Appendix: Reciprocity in Groups and the Limits to Social Capital.

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Existence of Equilibrium Values.

Proposition 1. *An equilibrium exists.*

Proof . Fix any value v in some large compact interval V containing all feasible payoffs. For any state ω and group S of size s, define

$$\Delta(\omega, v, S) = \max\{\min_i\{(1 - \delta)c(q, \alpha_i) - \delta[v - v^*(s)]\}, 0\}.$$

Obviously, $\Delta(\omega, v, S) > 0$ if and only if S has a strictly profitable deviation under v.

Define a *collection* $\phi^1(\omega, v)$ of symmetric probability distributions on (ω, v) by:

[i] the *singleton* set containing

(1) $$q(\omega) = 0 \text{ and } p(\omega, v, S) = \frac{\Delta(\omega, v, S)}{\sum_T \Delta(\omega, v, T)} \ \forall S$$

if some *strictly* profitable deviation exists at (ω, v), and

[ii] the collection of *all* symmetric probability distributions of the form (p, q) otherwise, under the restriction that $p(\omega, v, S)$ can have positive value only if S has a weakly profitable deviation.

Lemma 1. *For each ω, $\phi^1(\omega, v)$ is nonempty, convex-valued, and upper-hemi-continuous (uhc) in v.*

Proof. Obviously, $\phi^1(\omega, v)$ is nonempty and convex-valued for each (ω, v). Now we claim that it is uhc in v. To this end, let v^k be some sequence in V converging to v. Consider a corresponding sequence $(p^k, q^k) \in \phi^1(\omega, v^k)$ (we omit the explicit dependence on ω and S for notational ease) and extract a convergent subsequence converging to

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some (p, q) (but retain the original sequence notation). We claim that $(p, q) \in \phi^1(\omega, v)$.

This claim is obviously true if no strictly profitable deviation exists at (ω, v).[1] So suppose that a strictly profitable deviation does exist at (ω, v). But then a strictly profitable deviation must exist for k large enough, so that far enough out in the sequence, (p^k, q^k) must be uniquely pinned down by the condition (1). Because $\Delta(\omega, v, S)$ is obviously continuous in v, we must have that $(p^k, q^k) \to (p, q)$ in this case as well. ∎

We now construct a second map — this time, a function — that links symmetric probability systems to values. In line with condition [3] of an equilibrium (Consistent Values), simply define it by

(2)
$$\phi^2(p, q) = \frac{1}{1 - \delta E_\omega q(\omega)} E_\omega \left[q(\omega)(1 - \delta)u_i(\omega) + \sum_S p(\omega, S)v_i(\omega, S) \right].$$

The reason ϕ^2 is well-defined is precisely because p is symmetric, so that the subscript i on the right hand side of (2) no longer appears on the left hand side after integrating.

It is trivial to see that ϕ^2 is continuous in (p, q). Now compose the two correspondences, by defining a third correspondence $\phi : V \mapsto V$:

(3)
$$\phi(v) = \{v' \in V | v' = \phi^2(p, q) \text{ for } (p, q) \text{ with } (p(\omega), q(\omega)) \in \phi^1(\omega, v) \ \forall \omega\}.$$

Since ϕ^2 is a continuous function on a non-empty, covex and upper-hemi continuous correspondence (by Lemma 1), $\phi(v)$ is nonempty, convex-valued, and upper-hemi-continuous (uhc) in v. Hence, ϕ must have a fixed point, call it $v^*(n)$. Define an associated probability system (p, q) by the particular value of (p, q) in (3) that permits the fixed point to be attained. One can now check that all the five conditions for an equilibrium are satisfied. ∎

Fragility of Mutual Help Groups

Proposition 2. *For any $\epsilon > 0$, there exists $\overline{n}$ so that for all $n \geq \overline{n}$,* $\mathrm{Frag}(n) > 1 - \epsilon$.

Proof. Assume that the proposition is false. Then there exists $\epsilon > 0$ and an infinite set N of group sizes such that for all $n \in N$, $\mathrm{Frag}(n) \leq 1 - \epsilon$.

By our assumptions on the cost function, there exists a closed interval $I(p)$ containing p in its interior and a value $\underline{c} > 0$ such that $c(q, \alpha) \geq \underline{c}$ for all cost shocks α and $q \in I(p)$. Fix this interval and the lower bound on costs in what follows.

Note that per-capita group payoffs are obviously bounded. It follows that for every $\mu > 0$, there exists $n(\mu) \in N$ so that $v^*(n) - v^*(n(\mu)) < \mu$ for all $n > n(\mu)$, $n \in N$. Pick any such $\mu < \frac{1-\delta}{\delta}\underline{c}$, where $\underline{c}$ is described in the previous paragraph.

An application of the weak law of large numbers yields the following implication: For every $\epsilon > 0$ there exists a group size $\hat{n}$ such that for every $n \in N$ with $n > \hat{n}$, the joint event that

(a) the number of actual donors exceeds $n(\mu)$, and

(b) the proportion of those in need lies within $I(p)$

has probability exceeding $1 - \epsilon$. But in this event, we may use the definition of μ to conclude that for all individuals i in some subgroup of size $n(\mu)$,

$$c(q, \alpha_i) \geq \underline{c} > \frac{\delta}{1-\delta}\mu > \frac{\delta}{1-\delta}[v^*(n) - v^*(n(\mu))],$$

so that (**??**) holds for $s = n(\mu)$. We may therefore conclude that the fragility of all such $n \in N$ exceeds $1 - \epsilon$, which is a contradiction.

Proposition 3. *For any $\mu > 0$, there exists $\overline{\delta} < 1$ and $\overline{n}$ so that, for all $n \geq \overline{n}$ and all $\delta \geq \overline{\delta}$,* $\mathrm{Frag}(n) < \mu$.

Proof. Fix $\mu > 0$. Now choose a size $m > 1$ and a value ϵ with $\widetilde{v}(m) - \widetilde{v}(1) > \epsilon > 0$, and define

$$\overline{c}(\delta) \equiv \frac{\delta}{1-\delta}[\widetilde{v}(m) - \epsilon - \widetilde{v}(1)].$$

Denote as $I(c)$ the set of q so that $c(q,\alpha) \geq c$ for any cost shock α. It is easy to select $\bar{\delta}$ so that $max_\alpha c(p,\alpha) < \bar{c} \equiv \bar{c}(\bar{\delta})$.

The probability that $q \in I(\bar{c})$ is an upper bound on the i-fragility of a group with a value v of at least $\underline{v} = \tilde{v}(m) - \epsilon$.

It follows that for any group of size n, if $\hat{v}(n) \geq \underline{v}$, a lower bound on the utility of the agents is given by

$$v(n) = \frac{1}{[1 - \delta Pr(q \notin I(\bar{c}))]} \sum_{k=0}^{n-1} p(k, n-1) \left[p\, r\left(\frac{k+1}{n}\right) b - (1-p)\, d\left(\frac{k}{n}\right) Ec\left(\frac{k}{n}, \alpha\right) \right].$$

An application of the weak law of large numbers yields the following implication: for every $\eta > 0$ there exists a group size $\hat{n}(\eta)$ such that for every $n > \hat{n}(\eta)$, the probability that the proportion of those in need lies within $I(\bar{c})$ has probability less than η. It follows that we can choose η sufficiently small that (i) $v(n) \geq \underline{v}$ for all $n > \hat{n}(\eta)$, and (ii) $\eta < \mu$.

Since $\bar{c}(\delta)$ is increasing, it follows that, for $\delta > \bar{\delta}$, the fragility of all groups of size $n > \hat{n}(\eta)$ will be less than μ.

Social Networks.

Proposition 4. *For any graph $g \subset g^c$, $\delta(g^c) \leq \delta(g)$ and the inequality is strict for some g.*

Proof. Let $w_i(g, \omega)$ denote the net payoff of i at state ω. For any ϵ, let $\underline{p}$ be such that for all i, $\sum_{\omega | \omega \neq \omega_{ri}} \text{for some i}\ w_i(g, \omega) < \epsilon$. Then,

$$\tilde{v}_j(g) - \tilde{v}_i(g) \;>\; pb(\rho_j(g, \omega_{rj}) - \rho_i(\omega_{ri}) + pc(\sum_{k, ik \in g} \eta_i(g, \omega_{rk}) - \sum_{k, jk \in g} \eta_j(g, \omega_{rk})) - \epsilon$$

$$>\; pc(\sum_{k, ik \in g} \eta_i(g, \omega_{rk}) - \sum_{k, jk \in g} \eta_j(g, \omega_{rk})) - \epsilon.$$

Hence, one can choose ϵ and hence $\underline{p}$ so that $\tilde{v}_j(g) - \tilde{v}_i(g) > 0$.

By a similar argument, for any ϵ, define $\bar{p}$ so that for all i, i, $\sum_{\omega | \omega \neq \omega_{di}} \text{for some i}\ w_i(g, \omega) < \epsilon$. Then,

$$\widetilde{v}_i(g) - \widetilde{v}_j(g) \;>\; (1-p)b\Big(\sum_{k,ik\in g}\rho_i(g,\omega_{dk}) - \sum_{k,jk\in g}\rho_j(g,\omega_{dk})\Big) + (1-p)c(\eta_j(g,\omega_{dj}) - \eta_i(g,\omega_{di}))$$

$$>\; (1-p)b\Big(\sum_{k,ik\in g}\rho_i(g,\omega_{dk}) - \sum_{k,jk\in g}\rho_j(g,\omega_{dk})\Big) - \epsilon.$$

establishing the result. ∎

Proof. We show that $\widetilde{v}_i(g^c) \geq \min_i \widetilde{v}_i(g)$ for any graph g, with strict inequality for some graphs g.

Notice that for any state ω,

$$\sum_{i|i=R}\rho_i(g,\omega) = \sum_{i|i=D}\eta_i(g,\omega) \leq \min\{k(\omega), n-k(\omega)\} = \sum_{i|i=R}\rho_i(g^c,\omega) = \sum_{i|i=D}\eta_i(g^c,\omega),$$

Hence,

$$\sum_{i|i=R}\rho_i(g,\omega)b - \sum_{i|i=D}\eta_i(g,\omega)Ec(\alpha) \leq \sum_{i|i=R}\rho_i(g^c,\omega)b - \sum_{i|i=D}\eta_i(g^c,\omega)Ec(\alpha).$$

Taking expectations over ω,

$$\sum_i \widetilde{v}_i(g) \leq \sum_i \widetilde{v}_i(g^c).$$

By anonymity, $\widetilde{v}_i(g^c) = \frac{\sum_i \widetilde{v}_i(g^c)}{n}$.

This shows that $\widetilde{v}_i(g^c) \geq \min_i \widetilde{v}_i(g)$ for any graph g. Furthermore, if the graph g is not symmetric or disconnected, $\widetilde{v}_i(g^c) > \min_i \widetilde{v}_i(g)$.

Now consider $\overline{\delta}$ such that

$$\frac{\overline{\delta}}{1-\overline{\delta}}(\widetilde{v}_i(g^c) - \widetilde{v}(1)) = \max_\alpha c(\alpha).$$

For any $\delta \geq \overline{\delta}$, Frag($g^c$)=0 and as $\widetilde{v}_i(g^c) \geq \min_i \widetilde{v}_i(g)$ for all g with strict inequality for some g, the conclusion follows. ∎