Credit Traps
Web Appendix

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Proof of Lemma 1. (i) The equilibrium borrowing firm must have $B^* \leq L/(1+r^*)$, since otherwise it cannot borrow. Suppose $r^* > 0$. Then the loan market clears, so that $\int_0^{B^*} BdG(B) = C + \int_{B^*}^{I}(I - B)dG(B)$, which implies that $IG(B^*) - E[A] = C$. However, when $r^* > 0$, $B^* \leq L/(1+r^*) < L$, so $IG(L) > IG(B^*) = E[A] + C$, a contradiction. Therefore, $r^* = 0$. Since $X_1 > I$, every firm would like to borrow, and thus firms who are borrowing in equilibrium are those who can borrow. Hence, $B^* = L$, and the aggregate lending is $\int_0^L BdG(B)$.
(ii) The proof of (i) implies that in an equilibrium where $r^* = 0$, $B^* = L$. For this to be an equilibrium, we must have $\int_0^L BdG(B) \leq C + \int_L^I (I - B)dG(B)$, which implies that $IG(L) - E[A] \leq C$, a contradiction. Therefore, $r^* > 0$. The loan market clears when $r^* > 0$, so $\int_0^{B^*} BdG(B) = C + \int_{B^*}^{I}(I - B)dG(B)$, which implies that $G(B^*) = (C + E[A])/I$ and that the aggregate lending is $\int_0^{B^*} BdG(B)$. Now $\hat{r}$ is defined so that $IG(L/(1+\hat{r})) = E[A] + C$, so $B^* = L/(1+\hat{r})$. Since in equilibrium $B^* \leq L/(1+r^*)$, we have $r^* \leq \hat{r}$. If $r^* > \bar{r}$, then all firms strictly prefer not borrowing, and the loan market cannot clear. Therefore, $r^* \leq \min\{\hat{r}, \bar{r}\}$. It remains to show that $r^* \geq \min\{\hat{r}, \bar{r}\}$. Suppose that $r^* < \min\{\hat{r}, \bar{r}\}$. Then all firms strictly prefer borrowing, and thus $B^* = L/(1+r^*)$, which implies that $r^* = \hat{r}$, a contradiction.

Proof of Proposition 1. First assume that L^* satisfies $p(L^*; C) = L^*$. By definition, $B^*(L; C)$ and $r^*(L; C)$ are the associated equilibrium marginal borrowing firm and market clearing interest rate associated with the exogenous pair of liquidation values and bank capital (L, C). By construction, therefore, the vector (C, r^*, L^*, B^*) satisfies conditions (i) through (iii) of a market equilibrium. Further, we have that $L^* = p(L^*; C) = P(B^*(L^*; C), r^*(L^*; C))$, where the second equality results from the definition of the pricing function p. Thus, condition (iv) of the market equilibrium is satisfied as well, guaranteeing that C, r^*, L^*, B^* is indeed an equilibrium.

Suppose now that the vector (C, r^*, L^*, B^*) is a market equilibrium. By Section 3, it is easy to see that for every exogenous pair (L^*, B^*) there is a unique equilibrium marginal borrowing firm and market clearing interest rate. Thus, $B^*(L^*; C) = B^*$ and $r^*(L^*; C) = r^*$. Since (C, r^*, L^*, B^*) is a market equilibrium, condition (iv) implies that $L^* = P(B^*, r^*) = P(B^*(L^*; C), r^*(L^*; C)) = p(L^*; C)$. L^* is therefore a fixed point of p as required.

As a final point, note that existence of at least one equilibrium is guaranteed by the fact that p is continuous (since P is continuous) and bounded from above by X_1.

Proof of Proposition 2. Suppose that $r^* > 0$ in equilibrium. Let B^* be the marginal borrower in equilibrium. Then $B^* \le L/(1+r^*)$ as only firms with borrowing requirement not exceeding $L/(1+r^*)$ are able to borrow. Then the clearing of the loan market implies that

$$\int_0^{B^*} BdG(B) = C + \int_{B^*}^{I} (I-B)dG(B),$$

or equivalently,

$$C = \int_0^{B^*} BdG(B) - \int_{B^*}^{I} (I-B)dG(B).$$

The right hand side is strictly increasing in B^*, and $B^* \le L/(1+r^*) < \bar{B}(C)$, so the right hand side is less than $\int_0^{\bar{B}(C)} BdG(B) - \int_{\bar{B}(C)}^{I} (I-B)dG(B)$, which equals C by definition of $\bar{B}(C)$, a contradiction. Therefore, the equilibrium interest rate must be zero.

Note that it is assumed that $X_1 > I$, so when the equilibrium interest rate is zero, every firm *would like* to borrow and invest. Therefore, all the firms with borrowing requirement $B \le L$ are borrowing, while those with $B > L$ are not *able* to borrow. Therefore, at a zero interest rate the marginal borrowing firm will have a borrowing requirement of L; in other words, $B^* = L$. It is easy to check that this is indeed a market equilibrium. By definition, the indirect pricing function is therefore $p(L;C) = P(L,0)$. Finally, in this equilibrium, the level of loan supply actually lent out will equal the effective loan demand at the zero interest rate, $\int_0^{L} BdG(B)$. Since $L < \bar{B}(C)$,

$$C = \int_0^{\bar{B}(C)} BdG(B) - \int_{\bar{B}(C)}^{I} (I-B)dG(B) > \int_0^{L} BdG(B) + \int_{L}^{I} (I-B)dG(B),$$

which implies that the demand for loanable funds is smaller than the total loan supply.

When $L = \bar{B}(C)$, the above argument implies that a) there cannot be an equilibrium with $r^* > 0$, and b) $(r^*, B^*) = (0, L)$ is a market equilibrium, which obviously satisfies 2(i), (ii), and (iii).
Consider now the case where $L > \bar{B}(C)$. By the above argument, in an equilibrium with zero interest rate, the marginal borrower is L and $\int_0^{L} BdG(B) \le C + \int_{L}^{I}(I-B)dG(B)$, which cannot be true as $L > \bar{B}(C)$. Therefore, the equilibrium interest rate must be positive. It follows that the entire loan supply is lent out, and therefore the market clearing condition $\int_0^{B^*} \tilde{B}dG(\tilde{B}) = C + \int_{B^*}^{I}(I-\tilde{B})dG(\tilde{B})$ implies that $B^* = \bar{B}(C)$. Since the marginal borrower must be able to borrow, $\bar{B}(C) \le L/(1+r^*)$, or $r^* \le L/\bar{B}(C) - 1$. On the other hand, the marginal borrow has to be willing to borrow, so $r^* \le \bar{r}(L)$. Note that in all the proofs we will write $\bar{r}(L)$ to emphasize the dependence of $\bar{r}$ on L.

If the marginal borrower is indifferent between borrowing and not borrowing, then the analysis that proceeds the proposition implies that $r^* = \bar{r}(L)$. Otherwise, since a firm's participation constraint is independent of its borrowing requirement, all firms strictly prefer borrowing to not borrowing. Therefore, it must be the case that in equilibrium only firms with borrowing requirement at most $\bar{B}(C)$ can borrow, which implies that $\bar{B}(C) = L/(1+r^*)$, or $r^* = L/\bar{B}(C) - 1$. It is easy to check that we have obtained a market equilibrium in both cases. Finally, since the equilibrium marginal borrowing firm and interest rate are $\bar{B}(C)$ and r^* respectively, we have that the indirect pricing function satisfies $p(L;C) = P(\bar{B}(C), r^*)$.

Proof of Corollary 1. Consider first a liquidation value L with $L \leq \bar{B}(C)$. By Proposition 2, we have that in this region $p(L;C) = P(L,0)$. Since $P(L,0)$ is increasing in L, p will be increasing in L as well in this region (recall that, unless stated otherwise, throughout the paper 'increasing' refers to weak monotonicity.)

Similarly, by Proposition 2, if $L > \bar{B}(C)$ we have that

$$p(L;C) = P(\bar{B}(C), r^*), \tag{1}$$

with $r^* = \min\{\bar{r}(L), L/\bar{B}(C) - 1\}$ by Proposition 2. It is easy to see that $\bar{r}(L)$ is increasing in L. Therefore, r^* is increasing in L (recall that C is exogenous and therefore constant in this comparative static). Thus, since P is decreasing in r by (1) we have that p is decreasing in L.

Proof of Proposition 3. Assume first that $P(\bar{B}(C), 0) < \bar{B}(C)$ and assume by contradiction that there exists an equilibrium (C, r^*, L^*, B^*) in which the loan market clears. By Proposition 2, $p(\bar{B}(C);C) = P(\bar{B}(C), 0)$. Since by assumption $P(\bar{B}(C), 0) < \bar{B}(C)$, we therefore have that $p(\bar{B}(C);C) < \bar{B}(C)$. By Corollary 1, $p(L;C)$ is decreasing in L over the region $L > \bar{B}(C)$, so $p(L;C) < L$ for all $L \geq \bar{B}(C)$. Since any equilibrium liquidation value must satisfy $p(L^*;C) = L^*$, this implies that $L^* < \bar{B}(C)$. By Proposition 2(1)(i), however, this implies that the loan market does not clear, contrary to the original assumption.

Suppose now that $P(\bar{B}(C), 0) \geq \bar{B}(C)$ and assume that the level of bank capital is C. To show that there exists an equilibrium in which the loan market clears, note again that by Proposition 2, we have that $p(\bar{B}(C);C) = P(\bar{B}(C), 0)$. This implies that $p(\bar{B}(C);C) \geq \bar{B}(C)$. Since p is continuous and bounded from above by V there exists an $L^* \in [\bar{B}(C), V]$ which satisfies $p(L^*;C) = L^*$. By Proposition 1 this L^* is an equilibrium liquidation value. Further, by section (2)(i) of Proposition 2, since $L^* \geq \bar{B}(C)$ the loan market clears.

We therefore have that there exists an equilibrium in which the loan market clears if and only if $P(\bar{B}(C), 0) \geq \bar{B}(C)$.

We made the auxiliary assumption that there is an infinitesimal benefit of borrowing for firms with small borrowing requirement, so that the set of borrowing requirement of actual borrowers in equilibrium is always $[0, B^*]$ for some $B^* \in [0, I]$. Without this assumption, the indirect pricing function is not well-defined, but with some additional assumptions on the pricing function of the liquidated assets, Propositions 3 and 4 are still true, as shown below.
In the general setting, we allow the set of the borrowing requirement of borrowers to be a general measurable subset of $[0, I]$. (In an even more general setting, borrowers with the same borrowing requirement can make different borrowing decisions, but this does not add anything new to the model.) Let $\hat{P} : \mathcal{B}(\text{[0,I]}) \times \mathbb{R}^+ \to \mathbb{R}$ be the new direct pricing function, where $\mathcal{B}(\text{[0,I]})$ is the collection of Borel measurable subsets of $[0, I]$, and $P(B, r)$ is the price of liquidated assets when borrowers with borrowing requirement in B borrowed at Period 0 and the interest rate at Period

0 is r. Clearly, $\hat{P}([0,b],r) = P(b,r)$ where P is the pricing function introduced in the paper. To avoid confusion, in the remaining part of this discussion we use B to denote a measurable subset of $[0,I]$ and b a number in $[0,I]$, contrary to the notation used in other parts of the paper. For $B \in \mathcal{B}([0,\mathrm{I}])$, let $\mu(B) = \int_B dG(b)$, the probability measure of B.

Liquidity Pricing Assumption.
(i) $\partial P/\partial b \geq 0$ *for* $r^* = 0$.
(ii) $\hat{P}([0,b],0) \geq \hat{P}(B,0)$ *for all* $b \in [0,I]$ *and* $B \in \mathcal{B}([0,I])$ *with* $\mu(B) = G(b)$.
(iii) $\hat{P}(B,r^*)$ *is continuous in* r^* *and* $\partial\hat{P}/\partial r^* \leq 0$ *for all* $B \subset [0,I]$ *with* $\mu(B) \geq G(\bar{B}(0))$.

Assumption (i) is the same as before, and Assumption (iii) is a direct generalization of Assumption (ii) on P. Assumption (ii) states that sets of the form $[0,b]$ maximize the price of liquidated assets among all sets with the same probability measure. It is easy to check that all the three assumptions are satisfied by our micro-foundation, as in our micro-foundation

$$\hat{P}(B,r) = \min\left\{V, \max\left\{kV, \frac{1-\gamma}{\gamma}\left((X_1-(1+r)I)\mu(B) + (1+r)(I-E[b])\right)\right\}\right\},$$

which depends on B only through $\mu(B)$, and is increasing in $\mu(B)$ when $r = 0$, as $X_1 > I$.

Proof of Proposition 3.*(*under new assumptions) Let $\Pi(r,L) = x_2 + [X_1 - (1+r)L][\gamma + (1-\gamma)L^{-1}V]$. It is easy to see that a firm weakly prefers borrowing if and only if $\Pi(r,L) \geq 0$, and $\bar{r}(L)$ is such that $\Pi(\bar{r}(L),L) = 0$. Clearly, Π is continuous in both r and L, and $\Pi(0,L) > 0$ for all L. We will denote an allocation by (B,L,r) where $B \in \mathcal{B}([\prime,\mathcal{I}])$ is the set of borrowers, r is the interest rate, and L is the liquidation value of assets.
We first prove the "if" part. Since by the assumption $P(\bar{B}(C),r)/(1+r)$ is strictly decreasing in r and approaches zero when $r \to \infty$, there exists $r_1 \in [0,\infty)$ such that $P(\bar{B}(C),r_1)/(1+r_1) = \bar{B}(C)$. Let $L_1 = P(\bar{B}(C),r_1)$.

Case 1: $r_1 \leq \bar{r}(L_1)$. Then the allocation $([0,\bar{B}(C)],L_1,r_1)$ is an equilibrium. Indeed, Conditions 2 through 4 for the market equilibrium are satisfied by construction, and Condition 1 is satisfied since all firms weakly prefer borrowing as $r_1 \leq \bar{r}(L_1)$, but only those with borrowing requirement at most $L_1/(1+r_1) = \bar{B}(C)$ are able to borrow.

Case 2: $r_1 > \bar{r}(L_1)$. Then by the continuity of Π and $P(B,\cdot)$, $\Pi(r,P(\bar{B}(C),r))$ is continuous in r. Now $\Pi(r_1,P(\bar{B}(C),r_1)) < 0$ and $\Pi(0,P(\bar{B}(C),0)) > 0$, so there exists $r_2 \in (0,r_1)$ such that $\Pi(P(\bar{B}(C),r_2),r_2,\bar{B}(C)) = 0$. Let $L_2 = P(\bar{B}(C),r_2)$. Then by Assumption 1, $L_2 \geq L_1$, and thus $L_2/(1+r_2) > \bar{B}(C)$. Therefore, when the interest rate is r_2 and the liquidation value of assets is L_2, all firms with $B \in [0,L_2/(1+r_2)]$ can borrow, but all firms are indifferent between borrowing and not borrowing, so the allocation $([0,\bar{B}(C)],L_2,r_2)$ is an equilibrium.

Now we prove the "only if" part. In an equilibrium (B,L,r) in which the loan market clears, $\int_B b\,dG(b) = C + \int_{B^c}(I-b)dG(b)$, so $\mu(B) = \mu([0,\bar{B}(C)]) = G(\bar{B}(C))$. Also, the firm with borrowing requirement $\bar{B}(C)$ must be able to borrow, as borrowers with less borrowing requirement do not have enough mass. Therefore, $\bar{B}(C) \leq L/(1+r) = \hat{P}(B,r)/(1+r) \leq \hat{P}(B,0) \leq P(\bar{B}(C),0)$, where we have used the fact that $\mu(B) = G(\bar{B}(C))$ and Assumption (ii) in the last step.

The following corollary is immediate from the proof:

Corollary A. *When for a given set of parameters there exists an equilibrium in which the loan market clears, there must exist an equilibrium in which the loan market clears and firms with borrowing requirement in $[0, \bar{B}(C)]$ borrow for the same set of parameters.*

Also, note that in an equilibrium in which the loan market does not clear, the interest rate is zero and all firms strictly prefer borrowing, so the firms who borrow are those who are able to do so; in other words, in such an equilibrium firms with borrowing requirement in $[0, L]$ borrow. The condition for a market equilibrium then implies that $P(L, 0) = L$ if the loan market does not clear. This observation, together with Proposition 3, implies Proposition 4 under the new assumptions.

Proof of Proposition 4. The proposition is a direct result of Proposition 2 and Proposition 3. Consider first the case of the conventional equilibrium. Since $P(L, 0) > L$ for all $0 < L \leq I$, we know by Proposition 3 that monetary policy is effective at any $C \in [\bar{B}^{-1}(0), \bar{B}^{-1}(I)] = [0, C_{max}]$ where $C_{max} = \int_0^I BdG(B)$.

Next, consider the credit trap equilibrium described in case (ii) of the proposition. Since $P(L, 0) \geq L$ for $0 < L \leq L^*$, by Proposition 3 monetary policy is effective over the region $[\bar{B}^{-1}(0), \bar{B}^{-1}(L^*)]$. Further, since $P(L, 0) < L$ for $L > L^*$ by Proposition 3 monetary policy is ineffective for any $C > \bar{B}^{-1}(L^*)$. Maximal equilibrium lending is therefore $C^* = \bar{B}^{-1}(L^*)$.

Consider now a level of bank capital C greater than C^*. By Proposition 3, the loan market cannot clear, which implies that $r^* = 0$. Since all firms strictly prefer borrowing when $r^* = 0$, $B^* = P(B^*, 0)$, the equilibrium liquidation value of assets. However, the assumption implies that this equality holds only when $B^* = L^*$, so the equilibrium liquidation value is L^* and the aggregate lending is $\int_0^{L^*} BdG(B)$.

Case (iii) of the jump start equilibrium is proved in a similar manner. Since $P(L, 0) < L$ for $L_1 < L < L_2$, by Proposition 3 we have that monetary policy is ineffective at any $C \in (\bar{B}^{-1}(L_1), \bar{B}^{-1}(L_2)) = (C_1, C_2)$. Now, since $P(L_2, 0) = L_2$, it is easy to see that when $C = \bar{B}^{-1}(L_2)$, the following allocation is an equilibrium: the marginal borrowing firm is $B^* = L_2$, the interest rate is zero, and the liquidation value of assets is L_2. Clearly, in this equilibrium the aggregate lending is $\int_0^{L_2} BdG(B)$.

It remains to show that the equilibrium liquidation value of assets remains L_1 for $C \in (\bar{B}^{-1}(L_1), \bar{B}^{-1}(L_2))$ and that the aggregate lending remains $\int_0^{L_1} BdG(B)$ in this case. However, here we can apply the argument used in (ii).

Proof of Lemma 2. By assumption a firm liquidates its assets if and only if it is hit by the liquidity shock. Therefore, the total supply of liquidated assets is γ. Clearly, the price of liquidated assets cannot exceed V. When $P(B, r) < V$, all firms with liquidity strictly prefer buying assets, so for the market to clear it must be the case that all market liquidity has been used to buy liquidated assets. In other words, $\gamma P(B, r) = Q(B, r)$ in this case. If $\gamma P(B, r) > Q(B, r)$, then not all the liquidated assets are purchased, which implies that in equilibrium $P(B, r) = 0$, a contradiction.

Therefore, the market clearing condition can be written as

$$\gamma P(B,r) \leq Q(B,r), \text{ with equality if } P(B,r) < V,$$

or equivalently,

$$P(B,r) = \min\left\{V, \frac{Q(B,r)}{\gamma}\right\}.$$

Proof of Lemma 3. Note that

$$Q(B,r) = \int_0^B [X_1 - (1+r)\tilde{B}]dG(\tilde{B}) + \int_B^I (1+r)(I-\tilde{B})dG(\tilde{B}) = [X_1 - (1+r)I]G(B) + E[A], \quad (2)$$

where $E[A] = I - E[B]$ is the aggregate liquidity at Period 0. Therefore,

$$P(B,r) = \min\left\{V, \frac{Q(B,r)}{\gamma}\right\} = \min\left\{V, \frac{1-\gamma}{\gamma}[X_1 - (1+r)I]G(B) + (1+r)E[A]\right\}.$$

Since $\bar{B}(0)$ satisfies

$$\int_0^{\bar{B}(0)} BdG(B) = \int_{\bar{B}(0)}^I (I-B)dG(B),$$

$G(\bar{B}(0)) = E[A]/I$. Therefore, for $B \geq \bar{B}(0)$, $IG(B) - E[A] \geq 0$, and thus $P(B,r)$ is weakly decreasing in r. Furthermore, $X_1 > I$, so when $r = 0$, $P(B,r)$ is increasing in B.

Proof of Proposition 5. We first show that there exists some constant $W_0 > 0$ such that $Q(B,r)/(1-\gamma) \geq W_0$ in equilibrium for all parameter values, where $Q(B,r)$ is the total market liquidity at Period 1. Since the price of liquidated assets is at most V, a firm cannot promise to pay more than V, and thus $(1+r)\tilde{B} \leq V$ if a firm with borrowing requirement $\tilde{B}$ borrows in equilibrium. Therefore,

$$Q(B,r) = (1-\gamma)\int_0^B [X_1 - (1+r)\tilde{B}]dG(\tilde{B}) + (1+r)\int_B^I (I-\tilde{B})dG(\tilde{B}) \geq (1-\gamma)\int_0^I w(\tilde{B})dG(\tilde{B}),$$

where

$$w(\tilde{B}) \quad = \begin{cases} X_1 - V, & \text{if } \tilde{B} \leq B; \\ I - \tilde{B}, & \text{otherwise.} \end{cases}$$

Now $w(\tilde{B}) \geq 0$ for all $\tilde{B} \in [0, I]$, and also $w(\tilde{B}) \geq \min\left\{X_1 - V, \frac{I}{2}\right\}$ for $\tilde{B} \leq I/2$. Put $W_0 = G(I/2)\min\left\{X_1 - V, \frac{I}{2}\right\}$. Then

$$Q(B,r) \geq (1-\gamma)\int_0^{\frac{I}{2}} w(\tilde{B})dG(\tilde{B}) \geq (1-\gamma)W_0.$$

Now choose $\bar{\gamma}$ so that $(1-\bar{\gamma})W_0/\bar{\gamma} = V$. Then in equilibrium $Q(B,r)/\gamma \geq V$ for all $\gamma \leq \bar{\gamma}$, and thus $P(B,r) = V$. Clearly, when $I < V$ this also means that we are in a conventional equilibrium. (Note: when a general set of borrowers, which can be any measurable subset of $[0, I]$, is allowed in equilibrium, the proof goes essentially unchanged, except that now $w(\tilde{B}) = (X_1 - V - I + \tilde{B})1_{\mathcal{B}} + I - \tilde{B}$

where $\mathcal{B}$ is the set of borrowers in equilibrium.)

Proof of Proposition 6. We first show that $\bar{L}$ can be attained. An equilibrium is characterized by (B, L, r), where B is the borrowing requirement of the marginal borrower, L is the liquidation value of assets, and r is the interest rate. Since $L \leq V$ in all equilibrium, there exists a sequence of equilibriums (B_n, L_n, r_n) for bank capital C_n such that $L_n \to \sup_C L^*$. Since $B_n \in [0, I]$, $L_n \in [0, V]$, and $r_n \in [0, \bar{r}(V)]$ for all n, one can choose a subsequence $(B_{n_k}, L_{n_k}, r_{n_k})$ such that $(B_{n_k}, L_{n_k}, r_{n_k})$ converges to some $(\hat{B}, \hat{L}, \hat{r})$ as $k \to \infty$. Since all equilibrium conditions are continuous, $(\hat{B}, \hat{L}, \hat{r})$ is an equilibrium for some $\hat{C}$. Clearly, $\hat{L} = \sup_C L^*$.Therefore, the supremum of L^* can be attained, and thus $\bar{L} = \max_C\{L^*\}$ is well-defined. The following lemma will be also be useful later:

Lemma A1. *If $P(I, 0) < I$, then $\bar{L} = \max\{L \in [0, I] : P(L, 0) \geq L\}$. Furthermore, the maximum aggregate lending in equilibrium is $\int_0^{\bar{L}} BdG(B)$. (As before, the maximum is taken over all stances of monetary policy.)*

Proof. Let $\hat{L} = \sup\{L : P(L, 0) \geq L\}$. Then by the continuity of P, $\hat{L}$ can be attained. Since $P(I, 0) < I$, $\hat{L} < I$, and by the continuity of P again $P(\hat{L}, 0) = \hat{L}$. This implies that the following allocation is an equilibrium: firms with borrowing requirement in $[0, \hat{L}]$ borrows, the interest rate is zero, and the liquidation value of assets is $\hat{L}$. Therefore, $\bar{L} \geq \hat{L}$.
Let $(B^*, \bar{L}, r^*)$ be an equilibrium that attains the maximum liquidation value of assets. If $r^* > 0$, then $B^* = \bar{B}(C)$ for some $C \in [0, C_{max}]$. By Proposition 3 $P(B^*, 0) \geq B^*$. Moreover, $P(B^*, r^*) \leq P(B^*, 0)$. If $r^* = 0$, then it must be the case that all firms strictly prefer borrowing, and the definition of market equilibrium implies that $P(B^*, 0) = \bar{L} = B^*$. Therefore, we always have $P(B^*, r^*) \leq P(B^*, 0)$ and $P(B^*, 0) \geq B^*$. By the construction of $\hat{L}$, $B^* \leq \hat{L}$. Now by the monotonicity assumptions on P,

$$\bar{L} = P(B^*, r) \leq P(B^*, 0) \leq P(\hat{L}, 0) = \hat{L}.$$

Therefore, $\bar{L} = \hat{L}$.
Now we show that the maximum aggregate lending is $\int_0^{\bar{L}} BdG(B)$. We have seen that $(\bar{L}, \bar{L}, 0)$ is an equilibrium. (i.e. the marginal borrower and the liquidation value of assets are both $\bar{L}$, and the interest rate is zero.) In this equilibrium the aggregate lending is $\int_0^{\bar{L}} BdG(B)$. Suppose that there exists some equilibrium (B', L', r') in which the aggregate lending is more than $\int_0^{\bar{L}} BdG(B)$. Then since only borrowers with borrowing requirement at most $L'/(1+r')$ can borrow, $\int_0^{L'/(1+r')} BdG(B) > \int_0^{\bar{L}} BdG(B)$, implying that $L' > (1 + r')\bar{L} \geq \bar{L}$, contradicting the maximality of $\bar{L}$. Therefore, the maximum aggregate lending in equilibrium is $\int_0^{\bar{L}} BdG(B)$. □

Now we return to the proof of Proposition 6. Note that by Eq. (2) in any equilibrium (B, L, r) the market liquidity at Period 1 is

$$Q(B, r) = (1 - \gamma)\{[X_1 - (1 + r)I]G(B) + (1 + r)E[A]\},$$

where $E[A]$ is the expectation of firms' liquidity at Period 0. For $B < \bar{B}(0)$, r must be zero as otherwise the clearing of the loan market would imply a negative C, and for $B \geq \bar{B}(0)$, $Q(B, r)$ is weakly decreasing in r as in that case $IG(B) - E[A] \geq IG(\bar{B}(0)) - E[A] = 0$. Therefore,

$$Q(B, r) \leq Q(B, 0) \leq (1 - \gamma)(X_1 - I + E[A]).$$

Put $\gamma_1 = 1 - V/(X_1 - I + E[A])$. Then

$$\frac{Q(B,r)}{\gamma} \leq \frac{1-\gamma}{\gamma}(X_1 - I + E[A]) < V,$$

when $\gamma > \gamma_1$. This implies that $L = \min\{V, Q(B,r)/\gamma\} < V$, for all equilibrium (B, L, r). Consequently $\bar{L} < V$. Note also that when $\gamma > \gamma_1$,

$$P(I,0) = \frac{1-\gamma}{\gamma}(X_1 - I + E[A]).$$

Put

$$\gamma_2 = \frac{X_1 - I + E[A]}{X_1 + E[A]}.$$

Then $P(I, 0) < I$ for $\gamma > \max\{\gamma_1, \gamma_2\}$. Put

$$\bar{\gamma} = \max\{\gamma_1, \gamma_2\}. \tag{3}$$

In what follows, assume that $\gamma > \bar{\gamma}$.

(i) We have seen that $L < V$ in all equilibrium, and $\bar{L}$ can be attained, so $\bar{L} < V$. By Lemma A1, $P(L, 0) < L$ for all $L \in (\bar{L}, I]$. That the monetary policy is ineffective beyond $\bar{B}^{-1}(\bar{L})$ follows directly from Proposition 3.

(ii) In this part, we write $\bar{L}(\gamma)$ to emphasize the dependence of $\bar{L}$ on γ. Fixing $\gamma', \gamma" \in (\bar{\gamma}, 1)$ with $\gamma'' < \gamma'$, we show that $\bar{L}(\gamma'') > \bar{L}(\gamma')$. Let $(B^*, \bar{L}(\gamma'), r^*)$ be an equilibrium that attains the maximum liquidation value of assets when the probability of the liquidity shock is γ'. Then the definition of an equilibrium implies that r^* and $\bar{L}(\gamma')$ solves the following system of (r, L) with $\gamma = \gamma'$:

$$\begin{cases} r = \min\left\{\bar{r}(L;\gamma), \frac{L}{B^*} - 1\right\}; \\ L = P(B^*, r; \gamma), \end{cases}$$

where we have emphasized the dependence of $\bar{r}$ and P on γ. It is easy to see that $\bar{r}$ is strictly increasing in γ and P is strictly decreasing in γ. There are two cases:

Case 1: $B^* < \bar{B}(0)$. Then $r^* = 0$ and $\bar{L}(\gamma') = B^* = P(B^*, 0; \gamma')$. Since P is strictly decreasing in γ, $P(B^*, 0; \gamma'') > B^*$. Let $B' = \min\{\bar{B}(0), \sup\{\tilde{B} : P(\tilde{B}, 0; \gamma'') > \tilde{B}\}\}$. Then $B' > B^*$ and $P(B', 0; \gamma'') \geq B'$ with equality when $B' < \bar{B}(0)$. If $B' = \bar{B}(0)$, by Proposition 3 there exists an equilibrium (B', L', r') when $\gamma = \gamma''$. Then

$$\bar{L}(\gamma'') \geq L' \geq (1 + r')B' \geq \bar{B}(0) > B^* = \bar{L}(\gamma').$$

If $B' < \bar{B}(0)$, then $(B', B', 0)$ is an equilibrium, and thus $\bar{L}(\gamma'') \geq B' > B^* = \bar{L}(\gamma')$.

Case 2: $B^* \geq \bar{B}(0)$. Then $P(B^*, 0; \gamma'') > P(B^*, 0; \gamma') \geq B^*$, and by Proposition 3, there exists an equilibrium (B^*, L', r') when $\gamma = \gamma''$. Suppose $L' \leq \bar{L}(\gamma')$. Then

$$r' = \min\left\{\bar{r}(L';\gamma''), \frac{L'}{B^*} - 1\right\} \leq \min\left\{\bar{r}(\bar{L}(\gamma');\gamma'), \frac{\bar{L}(\gamma')}{B^*} - 1\right\} = r^*,$$

and thus

$$L' = P(B^*, r'; \gamma'') > P(B^*, r^*; \gamma') = \bar{L}(\gamma'),$$

a contradiction. Therefore, $L' > \bar{L}(\gamma')$ and thus $\bar{L}(\gamma'') > \bar{L}(\gamma')$.

(iii) We have seen that in all equilibrium (B, L, r),

$$L = \frac{Q(B,r)}{\gamma} \leq \frac{Q(B,0)}{\gamma} \leq \frac{1-\gamma}{\gamma}(X_1 - I + E[A]),$$

so

$$\bar{L} \leq \frac{1-\gamma}{\gamma}(X_1 - I + E[A]),$$

but the right hand side approaches zero as γ approaches one.

Proof of Proposition 7. The proofs of Propositions 7 and 8 rely on the following lemma.

Lemma A2. Given a distribution G and (X_1, V, I) with $\bar{B}(0) < V$, there exists $\gamma_1 < \gamma_2$ such that quantitative easing will be successful in increasing lending for $\gamma \in [\gamma_1, \gamma_2]$ if

$$\frac{\min\{V, I\}}{(X_1 - I)G(\min\{V, I\}) + E[A]} < \frac{I\bar{B}(0)}{X_1 E[A]}. \tag{4}$$

Proof. Let $f(L) = P(L, 0) - L$. Clearly, $f(L)$ is continuous in L. Clearly, quantitative easing will be successful if $f(\bar{B}(0)) < 0$ and $f(L) > 0$ for some $L \in (\bar{B}(0), \min\{V, I\}]$, by Proposition 3. Note that

$$f(\bar{B}(0)) = \min\left\{V, \frac{1-\gamma}{\gamma}[(X_1 - I)G(\bar{B}(0)) + E[A]]\right\} - \bar{B}(0).$$

Let

$$\alpha_1 = \frac{\bar{B}(0)}{(X_1 - I)G(\bar{B}(0)) + E[A]}.$$

Then $f(\bar{B}(0)) < \bar{B}(0)$ if $(1-\gamma)/\gamma < \alpha_1$. Using the definition of $\bar{B}(0)$, $G(\bar{B}(0)) = E[A]/I$, so $\alpha_1 = (\bar{B}(0)I)/(X_1 E[A])$.
Let

$$\alpha_2 = \frac{\min\{V, I\}}{(X_1 - I)G(\min\{V, I\}) + E[A]}.$$

Then by the continuity of G, when $(1-\gamma)/\gamma > \alpha_2$, one can find $L \in (\bar{B}(0), \min\{V, I\})$ such that

$$\frac{1-\gamma}{\gamma}[(X_1 - I)G(L) + E[A]] > \min\{V, I\},$$

which implies that

$$f(L) = \min\left\{V, \frac{1-\gamma}{\gamma}[(X_1 - I)G(L) + E[A]]\right\} - L > \min\left\{V, \frac{1-\gamma}{\gamma}[(X_1 - I)G(L) + E[A]]\right\} - \min\{V, I\} \geq 0.$$

Therefore, if $\alpha_2 < \alpha_1$, which is exactly Eq. (4), then quantitative easing will be successful for $\gamma \in (1/(1+\alpha_1), 1/(1+\alpha_2))$. □

Now we return to the proof of Proposition 7. Note that when $I < V$, the left hand side of Eq. (4) becomes $I/(X_1 - I + E[A])$, and thus Eq. (4) is equivalent to the following condition:

$$(\bar{B}(0) - E[A])X_1 > \bar{B}(0)(I - E[A]). \tag{5}$$

Let $A_0 = (X_1 - I)\delta/X_1$ and $\epsilon = (X_1 - I)/(X_1 I)$. If $\bar{B}(0) \geq \delta$, then Eq. (5) holds because

$$(\bar{B}(0) - E[A])/\bar{B}(0) > (\bar{B}(0) - A_0)/\bar{B}(0) \geq 1 - \frac{X_1 - I}{X_1} > (I - E[A])/X_1.$$

Now assume that $\bar{B}(0) < \delta$. Note that since $E[A] = IG(\bar{B}(0))$, we have

$$[\bar{B}(0) - E[A]]X_1 = [\bar{B}(0) - IG(\bar{B}(0))]X_1 > X_1\bar{B}(0)(1 - I\epsilon) = I\bar{B}(0) > \bar{B}(0)(I - E[A]),$$

where we have used the fact that $G'(L) = g(L) < \epsilon$ for all $L \in (0, \delta)$ in the second step. Therefore, Eq. (5) is satisfied in this case too.

Proof of Proposition 8. Again, under the assumption that $I < V$, Eq. (4) is equivalent to Eq. (5). Since G is strictly convex,

$$E[A] = IG(\bar{B}(0)) < I\left[\frac{I - \bar{B}(0)}{I}G(0) + \frac{\bar{B}(0)}{I}G(I)\right] = \bar{B}(0).$$

Let $\underline{X_1} = \bar{B}(0)(I - E[A])/(\bar{B}(0) - E[A])$. Then $\underline{X_1} > 0$ and Eq. (5) holds when $X_1 > \underline{X_1}$.

Proof of Proposition 9. Since $X_1 < I/(1 - \gamma)$,

$$\frac{1 - \gamma}{\gamma}(X_1 - I) < I.$$

Therefore, there exists $A_0 > 0$ and $C^* < C_{max} \equiv \bar{B}^{-1}(I)$ such that

$$\frac{1 - \gamma}{\gamma}(X_1 - I) + A_0 < \bar{B}(C^*).$$

Then for any distribution G with $E[A] \leq A_0$ and any $C \in [C^*, C_{\max}]$, we have

$$P(\bar{B}(C), 0) = \min\left\{V, \frac{1 - \gamma}{\gamma}(X_1 - I)G(\bar{B}(C)) + E[A]\right\} < \frac{1 - \gamma}{\gamma}(X_1 - I) + A_0 < \bar{B}(C^*) \leq \bar{B}(C).$$

Therefore, by Proposition 3, monetary policy is ineffective beyond C^*.

Proof of Proposition 10. To emphasize the dependence of P, $E[A]$ and $\bar{B}$ on the distribution G, we will write $P(B, r; G)$, $E_G[A]$ and $\bar{B}(C; G)$, respectively. By Proposition 3 and the assumption, $P(I, 0; G_1) < I$. By Lemma A1, $\bar{L}_1 = \max\{L \in [0, I] : P(L, 0; G_1) \geq L\}$. Consequently, $P(L, 0; G_1) < L$ for all $L \in (\bar{L}_1, I]$.
Since G_2 stochastically dominates G_1, $E_{G_2}[A] < E_{G_1}[A]$, and $G_2(L) \leq G_1(L)$ for all $L \in [0, I]$. Therefore, for all $L \in [0, I]$,

$$\begin{aligned} & P(L, 0; G_2) \\ = \;& \min\left\{V, \frac{1 - \gamma}{\gamma}((X_1 - I)G_2(L) + E_{G_2}[A])\right\} \\ \leq \;& \min\left\{V, \frac{1 - \gamma}{\gamma}((X_1 - I)G_1(L) + E_{G_1}[A])\right\} \\ = \;& P(L, 0; G_1). \end{aligned}$$

In particular, $P(L, 0; G_2) \leq P(L, 0; G_1) < L$ for all $L \in (L_1^*, I]$. Therefore, Lemma A1 applies, and $\bar{L}_2 = \max\{L \in [0, I] : P(L, 0; G_2) \geq L\} \leq \bar{L}_1$. The result on the aggregate lending also follows from Lemma A1.