delta_F^*, D_N^*, R_F^*\}$. Then, we analyze the country's choice between plan $\mathcal{N}$ and plan $\mathcal{F}$.

3.4 Equilibrium Analysis

First, we consider plan $\mathcal{N}$ and analyze country's decision to accumulate debt and reserves, and the lender choice of δ, the discount factor.

LEMMA 2. *At the equilibrium, the country accumulates $D_N^* \in (0, \bar{D})$ and the lender sets $\delta_N^* \in (0, 1)$. Moreover, the country accumulates $R_N^* > 0$ in the viable range if*

$$1 - (1 - \eta(D_N^*))Y'(\delta_N^*, D_N^*, R_N^*) \int_{\bar{z}(\delta_N^*,D_N^*,R_N^*)}^{\infty} zdF(z) = 0 \tag{9}$$

is satisfied.

Proof. See Appendix B.

Lemma 2 shows that, if (9) holds true, when the country chooses not to inject the accumulated liquidity in the event of a liquidity shock, there exists an equilibrium in pure strategies in which the sovereign accumulates a positive value of both debt and reserves and the lender sets the discount factor accordingly.[6] The expression in (9) corresponds to the first order condition of the problem in (8) with respect to R: it is satisfied if, at the equilibrium values, the increase in reserves' marginal rate of return (equal to 1) is equal to the marginal increase in the expected value of the production activity.

Lemma 2 also shows that the rationale behind the accumulation of sovereign reserves is independent from a precautionary motive: in plan $\mathcal{N}$ the country does not accumulates reserves to employ them in the event of a liquidity shock, but merely because it is profitable to do so. As in Lemma 1, the value of D results from the trade-off between an increase in the available resources and the increase of the probability of a liquidity shock ($\eta(.)$).

We now turn to the determination of the equilibrium under plan $\mathcal{F}$, that is when the country chooses to inject the needed liquidity (ϵ) after the liquidity shock occurs (at stage $t = 1$).

LEMMA 3. *At the equilibrium, the country accumulates $D_F^* \in (0, \bar{D})$ and the lender sets $\delta_F^* \in (0,1)$. Moreover, the country accumulates $R_F^* > 0$ if*

$$1 - Y'(\delta_F^*, D_F^*, R_F^*) \int_{\bar{z}(\delta_F^*, D_F^*, R_F^*)}^{\infty} z dF(z) = 0 \tag{10}$$

is satisfied and

$$\begin{cases} R_F^* \in [\epsilon, \delta_F^* D_F^*), \\ \int_{\bar{z}(\delta_F^*, D_F^*, R_F^*)}^{\infty} [zY(\delta_F^*, D_F^*, R_F^*) - D_F^*] dF(z) > \epsilon, \end{cases} \tag{11}$$

or

$$\begin{cases} R_F^* \in (\epsilon, \delta_F^* D_F^*), \\ \int_{\bar{z}(\delta_F^*, D_F^*, R_F^*)}^{\infty} [zY(\delta_F^*, D_F^*, R_F^*) - D_F^*] dF(z) = \epsilon. \end{cases} \tag{12}$$

Proof. See Appendix C.

Lemma 3 shows that two cases can arise under plan $\mathcal{F}$:[7] if (10) and (11) hold true, then there exists an equilibrium in pure strategies at which $R_F^* \geq \epsilon$ and the no-default constraint is slack. Instead, if (10) and (12) are satisfied, there exists an equilibrium in pure strategies at which the value of reserves results from a binding no-default constraint and a slack resource constraint ($R_F^* > \epsilon$).[8]

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[6] In the appendix we discuss the unicity of the equilibrium in Lemma 2.

[7] In the appendix we discuss the unicity of the equilibrium presented in Lemma 3.

[8] Therefore, there is no Nash equilibrium in pure strategies if both the resource and the no-default constraints are binding. The intuition for this outcome is given in the following. If the resource constraint is binding, then the country would like to accumulate a lower level of reserves than the necessary injection, but the resource constraint fixes R exactly at ϵ. In other words, the resource constraint sets a higher R than in an unconstrained optimum. The role of the no-default constraint is the opposite. Since the expected residual output after repaying the creditors in the event of no default (that is the left-hand-side of the no-default constraint) is decreasing in R, a binding no-default constraint implies that although the country would like to accumulate a higher level of reserves, it has to reduce R in order to satisfy the constraint with an equality. In other words, the impact of the level of reserves in the two

The optimal value of D is determined by two conflicting forces: on the one hand, a higher level of D raises the expected return of the production activity, on the other hand, if D increases also the probability to inject ϵ in the event of a liquidity crisis increases. However, in plan $\mathcal{F}$ the country employs the accumulated liquidity in the event of a liquidity crisis. This implies that the consequences of the liquidity shock should be less harsh for the sovereign with respect to the benchmark without reserves and we expect that the country accumulates a larger level of debt than in the no-reserve set-up.

Finally, since in plan $\mathcal{N}$ the country can bring the production process to the end only if it is not hit by a liquidity crisis, the value of the expected marginal return of the illiquid project in the first order condition for R is discounted by $1 - \eta(D)$ and is thus lower than in the first order condition under plan $\mathcal{F}$, *ceteris paribus*. This implies that the incentive to accumulate liquidity should be stronger under plan $\mathcal{N}$.

Summarizing, in Lemmata 2 and 3 we have shown that the country always accumulates a positive amount of debt (D) in the viable range and the lender sets the discount factor (δ) accordingly. When deciding the optimal value of reserves (R), the country trades-off the marginal return of reserves, equal to 1, with the marginal return of the illiquid project, equal to $\int_{\bar{z}(\delta,D,R)}^{\infty} zY'(\delta, D, R)dF(z)$ under plan $\mathcal{F}$ and $(1-\eta(D))\int_{\bar{z}(\delta,D,R)}^{\infty} zY'(\delta, D, R)dF(z)$ under plan $\mathcal{N}$. Overall, then, the sovereign accumulates a positive level of reserves provided the marginal return of the illiquid project is low enough.

The fact that the sovereign has incentive to employ reserves in the event of a liquidity shock implies that the level of accumulated debt should be higher under plan $\mathcal{F}$ than in the benchmark without reserves: as more debt makes a liquidity crisis more likely (via $\eta(D)$), the accumulation of a larger amount of reserves insures the country against the consequences of the same crisis.[9] Moreover, since under plan $\mathcal{N}$ the country reaches the final stage only if it is not hit by the liquidity shock (with probability $1 - \eta(D)$), the incentive to accumulate liquidity is higher in plan $\mathcal{N}$ than in plan $\mathcal{F}$, *ceteris paribus*.

4 Simulations

Here we present a numerical example that illustrates the results above.[10] More specifically, it is assumed that the production function is a Cobb-Douglas of the following type:

$$Y(z, g) = zY(g) = zY(\delta, D, R) = z(\delta D - R)^{\alpha}.$$

Also, we maintain the assumption for which the productivity shock is distributed as a uniform random variable:

$$z \sim U(1, c^2/3), z \in [1 - c; 1 + c].$$

constraints goes in two opposite directions: R should increase with respect to the unconstrained optimum in order to satisfy the resource constraint; R should decrease with respect to the unconstrained optimum to satisfy the no-default constraint. Overall, this implies that an equilibrium in which both constraints are contemporaneously binding cannot exist.

[9] More on this in the simulations.

[10] For the sake of the exposition, we focus on cases in which an equilibrium in pure strategies exists and is well-defined.

The probability of a liquidity shock, $\eta(D) \in (0,1)$, is such that $\lim_{D \to 0} \eta(D) = 0$, $\lim_{D \to \bar{D}} \eta(D) = 1$, $\eta(.)', \eta(.)'' > 0$ for all $D \in (0, \bar{D})$, and is modelled as an exponential function:

$$\eta(D) = 2^{D/\bar{D}} - 1.$$

The value of $\bar{D}$ is set high enough in order not to constrain the equilibrium level of debt in any of the cases analyzed. The income share of capital (α) is taken from the literature and set to 0.3. The needed capital infusion in the event of a liquidity shock (ϵ) is fixed at 10% of the average expected GDP, as proposed by Rodrik (2006) and Obstfeld et al. (2009).

The variance of the productivity process is arbitrary; we let it vary to analyze how it affects our choice variables (δ, D and R) and to check the robustness of our results.

[FIGURE 3 ABOUT HERE]

The simulations in Figure 3 plot the variables of interest and the final equilibrium of the model as the variance of the underlying productivity process (equal to $c^2/3$) rises from 1.5 to 5.5. In particular, the figure reports the rate of return chosen by lenders (r),[11] the equilibrium level of debt and reserves set by the country, the implied values of expected welfare, expected output and probability of the liquidity shock in the three cases, namely in a world where the country cannot accumulate reserves, under plan $\mathcal{F}$ and under plan $\mathcal{N}$.

In the lower-left panel we read the equilibrium chosen by the country: it is the one that generates the highest expected welfare. For low levels of the variance, the country chooses plan $\mathcal{N}$: hence, although the sovereign accumulates reserves, these resources are not employed in the event of a liquidity crisis. In this case, reserve accumulation is not due to a precautionary motive. Higher values of the variance imply that plan $\mathcal{F}$ is chosen at equilibrium. In this case, it becomes more profitable for the country to inject the accumulated liquidity in the event of a liquidity shock and bring the project to completion. The rationale for this result is that the expected output increases with the value of the variance. Finally, as the variance approaches 5.5, the expected welfare in the two cases is almost equalized.

With respect to the benchmark without reserves, the country always accumulates a higher level of debt and a positive level of reserves, well above ϵ (the amount of liquidity needed in the event of a liquidity shock). This result shows that reserves and debt can be strategic complements, especially in plan $\mathcal{F}$: although more debt increases the likelihood of a liquidity shock, by accumulating a larger amount of reserves the country can insure itself against the consequences of the same shock.

Debt increases with the variance since it ensures higher expected output. Indeed, as the variance becomes larger reserves increase less than debt, due to a growing opportunity cost of holding reserves,[12] which implies that a higher share of the issued debt is invested in the productive activity.

[11] Recall that the discount factor δ is equal to $1/(1+r)$. In our model, δ is inversely correlated to the cost of debt: an increase in the value of δ set by the lender reflects an increase in the probability that the lender expects the country to be solvent and stands for a decrease in the cost of debt.

[12] To assess the impact of the distribution of the shock, and in particular of its variance, on the opportunity cost of holding reserves ($\int_{\bar{z}(\delta,D,R)}^{\infty} zY'(\delta, D, R)dF(z)$), notice that ($i$) $Y'(.)$ is not affected by the variance and (ii) the assumption that z is distributed as a continuous uniform with support $[1-c, 1+c]$ implies that:

$$\int_{\bar{z}(\delta,D,R)}^{\infty} zdF(z) = [(1+c)^2 - (\bar{z}(\delta, R))^2]/4c,$$

and

$$\frac{\partial}{\partial c}\left(\frac{(1+c)^2 - (\bar{z}(\delta, D, R))^2}{4c}\right) = \frac{(\bar{z}(\delta, D, R))^2 - (1+c)(1-c)}{16c^2}.$$

5 Concluding Remarks

This paper contributes to the literature on the accumulation and the management of sovereign reserves, for countries that are subject to liquidity crises. We are able to model the sovereign choice on the accumulation of reserves and the level of issued debt, and the lenders' choice on the cost of debt. In our set-up, the country decides the level of international reserves and its debt exposure to maximize its expected welfare. The country takes into account that debt exposure increases the likelihood of a liquidity crisis and that it can always default on debt (with the cost in terms of output of this action). Competitive international lenders anticipate the country choice and set a discount factor over the lent resources that satisfies their zero-profit condition. In this way, we deliver a model that abstracts from the role of reserves in managing the exchange rate and that instead draws on the opportunity cost of holding reserves in terms of reduced expected output.

We find that the country can accumulate a positive value of both reserves and debt. Moreover, we show that both an equilibrium featuring the country using the accumulated reserves to self-insure against a liquidity shock and one with the country not employing reserves in the event of a shock can emerge, depending on the productivity process, the amount of liquid resources available and the probability and dimension of the liquidity crisis.

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Thus, if c rises above one (as in Figure 3) the incentive that a country has to invest an additional unit of resources in reserves decreases.

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A Proof of Lemma 1

To be completed

B Proof of Lemma 2

To be completed

C Proof of Lemma 3

To be completed

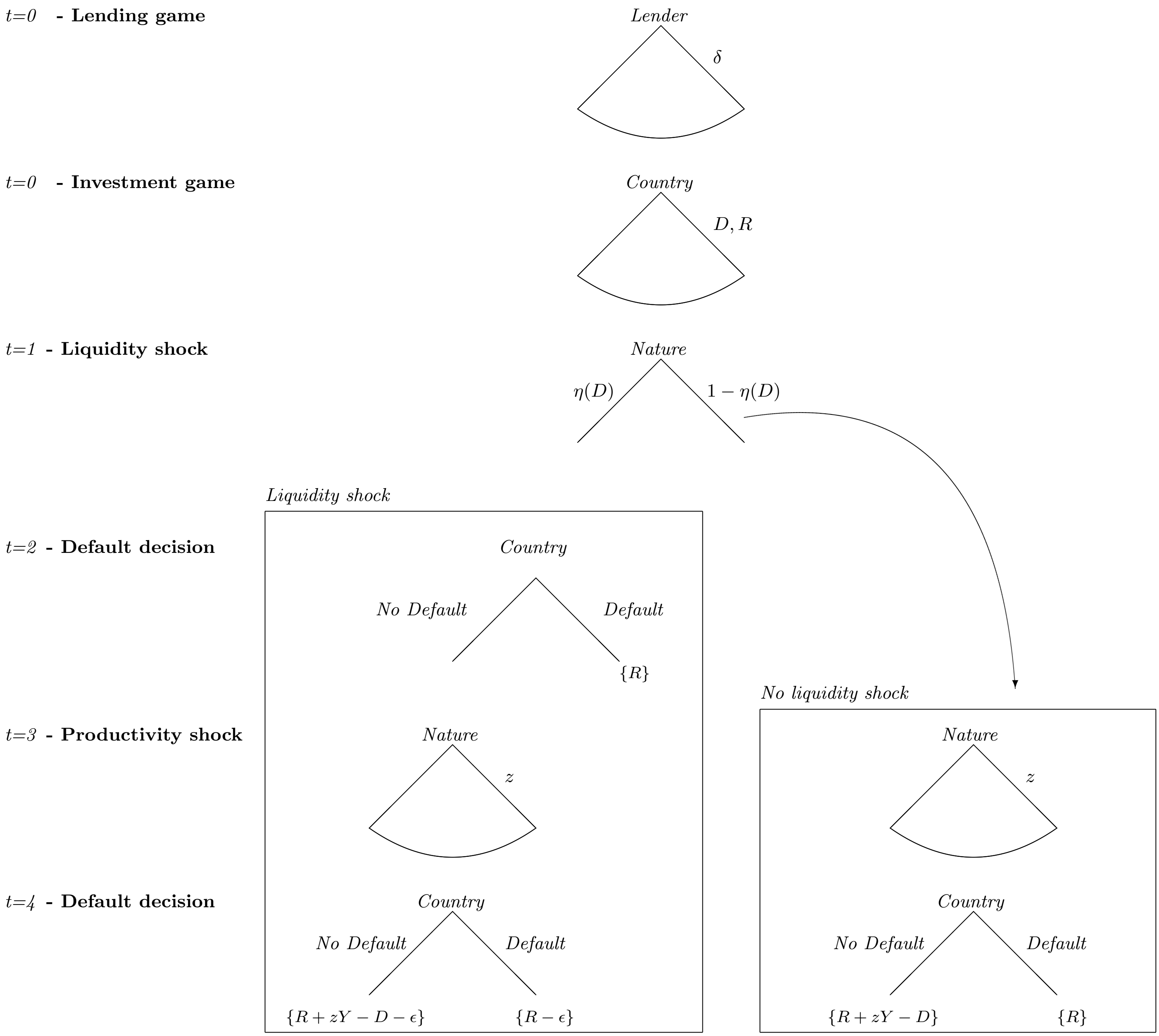

Figure 2: Game-tree with the country's payoffs

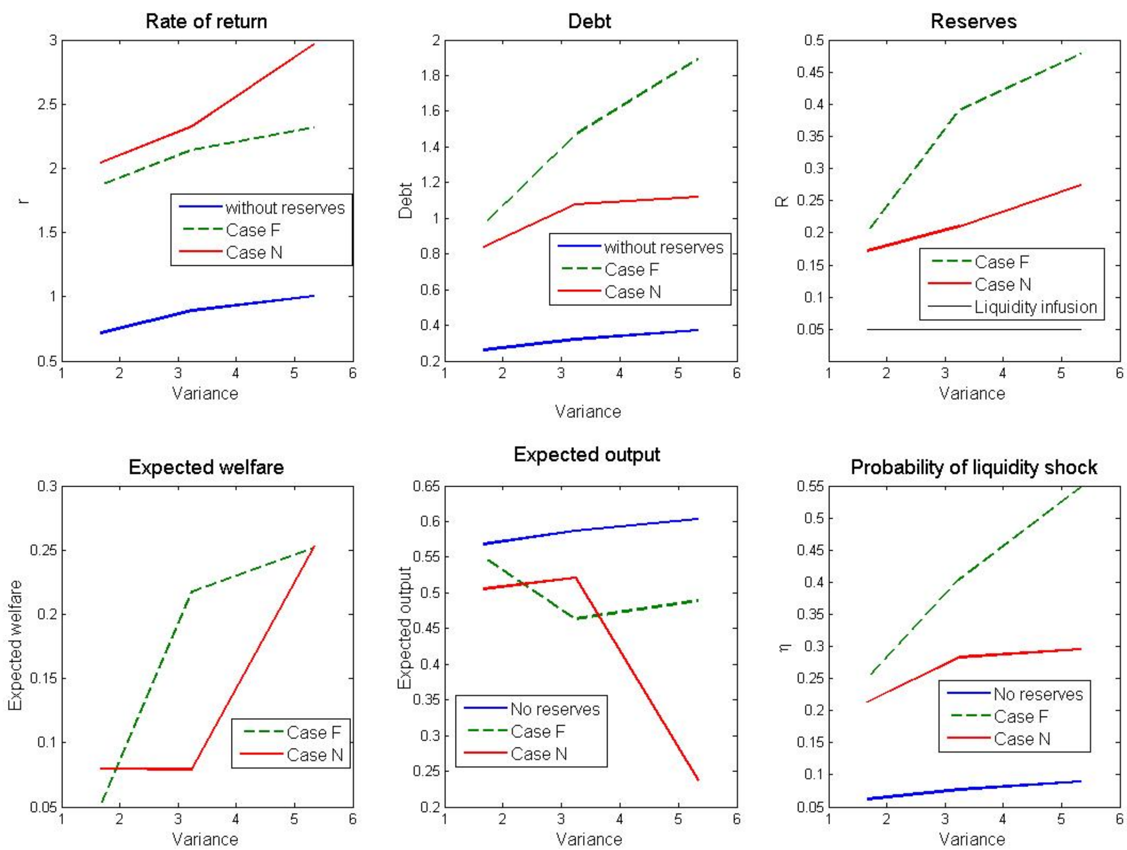

Figure 3: Numerical example - productivity shock variance ($c^2/3$)