

Supplementary Material for
“Ascending Auctions: Uniqueness and Robustness to Strategic Uncertainty”
by Dirk Bergemann and Stephen Morris

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The appendix contains the proofs of Proposition 1 and 2.

Proof of Proposition 1. The uniqueness result is an immediate consequence of Theorem 1 and Theorem 3 in general incomplete information environments in Bergemann, D. and Morris, S. (2009): “Robust Implementation in Direct Mechanisms,” *Review of Economic Studies*, **76**, 1175-1206. ■

The proof of Proposition 2 assumes that the payoff types of the bidders are distinct: $\theta_1 < \theta_2 < \dots < \theta_I$. In addition we allow the exit time of each agent to be in the interval $[0, I - 1]$ rather than $[0, 1]$. With a larger set of feasible exit times we avoid the possibility of corner solutions in the best response functions. The resulting set of rationalizable exit times will nonetheless be in $[0, 1]$. Proposition 2 continues to hold without distinct payoff types and with exit times restricted to the unit interval. The only consequence is a longer proof caused by the necessity of case distinctions due to corner solutions.

Proof of Proposition 2. The proof proceeds by induction on the number j of bidders still to leave the auction. We show that the best response of bidder $I - j$ to the reports of the bidders exiting before him is given by:

$$\beta_{I-j}(t_1, t_2, \dots, t_{I-j-1}) = \theta_{I-j} + \frac{\gamma}{1 + j\gamma} \sum_{i=1}^{I-j-1} (\theta_i - t_i). \quad (1)$$

We begin with the final bidder I and thus $j = 0$. We showed earlier in (6) that the best response function of bidder I is indeed given by:

$$\beta_I(t_1, t_2, \dots, t_{I-1}) = \theta_I + \gamma \sum_{i=1}^{I-1} (\theta_i - t_i).$$

We now proof the general inductive step. With the outcome function defined by (4), (9) and (10), we can write the payoff of agent $I - j$ for $j > 0$ as follows:

$$\varepsilon \left(\left(\theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i \right) t_{I-j} - \frac{1}{2} w_{I-j} t_{I-j}^2 - \gamma t_{I-j} \left(\sum_{i \neq I-j} t_i \right) \right).$$

We can rewrite the payoff function of bidder $I - j$ by separating the bidders who exited before and those who will exit after $I - j$:

$$\varepsilon \left(\left(\theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i \right) t_{I-j} - \frac{1}{2} w_{I-j} t_{I-j}^2 - \gamma t_{I-j} \left(\sum_{i < I-j} t_i + \sum_{i > I-j} t_i \right) \right).$$

By hypothesis, the inductive step holds for all $k < j$. For all $i > I - j$, we can therefore replace the report by the best response given by (1):

$$\varepsilon \left(\left(\theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i \right) t_{I-j} - \frac{1}{2} w_{I-j} t_{I-j}^2 - \gamma t_{I-j} \left(\sum_{i < I-j} t_i + \sum_{i > I-j} \beta_i(t_1, \dots, t_{i-1}) \right) \right). \quad (2)$$

In particular, as the inductive step holds for all $i > I - j$, we can rewrite the best response of every agent $i > I - j$ as follows:

$$\beta_i(t_1, t_2, \dots, t_{I-j}) = \theta_i + \frac{\gamma}{1 + j\gamma} \sum_{k=1}^{I-j} (\theta_k - t_k).$$

We can now insert $\beta_i(t_1, t_2, \dots, t_{I-j})$ for all $i > I - j$ into (2) to get:

$$\begin{aligned} & \varepsilon \left(\theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i \right) t_{I-j} \\ & - \varepsilon \left(\frac{1}{2} w_{I-j} t_{I-j}^2 + \gamma t_{I-j} \left(\sum_{i < I-j} t_i + \sum_{i > I-j} \left(\theta_i + \frac{\gamma}{1 + j\gamma} \sum_{k=1}^{I-j} (\theta_k - t_k) \right) \right) \right). \end{aligned}$$

We find the best response of bidder $I - j$ by determining the optimal exit time t_{I-j} in response to past exit times and differentiate the above payoff with respect to the exit time t_{I-j} :

$$\begin{aligned} & \theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i - w_{I-j} t_{I-j} \\ & - \gamma \left(\sum_{i < I-j} t_i + \sum_{i > I-j} \left(\theta_i + \frac{\gamma}{1 + j\gamma} \sum_{k=1}^{I-j} (\theta_k - t_k) \right) - \frac{j\gamma}{1 + j\gamma} t_{I-j} \right) = 0. \end{aligned}$$

We can collect terms to obtain:

$$\begin{aligned} & \left(\theta_{I-j} + \left(\gamma - \frac{j\gamma^2}{1 + (j-1)\gamma} \right) \sum_{i < I-j} (\theta_i - t_i) \right) - w_{I-j} t_{I-j} \\ & - \gamma \left(+ \frac{j\gamma}{1 + (j-1)\gamma} \theta_{I-j} - \frac{j\gamma}{1 + (j-1)\gamma} 2t_{I-j} \right) = 0, \end{aligned}$$

or

$$\begin{aligned} & \left(\theta_{I-j} \left(1 - \frac{j\gamma^2}{1 + (j-1)\gamma} \right) + \left(\gamma - \frac{j\gamma^2}{1 + (j-1)\gamma} \right) \sum_{i < I-j} (\theta_i - t_i) \right) \\ &= \left(w_{I-j} - \frac{2j\gamma^2}{1 + (j-1)\gamma} \right) t_{I-j}. \end{aligned}$$

We can then solve for t_{I-j} and find that:

$$t_{I-j} = \theta_{I-j} + \frac{\gamma}{1 + j\gamma} \sum_{i=1}^{I-j-1} (\theta_i - t_i),$$

thus proving the inductive step. ■