

**Supplementary Material for**  
**“An Ascending Auction for Interdependent Values:**  
**Uniqueness and Robustness to Strategic Uncertainty”**

by

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The appendix contains the proofs of Proposition 1 and 2.

**Proof of Proposition 1.** The uniqueness result is an immediate consequence of Theorem 1 and Theorem 3 in general incomplete information environments in Dirk Bergemann and Stephen Morris, “Robust Implementation: The Case of Direct Mechanism,” Cowles Foundation Discussion Paper 1561, Yale University (2006).■

The proof of Proposition 2 assumes that the payoff types of the bidders are distinct:  $\theta_1 < \theta_2 < \dots < \theta_I$ . In addition we allow the exit time of each agent to be in the interval  $[0, I - 1]$  rather than  $[0, 1]$ . With a larger set of feasible exit times we avoid the possibility of corner solutions in the best response functions. The resulting set of rationalizable exit times will nonetheless be in  $[0, 1]$ . Proposition 2 continues to hold without distinct payoff types and with exit times restricted to the unit interval. The only consequence is a longer proof caused by the necessity of case distinctions due to corner solutions.

**Proof of Proposition 2.** The proof proceeds by induction on the number  $j$  of bidders still to leave the auction. We show that the best response of bidder  $I - j$  to the reports of the bidders exiting before him is given by:

$$\beta_{I-j}(t_1, t_2, \dots, t_{I-j-1}) = \theta_{I-j} + \frac{\gamma}{1 + j\gamma} \sum_{i=1}^{I-j-1} (\theta_i - t_i). \quad (1)$$

We begin with the final bidder  $I$  and thus  $j = 0$ . We showed earlier in (??) that the best response function of bidder  $I$  is indeed given by:

$$\beta_I(t_1, t_2, \dots, t_{I-1}) = \theta_I + \gamma \sum_{i=1}^{I-1} (\theta_i - t_i).$$

We now proof the general inductive step. With the outcome function defined by (??), (??) and (??), we can write the payoff of agent  $I - j$  for  $j > 0$  as follows:

$$\varepsilon \left( \left( \theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i \right) t_{I-j} - \frac{1}{2} w_{I-j} t_{I-j}^2 - \gamma t_{I-j} \left( \sum_{i \neq I-j} t_i \right) \right). \quad (2)$$

We can rewrite the payoff function of bidder  $I - j$  by separating the bidders who exited before and those who will exit after  $I - j$ :

$$\varepsilon \left( \left( \theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i \right) t_{I-j} - \frac{1}{2} w_{I-j} t_{I-j}^2 - \gamma t_{I-j} \left( \sum_{i < I-j} t_i + \sum_{i > I-j} t_i \right) \right). \quad (3)$$

By hypothesis, the inductive step holds for all  $k < j$ . For all  $i > I - j$ , we can therefore replace the report by the best response given by (1):

$$\varepsilon \left( \left( \theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i \right) t_{I-j} - \frac{1}{2} w_{I-j} t_{I-j}^2 - \gamma t_{I-j} \left( \sum_{i < I-j} t_i + \sum_{i > I-j} \beta_i(t_1, \dots, t_{i-1}) \right) \right). \quad (4)$$

In particular, as the inductive step holds for all  $i > I - j$ , we can rewrite the best response of every agent  $i > I - j$  as follows:

$$\beta_i(t_1, t_2, \dots, t_{I-j}) = \theta_i + \frac{\gamma}{1 + j\gamma} \sum_{k=1}^{I-j} (\theta_k - t_k). \quad (5)$$

We can now insert  $\beta_i(t_1, t_2, \dots, t_{I-j})$  for all  $i > I - j$  into (4) to get:

$$\varepsilon \left( \left( \theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i \right) t_{I-j} - \frac{1}{2} w_{I-j} t_{I-j}^2 - \gamma t_{I-j} \left( \sum_{i < I-j} t_i + \sum_{i > I-j} \left( \theta_i + \frac{\gamma}{1 + j\gamma} \sum_{k=1}^{I-j} (\theta_k - t_k) \right) \right) \right).$$

We find the best response of bidder  $I - j$  by determining the optimal exit time  $t_{I-j}$  in response to past exit times and differentiate the above payoff with respect to the exit time  $t_{I-j}$ :

$$\left( \theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i \right) - w_{I-j} t_{I-j} - \gamma \left( \sum_{i < I-j} t_i + \sum_{i > I-j} \left( \theta_i + \frac{\gamma}{1 + j\gamma} \sum_{k=1}^{I-j} (\theta_k - t_k) \right) - \frac{j\gamma}{1 + j\gamma} t_{I-j} \right) = 0.$$

We can collect terms to obtain

$$\left( \theta_{I-j} + \left( \gamma - \frac{j\gamma^2}{1 + (j-1)\gamma} \right) \sum_{i < I-j} (\theta_i - t_i) \right) - w_{I-j} t_{I-j} - \gamma \left( + \frac{j\gamma}{1 + (j-1)\gamma} \theta_{I-j} - \frac{j\gamma}{1 + (j-1)\gamma} 2t_{I-j} \right) = 0,$$

or

$$\left( \theta_{I-j} \left( 1 - \frac{j\gamma^2}{1 + (j-1)\gamma} \right) + \left( \gamma - \frac{j\gamma^2}{1 + (j-1)\gamma} \right) \sum_{i < I-j} (\theta_i - t_i) \right) = \left( w_{I-j} - \frac{2j\gamma^2}{1 + (j-1)\gamma} \right) t_{I-j}. \quad (6)$$

We can then solve for  $t_{I-j}$  and find that

$$t_{I-j} = \theta_{I-j} + \frac{\gamma}{1 + j\gamma} \sum_{i=1}^{I-j-1} (\theta_i - t_i),$$

thus proving the inductive step. ■