V Appendix A: Extensions

A Stochastic Effort, Multiple Levels of Effort and Communication

So far we have assumed that the only decisions available for the Agent were wether to work or to shirk a natural question to explore is what would happen if we allowed for random effort choices and also if we allowed for more degrees in the effort choice. We start by discussing the extension to multiple effort levels. Then, we show that with two levels of effort it is without loss of generality to focus on deterministic effort choices by the Agent. Finally, we argue that when we allow for both multiple levels of effort and mixed strategies by the Agent new types of efficiency enhancing contracts should be considered.

If we were to allow for many effort levels but restrict our attention to contracts in which the Agent follows a pure strategy the structure of the contracts would be very similar to the ones presented in this paper. There is one interesting feature of this generalization worth mentioning and which has already been discussed in MacLeod (2003), the effort level with private monitoring might be above the first best level of effort. This is because although it will be harder to motivate the agent to exert high effort, low output will be observed less frequently and therefore, value will need to be destroyed less often. Hence, depending on the specific parametrization of the problem it can be the case that it is beneficial to have the agent work harder than in the full information case. This same type of results carry on to the repeated case.[45]

In the contracts we have analyzed so far, the Principal has been doing all the monitoring. The continuation values are conditional only on his observations. We might think there is potential to grant some monitoring power to the Agent. If the Agent follows a mixed strategy for effort, he would have private information on what the outcome distribution should look like. This information can then be compared to the Principal's output reports. We could conjecture that this might lead to an improvement. Although a complete analysis of these issues is left for future work we want to provide two results of interest illustrating whether this is or not the case.

We first prove a negative result showing that within the framework of our model and focusing on one-period review contracts it is optimal to have the Agent exert high effort deterministically.

A.1 No Mixing No Talking

Consider the following one period review contract with random effort. The Agent plays $e = 1$ with probability $\alpha \in (0, 1)$, the Principal then observes the output and they simultaneously announce the actual effort and the actual output.

The following table summarizes the parameters (other than α) of the self-enforcing contract conditional on the announcements.

Announcements (e, y)	Wage	Termination Probability
$(0, L)$	w_{0L}	β_{0L}
$(0, H)$	w_{0H}	β_{0H}
$(1, L)$	w_{1L}	β_{1L}
$(1, H)$	w_{1H}	β_{1H}

Continuation Values (including wages not including cost of effort)

Principal	$y = L$	$y = H$
$e = 1$	F_{1L}	F_{1H}
$e = 0$	F_{0L}	F_{0H}

Agent	$y = L$	$y = H$
$e = 1$	V_{1L}	V_{1H}
$e = 0$	V_{0L}	V_{0H}

In order to simplify our analysis and exposition, we define the following new variables:

$\Delta_1 = \alpha \left(F_{1H} - F_{1L} \right)$

$\Delta_0 = (1 - \alpha) \left(F_{0L} - F_{0H} \right)$

$\Delta_L = V_{1L} - V_{0L}$

$\Delta_H = V_{1H} - V_{0H}$

$\Delta_{cross} = V_{1H} - V_{0L}$

For the Principal we have the following truth-telling constraints:

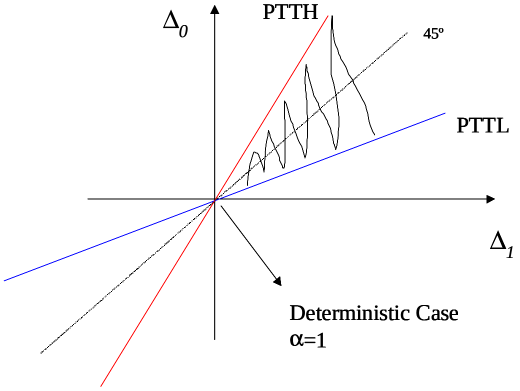
Given that he observed $y = H$:

$$\Delta_0 \leq \frac{p}{q} \Delta_1 \tag{PTTH}$$

Given that he observed $y = L$:

$$\Delta_0 \geq \frac{(1-p)}{(1-q)} \Delta_1 \tag{PTTL}$$

We can represent these constraints graphically:



These constraints imply that $\Delta_0, \Delta_1 \geq 0$

For the Agent we have the following two truth-telling constraints:

Given he exerted high effort:

$$\Delta_H \geq -\frac{(1-p)}{p} \Delta_L \tag{$ATT1$}$$

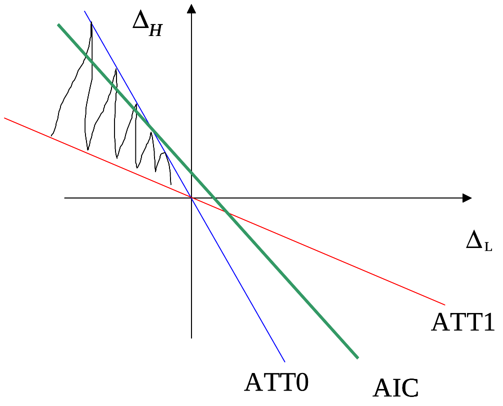
Given he exerted low effort:

$$(ATT0) \qquad \Delta_H \leq -\frac{q}{(1-q)}\Delta_L$$

We also need the Agent to be indifferent between high and low effort

$$(AIC) \qquad \Delta_H = \frac{c-(p-q)\,\Delta_{cross}}{q} - \frac{(1-p)}{q}\Delta_L$$

Graphically:



Note that (AIC) has always a slope between $(ATT0)$ and $(ATT1)$.

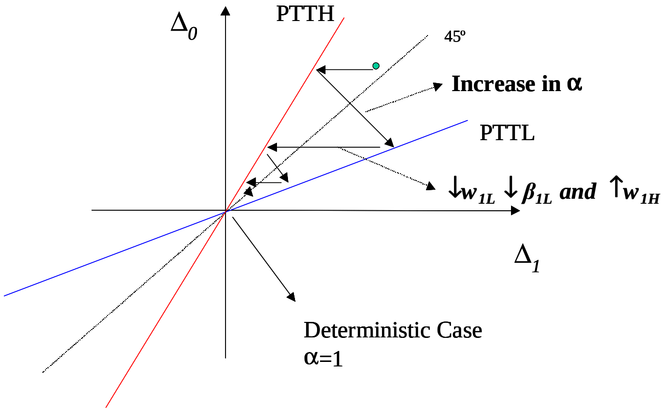
With these preliminaries we can now prove the following result:

Proposition 7 *Allowing for contracts in which the Agent randomizes and the continuation values are determined as a function of the simultaneous announcements by the Agent and the Principal does not provide any improvement over the one-period review contract in which the Agent exerts high effort deterministically.*

Proof. Suppose that the optimal contract has $\alpha < 1$ and $\Delta_1$and $\Delta_0 > 0$ and that $(PTTH)$ does not bind. Now consider the following: decrease w_{1L} and β_{1L} and increase w_{1H}. We can do this such a way that the resulting increase in V_{1H} is equal to $\frac{1-p}{p}$ times the decrease in

V_{1L}.As a result, the Agent's IC constraints continue to hold and the truth-telling constraints are relaxed. The decrease in β_{1L} implies an improvement in efficiency since there will be less value burning. Therefore, $(PTTH)$ must be binding in the optimal contract.

Now suppose that$(PTTH)$ binds, but that $(PTTL)$ does not bind. If we could decrease.Δ_0 we would relax $(PTTH)$ and improve. This can be achieved by simply increasing α. Changing α has no impact on the Agent's constraints, it relaxes $(PTTH)$ and we can do it till $(PTTL)$ starts binding. If we iterate between this two steps (see graph below) clearly we can drive $\Delta_1$and Δ_0 all the way to zero. This means that either $\alpha = 1$ or that $(F_{0L} - F_{0H}) = (F_{1H} - F_{1L}) = 0$ but if this is the case we can set $\alpha = 1$ anyway and this is preferable since lower $\alpha's$ implies an efficiency loss. ■



Mixing by the Agent fails as means to relax the Principal's truth-telling constraints when he observes a high realization of outcome. The intuition behind this negative result is that when the Agent mixes the Principal is actually getting some slack from claiming $y = L$ when $y = H$. This is because there is some chance that the Agent had actually done low effort and the Principal cannot be punished for claiming L when no effort effort was exerted. This intuition would seem to generalize when considering arbitrary contracts. If instead we allowed the Agent to mix a higher effort level (even if it were inefficient from a productive perspective) then mixing might work. We analyze this case in the next section.

A.2 Monitoring with high Effort

If instead the model allowed for an inefficiently high effort level, then things could work differently. The Agent can use mixing to a higher effort level to monitor the Principal. This higher effort level leads to high outcome with higher probability, mixing by the Agent to this higher effort level relaxes the Principal's truth-telling constraints. The intuition for this is clearer when we consider the case in which very high effort leads to a high output for sure. Suppose the Agent mixes between these two effort levels, now, when the Principal observes high output he less tempted to claim low. He fears that the Agent might have exerted very high effort and that he would be caught lying if he did.

To illustrate more precisely how this works consider the following simple example: let $e = \{l, m, h\}$ and $c(e) = \{0, c, 1\}$ with $p(e) = \{0, p, 1\}$ respectively.[46] $H = 1$ and $L = 0$. Medium effort is the only productively efficient effort level $p - c > 0$.

Suppose we wish to have the Agent mix between high and medium effort and we allow continuation values including wages to depend on the simultaneous announcements. Denote these by $V_{\tilde{e},\ \tilde{y}}$ for the Agent and $F_{\tilde{e},\ \tilde{y}}$ for the Principal. We face the following constraints:

For the Agent:

Thruth-telling:

Given he exerted m effort:

$$pV_{mH} + (1-p)\,V_{mL} \geq pV_{hH} + (1-p)\,V_{hL}$$

Given he exerted h effort:

$$V_{hH} \geq V_{mH}$$

Incentive compatibility for mixing:

$$pV_{mH} + (1-p)\,V_{mL} - c = V_{hH} - 1 \geq V_{mL}$$

For the Principal:

Thruth-telling:

Given he observed L:

$$(TTL) \qquad F_{mL} \geq F_{mH}$$

Given he observed H:

$$(TTH) \qquad \alpha F_{hH} + (1-\alpha)\, pF_{mH} \geq \alpha F_{hL} + (1-\alpha)\, pF_{mL}$$

Consider a contract in which the Principal gets all the residual surplus.

Announcements	Wage	Termination Probability
(m, L)	$w_{mL} = 0$	$\beta_{mL} = 0$
(m, H)	$w_{mH} = \frac{c}{p}$	$\beta_{mH} = 0$
(h, L)	$w_{hL} = -\frac{p}{1-p}$	$\beta_{hL} = 1$
(h, H)	$w_{hH} = 1$	$\beta_{hH} = 0$

The Agent has no incentives to deviate since when he is called to exert high effort he is compensated for sure with the cost he had to bare. When he is called to do medium effort he gets zero if he obeys. If he chooses to exert low effort the best he can get is also a payoff of zero.

We must now make sure that the Principal does not have incentives to claim L when he observes H.[47] Using the table above we can re-write (TTH) as follows:

$$\begin{aligned} & \alpha \left(-1 + \frac{\delta\left(\left(1-\alpha\right)\left(p-c\right)\right)}{1-\delta} \right) + (1-\alpha)\, p \left(-\frac{c}{p} + \frac{\delta\left(\left(1-\alpha\right)\left(p-c\right)\right)}{1-\delta} \right) \\ \geq \; & \alpha \frac{p}{1-p} + (1-\alpha)\, p \left(\frac{\delta\left(\left(1-\alpha\right)\left(p-c\right)\right)}{1-\delta} \right) \end{aligned}$$

We will use the following parameter values to illustrate the example: $p = \frac{1}{2}, \delta = .9$ and $c = \frac{1}{8}$. Using these particular values and doing some algebra, we see that satisfying (TTH) reduces to finding an α that satisfies:

$$-\frac{27}{4}\alpha^2 + 3\alpha - \frac{1}{4} \geq 0$$

Any $\alpha \in \left[\frac{1}{9}, \frac{1}{3}\right]$ is a solution to this, but since m is more productive than h, it is optimal to pick $\alpha = \frac{1}{9}$. In this case we get a total expected surplus of $3\frac{1}{3}$.The First Best surplus is $\frac{15}{4} = 3.75$ and with the deterministic one-period review contracts we get an expected surplus of only $\frac{2}{3}FB = 2.5$.This example shows that when considering a more general model, with many possible effort choices, ruling out mixed strategies by the Agent and communication might reduce efficiency.

B Tournaments

The role of tournaments as an incentive device for the agents has been largely studied in the literature starting with the seminal paper of Lazear and Rosen (1981). The literature on tournaments has generally assumed that output is contractable. By changing that assumption, this paper highlights a virtue of tournaments that, with the exception of Malcomson (1984), has not received much attention in the past. The tournament structure allows for contracts that provide a constant continuation value for the Principal for all output realizations and upward-sloping schedules for the Agents without the need to burn any surplus. This reminds us somewhat of the moral hazard in teams problem studied by Holmstrom (1982) in which the agents use the Principal as a budget breaker. In our setup, it is the Principal who uses a new agent to this effect.

Suppose the Principal can commit to give a prize b to the best performing Agent every period (or randomize in case of a draw). This way the Principal has no more constraints since he has to give a prize to some player but is really indifferent about which Agent he gives it to. The Agents have incentives exert high effort in order to win the prize.

For example for $T = 1$ with two Agents, to provide incentives for them to exert high effort we need:

$$\left(\frac{p-q}{2}\right) b - c \geq 0$$

The term multiplying the bonus is by assumption positive hence we can always find a bonus big enough to provide incentives for the Agents to exert high effort. Note that this might require the base wage to be negative depending on how much of the surplus is captured by the Principal and how much by the Agents.

Tournaments themselves are in turn susceptible to other problems. Examples of these shortcomings include: the Agents colluding against the Principal or the Principal colluding with one of the Agents against the other. The Agents might also engage in unproductive activities to try to win over the Principal's favor or directly to try to undermine the other Agent's work. Therefore, we would expect to observe tournaments in environments in which the information problem is of more importance than these other concerns. A full analysis of these issues is left for future work.

VI Appendix B: Ommited Proofs

The following definitions facilitate our analysis.

Definition 4 (Continuation Values) $V_t^+\left(\left\{h_{t-1}^P, y_t\right\}, \sigma\right)$ *is the promised continuation value to the Agent in period* t *after* e_t *has been exerted and the Principal has observed* y_t*. Similarly,* $F_t^+\left(\left\{h_{t-1}^P, y_t\right\}, \sigma\right)$ *is the principal's continuation value after observing and consuming* y_t*. Formally:*

$$V_t^+\left(\left\{h_{t-1}^P, y_t\right\}, \sigma\right) = \mathbb{E}\left[w_t + b_t + \sum_{j=t+1}^{T} \delta^{(j-t)}(w_j + b_j - c\left(e_j\right)) \middle| \left\{h_{t-1}^P, y_t\right\}, \sigma\right],$$

$$F_t^+\left(\left\{h_{t-1}^P, y_t\right\}, \sigma\right) = \mathbb{E}\left[-y_t + \sum_{j=t}^{T} \delta^{(j-t)}(y_j - w_j - b_j) | \left\{h_{t-1}^P, y_t\right\}, \sigma\right] \ .$$

The Agent can only form an expectation of this value:

$$v_t^+ = \mathbb{E}\left[V_t^+\left(\left\{h_{t-1}^P, y_t\right\}, \sigma\right) | \left\{h_{t-1}^A, e_t\right\}, \sigma\right] \quad .$$

Definition 5 *Given σ_ω and a history h_{t-1}^V such that $e\left(h_{t-1}^V, \sigma_\omega\right) = 1$, we say that σ_ω^A is incentive compatible with respect to effort for the Agent (ICA) after history h_{t-1}^V iff:*

$$\left(\begin{array}{c} \mathbb{E}\left[V_t^+\left(\left\{h_{t-1}^P, y_t\right\}, \sigma_\omega\right) | \left\{h_{t-1}^A, e_t = 1\right\}, \sigma_\omega^A, \sigma_\omega^P\right] - \\ \mathbb{E}\left[V_t^+\left(\left\{h_{t-1}^P, y_t\right\}, \sigma_\omega\right) | \left\{h_{t-1}^A, e_t = 0\right\}, \tilde{\sigma}_\omega^A, \sigma_\omega^P\right] \end{array}\right) \geq c \quad ,$$

for all $\tilde{\sigma}_\omega^A$ such that $\tilde{\sigma}_\omega^A \equiv \sigma_\omega^A$ for all $\tau < t$ and $\tilde{e}\left(h_{t-1}^V\right) = 0$.[48]

Definition 6 *Given σ^A and ω we say σ_ω^P is incentive compatible for the Principal (ICP) after history $\left\{h_{t-1}^P, y_t\right\}$ iff:*

$$F_t^+\left(\left\{h_{t-1}^P, y_t\right\}, \sigma\right) \geq F_t^+\left(\left\{h_{t-1}^P, y_t\right\}, \sigma^A, \tilde{\sigma}^P\right) \ ,$$

for all $\tilde{\sigma}_\omega^P$ such that $\tilde{\sigma}_\omega^P \equiv \sigma_\omega^P$ for all $\tau < t$.[49]

Proof of Proposition 1. Follows directly from *Lemmas* (5) and (6) stated below. ■

Lemma 5 *The ICA implies:*

$$\begin{aligned} (2) \qquad & \mathbb{E}\left[V_t^+\left(\left\{h_{t-1}^P, y_t = H\right\}, \sigma\right) | \left\{h_{t-1}^A, e_t = 1\right\}, \sigma\right] \\ > \ & \mathbb{E}\left[V_t^+\left(\left\{h_{t-1}^P, y_t = L\right\}, \sigma\right) | \left\{h_{t-1}^A, e_t = 1\right\}, \sigma\right] \quad , \end{aligned}$$

i.e. the expected payoff to the Agent is strictly increasing in the output when the Agent is supposed to exert effort.

Proof. Follows from the fact that effort increases the likelihood of $y_t = H$. ■

Lemma 6 *The IC for the Principal implies:*

$$F_t^+\left(\left\{h_{t-1}^P, y_t = H\right\}, \sigma\right) = F_t^+\left(\left\{h_{t-1}^P, y_t = L\right\}, \sigma\right) \quad \forall h_{t-1}^P \tag{3}$$

Proof. Suppose first that for some h_{t-1}^P :

$$F_t^+\left(\left\{h_{t-1}^P, y_t = H\right\}, \sigma\right) > F_t^+\left(\left\{h_{t-1}^P, y_t = L\right\}, \sigma\right) \ ,$$

and that $y_t = L$ is realized.

Now, consider the alternative strategy $\tilde{\sigma}_\omega^P$ for the Principal. $\tilde{\sigma}_\omega^P$ follows σ_ω^P except that it implies the same actions as σ_ω^P for histories with $y_t = L$ as it does for histories where $y_t = H$. This deviation is undetectable by the Agent and leads to an improvement for the Principal.[50]

A similar argument can be constructed if:

$$F_t^+\left(\left\{h_{t-1}^P, y_t = H\right\}, \sigma\right) < F_t^+\left(\left\{h_{t-1}^P, y_t = L\right\}, \sigma\right) \ .$$

Therefore, it must be the case that

$$F_t^+\left(\left\{h_{t-1}^P, y_t = H\right\}, \sigma\right) = F_t^+\left(\left\{h_{t-1}^P, y_t = L\right\}, \sigma\right) \ ,$$

for all h_{t-1}^P. ■

Lemmas 5 and 6 imply that for any strategy pair σ^* and history h_{t-1}^V such that σ^* induces $e\left(h_{t-1}^V\right) = 1$, the Agent's continuation value v_t^+ must depend on the outcome but the Principal's continuation value F_t^+ must be equal after either outcome. This implies that the total continuation surplus depends on the outcome realization.

Lemma 7 *Given σ^A and ω if for all $\left\{h_{t-1}^P, y_t\right\}$ σ_ω^P is incentive compatible for the Principal then σ_ω^P is a best response to σ^A given ω.*

Proof. Follows directly from the definition of best response. There is no $\tilde{\sigma}_{\omega}^{P} \in \Sigma_{\omega}^{P}$ that can achieve a higher value for the Principal. ■

Technically, we are abusing the definition of best response slightly because we are not explicitly considering detectable deviations. On the other hand these deviations could be dealt with in a simple way.

Proof of Proposition 2. When $T = 1$, since this is the last period of the game, the continuation values for both, the Agent and the Principal, solely consists on the current compensation. By Lemma 6 the total compensation the Principal pays must be the same for either realization of output. Hence, providing incentives for the Agent to exert effort, which requires compensation to be dependent on the output, is not possible.

Next consider any arbitrary finite horizon. Suppose there exists a contract that contemplates the Agent exerting effort in some periods. Consider the last period in which the Agent is supposed to exert effort. By the arguments given to prove the case $T = 1$ it follows that incentives cannot be provided to exert effort for this last period hence contradicting the claim that there can be a contract for finite T in which effort is exerted. ■

Proof of Lemma 1. Suppose that after some history $h_{t-1}^{V} \times m^{t} \times [0,1]$ a payment w_{t}^{p} was to be made by the Principal and $w_{t}^{A} \leq w_{t}^{P}$ to be received by the Agent. Instead of paying immediately the Principal can send a message that commits him to pay $\frac{w_{t}^{p}}{\delta^{T-t}}$ at time T. Also, any money burning that was supposed to take place at time t can also be delayed until time T. ■

Proof of Lemma 2.

Follows from Lemma (10) and noting that the destruction of value that can be achieved by termination can also be achieved by burning money. ■

Proof of Proposition 3. The proof to the first part of the Proposition is organized in four steps. First we write the problem, then we write a relaxed problem where only one-step deviations by the Agent are considered. The third step solves the relaxed problem and the last step proves that this is also a solution to the original problem.

i) To motivate effort for $T+1$ periods which we denote $\mathbf{e}^*$, we face the following money burning minimization problem:

$$\min_{\{Z(y^T)\}_{y^T\in Y^T}} \delta^T \sum_{y^t\in Y^T} Z\left(y^T\right) P\left(y^T|\mathbf{e}^*\right)$$

$$(4) \qquad s.t. \quad \mathbf{c}\left(\mathbf{e}^* - \mathbf{e}\right) + \delta^T \sum_{y^T\in Y^T} Z\left(y^T\right)\left(P\left(y^T|\mathbf{e}^*\right) - P\left(y^T|\mathbf{e}\right)\right) \leq 0 \ \ \forall \mathbf{e} \in \{0,1\}^{T+1}$$

$$Z\left(y^T\right) \geq 0 \quad \forall y^T \ .$$

Note that the Principal's incentive constraints are satisfied by construction since w^P is fixed.

ii) Instead of solving this problem directly, we first solve a relaxed problem in which we only consider one-step deviations by the Agent:

$$\min_{\{Z(y^T)\}_{y^T\in Y^T}} \delta^T \sum_{y^t\in Y^T} Z\left(y^T\right) P\left(y^T|\mathbf{e}^*\right)$$

$$(5) \qquad s.t. \quad \delta^t c + \delta^T \sum_{y^T\in Y^T} Z\left(y^T\right) P\left(y^T|\mathbf{e}^*\right)\left(\frac{P\left(y_t|e_t=1\right) - P\left(y_t|e_t=0\right)}{P\left(y_t|e_t=1\right)}\right) \leq 0 \ \ \forall t$$

$$Z\left(y^T\right) \geq 0 \quad \forall y^T \ .$$

From (5) it is clear that in order to provide incentives in period t, we need:

$$\left(\frac{P\left(y_t|e_t=1\right) - P\left(y_t|e_t=0\right)}{P\left(y_t|e_t=1\right)}\right) < 0 \ .$$

In words, to provide incentives for effort in period t we need to have burning if the outcome of that period is low which, is more likely to occur if the Agent deviates. Therefore, the constraints can be written as:

$$\delta^t c\frac{(1-p)}{p-q} \leq \delta^T \sum_{y^T \in Y^T \cap (y_t = L)} Z\left(y^T\right) P\left(y^T | \mathbf{e}^*\right) \quad \forall t$$

Clearly $t = 0$ is the most binding constraint if $\delta < 1$ which puts a lower bound on the necessary expected money burning $c\frac{(1-p)}{(p-q)}$.

iii) Next we show that:

$$Z^* = \begin{cases} \frac{c}{\delta^T (p-q)(1-p)^T} & if \quad y^T = \mathbf{L} \\ 0 & otherwise \end{cases}$$

achieves the lower bound and satisfies all the IC constraints of the relaxed problem.

First note that:

$$P\left(y^T = \mathbf{L} | \mathbf{e}^*\right) = (1-p)^{T+1}.$$

Hence,

$$\delta^T \sum_{y^T \in Y^T} Z^*\left(y^T\right) P\left(y^T | \mathbf{e}^*\right) = c\frac{(1-p)}{(p-q)} \geq \delta^t c\frac{(1-p)}{(p-q)} \quad \forall t .$$

iv) We now show that Z^* is also a solution to the original problem.

Using Z^*, the constraints of the original problem (4) take the following form:

$$\mathbf{c}\left(\mathbf{e}^* - \mathbf{e}\right) + \frac{c}{(p-q)(1-p)^T}\left((1-p)^{T+1} - P(\mathbf{L}|\mathbf{e})\right) \leq 0 \quad \forall \mathbf{e} \in \{0,1\}^{T+1}.$$

Letting n denote the number of periods the Agent deviates we can write the term that captures the expected cost of deviation as follows:

$$\frac{c}{(p-q)(1-p)^T}\left((1-p)^{T+1} - (1-p)^{(T+1)-n}(1-q)^n\right).$$

Now suppose the Agent was deviating in $n-1$ periods and now deviates for n periods. The increase in the probability that $y^T = \mathbf{L}$ is:

$$\left(\frac{1-q}{1-p}\right)^{n-1}(1-p)^T (p-q).$$

Since $\left(\frac{1-q}{1-p}\right) > 1$ the increase in the probability of punishment from an additional deviation is increasing in the number of previous deviations. Hence, if the Agent doesn't find it profitable to deviate once, he won't want to deviate at all. Given $\delta < 1$ and Z^*, the most profitable single deviation is in the first period. This is not profitable by step (iii).

Finally, the conditions for w^P guarantee that individual rationality constraints are satisfied so the players sign the contract ω at time zero.

Uniqueness proof for the case when $\delta = 1$:

Consider any other incentive compatible contract Z. Clearly, for Z to be optimal it must be that $Z\left(y^T = \mathbf{L}\right) < Z^*\left(y^T = \mathbf{L}\right)$ otherwise Z^* would imply strictly less expected money burnt. Denote all histories $y^T \neq \mathbf{L}$ for which $Z\left(y^T\right) > 0$ and $y_0 = L$ by $\dot{y}^T$. If there is no such history, then $Z\left(y^T = \mathbf{L}\right) \geq Z^*\left(y^T = \mathbf{L}\right)$ otherwise, the Agent would deviate in the first period. Hence, if Z is to be optimal it must be that there is some $\dot{y}^T$. Suppose there is one such $\dot{y}^T$. Let $j = min\ \{t : \dot{y}_t = H\}$. The incentive constraint for period j effort is given by:

$$\delta^j c + \delta^T \left(Z\left(y^T = \mathbf{L}\right)(1-p)^{T+1}\left(\frac{q-p}{1-p}\right) + Z\left(\dot{y}^T\right) P\left(\dot{y}^T\right)\left(\frac{p-q}{p}\right)\right) \leq 0$$

Now, let $\delta \to 1$

$$c + Z\left(y^T = \mathbf{L}\right)(1-p)^{T+1}\left(\frac{q-p}{1-p}\right) + Z\left(\dot{y}^T\right) P\left(\dot{y}^T\right)\left(\frac{p-q}{p}\right) \leq 0$$

This is the IC for period 0 effort. Note this implies that Z cannot be incentive compatible since this constraint held with equality for Z^* and $Z\left(y^T = \mathbf{L}\right) < Z^*\left(y^T = \mathbf{L}\right)$ and additionally, the last term in the left hand side of the inequality is positive. Therefore, Z^* is the only contract that is optimal for all δ. ■

Proof of Proposition 4 .

All of the Agent's IC constraints are holding with equality when he is completely uninformed. As soon as he can condition on any information, the probability he assigns to his effort being pivotal in any given period would change. Some IC constraints would therefore be violated when he updates this probability given the messages. ■

Restatement of Theorem 1

Given any contract (ω, σ_ω) *that generates values* V *and* F*, we can construct a payoff equivalent contract* $(\hat{\omega}, \hat{\sigma}_\omega)$ *with the following properties:*

(i) The Agent receives a constant wage until he is fired and no bonuses.

(ii) The Agent exerts effort every period until he is fired.

(iii) The Principal gives no feedback (sends no messages) to the Agent.

Proof. Properties (i) and (ii) follow from applying successively Lemmas (8), (9) and (10). Given properties (i) and (ii) Lemma (11) delivers property (iii). ■

Lemma 8 *For any contract* $[\omega, \sigma^*]$ *that generates values* V *and* F *there is a payoff-equivalent contract* $[\tilde{\omega}, \tilde{\sigma}^*]$ *such that for any history* h_{t-1}^V *with the property that* $e\left(h_{t-1}^V\right) = 0$ *all future actions are independent of the outcome* y_t*.*

Proof. Suppose the Principal conditioned his action on y_t. Now, instead let him condition his strategy $\tilde{\sigma}^P$ on ϕ_t in the following way: if $\phi_t < q$ let the Principal take the same actions that he did for σ^P when $y_t = H$ and if $\phi_t > q$ take the same actions he did for σ^P when $y_t = L$. The Principal had to be indifferent between the outcomes of y_t by Lemma 6 hence he will not have incentives to deviate from $\tilde{\sigma}$ if he didn't have incentives to deviate from σ. If the Agent follows the equilibrium strategy, his payoffs and information are the same as in the original equilibrium. Therefore, if he follows $e\left(h_{t-1}^V\right) = 0$ the incentives to deviate in later or earlier periods are unchanged. If he deviates in the current period to $e\left(h_{t-1}^V\right) = 1$ he is not affecting future payoffs nor obtaining any information hence such deviation costs him c without any benefit. ■

In words, to induce no effort we don't need to provide any incentives to the Agent. Therefore, we can always make his payoff unconditional of the outcome without inducing him to put the effort.

Definition 7 *A contract* $[\omega, \sigma^*]$ *is an input based compensation contract if:*

$$b_t + w_t = E\left[y_t|e_t\right] \;\; for \;\; all \;\; t > 0.$$

Lemma 9 (Input based pay) *For every contract* $[\omega, \sigma^*]$ *that generates values* V *and* F *there exists a payoff equivalent input based compensation contract* $[\tilde{\omega}, \tilde{\sigma}^*]$*.*

Proof. Modify $[\omega, \sigma^*]$ in the following way: let $\tilde{b}_t + \tilde{w}_t = E\left[y_t|e_t\right] \;\; for \;\; all \;\; t > 0$ and $\tilde{w}_0 = E\left[y_0|e_0\right] - F$. Also, let $\tilde{a}_t = a_t k_t$ and $\tilde{k}_t = 1 \; \forall t$. This changes imply that the Principal gets F as in the initial contract but also, that for all t the Principal's continuation value is zero (equal to his outside option). Therefore, there are no incentive issues having the Principal, who is indifferent on whether to terminate or not, do all the termination. Additionally, there are no incentive problems regarding the principal's announcements. By construction, his announcements might change effort level and compensation, but they do not change his expected surplus. The Agent won't want to deviate with respect to $\{\tilde{k}\}$ since terminating when he is not supposed to, would make him worse off as he gets all the surplus in the continuation game. Lemma 8 implies we only need to check that there will be no deviations with respect to effort for those histories where effort is supposed to be exerted. Note that under the original contract, when the Agent was supposed to exert effort there had to be value burning in case of a bad outcome to ensure that the right incentives were provided for the Agent to exert effort. We showed already that the new contract will destroy the same amount of value after the same histories than the original contract, hence the Agent will have the same incentives to exert effort under the new contract as he had under the old. ■

Definition 8 *A contract* $[\omega, \sigma^*]$ *is a termination contract if it is an input based compensation contract and the Agent exerts effort every period until termination.*

Lemma 10 (Termination) *For any input based compensation contract* $[\omega, \sigma^*]$ *that generates values* V *and* F *there is a payoff equivalent termination contract* $[\tilde{\omega}, \tilde{\sigma}^*]$*.*

Proof. First we will use *Lemma* 8 to transform $[\omega, \sigma^*]$ into payoff-equivalent contract $[\hat{\omega}, \hat{\sigma}^*]$ such that for any history h_{t-1}^V with the property that $e\left(h_{t-1}^V\right) = 0$ all future actions are independent of the outcome y_t. When there is a period of no effort, value is being destroyed because there is a delay of one period until any future surplus is realized. Therefore if we let $\tilde{a}\left(h_{t-2}^P, y_t, x_t\right) = \hat{a}\left(h_{t-2}^P, y_t, x_t\right)\delta$ we destroy in expectation the same amount of value. Finally, we must adjust all strategies by one period since we are eliminating the slack period.

■

This shows that incentives for the Agent to exert effort can be provided via efficiency wages and the threat of termination.

Definition 9 *A termination contract* $[\omega, \sigma^*]$ *is a no communication contract iff the messages* m_t *are completely uninformative* $\forall$ t*.*

Lemma 11 (No communication) *For any termination contract* $[\omega, \sigma^*]$ *there is a payoff equivalent no communication contract* $[\tilde{\omega}, \tilde{\sigma}^*]$*.*

Proof. First note that in a termination contract on the equilibrium path the Agent's actions are independent of the messages m_t. Having the Agent completely uninformed does not give the Principal any profitable deviation possibilities so he will still follow the original termination rule $\{a\}$. Finally, if for every message m_t that the Agent could have received, he chose to exert effort that means that if now he has no message on which to condition his action he would still choose to exert effort. ■

Restatement of Proposition 5 *A sequence of termination rules* $\{a(h^t)\}_{t=0}^{\infty}$ *that does not have each of the following properties, can be weakly improved upon.*

(i) *For all* h^t*, if* $a(h^t, H) < 1$ *then* $a(h^t, L) = 0$*.*

(ii) *For all* $h^t \times h^j$, *if* $a(h^t, H, h^j) < 1$ *then* $a(h^t, L, h^j) = 0$.

(iii) *For all* t, *if* $a(L^t) > 0$ *then* $a(h^t) = 1$ *for all* h^t.

(iv) *For all* h^t, $V(h^t, H) \geq V(h^t, L)$.

(v) *For all* h^t, *if* $a(h^t, L) = a(h^t, H) = 0$ *then* $a(h^t) = 0$.

Proof. (i) Suppose $a(h^t, H) < 1$ and $a(h^t, L) > 0$ for some h^t. Let $\tilde{a}(h^t, L) = a(h^t, L) - \varepsilon \geq 0$ for some small $\varepsilon > 0$. Also, instead of terminating with probability $1 - a(h^t, H)$, after history (h^t, H).Let the Principal terminate with probability $1 - a(h^t, H) - \left(\frac{1-p}{p}\right)\varepsilon \geq 0$ and with probability $\left(\frac{1-p}{p}\right)\varepsilon$ switch for history (h^t, L) upon observing the high outcome at $t+1$. That is, after history (h^t, H) start the Agent at $t+2$ with history (h^t, L) with probability $\left(\frac{1-p}{p}\right)\varepsilon$. We must make sure these changes respect two constrains: first the expected values at time t must not be changed. This guarantees that incentives for all periods prior to $t+1$ are unaffected. Formally:

$$\begin{aligned} & p \times a\left(h^t, H\right) \times V\left(h^t, H\right) + (1-p) \times a\left(h^t, L\right) \times V\left(h^t, L\right) \\ = \; & p\left(a\left(h^t, H\right) \times V\left(h^t, H\right) + \left(\frac{1-p}{p}\right)\varepsilon V\left(h^t, L\right)\right) + (1-p) \times \left(a\left(h^t, L\right) - \varepsilon\right) \times V\left(h^t, L\right) \end{aligned}$$

Second, the probability distribution on the tree is unaffected so all incentive constraints for periods after $t+1$ are unaffected.

$$(1-p) \times \left(a\left(h^t, L\right) - \varepsilon\right) + p\left(\frac{1-p}{p}\right)\varepsilon = (1-p) \times a\left(h^t, L\right)$$

The key is that incentive constraints at period $t+1$ have been relaxed. Since the change has increased $V(h^t, H)$ and decreased $V(h^t, L)$ the Agent now has more incentives to exert effort at $t+1$.

(ii) Suppose $\exists\ h^t, h^j \quad s.t.\ a(h^t, H, h^j) < 1$ and $a(h^t, L, h^j) > 0$ and consider the following changes to $\{a(h^t)\}$.Reduce the termination probability after history (h^t, H, h^j) to $1 - a(h^t, H, h^j) - \varepsilon$ where $\varepsilon > 0$. With probability ε instead of terminating assume the history is (h^t, L, h^j) and treat the Agent accordingly. Let $\tilde{a}(h^t, L, h^j)$ be the new continuation probability after $\left(h^t, L, h^j\right)$ .Before the changes, the probability an Agent was hired to begin work in period $t+j+2$ with history $\left(h^t, L, h^j\right)$ was:

$$P\left(h^t, L, h^j\right)\left(\prod_{\varnothing \leq h^u \leq \left(h^t, L, h^j\right)} a\left(h^u\right)\right)$$

Now that probability is:

$$\begin{aligned}
& P\left(h^t, L, h^j\right)\left(\prod_{\varnothing \leq h^u < \left(h^t, L, h^j\right)} a\left(h^u\right) \times \tilde{a}\left(h^t, L, h^j\right)\right) \\
& +P\left(h^t, H, h^j\right)\left(\prod_{\varnothing \leq h^u < \left(h^t, H, h^j\right)} a\left(h^u\right)\right) \times \varepsilon \times \tilde{a}\left(h^t, L, h^j\right) \\
= \; & \tilde{a}\left(h^t, L, h^j\right) P\left(h^t, L, h^j\right)\left(\prod_{\varnothing \leq h^u < \left(h^t, L, h^j\right)} a\left(h^u\right)+\varepsilon \frac{p}{1-p}\left(\prod_{\varnothing \leq h^u < \left(h^t, H, h^j\right)} a\left(h^u\right)\right)\right)
\end{aligned}$$

This probability must be kept constant to guarantee incentive constraints in the future hold hence:

$$\tilde{a}\left(h^t, L, h^j\right)=a\left(h^t, L, h^j\right) \frac{\left(\prod_{\varnothing \leq h^u < \left(h^t, L, h^j\right)} a\left(h^u\right)\right)}{\left(\prod_{\varnothing \leq h^u < \left(h^t, L, h^j\right)} a\left(h^u\right)+\varepsilon \frac{p}{1-p}\left(\prod_{\varnothing \leq h^u < \left(h^t, H, h^j\right)} a\left(h^u\right)\right)\right)}$$

This choice of $\tilde{a}\left(h^t, L, h^j\right)$ also leaves the continuation value $V\left(h^t\right)$ constant assuring that IC constraints for $\tau \leq t$ are unaffected. Incentive constraints between $t+2$ and $t+j+1$ are not affected either since the changes realized are only contingent on period $t+1$ action.

Finally, $V\left(h^t, H\right)$ has been increased and $V\left(h^t, L\right)$ decreased as a result of these changes so incentives in period $t+1$ have been relaxed. This implies at least a weak improvement can be achieved.

(iii)Follows from the proof of (ii)

(iv) Since $\forall h^t \times h^j \; a\left(h^t, H, h^j\right) \geq a\left(h^t, L, h^j\right)$ then since:

$$V\left(h^t\right)=a\left(h^t\right) \sum_{j=1}^{\infty} s\left(\delta^j\left(\sum_{h^j} P\left(h^t, h^j\right) \prod_{h^\tau \in \left(h^t, h^j\right)} a\left(h^\tau\right)\right)\right)$$

it follows that $V(h^t, H) \geq V(h^t, L)$.

(v) Suppose $a(h^t, L) = a(h^t, H) = 0$ and $a(h^t) > 0$. Let $\tilde{a}(h^t) = a(h^t) - \varepsilon$ for some small $\varepsilon > 0$. Also with probability $\gamma\left(\tilde{h}^t\right)$ after h^t let the Principal pretend that all other histories $\tilde{h}^t$ occurred where the relative weights are given according to the likelihood of each alternative history $\tilde{h}^t$ occurring on the equilibrium path. γ must also be chosen so that the value after h^t is kept constant. This change has kept values constant so past IC are still satisfied. The only change that we have made is to increase the probability that when deciding effort at $t+1$ the Agent is somewhere in the outcome tree where the current outcome realization affects his continuation value. Therefore, the incentive constraint for effort at $t+1$ has been relaxed. Incentives in all future periods are unaffected since the Agent's beliefs are the same that under the original contract. ■

Proof of Proposition 6. The incentive constraint can alternatively be written as :

$$(1-p)^T \delta^T \left(\frac{(py_H + (1-p)y_L - c)(p-q)}{(1-\delta)(1-p)} - \frac{c}{1-\delta^T} \right) - \frac{c}{\beta} \geq 0$$

Since V is increasing in T, the optimal T must therefore be the largest feasible $T \in \mathbb{N}$ such that the innequality above is satisfied for $0 < \beta \leq 1$. For all T the left hand side can be shown to be decreasing in c and q and increasing in y_H and δ. Which implies that the largest feasible (optimal) T is decreases for larger c and q and increases for larger y_H and δ. ■

Proof of Lemma 4. It follows from noting that as $\delta \to 1$ the incentive compatibility constraint (ICA T) becomes $\frac{(p-q)}{(1-p)} sT \geq c$. Since the left hand side is a positive number we can always find a T large enough so that the condition is satisfied. Note as well that for any fixed T as $\delta \to 1$, $\beta \to 0$ so there is a feasible β^* to implement the contract. ■

Proof of Theorem 2 (Folk Theorem). The inefficiency λ is given by:

$$\lambda_T = \frac{c(1-p)}{(p-q)} \frac{1}{S_T}$$

Taking the limit as $\delta \to 1$

$$\lim_{\delta \to 1} \lambda_T = \frac{c\left(1-p\right)}{\left(p-q\right)T\left(\varepsilon, s\right)s}$$

Now let $T\left(\varepsilon, s\right)$ be greater than $\bar{T}$, where:

$$\bar{T} = \frac{c\left(1-p\right)}{\varepsilon\left(p-q\right)s}$$

Therefore the inefficiency is:

$$\lambda_T = \frac{c\left(1-p\right)}{T\left(\varepsilon, s\right)\left(p-q\right)s} < \frac{c\left(1-p\right)}{\frac{c(1-p)}{\varepsilon(p-q)s}\left(p-q\right)s} = \varepsilon$$

■

[41] The proof, which is ommited, follows the same arguments as Part (iv) of Proposition 3.

[42] This is seen clearly when we simplify V to: $V = \frac{s}{1-\delta} - \frac{c(1-p)}{(p-q)}\frac{1}{1-\delta^T}$.

[43] The $(ICAT)$ constraint allows us to conjecture that if we allowed for limited amounts money burning, both instruments would play the same role in the provision of incentives. Furthermore, that they would be used together to extend the review length as much as possible.

[44] MacLeod (2003) studies this environment for the static case.

[45] Some more structure needs to be imposed on the cost function and the probability of success to keep the number of deviations to consider at bay.

[46] Similar examples could be constructed for less extreme cases, we pick these parameters for expositional convenience.

[47] It is easily verified that (TTL) is satisfied.

[48] Note that $\tilde{\sigma}^A_\omega$ can include arbitrary future deviations both in the effort and in the termination, but agrees with equilibrium until $t-1$.

[49] In words, $\tilde{\sigma}^P_\omega$ is consistent with σ^P_ω up to $t-1$ but can include arbitrary future deviations both in the messages and in the termination.

[50] For the case when $p = 1$ and $q > 0$. We have to be careful when $e_t = 1$ because now the Agent can detect a deviation if the Principal claims $y_t = L$.