Endogenous Trading Constraints with Incomplete Asset Markets*

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Abstract. The present paper endogeneizes the borrowing constraints on capital holdings in an infinite horizon incomplete markets model with capital accumulation. In particular, it assumes that two types of households can break their trading arrangements by going into financial autarky, in which case they are seized from any positive asset holdings and excluded from future asset trade forever. Since default cannot be an equilibrium outcome with free entry into a competitive financial intermediation sector, the endogenous trading constraints are then chosen at the loosest possible level that prevents default in equilibrium. Our results show that these limits are significantly different from zero, which is the ad hoc value often introduced in the literature. Further, they are monotone in the aggregate capital stock, since a higher capital reduces the interest rate on the financial obligations and decreases the incentive to default. Finally, in contrast with the existing literature, the limits are non monotone in the idiosyncratic and aggregate shocks due to the presence of incomplete markets and capital accumulation effects.

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1. Introduction

The present work endogeneizes the borrowing constraints through the introduction of asset market participation constraints in an infinite horizon incomplete markets model. In particular, it studies an heterogeneous agent version of the stochastic growth model where households can break their trading arrangements by going into financial autarky. In this case, they are seized from any positive asset holdings and excluded for future asset trade forever. Since default cannot be an equilibrium outcome with free entry into a competitive financial intermediation sector, the endogenous trading limits are then chosen at the level where the participation constraint is satisfied with equality at each possible date and state. Using this setup, the main objectives of the present work are to quantify and characterize the endogenous limits resulting from the option of default and to study their real and financial implications along the growth path and in the stationary distribution.

Our work is of considerable relevance, since it builds a bridge between several important strands of literature. First, it contributes to an active and increasingly growing literature where a number of authors have introduced limited enforceability of contracts, resulting in agent and state specific trading constraints. In particular, Kehoe and Levine (1993) and Alvarez and Jermann (2000) introduce these type of limits in exchange economies with a finite number of agents, Krueger and Perri (2001) and (2003) study a similar model with a continuum of agents, and Kehoe and Perri (2002) study a two agent production economy where investors are interpreted as countries. These authors, however, introduce the possibility of default into an otherwise complete markets context, and this allows them to solve a centralized problem without explicitly characterizing the trading limits (on Arrow securities).

While our competitive equilibrium does not have a clear-cut central planning problem yielding the same allocation, the decentralized solution provides a complete characterization of the endogenous trading limits (on a limited set of assets) that prevent default in equilibrium. Thus, our work sheds light on important issues such as how the limits depend on the individual characteristics and on the aggregate state of the economy. In addition, we extend the previous models to the presence of capital accumulation, showing that this outweighs the benefits of default when there is a complete set of assets, resulting in a perfect risk sharing allocation in the long run. Note that this is in sharp contrast with the empirical findings on consumption risk sharing, suggesting that an incomplete markets framework with imperfect risk sharing is more in line with the data. Further, since capital accumulation affects the incentives to default in a non-trivial way when markets are incomplete, our characterization of the endogenous limits also provides an overview of the different forces affecting the default decision in the presence of a production sector.

Our work is also closely related to the traditional incomplete market models where occasionally binding short-selling or borrowing limits on the different assets are ad hoc. Among others, Heaton and Lucas (1996), Marcet and Singleton (2001), and Telmer (1993) have studied asset prices in two agent endowment economies, and Aiyagari (1994) or Krusell and Smith (1997, 1998) have studied production economies with a large number of households. While the previous authors have often argued that the ad hoc trading constraints are tighter than the natural borrowing limits to avoid default in equilibrium, the present work is to our knowledge the first one (with the exception of the papers by Zhang that we discuss in what follows) formalizing this argument. Given this, it provides a deeper foundation of the limits. Finally, our work is also closely related to the papers of Zhang (1997a, 1997b), where the author derives an endogenous but fixed borrowing limit resulting from the possibility of default in a Lucas type exchange economy with trade in one asset. In contrast to this, our framework allows for the possibility of capital accumulation, an assumption which modifies the incentives to default under complete and incomplete markets. Further, it derives the state-dependent trading limits that avoid equilibrium default.

Our main results can be summarized as follows. We show that, in a model where two types of households can trade in physical capital, the endogenous limits that prevent default are significantly different from zero, which is the ad hoc limit often introduced in the literature. In addition, they are monotone in the aggregate capital stock, since a higher aggregate capital reduces the interest rate on the financial obligations and decreases the incentive to default. Finally, they are non monotone in the labor income and technology shocks for the following reasons. An increase in the labor income shock increases both the contract and autarky values, but it also modifies the factor prices through its effect on the aggregate capital stock. On the one hand, when capital is low and its return is relatively high, the latter capital accumulation effects are relatively small, while the marginal utility of consumption in the contract is very high. This leads to an increase in the contract value above the autarky value, and a household with a higher labor shock is therefore subject to a looser limit. On the other hand, the reverse is true when capital is high, in which case the limit gets tighter with a higher income shock. Similar arguments can be used to explain the non-monotonicity of the limits with respect to the aggregate technology shock.

Finally, note that the presence of endogenous trading limits considerably complicates our computations, since we have to extend usual policy function iteration algorithm to incorporate a state dependent and non rectangular grid for some of the endogenous states, introducing an additional fixed point problem. In addition, given the typical nonlinearities of the policy and value functions around the limits, and given that fact that these limits are endogenously determined at the level where the value function from staying in the contract is at least as large as the autarky value, it becomes clear that a good approximation of the value functions close to the limits is needed to obtain reliable results. In this respect, our contribution is also methodological, and it will therefore enhance the understanding of the numerical methods that can be used to solve these type of problems.

The paper is organized as follows. The following section presents model with incomplete markets, and section three discusses an intermediation structure that supports the endogenous limits preventing default in equilibrium. Further, the complete markets allocation and the solution methodology are discussed in sections four and five, and section six presents the quantitative results.

2. The Model

We consider several variants of an infinite horizon economy with aggregate uncertainty, idiosyncratic labor income shocks and sequential trading. Time is discrete and indexed by $t = 0, 1, 2....$ Further, the resolution of uncertainty is represented by an information structure or event-tree S. Each node or date-state $s^t \in S$, summarizing the history of the environment through and including date t, has a finite number of immediate successors, denoted by $s^{t+1}|s^t$. We use the notation $s^r|s^t$ with $r \geq t$ to indicate that node s^r belongs to the sub-tree with root s^t. Further, with the exception of the unique root node s^0 at date $t = 0$, each node has a unique predecessor, denoted by s^{t-1}. The probability as of period 0 of date-event s^t is denoted by $\pi(s^t)$, with $\pi(s^0) = 1$, since the initial realization s^0 is given. In addition, $\pi(s^r|s^t)$ denotes the probability of s^r given s^t, with $\pi(s^t|s^t) = 1$.

Households. The economy is populated by a firm and by a set I of infinitely lived households, indexed by $i \in I$, where I can be either finite or infinite and continuous. Households have identical additively separable preferences over sequences of consumption $\left\{c_i(s^t)\right\}_{s^t \in S}$ of the form:

$$U(c_i) = \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t)\beta^t u\left(c_i(s^t)\right) = E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_i(s^t)\right) \tag{1}$$

where $\beta \in (0, 1)$ is the subjective discount factor and E_0 denotes the expectation conditional on information at date $t = 0$. The period utility function u is assumed to be strictly increasing, strictly concave and continuously differentiable, with $\lim_{c_i \to 0} u'(c_i) = \infty$, and $\lim_{c_i \to \infty} u'(c_i) = 0$.

At each date-state s^t, households can trade in physical capital to insure against uncertainty[1]. Capital depreciates at a constant rate δ, and its one period payoff net of depreciation is denoted by $r(s^{t+1})$ for each $s^{t+1}|s^t$. At each node, household $i \in I$ receives asset income from his previous period investments $k_i(s^{t-1})$. In addition, he also receives a stochastic labour endowment $\epsilon_i(s^t)$, following a stationary Markov chain with S_ϵ values, where $0 < \epsilon^1 ... < \epsilon^{S_\epsilon} < \infty$. Given this, his individual labor income is equal to $w(s^t)\epsilon_i(s^t)$, where $w(s^t)$ is the aggregate wage rate paid by the firm, and his date-state s^t budget constraint is given by:

$$c_i(s^t) + k_i(s^t) = \omega_i(s^t) \tag{2}$$

[1] The present framework can be easily extended to trade in many assets. Further, the next section shows how the trading arrangements can be supported by a competitive financial intermediation sector.

$$\omega_i(s^{t+1}) \equiv w(s^{t+1})\epsilon_i(s^{t+1}) + r(s^{t+1})k_i(s^t) \tag{3}$$

where $\omega_i(s^t)$ represents the individual level of wealth. At period $t = 0$, the budget constraint takes the same form with $\omega_i(s^0) = w(s^0)\epsilon_i(s^0) + r(s^0)k_i(s^{-1})$, where the initial shock $\epsilon_i(s^0)$ and the initial capital holdings $k_i(s^{-1}) \in \mathbb{R}_+$ are given. Apart from the budget constraint, household $i \in I$ also faces a possibly endogenous and state-dependent trade restriction on capital holdings of the form:

$$k_i(s^t) \geq \underline{k}_i(s^t). \tag{4}$$

As to the previous trading restriction, we consider two different cases. In the first case, we assume that households cannot commit on the trading contracts, and the limits on capital holdings are endogenously determined at the level that prevents default in equilibrium. While these limits are simply imposed here, next section shows that they would arise in equilibrium in the presence of a competitive intermediation sector. In the second case, we assume that the restrictions are of the form $k_i(s^t) \geq \underline{k}_i$, where $\underline{k}_i$ is some exogenously fixed level that is chosen ad hoc, as usual in the incomplete markets literature (see e.g. Heaton and Lucas (1996), Marcet and Singleton (2001) or Telmer (1993)).

Production and Market Clearing. At each date-state s^t, the representative firm uses capital $K(s^t) \in \mathbb{R}_+$ and aggregate labor $L(s^t) \in (0, 1)$ to produce a single good $y(s^t) \in \mathbb{R}_+$ with the constant returns to scale technology:

$$y(s^t) = f(z(s^t), K(s^t), L(s^t)) \tag{5}$$

where $z(s^t)$ is an aggregate productivity shock following a stationary Markov chain with S_z values, where $0 < z^1 ... < z^{S_z} < \infty$. In what follows, we denote total output including undepreciated capital by $F(z(s^t), K(s^t), L(s^t)) = f(z(s^t), K(s^t), L(s^t)) + (1 - \delta)K(s^t)$.

Each period, after observing the realization of the productivity shock, the firm rents capital and labor to maximize period profits:

$$F(z(s^t), K(s^t), L(s^t)) - w\left(s^t\right) L(s^t) - r(s^t)K(s^t) \tag{6}$$

leading to the following first order conditions:

$$w(s^t) = f_L(z(s^t), K(s^t), L(s^t)) \tag{7}$$

$$r(s^t) = f_K(z(s^t), K(s^t), L(s^t)) + 1 - \delta \tag{8}$$

Finally, labor and asset market clearing require that the sum over households of the labor income shocks and the capital asset holdings are equal to the total labor supply and to the aggregate capital stock respectively. Further, the good's market clearing condition requires that the aggregate consumption and investment are equal to the aggregate output.

Recursive Competitive Equilibrium. In the present model, the aggregate state of the economy is given by $S = (\Psi(\epsilon, k), z)$, where $\Psi(\epsilon, k)$ represents the joint distribution of consumers over individual capital holdings k and idiosyncratic productivity status ϵ. Note that, in the presence of only one asset, we can either include the individual wealth ω or the individual capital holdings k as a state variable, but we choose k for expositional simplicity. Further, introducing the aggregate capital K as a separate state becomes unnecessary, since it is just a particular moment of the wealth-productivity distribution.

The aggregate state vector is used by the households to predict future prices. Here, note that the law of motion of the two shocks is exogenously given by a joint discrete Markov process with transition matrix $\Pi(\epsilon', z'|\epsilon, z)$, which can possibly allow the shocks to be correlated. Further, households expect the law of motion of Ψ to evolve according to:

$$\Psi(\epsilon', k') = \Gamma[\Psi(\epsilon, k), z]$$

where Γ represents the stochastic mapping from the current state into tomorrow's wealth-productivity distribution. Finally, the individual state vector includes the individual labour productivity and the individual capital holdings (ϵ, k). Given this, the relevant state variables for a household are summarized by the vector $(\epsilon, k; \Psi(\epsilon, k), z)$.

Definition 2.1: Given a vector of initial asset holdings $k_{-1} \equiv (k_i(s^{-1}))_{i \in I}$, a vector of initial shocks $(z_{-1}, \epsilon_{-1}) \equiv (z_0, (\epsilon_{i,0})_{i \in I})$ and a transition matrix Π, a *recursive competitive equilibrium* relative to the trading limits $\underline{k}(\epsilon; S)$ is defined by a law of motion Γ, a vector of factor prices $(r, w) = (r(S), w(S))$, value functions $W = W(\epsilon, k; S)$ and individual policy functions $(c, k') = (c(\epsilon, k; S), k(\epsilon, k; S))$ such that[2]:

(i) *Utility Maximization:* For each $i \in I$, W and (c, k') solve the following problem given k^{-1}, (z_{-1}, ϵ_{-1}), Π, Γ and (r, w):

$$W(\epsilon, k; \Psi(\epsilon, k), z) = \max_{c,k'} \left\{ u(c) + \beta \sum_{\epsilon', z'} \Pi(\epsilon', z'|\epsilon, z) W(\epsilon', k'; \Psi(\epsilon', k'), z') \right\} \text{ s.t.}$$

$$c + k' = w(S)\epsilon + r(S)k$$

$$\Psi(\epsilon', k') = \Gamma[\Psi(\epsilon, k), z]$$

$$k' \geq \underline{k}(\epsilon'; \Psi(\epsilon', k'), z') \text{ for all } (\epsilon', z')|(\epsilon, z) \text{ with } \Pi(\epsilon', z'|\epsilon, z) > 0.$$

(ii) *Profit Maximization:* Factor prices satisfy the firm's optimality conditions, i.e., $w(S) = w(z, K) = f_L(z, K, L)$ and $r(S) = r(z, K) = f_K(z, K, L) + 1 - \delta$.

[2] Note that we have assumed that households have the same preferences and processes for the idiosyncratic shocks, implying that the policy rules as functions of the states are identical across them.

(iii) *Market Clearing:*

$$\int k(\epsilon, k; S)\Psi(\epsilon, k)d\epsilon dk = K'$$

$$\int \epsilon\Psi(\epsilon, k)d\epsilon dk = L$$

$$\int (k(\epsilon, k; S) + c(\epsilon, k; S))\Psi(\epsilon, k)d\epsilon dk = F(z, K, L) + (1 - \delta)K.$$

(iv) *Consistency:* Γ is consistent with the agent's optimal decisions.

As reflected in (i), we have imposed endogenous limits on the individual capital holdings that depend on the two shocks and on the aggregate wealth-productivity distribution. Further, the limits have to hold for all possible contingencies, and we therefore assume that they have to be satisfied for all possible continuation states with positive probability. The exact specification and the determination of these limits will be discussed below.

Note also that the previous definition of equilibrium allows for the presence of a finite or an infinite number of households. For comparison with a significant portion of the incomplete markets literature, however (see e.g. Heaton and Lucas (1996), Alvarez and Jermann (2000) and Zhang (1997a, 1997b)), our benchmark model version assumes two types of households and a continuum of identical households of each type. Further, the individual labor income shocks are assumed to be perfectly negatively correlated across types, i.e., they are not truly idiosyncratic but simply redistribute labor income shocks across types.

These assumptions have the following implications. First, the aggregate labor supply is constant and it can therefore be normalized to one without loss of generality. Second, the aggregate capital stock K and the vector (ϵ_i, k_i) for $i = 1$ or 2 provide a complete description of the wealth-productivity distribution, since we can infer (ϵ_{-i}, k_{-i}) from $\epsilon_{-i} = 1 - \epsilon_i$ and $k_{-i} = K - k_i$. The aggregate state vector for a household in the presence of two types is therefore given by $S = (z, K, \boldsymbol{\epsilon}, \mathbf{k})$, where the variables in bold represent the states that households regard as beyond their control when making their individual decisions. Here, it is important to note that we distinguish (k, ϵ) and $(\mathbf{k}, \boldsymbol{\epsilon})$ only when posting the individual decisions, but we set $k = \mathbf{k}$ and $\epsilon = \boldsymbol{\epsilon}$ to impose the equilibrium. Given this, the law of motion Γ simplifies to:

$$\left(K', \boldsymbol{\epsilon}', \mathbf{k}'\right) = \Gamma[\left(K, \boldsymbol{\epsilon}, \mathbf{k}\right), z].$$

In addition, the optimization problem in the previous definition of equilibrium can now be expressed as:

$$W(\epsilon, k; \left(K, \boldsymbol{\epsilon}, \mathbf{k}\right), z) = \max_{c,k'} \left\{ u(c) + \beta \sum_{(\epsilon',z')} \Pi(\epsilon', z'|\epsilon, z) W(\epsilon', k'; \left(K', \boldsymbol{\epsilon}', \mathbf{k}'\right), z') \right\} \text{ s.t.} \quad (9)$$

$$c + k' = w(S)\epsilon + r(S)k \quad (10)$$

$$\left(K', \boldsymbol{\epsilon}', \mathbf{k}'\right) = \Gamma[(K, \boldsymbol{\epsilon}, \mathbf{k}), z] \tag{11}$$

$$k' \geq \underline{k}(\epsilon'; K', z') \text{ for all } (\epsilon', z')|(\epsilon, z) \text{ with } \Pi(\epsilon', z'|\epsilon, z) > 0. \tag{12}$$

while the market clearing conditions are now given by:

$$k(\epsilon, k; S) + k(1 - \epsilon, K - k; S) = K' \tag{13}$$

$$L = 1 \tag{14}$$

$$K' + c(\epsilon, k; S) + c(L - \epsilon, K - k; S) = F(z, K, L) + (1 - \delta)K \tag{15}$$

As stated earlier, consistency requires that $k' = \mathbf{k}'$ and $\epsilon' = \boldsymbol{\epsilon}'$, and we denote the reduced form or equilibrium policy and value functions by $W(\epsilon, k; K, z)$, $c(\epsilon, k; K, z)$ and $k(\epsilon, k; K, z)$. In addition, note that the endogenous trading restriction with $I = 2$ only depends on the aggregate capital stock and on the two exogenous shocks. A detailed description of the determination of this limit is provided in what follows[3].

The Endogenous Trading Restriction. As stated earlier, our benchmark model version assumes that households cannot commit on the trading contracts. Further, a household will be seized from his asset holdings and from future asset trade forever upon default, implying that his only source of income from the default period on will be his labor income. The autarky value V can therefore be expressed recursively as:

$$V(\epsilon, k; (K, \boldsymbol{\epsilon}, \mathbf{k}), z) = u(w(S)\epsilon) + \beta \sum_{(\epsilon', z')} \Pi(\epsilon', z'|\epsilon, z) V(\epsilon', k'; \left(K', \boldsymbol{\epsilon}', \mathbf{k}'\right), z'). \tag{16}$$

Several remarks are worth noting. First, in contrast with some of the literature under complete markets and no commitment, where the autarky value V is exogenous, the previous equation reflects that the autarky value in the present framework is a function of the wealth-productivity distribution. Note that this is due to the fact that the distribution influences the aggregate capital accumulation, which in turn affects future wages and therefore the future value of financial autarky. In particular, as we will see later, a higher dispersion of the wealth-productivity distribution, represented by a lower $\mathbf{k}$, leads to a higher capital accumulation and to lower future wages. This clearly implies that $V_k(\epsilon, k; (K, \epsilon, k), z) < 0$, at least for negative levels of k.

Second, as shown in the next section, default cannot be an equilibrium outcome in the presence of a competitive intermediation sector. Further, we consider the loosest possible

[3] Since the trading restriction has to be satisfied for all possible continuation states $(\epsilon', z')|(\epsilon, z)$, note that the effective limit faced by the households will only be a function of K if the probability of all future states is strictly positive. While this is not the case with our present calibration, this is a particular feature of the one asset environment, and the limit will therefore be state dependent in the presence of trade in more than one asset.

limits such that there is no default in equilibrium. In other words, we will analyze the economy with limits that are not *too tight*, in the sense that they are determined by the individual capital holdings $\underline{k}$ at which the agent is indifferent between defaulting or paying back the debt, i.e.,

$$\underline{k}(\epsilon; K, z) = \{\underline{k} : W(\epsilon, \underline{k}; (K, \epsilon, \underline{k}), z) = V(\epsilon, \underline{k}; (K, \epsilon, \underline{k}), z)\}. \tag{17}$$

It is important to note that the previous limit has to be negative, since no agent would default with a positive level of asset holdings. Clearly, he could then afford a higher current consumption than in autarky and at least as high of a life-time utility as in autarky from next period on. Finally, as we will see later, an increase in k leads to a higher contract value due to its positive first order effect on the individual level of wealth. Given this, the previous wealth restriction defines a unique limit for every date-state, and it implies that:

$$W(\epsilon, k; (K, \epsilon, k), z) \geq V(\epsilon, k; (K, \epsilon, k), z) \; \forall k \geq \underline{k}(\epsilon; K, z) \tag{18}$$

at least for negative values of k.

The next section shows that the previous endogenous limits will arise in equilibrium in the presence of a competitive intermediation sector. As stated earlier, the allocation with these limits will first be compared to the allocation resulting in the presence of fixed and exogenous limits. Further, for comparison with the literature with enforcement constraints (see e.g. Alvarez and Jermann (2000) or Kehoe and Perri (2002)), it will also be compared to the allocation resulting when markets are complete. This framework is discussed in section four.

3. The Intermediation Sector

The present section shows that the endogenous limits specified in the previous section would arise in equilibrium in the presence of a competitive intermediation sector. To do this, assume that, at each node s^t, households can trade through financial intermediaries in a risky asset with an endogenous one period payoff of $R(s^{t+1})$ for each future state $s^{t+1}|s^t$. In particular, if a household invests in the asset ($a_i(s^t) > 0$), for each unit of consumption he or she gives up at node s^t, the intermediary promises to pay back $R(s^{t+1})$ units of the consumption good if date-state $s^{t+1}|s^t$ is realized. Further, if the household borrows ($a_i(s^t) < 0$), for each unit of consumption he or she receives, he or she promises to pay back $R(s^{t+1})$ units to the intermediary if the continuation state $s^{t+1}|s^t$ is realized. We assume that all intermediaries take the one period payoffs as given. Further, they cannot price discriminate, in the sense that they have to pay the same return to investors as they charge to borrowers. Under these assumptions, the budget constraint and wealth accumulation equation of household $i \in I$ can be written as follows:

$$c_i(s^t) + a_i(s^t) = \omega_i(s^t)$$

$$\omega_i(s^{t+1}) \equiv w(s^{t+1})\epsilon_i(s^{t+1}) + R(s^{t+1})a_i(s^t).$$

The intermediaries live for two periods. At the beginning of the first period, they set limits on the agents' capital holdings by only allowing them to take asset positions that do not violate lower bounds on their asset levels at any possible continuation state. Further, the limits are set in a way that no intermediary has an incentive to deviate from them, and this implies that households are also subject to the following constraint:

$$a_i(s^t) \geq \underline{a}_i(s^t).$$

The cash flows of the intermediaries can be described as follows. During the first period, they trade consumption goods with the households and collect a total amount of $A(s^t) \geq 0$ goods. Further, they transform this (or some portion of it) into physical capital $k(s^t) \leq A(s^t)$, which is rented to the representative firm. For simplicity, we assume that this transformation is one-to-one. In the second period, they a receive rental income of $r(s^{t+1})k(s^t)$ from the firm if state $s^{t+1}|s^t$ is realized, and they have to honor the trading contracts with the households by paying back $R(s^{t+1})A(s^t)$. We assume that all the intermediaries can commit to repay back their debt to the households at any possible contingency, but they cannot be forced to lend or borrow from any household. Further, we assume that $A(s^t) > 0$, implying that intermediaries cannot be solely making pure arbitrage profits but have to mediate between the households and the production sector[4].

We focus on symmetric equilibria where all intermediaries hold the same portfolio, leading to the following restrictions. First, an equilibrium implies that $k(s^t) = K(s^t)$, where $K(s^t)$ is the demand for capital of the representative firm. Second, the following condition has to hold:

$$\sum_{s^{t+1}} \pi(s^{t+1} \mid s^t) r(s^{t+1}|s^t) \leq \sum_{s^{t+1}} \pi(s^{t+1} \mid s^t) R(s^{t+1}|s^t). \tag{19}$$

To see why this is the case, note that the intermediaries could otherwise make arbitrarily large profits by demanding arbitrarily large funds $A(s^t)$ and by setting $k(s^t) = A(s^t)$. Third, since the intermediary can commit to repay back its debt to the households at each node s^t, solvency requires that $R(s^{t+1})A(s^t) \leq r(s^{t+1})k(s^t)$ for all $s^{t+1}|s^t$. This, together with the fact that $k(s^t) \leq A(s^t)$, implies that $R(s^{t+1}) \leq r(s^{t+1})$ for all $s^{t+1}|s^t$. Finally, combining the last condition with (19), it becomes clear that the only possible equilibrium is to have $R(s^{t+1}) = r(s^{t+1})$ at all $s^{t+1}|s^t$ and $k(s^t) = A(s^t) = K(s^t)$ at all s^t. Clearly, this implies that all the intermediaries make zero profits.

While there are many symmetric equilibria, we consider the one that has the loosest possible limits or limits that are *not too tight*, in the sense that some agent would default

[4] This assumption is necessary to guarantee the existence of symmetric equilibria where all intermediaries hold the same portfolio.

under some continuation state if the limit was loosened. Formally, the limits $\underline{a_i}(s^t)$ have to satisfy:

$$W(\epsilon_i(s^t), \underline{a_i}(s^t); S(s^t)) = V(\epsilon_i(s^t), \underline{a_i}(s^t); S(s^t))$$

where $S(s^t) = ((K(s^t), \underline{a_i}(s^t), \epsilon_i(s^t)), z(s^t))$ and the contract and autarky values W and V are defined as in the previous section. As we have just shown, agents face the same asset return on physical capital as in the previous section. Given this, we are also entitled to use the same value functions. The following proposition shows that there does not exist any symmetric equilibrium with limits that allow for default. Further, it shows that there do not exist profitable deviations from the symmetric equilibrium with limits that are not too tight.

Proposition 3.1 *Under the previous assumptions, a symmetric equilibrium with default does not exist. Further, there do not exist profitable deviations from the symmetric equilibrium with limits that are not too tight.*

Proof. We first show that symmetric equilibria with default cannot exist. To do this, assume first that the limits are such that, at a given current state $\widetilde{s}$, there exists at least a future state $\widehat{s}$ (or more generally a set of states) where agents of the first type will default. We now show that this equilibrium cannot exist, since the intermediaries will be able to make positive profits by not "lending" to the types with positive default probabilities. To see this, note first that the following condition has to hold in equilibrium:

$$\sum_{s^{t+1}|\widetilde{s}} \pi(s^{t+1}|\widetilde{s}) r(s^{t+1}) \leq \sum_{s^{t+1}|\widetilde{s}} \pi(s^{t+1}|\widetilde{s}) R(s^{t+1}) - \pi(\widehat{s}|\widetilde{s}) R(\widehat{s}) \frac{a_1(\widetilde{s})}{A(\widetilde{s})} \tag{20}$$

where $a_1(\widetilde{s})$ are the asset holdings of the type one agents and $A(\widetilde{s})$ are the total funds collected by the intermediaries. Further, r and R represent the prices in the symmetric equilibrium with default. Note that, if the previous condition was not satisfied, the intermediaries could set $k(s^t) = A(s^t)$ and make arbitrarily large profits, since their cash flows at $s^{t+1}|\widetilde{s}$ would then be given by:

$$\sum_{s^{t+1} \neq \widehat{s}|\widetilde{s}} \pi(s^{t+1}|\widetilde{s}) \left[r(s^{t+1}) - R(s^{t+1}) \right] A(\widetilde{s}) + \pi(\widehat{s}|\widetilde{s}) r(\widehat{s}) A(\widetilde{s}) - \pi(\widehat{s}|\widetilde{s}) R(\widehat{s}) a_2(\widetilde{s}) > 0$$

Second, since $k(s^t) \leq A(s^t)$ at all nodes and the intermediaries have to be solvent at every possible contingency, the only equilibrium with $A(s^t) > 0$ is given when $k(s^t) = A(s^t)$ and (20) is satisfied with equality. In this case, the intermediaries make zero profits. On the other hand, since the agents of type one do not want to default unless $a_1(\widetilde{s}) < 0$, it also follows that:

$$\sum_{s^{t+1}|\widetilde{s}} \pi(s^{t+1}|\widetilde{s}) r(s^{t+1}) > \sum_{s^{t+1}|\widetilde{s}} \pi(s^{t+1}|\widetilde{s}) R(s^{t+1}) \tag{21}$$

As we see, condition (21) implies that an intermediary could make positive profits under the current prices by only accepting deposits from the type two agents and by not lending to the type one agents, i.e., by setting $a_1(\widetilde{s}) = 0$. Under this profitable deviation, any intermediary offering the original contract would be driven out of the market. Further, there would not be any lending to the type one agents, contradicting the existence of an equilibrium with default of the agents of type one.

We now consider a symmetric equilibrium with the loosest possible limits avoiding default. Further, we show that no intermediary can achieve positive or zero profits by loosening these limits. To do this, assume that the participation constraint is binding for the agents of type one at some contingency $\widetilde{s}$, implying that $W(\epsilon_1(\widetilde{s}), a_1(\widetilde{s}); S(\widetilde{s})) = V(\epsilon_1(\widetilde{s}), a_1(\widetilde{s}); S(\widetilde{s}))$. We now show that no intermediary can build a portfolio $\overline{A}(\widetilde{s}) = \overline{a}_1(\widetilde{s}) + \overline{a}_2(\widetilde{s}) > 0$ which gives him at least zero profits and which involves more lending $\overline{a}_1(\widetilde{s}) < a_1(\widetilde{s})$ to the agents of type one. To see this, note first that the intermediaries need to satisfy the following condition to be solvent at each $s^{t+1}|\widetilde{s}$:

$$\overline{R}(s^{t+1})\overline{A}(\widetilde{s}) \leq r(s^{t+1})\overline{k}(\widetilde{s})$$

where $\overline{R}(s^{t+1})$ is the return offered by the intermediary at $s^{t+1}|\widetilde{s}$ and $\overline{k}(\widetilde{s})$ is the total capital rented to the firm at $\widetilde{s}$. Second, since $\overline{k}(\widetilde{s}) \leq \overline{A}(\widetilde{s})$, it has to be the case that $\overline{R}(s^{t+1}) \leq r(s^{t+1}) = R(s^{t+1})$ for all $s^{t+1}|\widetilde{s}$, where the last equality follows from the fact that R and r satisfy the conditions for a symmetric equilibrium with no default discussed above. Finally, note that the agents of type one will default at some contingency $\widehat{s}|\widetilde{s}$ with $\pi(\widehat{s}|\widetilde{s}) > 0$. Given this, it also follows that $\overline{R}(\widehat{s}) < r(\widehat{s}) = R(\widehat{s})$ for some $\widehat{s}|\widetilde{s}$ with $\pi(\widehat{s}|\widetilde{s}) > 0$, implying that this is a strictly worse deal for the type two agents. Thus, the intermediary will not be able to to build this portfolio. To see that the last condition has to hold, note that the intermediary has at most $r(\widehat{s})\overline{A}(\widetilde{s})$ goods available at node $\widehat{s}|\widetilde{s}$. On the other hand, he has to pay out $\overline{R}(\widehat{s}|\widetilde{s})\left(\overline{A}(\widetilde{s}) - \overline{a}_1(\widetilde{s})\right)$ to the type two agents. Since the type one agents will only default if $\overline{a}_1(\widetilde{s}) < 0$, however, this can only be done if $\overline{R}(\widehat{s}) < r(\widehat{s}) = R(\widehat{s})$ at $\widehat{s}|\widetilde{s}$. Alternatively, the intermediary could decrease $\overline{R}(\widehat{s})$ so as to make the type one agents not want to default any more, but this would directly imply that $\overline{R}(\widehat{s}) < r(\widehat{s}) = R(\widehat{s})$ at $\widehat{s}|\widetilde{s}$, and the type two agents would again not be willing to accept the deal.■

4. The Complete Markets Model

The present section discusses the model when markets are complete. To implement this allocation, we assume that households can trade in a complete set of one period ahead state contingent claims (or Arrow securities) through financial intermediaries that set the trading limits. Under these assumptions, the constraints faced by the households are given by:

$$c_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)a_i(s^{t+1}) \leq \omega_i(s^t) \tag{22}$$

$$\omega_i(s^{t+1}) = w(s^{t+1})\epsilon_i(s^{t+1}) + a_i(s^{t+1}) \tag{23}$$

$$a_i(s^{t+1}) \geq \underline{a}_i(s^{t+1}) \tag{24}$$

where $a_i(s^{t+1})$ represents the amount of state contingent claims chosen by household $i \in I$ at the end of period t, with $a_i(s^0)$ given. Further, $q(s^{t+1}|s^t)$ denotes the price of one unit of consumption good delivered at $t+1$ contingent on the realization $s^{t+1}|s^t$. As before, we consider the loosest possible limits on the Arrow security holdings that satisfy the enforcement constraints. Thus, we assume that the limits in (24) are *not too tight*, in the sense that they satisfy the following condition:

$$W^{CM}(\epsilon_i(s^t), \underline{a}_i(s^t); S(s^t)) = V^{CM}(\epsilon_i(s^t), \underline{a}_i(s^t); S(s^t)) \tag{25}$$

where $S(s^t) = (K(s^t), \epsilon_i(s^t), \underline{a}_i(s^t)), z(s^t))$ and $W^{CM}$and V^{CM} represent the contract and autarky value functions under complete markets, defined similarly to the the value functions in the incomplete market economies of section 2.

Each period, after observing the realization of the productivity shock, the firm rents labor from the households and physical capital from the intermediaries to maximize the period profits. Further, the intermediary purchases capital $k(s^{t+1})$ and rents it to the firm, earning a rental revenue of $r(s^{t+1})k(s^{t+1})$ and a liquidation value of $(1-\delta)k(s^{t+1})$ in the following period. To finance the capital purchases, the intermediary sells the future consumption goods in the spot market for one period ahead contingent claims. Given this, the zero profit condition implied by the presence of perfect competition in the intermediary sector requires that:

$$1 = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)[r(s^{t+1}) + (1-\delta)] \tag{26}$$

Further, for the state contingent debt issued by the intermediary to match the demand from the households it must be the case that:

$$\sum_i a_i(s^{t+1}) = [r(s^{t+1}) + (1-\delta)]k(s^{t+1}) \tag{27}$$

Finally, the resource constraint of this economy is given by:

$$\sum_{i \in I} c_i(s^t) + K(s^{t+1}) = F(z(s^t), K(s^t), L(s^t)) + (1-\delta)K(s^t) \tag{28}$$

Consider an allocation satisfying (22)-(28). In spite of the fact that markets are complete, it is important to note that it is not constrained efficient. To see why this is the case, note that the constrained efficient allocations of the previous economy can be calculated by solving the following central planning problem:

$$Max_{\{c_i, K\}} \sum_{i \in I} \alpha_i \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t)\beta^t u\left(c_i(s^t)\right) \text{ s.t.}$$

$$\sum_{i \in I} c_i(s^t) + K(s^{t+1}) = F(z(s^t), K(s^t), L(s^t)) + (1-\delta)K(s^t) \tag{29}$$

$$W^{CM}(\epsilon_i(s^t); K(s^t), z(s^t)) \geq V^{CM}(\epsilon_i(s^t); K(s^t), z(s^t)) \tag{30}$$

where α_i is the initial Pareto weight assigned by the planner to each household. Clearly, standard dynamic programming is inapplicable to the present framework, since future decision variables appear in the enforcement constraints. On the other hand, following Marcet and Marimon (1999), we can expand the state space with a pseudo state variable to formulate the problem recursively. In particular, if $\beta^t \gamma_i(s^t)$ is the Lagrange multiplier of the time t participation constraint of household i, and $\mu_i(s^{t-1})$ denotes the period t pseudo state variable of household i, the previous optimization problem can be transformed into the following one:

$$\inf_{\{\gamma_i\}} \sup_{\{c_i, K\}} H \equiv \sum_{i\in I} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t)\beta^t \left\{ \frac{c_i(s^t)^{1-\sigma}}{1-\sigma}(\mu_i(s^t) + \alpha_i) - \gamma_i(s^t) V_i^{CM}(s^t) \right\}$$

subject to the resource constraint in (29) and to the law of motion of $\mu_i(s^t)$, defined recursively by:

$$\mu_i(s^t) = \mu_i(s^{t-1}) + \gamma_i(s^t),\ \mu_i(s^{-1}) = 0 \tag{31}$$

It is easy to see that the solution to the previous problem can be characterized by equations (29)-(31) and by the following first order conditions:

$$\frac{c_1(s^t)^{-\sigma}}{c_2(s^t)^{-\sigma}} = \lambda(s^t) = \frac{(1 + v_2(s^t))}{(1 + v_1(s^t))} \lambda(s^{t-1}) \tag{32}$$

$$\begin{aligned} 1 \;=\; & \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{c_i(s^{t+1})^{-\sigma}}{c_i(s^t)^{-\sigma}} (1 + v_i(s^{t+1}))[\alpha z(s^{t+1}) K(s^{t+1})^{\alpha-1} + (1-\delta)] \right\} \\ & -\beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{v_i(s^{t+1})}{c_i(s^t)^{-\sigma}} V_i^{CM\prime}(s^{t+1}) + \frac{v_{-i}(s^{t+1})}{c_{-i}(s^t)^{-\sigma}} V_{-i}^{CM\prime}(s^{t+1}) \right\} \end{aligned} \tag{33}$$

where the normalized multipliers v_i and λ are given by $v_i(s^t) = \frac{\gamma_i(s^t)}{\mu_i(s^{t-1})+\alpha_i}$ and $\lambda(s^t) = \frac{\mu_2(s^t)+\alpha_2}{\mu_1(s^t)+\alpha_1}$, with $\lambda(s^{-1}) = \frac{\alpha_2}{\alpha_1}$. Further, the terms $V_i^{CM\prime}(s^{t+1})$ for $i = 1, 2$ on the right hand of the previous equation represent the change in the autarky values due to a change in the aggregate capital stock.

As shown by Abraham and Carceles-Poveda (2004), the presence of the autarky effects in equation (33) implies that the optimal central planning problem cannot be decentralized as above without imposing an upper bound on capital holdings[5]. Further, the authors show that an allocation that satisfies (29)-(33) without the autarky effects in (33) also satisfies equations (22)-(28) and viceversa. In addition, they show that the autarky effects are quantitatively unimportant, and that the intermediaries cannot make profitable deviations by

[5] See also Kehoe and Perri (2004) for a formal proof of this statement in a related model.

further loosening the limits that are not too tight. Given this, and the fact that we do not impose any upper bounds on the capital holdings in the incomplete market economies, we will analyze the allocation that satisfies (29)-(32) and the following Euler condition:

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{c_i(s^{t+1})^{-\sigma}}{c_i(s^t)^{-\sigma}} (1 + v_i(s^{t+1}))[\alpha z(s^{t+1}) K(s^{t+1})^{\alpha-1} + (1-\delta)] \right\} \tag{34}$$

5. Solution Method

The present section provides a detailed description of the algorithm used to solve for the equilibrium allocations of the previous economies. To find the solution to the incomplete markets economy with endogenous trading limits, we use a policy function iteration algorithm that is modified to include an endogenous and not rectangular grid for the states. It is important to note that the system of equations (9)-(15) could also be solved with a value function iteration algorithm. This solution, however, would involve an extended state space including the variables that agents regard as beyond their control as well as iterations on Γ to satisfy the consistency requirements. Since the method of policy function iterations imposes the key equilibrium conditions and uses directly the reduced policy and value functions, we opt for using using policy iterations.

The solution method is implemented as follows. First, we define a grid on the endogenous state space, given by the individual level of capital k and by the aggregate capital stock K. Note that the grid on the exogenous state space (ϵ, z) is implicitly defined by our Markov assumption. Second, we fix some initial limits $\underline{k}(\epsilon; K, z)$. Given the grid on k, K and (ϵ, z), our procedure finds then continuous equilibrium policy functions for the individual consumptions $c = c(\epsilon, k; K, z)$, the individual asset holdings k'=k$(\epsilon, k; K, z)$, and the law of motion for the aggregate capital stock $K' = K(\epsilon, k; K, z)$, such that all the conditions of the recursive competitive equilibrium defined earlier are satisfied for the given set of limits, $\underline{k}(\epsilon; K, z)$. Here, we have to make sure that the following Euler equation coming from the household's optimization problem is satisfied:

$$u'(c) \geq \beta \sum_{(\epsilon', z')} \Pi(\epsilon', z'|\epsilon, z) u'(c') \left[f_K(z', K', L') + 1 - \delta \right].$$

As usual, the above condition holds with equality only if the trading constraint is not binding, i.e., only if $k' > \underline{k}(\epsilon; K, z)$ for all $(\epsilon', z') | (\epsilon, z)$ such that $\Pi(\epsilon', z'|\epsilon, z) > 0$. Third, using the equilibrium policy functions, the value functions $W = W(\epsilon, k; K, z)$ and $V = V(\epsilon, k; K, z)$ are calculated recursively. Finally, the endogenous limits are loosened if $W(\epsilon, \underline{k}(\epsilon; K, z); K, z) > W(\epsilon, \underline{k}(\epsilon; K, z); K, z)$ and they will be tightened otherwise. All the previous objects are approximated with continuous functions using linear interpolation over the finite and endogenous grid. The previous procedure is repeated until convergence.

More precisely, given a set of endogenous limits $\underline{k}$, let h be the vector consisting of the policy functions of interest, i.e., $h = [c, k', K']$. Further, let T be a non-linear operator such that $T[h\ W\ V\ \underline{k}\ ]$ satisfies the equilibrium system of equations and the participation constraints determining the limits. The solution to our problem is then a fixed point of T, i.e., a vector $[h\ W\ V\ \underline{k}]$ such that $[h\ W\ V\ \underline{k}] = T[h\ W\ V\ \underline{k}]$. To approximate the fixed point, we follow the steps below.

Step 1: Guess an initial vector $[h_0\ W_0\ V_0\ \underline{k}_0]$, where $h_0 = [c_0, k_0', K_0']$

Step 2: For each iteration $n \geq 1$, use the previous guess $[h_{n-1}\ W_{n-1}\ V_{n-1}]$ and $\underline{k}_{n-1}$ to compute the new vector $[h_n\ W_n\ V_n]$ that satisfies the equilibrium conditions. Further, use the new vector $[h_n\ W_n\ V_n]$ to find the new lower bound $\underline{k}_n$ such that $W_n(\epsilon, \underline{k}_n; K_n, z) \approx V_n(\epsilon, \underline{k}_n; K_n, z)$ and update the grid accordingly.

Step 3: Use $[h_n\ W_n\ V_n]$ and $\underline{k}_n$ as the next initial guess and iterate until $[h_n\ W_n\ V_n\ \underline{k}_n]$ converges.

As reflected by the previous algorithm, solving the different incomplete market models with endogenous trading limits involves taking into account several computational difficulties. First, our state space endogenous. We address this problem by incorporating an additional fixed point problem to the one used to find the different policy functions in order to find the state-dependent limits on individual capital holdings. This also implies, that our policy functions have to be calculated over a non-rectangular grid. Second, given the typical nonlinearities of the policy and value functions around the limits, and given that the limits in our model are endogenously determined at the level where the value function from staying in the contract is at least as large as the autarky value, it becomes clear that a good approximation of the value functions close to the limits is needed to obtain reliable results. To address this issue, we use a relatively high number of grid points and allow the limits take values between grid points as well..

Finally, to solve for the complete markets and the incomplete markets allocations with ad hoc limits described earlier, we use a simpler version of the algorithm described above. In particular, if the limits are fixed, we iterate on the vector $[h\ W\ V]$, where $h = [c, k', K']$. Note that this considerably simplifies our computations, since we do not have to iterate on the ad hoc trading limits. Further, if markets are complete, the state space is given by $(\lambda, \epsilon; z, K)$ and we iterate on the vector $[h\ W\ V]$, where $h = [c, \lambda', K']$.

6. Quantitative Results

The present section discusses the quantitative results obtained for the benchmark model, which is calibrated following the asset pricing and real business cycle literature. The time period is assumed to be one quarter, and the discount factor and depreciation rate are therefore set to $\beta = 0.99$ and $\delta = 0.025$. Concerning the functional forms, we assume that the production function is Cobb Douglas, with a constant capital share of $\alpha = 0.36$. Further, the utility function of the households is assumed to be of the constant relative risk aversion class, with a risk aversion parameter of $\gamma = 1$. Finally, the exogenous shock processes are assumed to be independent. In particular, the aggregate technology shock follows a two state Markov chain with a high persistence. Further, the idiosyncratic shock process is calibrated from labor income data and it is assumed to follow a seven state Markov chain that is negatively correlated across the two types, as usual in the literature with a finite number of agents.

Incomplete Markets. Before presenting our numerical results, we provide an approximate analytical characterization of the behavior of the equilibrium endogenous limits using the reduced form value functions. To do this, we assume a lognormal process for the aggregate shock z and a truncated lognormal process for the idiosyncratic shock of the first type. Note that this will allow us to express the differential effect of a change in the shocks on the limits. Along the section, we denote the period t state vector of a household by $s_t = (k, \epsilon, z, K)$ and the period t policy functions by $k' = g^k(s_t)$, $K = g^K(s_t)$ and $c = g^c(s_t)$. Further, we let $s_{t+1} = \left(g^k(s_t), \rho_\epsilon \epsilon + \varepsilon_{\epsilon t+1}, \rho_z z + \varepsilon_{zt+1}, g^K(s_t)\right)$. Using this notation, the period t contract and autarky functions can be written as:

$$W(s_t) = u(g^c(s_t)) + \beta \int_{\varepsilon_\epsilon} \int_{\varepsilon_z} W(s_{t+1}) d\varepsilon_\epsilon d\varepsilon_z$$

$$V(s_t) = u(w(K, z)\epsilon) + \beta \int_{\varepsilon_\epsilon} \int_{\varepsilon_z} V(s_{t+1}) d\varepsilon_\epsilon d\varepsilon_z$$

As shown earlier, the endogenous trading restriction is determined by the following condition:

$$W(\underline{k}(\epsilon, z, K), \epsilon, z, K) = V(\underline{k}(\epsilon, z, K), \epsilon, z, K)$$

Further, we can differentiate the previous equation to express the effects of a change in K, ϵ or z on the trading limits as follows:

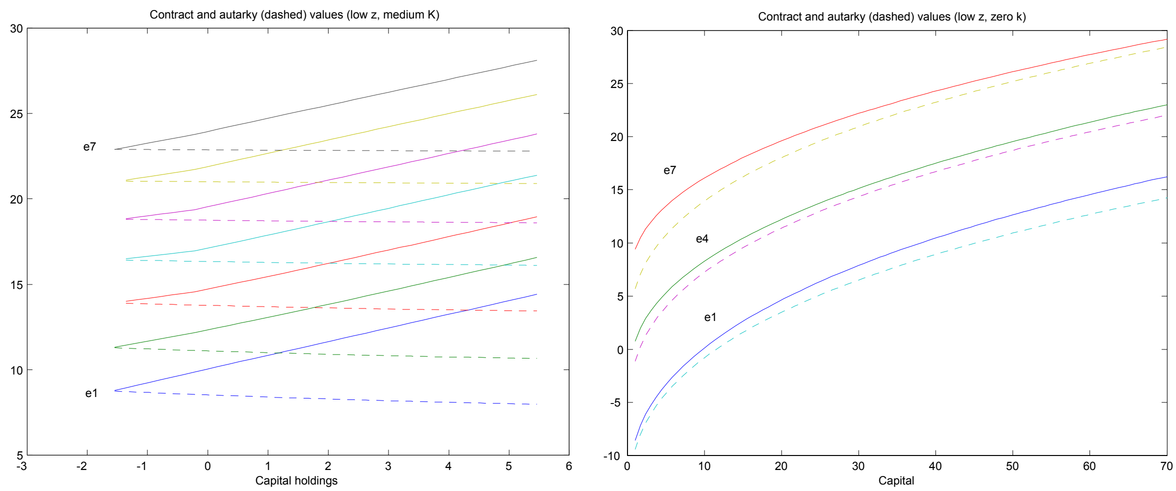
$$\frac{\partial \underline{k}(\epsilon, z, K)}{\partial K} = -\frac{W_K(k, \epsilon; z, K) - V_K(k, \epsilon; z, K)}{W_k(k, \epsilon; z, K) - V_k(k, \epsilon; z, K)} \tag{35}$$

$$\frac{\partial \underline{k}(\epsilon, z, K)}{\partial \epsilon} = -\frac{W_\epsilon(k, \epsilon; z, K) - V_\epsilon(k, \epsilon; z, K)}{W_k(k, \epsilon; z, K) - V_k(k, \epsilon; z, K)} \tag{36}$$

$$\frac{\partial \underline{k}(\epsilon, z, K)}{\partial z} = -\frac{W_z(k, \epsilon; z, K) - V_z(k, \epsilon; z, K)}{W_k(k, \epsilon; z, K) - V_k(k, \epsilon; z, K)} \tag{37}$$

Looking at the previous equations, we see that the sign of the limit derivatives crucially depends on the sign of $W_j(k, \epsilon; z, K) - V_j(k, \epsilon; z, K)$, where $V_j(k, \epsilon; z, K)$ and $W_j(k, \epsilon; z, K)$ represent the derivatives of the autarky and contract values with respect to the corresponding state variable $j \in \{k, \epsilon, z, K\}$. The two value functions are displayed below for different values of the state vector[6].

Figure 1: Contract and autarky values



The left hand-side figure depicts the value functions for a medium level of capital inside the stationary distribution and for different values of ϵ and k. As we see, for a given state vector $(\epsilon; z, K)$, the contract value increases in k. We therefore have that $W_k(k, \epsilon; z, K) > 0$. In addition, the autarky value decreases in k for negative values of the initial capital holdings, implying that $V_k(k, \epsilon; z, K) < 0$ if $k < 0$. Given this, there exists a unique endogenous limit $\underline{k}(\epsilon; z, K)$ satisfying $W(\underline{k}(\epsilon; z, K), \epsilon; z, K) = V(\underline{k}(\epsilon; z, K), \epsilon; z, K)$, as reflected on the graph. Further, the denominator of equations (35)-(37) is clearly positive.

The right hand-side figure depicts the contract and autarky values for zero initial asset holdings and for different values of K and ϵ. As we see, both the autarky and contract values are increasing in the two state variables. In addition, results not depicted here show that both functions are increasing in the aggregate technology shock z. Given this, the sign of the limit derivatives with respect to $j \in \{\epsilon, z, K\}$ entirely depends on the relative magnitude of the effects of a change in the corresponding state variable on the two value functions.

To get some intuition for the previous findings, we can first express the derivatives of the value functions with respect to k as infinite sums. As shown in the appendix, they are given

[6] Unless otherwise specified, all the graphs along the section set the technology shock to the low value, but similar graphs can be obtained for the high value of the shock.

by:

$$\begin{aligned} W_k(s_t) &= u'(c_t)(1 + r(K, z) - \delta) + \\ &+ E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1}) \\ &+ E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1}) \end{aligned} \tag{38}$$

$$V_k(s_t) = E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \epsilon_\tau u'(c_\tau^{au}) \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1}) \tag{39}$$

where $c_\tau^{au} = \epsilon_\tau w(K_\tau, z_\tau)$ and $c_\tau = \epsilon_\tau w(K_\tau, z_\tau) + (r(K_\tau, z_\tau) + 1 - \delta)k_\tau - k_{\tau+1}$ represent the consumptions in autarky and the contract respectively. Further, the terms $G^k(s_{\tau-1})$ capture the second order effects of a change in k through its effect on the evolution of the aggregate capital stock. As shown in the Appendix, these terms consist of products of the policy function derivatives g_k^k, g_K^k, g_k^K and g_K^K, which have the following sign according to our numerical results. First, $g_K^K > 0$ due to the fact that a higher initial aggregate capital leads to a higher aggregate investment. Second, $g_k^K < 0$ for low levels of k due to the following general equilibrium effect. Given a relatively low level of initial asset holdings, a further decrease in the asset holdings brings the household closer to the borrowing limit. In this case, the aggregate capital, representing the implicit price of a perfectly divisible capital share, has to increase to clear the markets[7]. Third, $g_k^k > 0$ due to the fact that a higher level of initial asset holdings increases the individual wealth and leads to a higher individual investment. Finally, as we will see later, we have that $g_K^k > 0$ for very low levels of the aggregate capital stock and $g_K^k < 0$ otherwise.

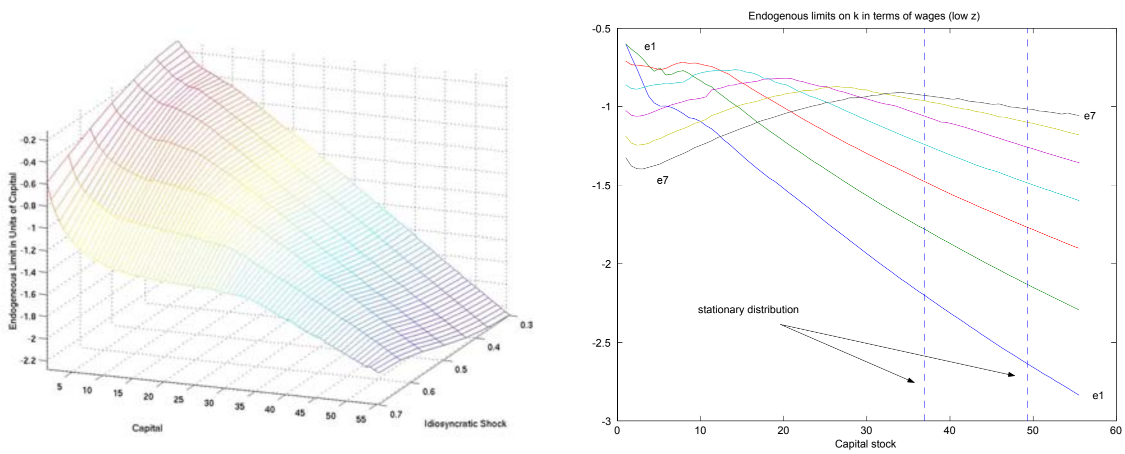
Assume now that a household is close to the limit, implying that k is negative and that it will stay negative for a while. In this case, the definition of $G^k(s_{\tau-1})$ in the Appendix implies that we expect the elements of the sequence $\left\{G^k(s_{\tau-1})\right\}_{\tau=t+1}^{\infty}$ to be negative for a medium level of capital inside the stationary distribution. Further, since the wage rate is increasing in capital, this clearly implies that $V_k(k, \epsilon; z, K) < 0$, confirming the findings above. Intuitively, an increase in the individual capital holdings leads to a lower aggregate capital stock and to higher future initial asset holdings. In addition, these two effects translate into a lower future aggregate capital and a lower aggregate wage rate, which also decreases the future autarky value. On the other hand, equation (38) reflects that an increase in k has three effects on the contract value. The first is a direct and positive effect due to the fact that a higher k increases the current individual wealth. Further, the last two terms are second order

[7] Note that the equilibrium price of a share is equal to the present discounted value of its dividends. Thus, the price of a capital share is equal to the present discounted value of the aggregate capital income net of investment, which is also the equilibrium value of the aggregate capital stock.

negative effects due to a change in the factor prices through the change in K. As reflected by the previous graph, however, the first order positive effect dominates, and $W_k(k, \epsilon; z, K)$ is therefore positive.

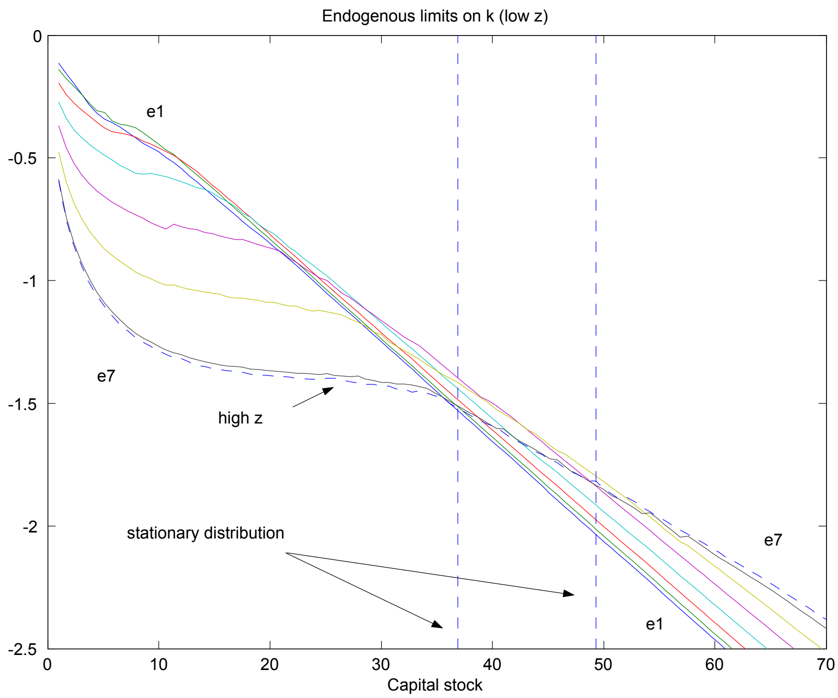
Using similar arguments, it is easy to show that $W_j > 0$ and $V_j > 0$ for $j \in \{\epsilon, z, K\}$. Thus, the signs of the limit derivatives with respect to $j \in \{\epsilon, z, K\}$ entirely depend on the signs of $W_j(k, \epsilon; z, K) - V_j(k, \epsilon; z, K)$. In particular, if the contract value increases by more than the autarky value after an increase in the relevant state variable, a household will not have an incentive to default, and his limit will become looser. On the other hand, if the autarky value increases by more than the contract value, his limit will become tighter. The endogenous limits $\underline{k}(\epsilon; z, K)$ on the total amount of borrowing are depicted below for different values of the state vector.

Figure 2: Endogenous limits on total borrowing



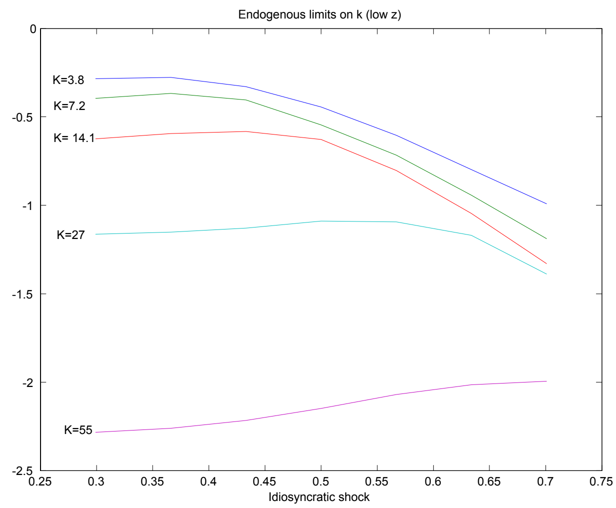
The left hand side figure displays the value of the limits as a function of K and ϵ, while the right hand side figure depicts their value in terms of the individual labor incomes, that is, $\underline{k}(\epsilon; z, K)/(w(z, K)\epsilon)$. Several important facts are worth noting. First, the endogenous trading limits on the total amount of borrowing do vary to a great extent with the aggregate state of the economy. As we see on the right hand-side figure, households can borrow approximately between 90 and 250 percent of their individual labor incomes in the stationary distribution. Thus, the endogenous limits on physical capital that prevent default in equilibrium are significantly different from the ad hoc zero limit usually assumed in the incomplete markets literature. Second, the left hand-side figure reflects that the limits become looser with the aggregate capital stock, while they are not monotone in the idiosyncratic shock. This can also be seen on the following graphs, representing the two dimensional cuts of the previous left hand side figure.

Figure 3: Endogenous limits on total borrowing



The previous figure depicts the limits on the total amount of borrowing for different values of K and ϵ. Further, the dashed line at the bottom of the graph represents the limits for the high income and technology shocks. As we see, the limits become looser with a higher aggregate capital stock for all levels of the idiosyncratic shock. In addition, they are not monotone in the aggregate and idiosyncratic shocks. In particular, when capital is scarce and its return is relatively high, the limits are looser for the household with the highest income shock (or for the highest aggregate shock), while the reverse is true when capital is high and its return is relatively low. This can also be seen on the following figure, depicting the limits on total borrowing as a function of the idiosyncratic shock for different values of the aggregate capital stock.

Figure 4: Endogenous limits on total borrowing



As stated earlier, the figure reflects that the limit is getting looser (tighter) with a higher income shock when capital is low (high). Further, we see hump shape for medium levels of capital, since the limits become tighter for low to medium values of the idiosyncratic shock, while they become looser for higher shock values. To get more intuition for these findings, we can look at the expressions for the differences of the contract and autarky value derivatives with respect to the aggregate capital stock and the two shocks. As shown in the appendix, these are given by:

$$
\begin{aligned}
W_K(s_t) - V_K(s_t) &= E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^K(s_{\tau-1})(u'(c_\tau) - u'(c_\tau^{au})) \\
&\quad + E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u'(c_\tau) k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} G^K(s_{\tau-1})
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
W_\epsilon(s_t) - V_\epsilon(s_t) &= E_t \sum_{\tau=t}^{\infty} (\beta\rho_\epsilon)^{\tau-t}\, w(K,z)(u'(c_\tau) - u'(c_\tau^{au})) \\
&\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} G^\epsilon(s_{\tau-1}) \\
&\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} (u'(c_\tau) - u'(c_\tau^{au})) \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^\epsilon(s_{\tau-1})
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
W_z(s_t) - V_z(s_t) \quad = \quad & E_t \sum_{\tau=t}^{\infty} \left((\beta\rho_z)^{\tau-t} \, \epsilon_\tau \frac{\partial w(K,z)}{\partial z} (u'(c_\tau) - u'(c_\tau^{au})) \right) \\
& + E_t \sum_{\tau=t+1}^{\infty} (\beta\rho_z)^{\tau-t} \, u'(c_\tau) k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial z} \\
& + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} (u'(c_\tau) - u'(c_\tau^{au})) \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^z(s_{\tau-1}) \\
& + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(s_\tau) k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} G^z(s_{\tau-1})
\end{aligned}
\tag{42}
$$

As before, the terms G^j for $j \in \{\epsilon, z, K\}$ in the previous expressions capture the second order effects of a change in the relevant state variable through its effect on K. In addition, looking at their definition, it is easy to see that the terms $G^K(s_{\tau-1})$ and $G^z(s_{\tau-1})$ are expected to be positive and the terms $G^\epsilon(s_{\tau-1})$ are expected to be negative.

Consider first an increase in the aggregate capital stock. Looking at equation (40), we see that this has two different effects. First, it increases the present and future labor income in both the contract and autarky values. This is reflected by the first term on the right hand side of the previous equation, which is positive on average due to the fact $u'(c_\tau) - u'(c_\tau^{au}) > 0$ for a while. Note that the last inequality has to hold, since a household will only want to default in the presence of negative asset holdings, in which case it follows that $c_t < c_t^{au}$ and $c_\tau < c_\tau^{au}$ for a while due to the persistence of the income and consumption processes. Second, a higher aggregate capital generates a lower present and future rental price, reducing the asset liabilities and increasing the contract value above the autarky value. This is reflected by the second term on the right hand side of the equation, and the effect is also positive on average due to the fact that k_τ is negative for a while. Given this, we have that $W_K(s_t) - V_K(s_t) > 0$ and $\frac{\partial \underline{k}(\epsilon,z,K)}{\partial K} < 0$, as reflected by the previous graphs.

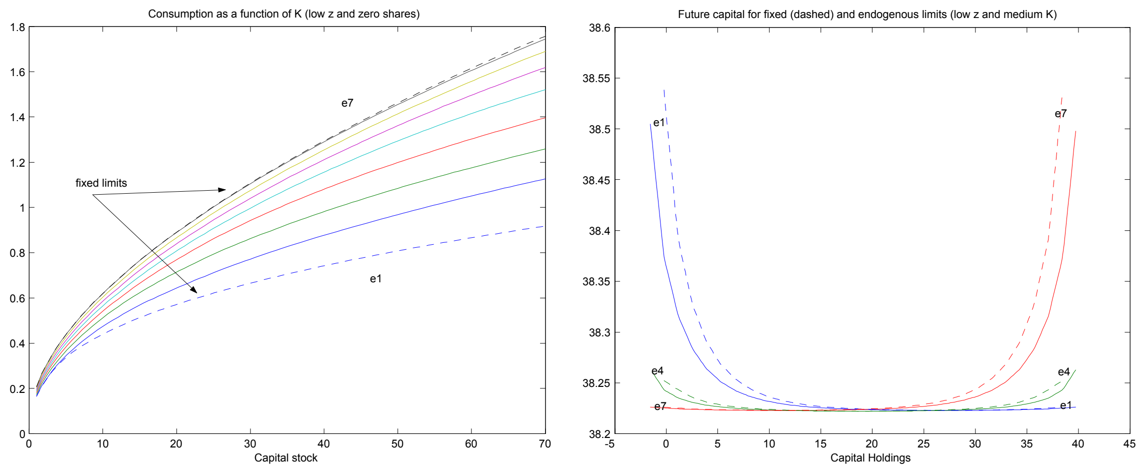
Consider now an increase in the labor income shock. As shown by equation (41), this has also several effects. On the one hand, a higher shock increases the labor income in the autarky value. This direct effect is reflected by the negative part of the first term on the right hand side in the equation, which we denote as Alvarez and Jermann (AJ) effect. Note that, under complete markets, a change in the labor income shock would leave the consumption in the contract unaffected, and this would imply that a household with a higher income shock would have a higher incentive to default and therefore a looser limit. On the other hand, a higher income shock also leads to a consumption change in the contract when markets are incomplete. This effect is reflected by the positive part of the first term on the right hand side of the equation, which we denote as incomplete markets (IM) effect. As reflected by the equation, the sum of these two direct effects is positive on average. Finally, a higher income shock leads to a change in the two factor prices through its effect on K. This is what we call the capital accumulation effect (CA), arising due to the presence of a production sector. In particular, a higher income shock leads to a lower aggregate capital and to higher interest payments that decrease the contract value. This negative effect is reflected by the second term on the right hand-side of equation (41). In addition, it also leads to lower wages in the contract and autarky values. This effect is represented by the third term on the right hand-side of (41), which is also negative on average.

Suppose first that the aggregate capital stock is very low. In this case, the relatively high return implies that every household wants to save. In other words, a change in the labor income shock has a very little effect on the aggregate capital distribution, and the G^{ϵ} terms multiplying the negative CA effects are therefore very small. Note also that consumption in the contract value is relatively low, leading to a high marginal utility and to a very high and positive IM effect that outweights the negative CA and AJ effects. Thus, if capital is very low, we should expect a household with a higher income shock to have a lower incentive to default and therefore a looser limit. On the other hand, the opposite situation would arise with a high level of capital. In this case, consumption would be high and the positive IM effect would therefore be outweighed by the negative AJ and CA effects. Clearly, this is a particularly appealing feature of the production economy with incomplete markets, since, at least for some stages of the transition to the stationary distribution, it can lead to the empirically more plausible case in which the borrowing constraints are looser for the high income agents. Finally, consider a medium level of capital inside the stationary distribution. As we will see later, the negative second order CA effects are very small when the idiosyncratic shock is relatively high. Thus, a household experiencing an increase of his labor income shock from a medium to a high level should have a lower incentive to default and a looser limit. On the other hand, since the CA effects are relatively high with a low labor income shock, the opposite should happen when the household experiences an increase in the labor income shock from a low to a medium level. This explains the hump shape of the limits that we observe on the previous figure for a medium level of capital.

Finally, consider an increase in the aggregate technology shock. As we see, equation (42) consists of four terms. The first term captures the direct effect of an increase in z, leading to a higher present and future labor income in both the contract and autarky. This effect is positive on average. The second term is negative on average, and it captures the direct effect of an increase in z on the rental price, which increases with the aggregate shock leading to higher asset liabilities in the contract. Finally, the third and fourth terms are second order effects. The first is positive on average, and it captures the fact that an increase in z leads to a higher aggregate capital and to higher future wages. The second is positive on average, and it captures the fact that an increase in z will lead to a higher capital and to lower present and future asset liabilities. Given this, an increase in z leads to positive second order effects and to positive and negative first order effects. Suppose now that capital is low. In this case, the second order positive effects are very high, and we expect $W_z(s_t) - V_z(s_t)$ to be positive. Further, the reverse behavior is expected when capital is very high. In other words, an increase in z will generate looser limits when capital is low and tighter limits when capital is high.

As stated earlier, the model with incomplete markets and endogenous limits will be compared to a model with exogenously fixed trading limits, which we set to zero for comparison with the existing literature. The policy functions for individual consumption and capital under fixed and endogenous limits are depicted below.

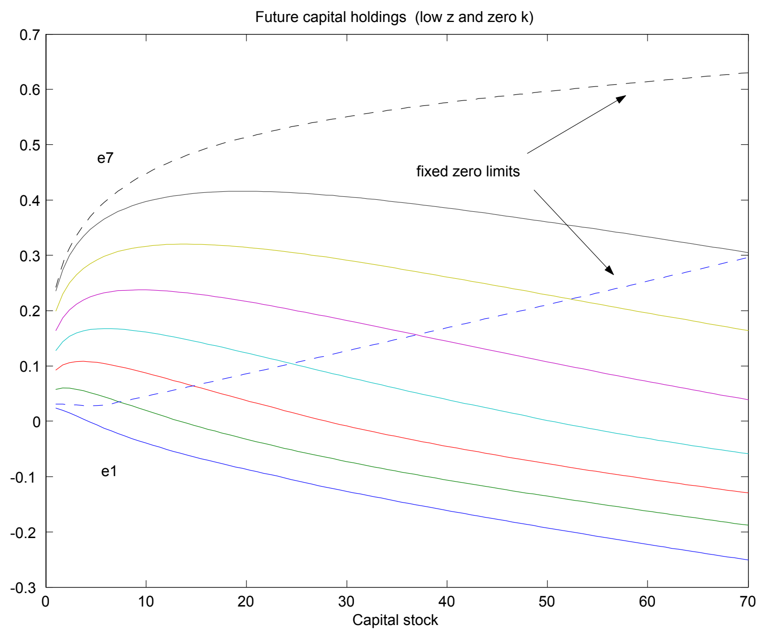
Figure 5: Policy Functions for consumption and capital



The left hand side figure depicts the individual consumption as a function of the aggregate capital stock for all levels of the idiosyncratic shock. Individual consumption is increasing in both the aggregate capital and the initial capital holdings, which are not shown here, since both variables lead to a higher level of individual wealth. Further, consumption inequality is higher under the tighter fixed zero limit, since households receiving a low income shock can borrow less in this case.

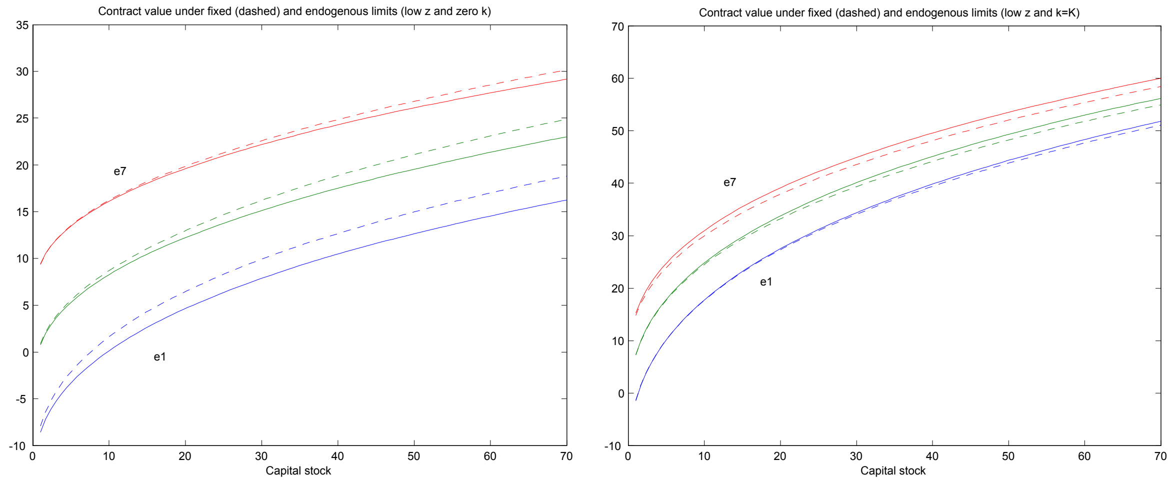
The right hand side figure depicts the policy functions for different values of the future capital holdings and the future capital stock. As we see, the figure reflects that a higher dispersion of capital holdings leads to a higher capital accumulation under endogenous and fixed zero limits. As stated earlier, if a household has a low income shock and a low level of initial shares, he will be close to the limit, and the aggregate capital (or implicit price of capital shares) will have to increase to clear the markets. On the other hand, we observe the same effect but of a smaller magnitude when the household has an intermediate income shock and a low level of initial capital holdings. Finally, if the household has a low income shock and a high level of initial holdings, his desire to borrow in order to smooth consumption is considerably smaller, and so is the desire to save of the symmetric household, implying that no one is closer to the limit. This is reflected by the rather linear shape of the capital policy function in this case. Note that this confirms the fact that $g_{\epsilon}^{K} < 0$ and $g_{k}^{K} < 0$. Further, the same pattern appears under fixed zero limits, but capital accumulation is higher in this latter case due to the fact that households have higher precautionary savings motives due to the fact that they are likely to become constrained earlier. The following figure reflects the future capital holdings under fixed and endogenous limits, where the differences between the two models become more apparent.

Figure 6: Policy Functions for capital holdings



As we see, the two models predict very different results concerning the future capital holdings. In particular, the model with fixed zero limits predicts that an increase in the aggregate capital stock leads to a higher level of future asset holdings. In particular, the high income shock households start increasing their savings as capital increases, while the low income shock households are constrained at the beginning but also start saving afterwards due to precautionary motives. In contrast to this, low income households start borrowing as capital increases when the limits are endogenous, while the higher income households save at the beginning but start decreasing their investments for moderate capital levels. As stated earlier, this implies that $g_K^k > 0$ when capital is low while $g_K^k < 0$ otherwise. Further, this is clearly due to the fact that an increase in the aggregate capital leads to a looser limit. In other words, households can borrow more as capital increases without getting closer to the limits when these are endogenous. For comparison, the level of welfare in the two economies is depicted below.

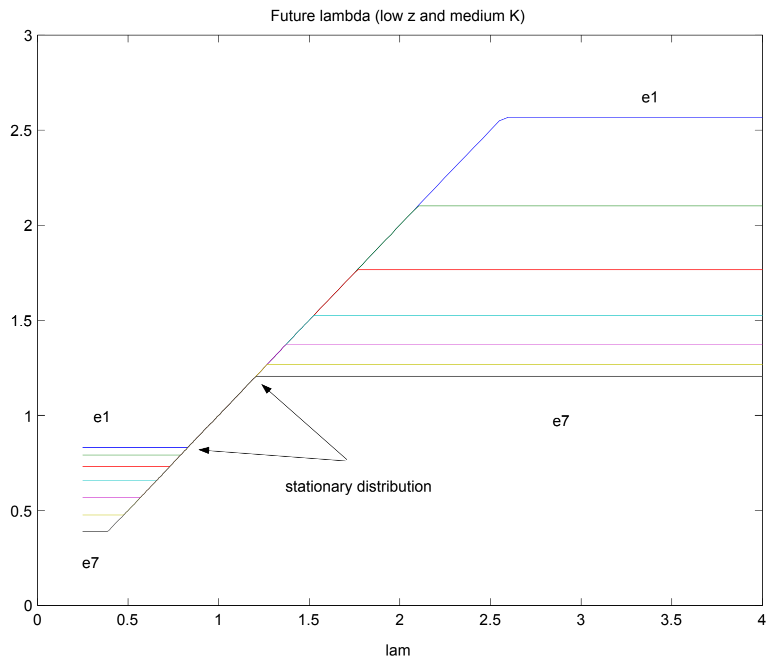
Figure 7: Welfare as a function of capital



The left hand side (right hand side) panel of the previous figure depicts the value function $W(k, \epsilon; z, K)$ of a household with zero capital holdings and different values of the idiosyncratic shock. As we see, the value under fixed zero limits is higher for a household with zero capital holdings in spite of the fact that risk sharing is lower in this case. Further, the value of the symmetric household owning the entire share and facing the opposite shock is lower under fixed zero limits. Thus, relaxing the limits from the fixed no-shortselling value to their endogenous value is not necessarily Pareto improving. While this seems to be counter-intuitive at first sight, note that it arises due to the general equilibrium effect discussed above, leading to a higher aggregate capital stock under fixed zero limits. In particular, we see that the effect is positive and stronger for the low wealth households, since they are the ones who rely more on labour income, which is increasing in aggregate capital. On the other hand, a higher capital stock, leading to a lower capital return, hurts the high wealth households, since they undertake most of the investment. Note that these considerations can have interesting implications for the optimal way of setting the endogenous limits, an issue that we leave for future research.

Complete Markets. As mentioned earlier, the incomplete markets allocation with endogenous limits is also compared to the case in which markets are complete but households cannot commit on the trading contracts. In this case, we use as a state space the vector $(\lambda, \epsilon; z, K)$, where λ corresponds to the ratio of marginal utilities of types one and two. The following graph depicts the policy function for λ' as a function the initial λ for a medium level of capital inside the stationary distribution and for different values of the idiosyncratic labor income shock.

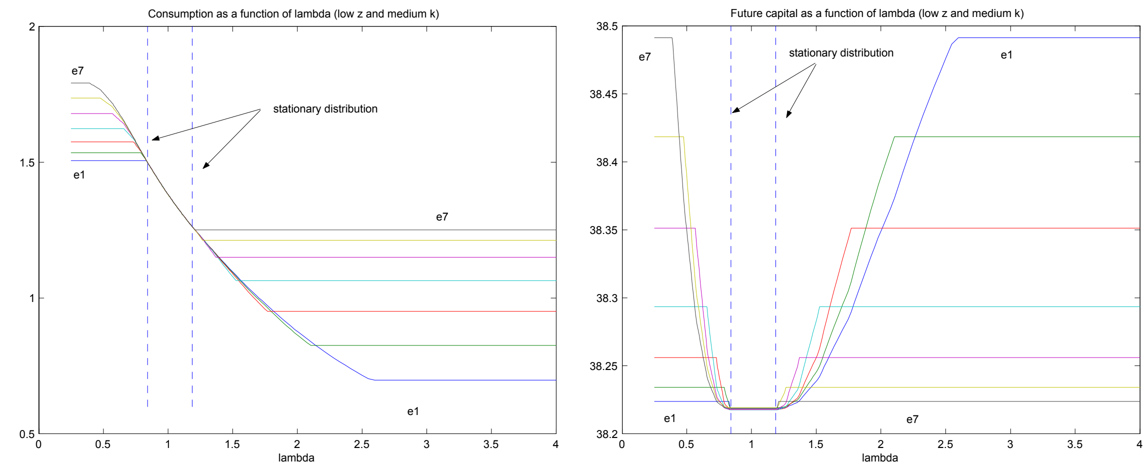
Figure 8: Future Lambda



To understand the graph, note that the initial value of λ is only be different from one if the initial wealth or the initial idiosyncratic shock differs across the two types. In addition, the multiplier stays constant and equal to one as long as the enforcement constraint (or the trading limit on Arrow securities) is not binding for any household. This is represented by the regions where $\lambda' = \lambda$. Note that, in this case, the ratio of marginal utilities does not change, and the allocation allows for full risk sharing as in the standard complete markets case. On the other hand, if the enforcement constraint (or limit on Arrow securities) becomes binding for a household, his consumption will have to increase to keep him in the contract. In this case, however, λ' becomes independent of the initial ratio λ, as reflected by the flat regions on the graph. Intuitively, the new λ' is the one that makes the agent indifferent between staying in the contract and defaulting, and this does not depend on the old λ.

As reflected by the figure, the regions with full risk sharing, implying that λ' coincides with the 45 degree line, vary with the level of the idiosyncratic shock. In particular, a household with a low idiosyncratic shock will not have an incentive to default except for relatively high levels of λ that are well above one, since his autarky value is relatively small if he breaks the contract. On the other hand, a household with a high idiosyncratic shock will want to default even for levels of λ that are not much above one. In this case, he will have to transfer income to the other household if he stays in the contract, while he will enjoy a relatively high autarky value upon default. Finally, the graph illustrates that a full risk sharing allocation will always result for any initial value of λ. In particular, if the initial value of λ is higher (lower) than the upper (lower) bound of the stationary distribution, it will always (more precisely almost surely) converge to the highest (lowest) point of the segment where $\lambda' = \lambda$ for all income levels. In addition, if the initial ratio is already inside the stationary distribution, it will stay there afterwards. The policy functions for consumption and capital are depicted below.

Figure 9: Policy functions for consumption and capital



As we see, the allocations are different from a full risk sharing allocation only outside the stationary distribution of λ. As shown above, however,the economy will be inside the stationary distribution in the long run. This is due to the fact that the economy with complete markets accumulates enough capital stock to outweigh the gains from default. Given this, it becomes clear that an incomplete markets framework is a more interesting case, since it gives empirically more plausible consumption patterns and borrowing limits.

APPENDIX

Appendix 1. The contract and autarky values can be expressed as:

$$W(s_t) = u(g^c(s_t)) + \beta \int_{\varepsilon_\epsilon} \int_{\varepsilon_z} W(s_{t+1}) d\varepsilon_\epsilon d\varepsilon_z$$

$$V(s_t) = u(w(K, z)\epsilon) + \beta \int_{\varepsilon_\epsilon} \int_{\varepsilon_z} V(s_{t+1}) d\varepsilon_\epsilon d\varepsilon_z$$

Differentiating the autarky value with respect to $j \in \{k, \epsilon, z, K\}$, the derivatives of the autarky value with respect to these variables can be expressed as an infinite sum as follows:

$$V_k(s_t) = E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \epsilon_\tau u'(c_\tau^{au}) \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1})$$

$$V_\epsilon(s_t) = E_t \sum_{\tau=t}^{\infty} (\beta\rho_\epsilon)^{\tau-t} w(K_\tau, z_\tau) u'(c_\tau^{au}) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \frac{\partial w(K_\tau, z_\tau)}{\partial K} u'(c_\tau^{au}) \epsilon_\tau G^\epsilon(s_{\tau-1})$$

$$V_z(s_t) = E_t \sum_{\tau=t}^{\infty} (\beta\rho_z)^{\tau-t} \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial z} u'(c_\tau^{au}) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} u'(c_\tau^{au}) G^z(s_{\tau-1})$$

$$V_K(s_t) = \epsilon_t \frac{\partial w(K_t, z_t)}{\partial K} u'(c_t^{au}) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} u'(c_\tau^{au}) G^K(s_{\tau-1})$$

where $c_\tau^{au} = \epsilon_\tau w(K_\tau, z_\tau)$ represents the consumption in autarky. To define $G^k(s_{\tau-1})$, we let Γ_τ^k be a set of sequences of indices of k or K with length $\tau - t + 1$ for every $\tau > t$. The first element of the sequence is always k, since we differentiate with respect to k, and the last one is K, since the long-lasting effects of a change in k are through the effect of K on wages. The set of all possible sequences for a given τ can then be defined as:

$$\Gamma_\tau^k = \{i_0, i_1, i_2, \ldots, i_{\tau-t} : i_j \in \{k, K\} \text{ for } \forall j\text{: } 0 < j < \tau - t,\ i_0 = k,\ i_{\tau-t} = K\}.$$

Given this, it follows that,

$$G^k(s_{\tau-1}) = \sum_{\mathbf{i}\in\Gamma_\tau^k} g_{i_0}^{i_1}(s_t) g_{i_1}^{i_2}(s_{t+1}) \ldots g_{i_{\tau-t-1}}^{i_{\tau-t}}(s_{\tau-1}).$$

To define $G^\epsilon(s_{\tau-1})$, we let $\Gamma_\tau^{\epsilon_i}$ be the set of sequences of indices of k, K and ϵ with length $\tau - t + 1$ for every $\tau > t$, such that the first element of the sequence is ϵ, since we differentiate with respect to ϵ, and the last one is K, since the long-lasting indirect effects are through the effect of K on wages. Furthermore, no ϵ_i in the sequence can follow any k or K. The set of all possible sequences for a given τ can then be defined as:

$$\begin{aligned} \Gamma_\tau^{\epsilon_i} &= \{i_0, i_1, i_2, \ldots, i_{\tau-t} : i_j \in \{k, K, \epsilon_i\} \text{ for } \forall j\text{: } 0 < j < \tau - t, \\ i_0 &= \epsilon_i, i_{\tau-t} = K \text{ and if } i_j \neq \epsilon_i \text{ then } \forall l > j\ i_l \neq \epsilon_i\}. \end{aligned}$$

Given this, we have that,

$$G^\epsilon(s_{\tau-1}) = \sum_{\mathbf{i}\in\Gamma_\tau^{\epsilon_i}} g_{i_0}^{i_1}(s_t) g_{i_1}^{i_2}(s_{t+1}) \ldots g_{i_{\tau-t-1}}^{i_{\tau-t}}(s_{\tau-1}).$$

Notice that the first l elements in the above product can be replaced by ρ_ϵ^l whenever $i_0 = i_1 = \ldots = i_l = \epsilon_i$. To define $G^z(s_{\tau-1})$, we let Γ_τ^z be the set of sequences of indices of k, K and z with length $\tau - t + 1$ for every $\tau > t$ such that the first element of the sequence is z, since we differentiate with respect to z, and the last one is K, since the long-lasting indirect effects are through the effect of K on wages. Further, no z in the sequence can follow any k or K. The set of all possible sequences for a given τ can then be defined as:

$$\begin{aligned} \Gamma_\tau^z &= \{i_0, i_1, i_2, \ldots, i_{\tau-t} : i_j \in \{k, K, z\} \text{ for } \forall j\text{: } 0 < j < \tau - t, \\ i_0 &= z, i_{\tau-t} = K \text{ and if } i_j \neq z \text{ then } \forall l > j\ i_l \neq z\}. \end{aligned}$$

Further,

$$G^z(s_{\tau-1}) = \sum_{\mathbf{i}\in\Gamma_\tau^z} g_{i_0}^{i_1}(s_t) g_{i_1}^{i_2}(s_{t+1}) \dots g_{i_{\tau-t-1}}^{i_{\tau-t}}(s_{\tau-1})..$$

Notice that the first l elements in the above product can be replaced by ρ_z^l whenever $i_0 = i_1 = \ldots = i_l = z$.

Finally, to define $G^K(s_{\tau-1})$ we let Γ_τ^K be the set of sequences of indices of k and K with length $\tau - t + 1$ for every $\tau > t$ such that the first element is K, since we differentiate with respect to K, and the last one is K, since all these long-lasting effects are through the effect of K on wages. The set of all possible sequences for a given τ can then be defined as:

$$\Gamma_\tau^K = \{i_0, i_1, i_2, \ldots, i_{\tau-t} : i_j \in \{k, K\} \text{ for } \forall j\text{: } 0 < j < \tau - t,\ i_0 = K,\ i_{\tau-t} = K\}.$$

Given this, we have that:

$$G^z(s_{\tau-1}) = \sum_{\mathbf{i}\in\Gamma_\tau^K} g_{i_0}^{i_1}(s_t) g_{i_1}^{i_2}(s_{t+1}) \dots g_{i_{\tau-t-1}}^{i_{\tau-t}}(s_{\tau-1}).$$

Differentiating the contract value with respect to $j \in \{k, \epsilon, z, K\}$, the derivatives of the autarky value with respect to these variables can be expressed as an infinite sum as follows:

$$\begin{aligned} W_K(s_t) &= u'(c_\tau)\left[\epsilon_t \frac{\partial w(K_t, z_t)}{\partial K} + k_t \frac{\partial r(K_t, z_t)}{\partial K}\right] + \\ &\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau)\left[\epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K}\right] G^K(s_{\tau-1}) \\ &\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau)\left[k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K}\right] G^K(s_{\tau-1}) \end{aligned}$$

$$\begin{aligned} W_k(s_t) &= u'(s_t)(1 + r(K, z) - \delta) + \\ &\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau)\epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1}) \\ &\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1}) \end{aligned}$$

$$\begin{aligned} W_\epsilon(s_t) &= E_t \sum_{\tau=t}^{\infty} (\beta\rho_\epsilon)^{\tau-t} u'(c_t) w(K, z) + \\ &\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau)\left[\epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} + k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K}\right] G^\epsilon(s_{\tau-1}) \end{aligned}$$

$$W_z(s_t) = E_t \sum_{\tau=t}^{\infty} \left((\beta\rho_z)^{\tau-t}\, u'(c_t) \left[\epsilon_\tau \frac{\partial w(K,z)}{\partial z} + k_t \frac{\partial r(K,z)}{\partial z} \right] \right)$$
$$+E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) \left[\epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} + k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} \right] G^z(s_{\tau-1})$$

where $c_\tau = \epsilon_\tau w(K_\tau, z_\tau) + (r(K_\tau, z_\tau) + 1 - \delta)k_\tau - k_{\tau+1}$ represents the consumption in the contract and the terms G^j for $j \in \{k, \epsilon, z, K\}$ are defined as above.

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