Supplementary Appendices

(to be posted on the web only)

Appendix S1: The Consumption-Savings Model

I. We first give a result for general per-period utility functions in the simple savings model. Consider the problem of maximizing

$$U(\vec{a}) = \sum_{t=1}^{\infty} \delta^{t-1}\left[(1+\gamma)u((1-a_t)y_t) - \gamma u(y_t)\right]$$

over all feasible plans $\vec{a}$, i.e. plans that satisfy $a_t \in [0,1]$ and the wealth equation $y_t = Ra_{t-1}y_{t-1}$. We suppose that u is non-decreasing and continuous on $(0,\infty)$; we do not require continuity on $[0,\infty)$ because we want to allow for the logarithmic case where $u(0) = \lim_{c\to 0} u(c) = -\infty$. Let $\bar{U}$ be the supremum in this problem.

Proposition S1.1: *Suppose* $R > 1$ *and*

(A.1) $$\sum_{t=1}^{\infty} \delta^{t-1} u(R^{t-1}y_1) < \infty .$$

Then: (i) For any feasible plan the sum defining *U* has a well defined value in the sense that either the sum converges absolutely or converges to $-\infty$.

(ii) The supremum $\bar{U}$ of the feasible values satisfies $-\infty < \bar{U} < \infty$

(iii) If feasible $\vec{a}^n \to \vec{a}^*$ in the product topology then $\vec{a}^*$ is feasible. If in addition $U(\vec{a}^n) \to \bar{U}$ then $U(\vec{a}^*) = \bar{U}$.

(iv) An optimal plan exists. That is, there is a feasible plan that attains $\bar{U}$.

Proof: (i) Define $\chi_+(x) = \max\{x, 0\}$ and $\chi_-(x) = -\chi_+(-x)$.

We can write any finite sum as the sum of negative and positive parts, so for any sequence $(\vec{a}, \vec{y})$ we have

$$\begin{aligned}
&\sum_{t=1}^{T} \delta^{t-1} \left[(1+\gamma)u((1-a_t)y_t) - \gamma u(y_t)\right] = \\
&\sum_{t=1}^{T} \delta^{t-1} \chi_+ \left[(1+\gamma)u((1-a_t)y_t) - \gamma u(y_t)\right] + \\
&\sum_{t=1}^{T} \delta^{t-1} \chi_- \left[(1+\gamma)u((1-a_t)y_t) - \gamma u(y_t)\right]
\end{aligned}$$

The positive part of the sum is summable from (A.1), since

$$(1+\gamma)u((1-a_t)y_t) - \gamma u(y_t) \le u((1-a_t)y_t) \le u(R^{t-1}y_1).$$

The negative part is monotone decreasing in *T,* so it either converges absolutely or converges to $-\infty$. In the former case the entire sum converges absolutely; in the latter case the sum converges to $-\infty$.

(ii) Part (i) already shows that $\overline{U} < \infty$. To see that $\overline{U} > -\infty$, note that it is feasible to set $a_t = 1/R$ for all *t*, and that for $R > 1$ this plan yields a finite value.

(iii) Consider a sequence of feasible plans $\vec{a}^n \to \vec{a}^*$. Because the constraints are period by period and closed, it is clear that $\vec{a}^*$ satisfies the constraints, so it is feasible. Now suppose in addition that $U(\vec{a}^n) \to \overline{U}$. Choose any $\varepsilon > 0$ and pick N large enough that $\overline{U} - U(\vec{a}^n) < \varepsilon / 2$ for all $n > N$. If we now pick T that

$$\sum_{t=T+1}^{\infty} \delta^{t-1} \left| u(R^{t-1} y_1) \right| < \varepsilon / 2 \,,$$

we know that

$$\bar{U} - \sum_{t=1}^{T} \delta^{t-1} \left[(1+\gamma) u((1-a_t^n) y_t^n) - \gamma u(y_t^n) \right] \le \varepsilon$$

for all $n > N$ and $\tau > T$. Since per-period payoffs are continuous at any (a, y) with $a > 0$, $\vec{a}^n \to \vec{a}^*$, and $y_t^n \to y_t^*$, it follows that

$$\bar{U} - \sum_{t=1}^{\tau} \delta^{t-1} \left[(1+\gamma) u((1-a_t^*) y_t^*) - \gamma u(y_t^*) \right] \le \varepsilon \text{ for all } \tau > T \,.$$

Since this is true for any $\varepsilon > 0$ and we know that $U(\vec{a}^*) \le \bar{U}$, we conclude that $U(a^*) = \bar{U}$.

(iv) Now consider a feasible sequence $(\vec{a}^n, \vec{y}^n)$ with $U(\vec{a}^n) \to \bar{U}$. Each savings rate a_t must lie in the compact interval $[0,1]$ and each y_t must lie in the compact interval $[0, R^{t-1} y_1]$, so the sequence $(\vec{a}^n, \vec{y}^n)$ has an accumulation point $(\vec{a}^*, \vec{y}^*)$ in the product topology. This accumulation point is a maximum by part (iii).

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II. Now we specialize to the CRRA utility functions

$$u(c) = \frac{c^{1-\rho} - 1}{1-\rho}$$

and $u(c) = \ln(c)$, which corresponds to the case $\rho = 1$. Assuming $\delta < R^{\rho-1}$ implies

$$\sum_{t=1}^{\infty} \delta^{t-1} u(R^{t-1} y_1) < \infty .$$

It follows from Proposition S1.1 that an optimum $\vec{a}^*$ exists.

Proposition S1.2: *With CRRA utility a stationary optimum with $a_t = a$ exists.*

Proof: Suppose that $\vec{a}^*$ is an optimal plan. By homogeneity of the objective function, and the fact that plans are defined in terms of savings rates, $\vec{a}^*$ is also an optimal plan starting in period 2 (for any initial condition). Note that the plan $\vec{a}^2 = (a_1^*, a_1^*, a_2^*, a_3^*, \ldots)$ yields wealth in period 2 of $a_1^* R y_1$, and let $\bar{U}(y_1)$ denote the maximized utility when starting in the second period with wealth y_1. Then

$$U(\vec{a}^2) = (1+\gamma)u((1-a_1^*)y_1) - \gamma u(y_1) + \delta \bar{U}(a_1^* R y_1) = \bar{U}$$

where the first equality follows because $\vec{a}^*$ is optimal from period 2 on, and the second equality because $\vec{a}^*$ is optimal from the first period. Proceeding in this way we can construct sequence of feasible plans $\vec{a}^n = (a_1^*, a_1^*, \ldots, a_1^*, a_2^*, a_3^*, \ldots)$ that play a_1^* for the first n periods such that $U(\vec{a}^n) = U(a^*) = \bar{U}$. Clearly $\vec{a}^n$ converges in the product topology to the plan of choosing the fixed savings rate a_1^*. From Proposition A.1 (iii), this limiting plan is feasible and gives utility $\bar{U}$; that is, it is optimal.

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III. We have shown that it is sufficient to compute the present value utility from a fixed savings rate a, and maximize over savings rates. We have present value utility

$$U = \frac{y_1^{1-\rho}}{1-\rho}\sum_{t=1}^{\infty}(\delta(Ra)^{1-\rho})^{t-1}\left[(1+\gamma)(1-a)^{1-\rho}-\gamma\right] - \frac{1}{(1-\delta)(1-\rho)}$$
$$= \frac{y_1^{1-\rho}}{1-\rho}\frac{\left[(1+\gamma)(1-a)^{1-\rho}-\gamma\right]}{1-\delta(Ra)^{1-\rho}} - \frac{1}{(1-\delta)(1-\rho)}$$

Since the optimal savings rate cannot be 0 or 1, we may differentiate with respect to the saving rate to find

$$dU\,/\,da \propto [(1+\gamma)(1-a)-\gamma(1-a)^{\rho}]\,\delta R(Ra)^{-\rho} - (1+\gamma)\left(1-\delta(Ra)^{1-\rho}\right)$$

which gives the first-order condition for an optimum[i]

$$(1+\gamma)a_{\gamma}^{*\rho} = R^{1-\rho}\delta((1+\gamma)-\gamma(1-a_{\gamma}^{*})^{\rho}).$$

When $\gamma = 0$ we get the usual solution $a^{*} = R^{(1-\rho)/\rho}\delta^{1/\rho}$. Thus we can rewrite the first order condition as

$$(a_{\gamma}^{*}\,/\,a^{*})^{\rho} = ((1+\gamma)-\gamma(1-a_{\gamma}^{*})^{\rho})/(1+\gamma).$$

IV: Turning to the simple banking model, utility starting in the second period is the $\gamma = 0$ solution

$$U_2(y_2) = \frac{y_2^{1-\rho}}{1-\rho}\frac{(1-a^{*})^{1-\rho}}{1-\delta(Ra^{*})^{1-\rho}} - \frac{1}{(1-\delta)(1-\rho)}$$
$$= \frac{y_2^{1-\rho}}{1-\rho}\frac{1}{\left(1-a^{*}\right)^{\rho}} - \frac{1}{(1-\delta)(1-\rho)}$$
$$= \frac{y_2^{1-\rho}}{1-\rho}\frac{1}{\left(1-\delta^{1/\rho}R^{(1-\rho)/\rho}\right)^{\rho}} - \frac{1}{(1-\delta)(1-\rho)}$$

The utility of both selves in the first period is

$$(1+\gamma)\frac{c_1^{1-\rho}-1}{1-\rho}-\gamma\frac{(x_1+z_1)^{1-\rho}-1}{1-\rho},$$

and so the overall objective of the long-run self is to maximize

$$\begin{aligned}&(1+\gamma)\frac{c_1^{1-\rho}-1}{1-\rho}-\gamma\frac{(x_1+z_1)^{1-\rho}-1}{1-\rho}\\&+\frac{(R(y_1+z_1-c_1))^{1-\rho}}{1-\rho}\frac{\delta}{\left(1-\delta^{1/\rho}R^{(1-\rho)/\rho}\right)^{\rho}}-\frac{\delta}{(1-\delta)(1-\rho)}\end{aligned}$$

The first order condition for optimal consumption is

$$\frac{c_1^*}{R(y_1+z_1-c_1^*)}=\frac{(1+\gamma)^{1/\rho}\left(1-\delta^{1/\rho}R^{(1-\rho)/\rho}\right)}{(\delta R)^{1/\rho}}.$$

If there are one or more solutions that satisfy the constraint $c_1^* \le x_1+z_1$ then one of them represents the optimum; otherwise the optimum is to consume all pocket cash.

Note that x_1 is the solution for $\gamma=0$, so it satisfies

$$\frac{x_1}{R(y_1-x_1)}=\frac{\left(1-\delta^{1/\rho}R^{(1-\rho)/\rho}\right)}{R^{1/\rho}}.$$

Thus we can write the first order condition as

$$\frac{c_1^*}{y_1+z_1-c_1^*}=(1+\gamma)^{1/\rho}\frac{x_1}{y_1-x_1}$$

or

$$
\begin{aligned}
c_1^* &= \frac{(1+\gamma)^{1/\rho} x_1}{y_1 - x_1 + (1+\gamma)^{1/\rho} x_1} (y_1 + z_1) \\
&= \frac{1 + \left[(1+\gamma)^{1/\rho} - 1\right]}{1 + \left[(1+\gamma)^{1/\rho} - 1\right](1 - a^*)} (1 - a^*)(y_1 + z_1) \\
&= B(1 - a^*)(y_1 + z_1).
\end{aligned}
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Appendix S2: Hyperbolic Procrastination and Delay

Theorem 4:

a) If $v > \mu$ *and* $\overline{x} > \delta\beta v$*, the "sophisticated quasi-hyperbolic model" has a stationary equilibrium with* $\underline{x} < x^{**} < \overline{x}$*.*

b) There is an open set of parameters that satisfies the restrictions of part a) and for which there are other equilibria.

proof: Recall equations (10) into (11) from the paper:

$$W^{**} = \frac{P(x^{**})(\delta V) + (1 - P(x^{**}))E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})}. \tag{10}$$

$$x^{**} = \delta\beta\left(V - W^{**}\right) \tag{11}.$$

Substituting (10) into (11) we see that

$$x^{**} = \delta\beta\left(\frac{V - \delta V + \delta P(x^{**})V - P(x^{**})(\delta V) - (1 - P(x^{**}))E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})}\right)$$
$$=$$
$$\delta\beta\left(\frac{v - (1 - P(x^{**}))E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})}\right).$$

Let $F(x^{**}) = x^{**} - \delta\beta\left(\frac{v - (1 - P(x^{**}))E(x \mid x > x^{**})}{1 - \delta + \delta P(x^{**})}\right).$

To prove the theorem it suffices to show that there is an x^{**} where $F(x^{**}) = 0$. The assumption that $v > \mu$ implies that $F(0) = 0 - \delta\beta\left(\frac{v-\mu}{1-\delta}\right) < 0$, and the assumption that $\overline{x} > \delta\beta v$ implies that $F(\overline{x}) = \overline{x} - \delta\beta v > 0$.

b) Suppose that $x^1 = \underline{x}$ and $x^2 = \overline{x}$ so that the odd-numbered agents never act an even ones always act. Then the equilibrium payoff of an even-numbered agent is $\delta\beta V$, and the payoff of an even-numbered agent with cost x_t who chooses to wait is $x_t + \delta\beta\mu + \delta^3\beta V$, so the even agents' strategy is a best response for all x_t if $\delta\beta V(1-\delta^2) > \overline{x} + \delta\beta\mu$, or equivalently

$$\delta\beta\,(\,v(1+\delta) - \mu\,) > \overline{x}\,. \tag{S2.1}$$

The payoff of the odd-numbered agents who wait is $x_t + \delta^2\beta V$, and the payoff to acting is $\delta\beta V$, so waiting is better for all x_t if

$$\underline{x} > \delta\beta(1-\delta)V = \delta\beta v\,. \tag{S2.2}$$

To complete the proof we must show that there is an open set of parameters such that (S2.1) and (S2.2) and the restrictions $v > \mu$ and $\overline{x} > \delta\beta v$. To do this, fix $v > 0$ and $\delta, \beta \in (0,1)$, and some $\varepsilon > 0$. Set , $\underline{x} = (1+\varepsilon)\delta\beta v$, $\mu = (1+2\varepsilon)\delta\beta v$, and $\overline{x} = (1+3\varepsilon)\delta\beta v$. (Note that these conditions are consistent with a range of distributions, including the uniform.) By construction this satisfies $\overline{x} > \delta\beta v$ and (S2.2).

If $\varepsilon < \dfrac{1-\delta\beta}{2\delta\beta}$ then $\mu = (1+2\varepsilon)\delta\beta v < (1+\dfrac{1-\delta\beta}{\delta\beta})\delta\beta v = v$, and for

$\varepsilon < \dfrac{\delta(1-\beta)}{(3+2\delta\beta)}$ we compute that

$$\begin{aligned}
&\delta\beta\,(\,v(1+\delta)-\mu\,) = \delta\beta\,(\,v(1+\delta)-(1+2\varepsilon)\delta\beta v\,) \\
&= \delta\beta v\,(\,1+\delta(1-\beta)-2\varepsilon\delta\beta\,) \\
&> \\
&\delta\beta v\,(\,1+3\varepsilon\,) \\
&= \overline{x}.
\end{aligned}$$

This shows that there is a '2-cyle equilibrium" whenever ε is sufficiently small. Since the inequalities in (12) and (13) hold strictly for the specified relationship between the parameters, they hold for an open set of β, δ, V, μ; the inequalities also hold for a range of distributions with the given mean and endpoints. ☑

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[i] We do not know if the first-order condition has a unique solution, except in the logarithmic case.