## Appendix A

LEMMA A1: *Incentive compatible equilibrium bidding strategy in a first price sealed bid auction with winner regret is a strictly increasing function.*

PROOF:

Let $b(.) : [\underline{v}, \overline{v}] \rightarrow \mathbb{R}$ be the equilibrium bidding function for a first price auction with winner regret. Let two valuations $v_1$, and $v_2$ be such that $v_2 > v_1$, $b_1$ and $b_2$ be the corresponding bids, and $b^2$ be the second highest bid. Then since we are interested in incentive compatible bids, we have

$$P(win\ with\ b_1)(v_1 - b_1 - h(b_1 - b^2)) \geq P(win\ with\ b_2)(v_1 - b_2 - h(b_2 - b^2))$$

and

$$P(win\ with\ b_2)(v_2 - b_2 - h(b_2 - b^2)) \geq P(win\ with\ b_1)(v_2 - b_1 - h(b_1 - b^2)).$$

By adding up these two inequalities and rearranging the terms, we have

$$\left[P(win\ with\ b_1) - P(win\ with\ b_2)\right](v_1 - v_2) \geq 0$$

Since $v_1 - v_2 < 0$, then $P(win\ with\ b_1) - P(win\ with\ b_2) \leq 0$. This gives $b_1 \leq b_2$. Moreover, $b_1 < b_2$ since otherwise if there exists an interval $[v_1, v_2]$ such that $b_1 = b_2 = b(v)$ for any $v \in [v_1, v_2]$ then $\widetilde{b}(v) = b(v) + \varepsilon$ for $v \in (v_1, v_2)$ is a profitable deviation given that all the opponents are bidding $b(v)$.

LEMMA A2: *In a first price sealed bid auction with winner regret, local and global incentive constraints (IC) are equivalent.*

PROOF:

It is trivially the case that global IC implies local IC.

The expected utility of a bidder who has valuation $v_1$ and bids as if her valuation is $z$ is

$$EU(v_1, z) = P(win\, with\, b(z))\, [v_1 - b(z) - E[h(b(z) - b(X))|X < z]]$$
$$= F^{N-1}(z)[v_1 - b(z)] - \int_0^z h(b(z) - b(X))(N-1)F^{N-2}(X)f(X)dX.$$

Observe that the cross derivative of this $EU$ is

$$EU_{zv} = \frac{\partial}{\partial z}F^{N-1}(z) = (N-1)F^{N-2}(z)f(z) > 0$$

To prove the converse of the statement, let the local IC constraint hold, then $\frac{\partial EU(v_1,z)}{\partial z}|_{z=v_1} = 0$.

Then for $y < v_1$

$$EU(v_1, v_1) - EU(v_1, y) = \int_y^{v_1} \frac{\partial EU}{\partial z}(v_1, z)dz$$
$$= \int_y^{v_1} (EU_z(v_1, z) - EU_z(z, z))\, dz \qquad \text{since } EU_z(z, z) = 0 \text{ by local IC}$$
$$= \int_y^{v_1}\int_z^{v_1} EU_{zv}(k, z)dkdz > 0 \qquad \text{since } EU_{zv} > 0.$$

For $y > v_1$

$$EU(v_1, v_1) - EU(v_1, y) = -\int_{v_1}^{y} \frac{\partial EU}{\partial z}(v_1, z)dz$$
$$= -\int_{v_1}^{y} (EU_z(v_1, z) - EU_z(z, z))\, dz \qquad \text{since } EU_z(z, z) = 0 \text{ by local IC}$$
$$= \int_{v_1}^{y}\int_{v_1}^{z} EU_{zv}(k, z)dkdz > 0 \qquad \text{since } EU_{zv} > 0.$$

Therefore, for every $y$ $EU(v_1, v_1) > EU(v_1, y)$, i.e. global IC holds.

PROOF OF THEOREM 2:

Consider any representative bidder motivated by winner regret and participating in a first price auction. Let $b(\cdot)$ be her optimum incentive compatible bidding strategy. If we consider the symmetric equilibrium (hence the identity index of bidder can be dropped) and solve the problem in an incentive compatible way then the solution to the following problem gives the optimal bid:

$$\begin{aligned}\max_w EU(v,b(w)) &= \max_w P(win)[v-b(w)-E[h(b(w)-b(X))|X<w]] \\ &= \max_w G(w)\{v-b(w)-E[h(b(w)-b(X))|X<w]\} \\ &= \max_w G(w)\left\{ v-b(w)-\frac{\int\limits_{\underline{v}}^{w}[h(b(w)-b(X))G'(X)]d(X)}{G(w)} \right\}\end{aligned}$$

where $G(w) = F^{N-1}(w)$. Above $P(win) = G(w)$ because the equilibrium bid function is increasing by Lemma A1.

Since the local and global IC are equivalent in this setting (by Lemma A2), the corresponding first order condition is: $\left.\dfrac{\partial EU(v,b(w))}{\partial w}\right|_{w=v} = 0.$

$$G'(v)[v-b(v)] - G(v)b'(v) - \int\limits_{\underline{v}}^{v}[h'(b(v)-b(X))b'(v)G'(X)]d(X) = 0$$

$$G'(v)v = G'(v)b(v) + b'(v)G(v) + b'(v)\int\limits_{\underline{v}}^{v}[h'(b(v)-b(X))G'(X)]d(X)$$

The solution of the above differential equation implicitly solves

$$E[X|X<v] = b^{FP_{wr}}(v) + E_X[h(b^{FP_{wr}}(v) - b^{FP_{wr}}(X))|X<v].$$

LEMMA A3: *Incentive compatible equilibrium bidding strategy in a first price sealed bid auction with loser regret is a strictly increasing function.*

PROOF:

Let $b(.) : [\underline{v}, \overline{v}] \to \mathbb{R}$ be the equilibrium bidding function for a first price auction with loser regret. Let two valuations $v_1$, and $v_2$ be such that $v_2 > v_1$, $b_1$ and $b_2$ be the corresponding bids, and

$b^w$ be the winning bid. Since we are interested in incentive compatible bids, we have

$$P(winwithb_1)(v_1 - b_1) + P(feelingloserregretwithb_1|b^w)(-g(v_1 - b^w))$$
$$\geq P(winwithb_2)(v_1 - b_2) + P(feelingloserregretwithb_2|b^w)(-g(v_1 - b^w))$$

and

$$P(winwithb_2)(v_2 - b_2) + P(feelingloserregretwithb_2|b^w)(-g(v_2 - b^w))$$
$$\geq P(winwithb_1)(v_2 - b_1) + P(feelingloserregretwithb_1|b^w)(-g(v_2 - b^w)).$$

By adding up these two inequalities and rearranging the terms, we have

$$[P(win\ with\ b_2) - P(win\ with\ b_1)]\,(v_2 - v_1)$$
$$+ [P(feelingloserregretwithb_2|b^w) - P(feelingloserregretwithb_1|b^w)]$$
$$\cdot\,(g(v_1 - b^w) - g(v_2 - b^w)) \geq 0$$

then

$$[P(win\ with\ b_2) - P(win\ with\ b_1)]\,(v_2 - v_1)$$
$$+ [(1 - P(win\ with\ b_2) - P(b^w > v)) - (1 - P(win\ with\ b_1) - P(b^w > v))]$$
$$\cdot (g(v_1 - b^w) - g(v_2 - b^w)) \geq 0$$

then

$$[P(win\ with\ b_2) - P(win\ with\ b_1)]\,[(v_2 - v_1) + (g(v_2 - b^w) - g(v_1 - b^w))] \geq 0$$

Since $v_2 - v_1 > 0$ and $g(v_2 - b^w) - g(v_1 - b^w) > 0$ then $P(win\ with\ b_2) - P(win\ with\ b_1) \geq 0$. This gives $b_2 \geq b_1$. Moreover, $b_2 > b_1$ since otherwise if there exists an interval $[v_1, v_2]$ such that $b_1 = b_2 = b(v)$ for any $v \in [v_1, v_2]$ then $\widetilde{b}(v) = b(v) + \varepsilon$ for $v \in (v_1, v_2)$ is a profitable deviation given that all the opponents are bidding $b(v)$.

LEMMA A4: *In a first price sealed bid auction with loser regret, local and global incentive constraints (IC) are equivalent.*

PROOF:

The proof of this statement is exactly the same as the proof of Lemma A2 once we show that under loser regret condition the cross derivative of expected utility is still positive, i.e. $EU_{zv}(v_1, z) > 0$. The expected utility of a bidder who has valuation $v_1$ and bids as if her valuation is $z$ is

$$\begin{aligned} EU(v_1, z) &= P(win\ with\ b(z))[v_1 - b(z) - E[P(feeling\ loser\ regret\ with\ b(z)|b^w)g(v_1 - b^w)]] \\ &= F^{N-1}(z)(v_1 - b(z)) - \int\limits_{z}^{b^{-1}(v_1)} g(v_1 - b(s))NF^{N-1}(s)f(s)ds. \end{aligned}$$

Then

$$EU_{zv} = \frac{\partial}{\partial v}\big((N-1)F^{N-2}(z)f(z)(v_1 - b(z)) - b'(z)F^{N-1}(z) + g(v_1 - b(z))NF^{N-1}(z)f(z)\big)$$
$$= (N-1)F^{N-2}(z)f(z) + g'(v_1 - b(z))NF^{N-1}(z)f(z) > 0.$$

The last inequality holds since $g$ is assumed to be increasing.

Now applying the same argument as in the proof of Lemma A2, one can immediately show that global and local ICs are equivalent in a first price sealed bid auction with loser regret.

PROOF OF THEOREM 1:

Any representative bidder with loser regret in FP solves the following expected utility maximization problem to decide on the optimal incentive compatible bidding strategy:

$$\begin{aligned}
\max_s EU(v, b(s)) &= \max_s \{P(win) \cdot [v - b(s)] \\
&\quad - P(\text{feeling loser regret}) \cdot E[g(v - b^w)|b(s) < b^w < v]\} \\
&= \max_s \{F^{N-1}(s) \cdot [v - b(s)] \\
&\quad - P(b(s) < b^w < v) \cdot E[g(v - b^w))|b(s) < b^w < v]\} \\
&= \max_s \{F^{N-1}(s) \cdot [v - b(s)] \\
&\quad - \int_s^{b^{-1}(v)} [\ g(v - b(y))(N-1)F^{N-2}(y)f(y)]d(y)\}
\end{aligned}$$

where $b^w$ is the winning bid. Here $P(win) = F^{N-1}(s)$ since the equilibrium bid function is increasing by Lemma A3.

Since the local and global IC are equivalent in this setting (by Lemma 4), the corresponding first order condition is: $\left.\dfrac{\partial EU(v, b(s))}{\partial s}\right|_{s=v} = 0.$

$$(N-1)F^{N-2}(v)f(v)[v - b(v)] - F^{N-1}(v)b'(v) + g(v - b(v))(N-1)F^{N-2}(v)f(v) = 0$$

$$
\begin{aligned}
(N-1)F^{N-2}(v)f(v)v &= b(v)(N-1)F^{N-2}(v)f(v)+b'(v)F^{N-1}(v) \\
&\quad -g(v-b(v))(N-1)F^{N-2}(v)f(v)
\end{aligned}
$$

The solution of above differential equation implicitly solves

$E_X[X|X<v] = b^{FP_{lr}}(v) - E_X[g(X-b^{FP_{lr}}(X))|X<v]$

where X is a random variable which is a maximum of N-1 random variables.

PROOF OF THEOREM 3:

For any bidder i the bid $b_i = v_i$ is a dominant strategy. Consider another action of player $i$ and call it $x_i$. If $\max_{j\neq i} b_j \geq v_i$ then by bidding $x_i$, bidder $i$ either gets the object and receives a nonpositive payoff or does not get the object and her payoff is $-g(v_i - b^w) = 0$, since $b^w = \max_{j\neq i} b_j \geq v_i$. While by bidding $b_i$, she guarantees herself a payoff of zero (observe that if she loses by bidding $b_i$, this will not create loser regret since $v_i > b^w > b_i$ is never a case). If $\max_{j\neq i} b_j < v_i$ then by bidding $b_i$, player $i$ obtains the good at the price of $\max_{j\neq i} b_j$, while bidding $x_i$ either she wins and gets the same utility or loses and gets non positive utility because of loser regret ($-g(v_i - b^w) \leq 0$ since $v_i > b^w > x_i$).