Mathematical Appendix

This appendix presents the solution to our version of the Holt, Modigliani, Muth and Simon (1960) model, and shows how to calculate variances and covariances of the decision rules. The derivation is based on Chapter 4 of their book, modified to allow for a discount factor and for an AR(1) sales process.

As stated in the text, the decision rules for the cost-minimization problem when sales are AR(1) take the form:

$$\begin{pmatrix} N_t \\ I_t \end{pmatrix} = C \begin{pmatrix} N_{t-1} \\ I_{t-1} \end{pmatrix} + \vec{d} \cdot S_t \tag{A.1}$$

The actual decision rules are

$$\begin{aligned} N_t \quad = \quad & \left[\frac{b_3 \left(\lambda_1 - \lambda_2 \right)}{\det(A)} \left(b_6 - \beta b_4 - \beta^2 \, \frac{b_7 - b_3 \left(\lambda_1 + \lambda_2 \right)}{\lambda_1 \lambda_2} \right) \right] N_{t-1} \\ & + \left[\frac{\beta^2 b_3}{\det(A)} \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2} \right) \right] I_{t-1} \\ & + \left[\frac{b_6 - \beta b_4 + \beta^2 b_3 \lambda_1^{-1}}{\det(A)} \left(\alpha_2 + \frac{1 - \alpha_2 (1 - \rho)}{1 - \lambda_2 \rho} \right) \right. \\ & \left. - \frac{b_6 - \beta b_4 + \beta^2 b_3 \lambda_2^{-1}}{\det(A)} \left(\alpha_2 + \frac{1 - \alpha_2 (1 - \rho)}{1 - \lambda_1 \rho} \right) \right] S_t \end{aligned} \tag{A.2}$$

and

(A.3)

$$
\begin{aligned}
I_t \quad = \quad & \left[\frac{b_2 b_3\left(\lambda_1-\lambda_2\right)}{\det(A)}\left(b_6-\beta b_4-\beta^2 \frac{b_7-b_3\left(\lambda_1+\lambda_2\right)}{\lambda_1\lambda_2}-\frac{b_1}{\beta}\right)\right. \\
& \left. -\frac{b_1\left(\lambda_1-\lambda_2\right)}{\det(A)}\left(b_3 b_5-b_4 b_7+b_3 b_7\left(\lambda_1+\lambda_2\right)-b_3^2\lambda_1\lambda_2\right)\right] N_{t-1} \\
& +\left[\frac{\beta^2 b_2 b_3}{\det(A)}\left(\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}\right)-\frac{b_1\left(\lambda_1-\lambda_2\right)}{\det(A)}\left(b_4-b_3\left(\lambda_1+\lambda_2\right)\right)+1\right] I_{t-1} \\
& +\left[\frac{\left(b_1\left(b_5-b_4\lambda_2+b_3\lambda_2^2\right)-b_2\left(b_6-\beta b_4+\beta^2 b_3\lambda_2^{-1}\right)\right)}{\det(A)}\left(\alpha_2+\frac{1-\alpha_2(1-\rho)}{1-\lambda_1\rho}\right)\right. \\
& \left. -\frac{\left(b_1\left(b_5-b_4\lambda_1+b_3\lambda_1^2\right)-b_2\left(b_6-\beta b_4+\beta^2 b_3\lambda_1^{-1}\right)\right)}{\det(A)}\left(\alpha_2+\frac{1-\alpha_2(1-\rho)}{1-\lambda_2\rho}\right)-1\right] S_t,
\end{aligned}
$$

where

(A.4) $$[A]=\begin{bmatrix} b_5-b_4\lambda_1+b_3\lambda_1^2 & -\left(b_6-\beta b_4+\beta^2 b_3\lambda_1^{-1}\right) \\ b_5-b_4\lambda_2+b_3\lambda_2^2 & -\left(b_6-\beta b_4+\beta^2 b_3\lambda_2^{-1}\right) \end{bmatrix}$$

and

(A.5) $$b_1=\left(\frac{\beta\gamma_4}{\theta\gamma_2}\right)$$

(A.6) $$b_2=\left(\theta+b_1\left(\frac{1+\beta}{\beta}\right)\right)$$

(A.7) $$b_3=\frac{b_1\gamma_2}{\alpha_1\beta^2}$$

(A.8) $$b_4 = b_1\left(\frac{1}{\beta}\right) + 2b_3\left(\beta+1\right)$$

(A.9) $$b_5 = \theta + b_1\left(\frac{\beta+1}{\beta}\right) + b_3\left(1+2\beta\right)$$

(A.10) $$b_6 = b_1 + b_3\left(2\beta+\beta^2\right)$$

(A.11) $$b_7 = \frac{b_1}{\alpha_1\beta^2}\left(\alpha_1\beta+\gamma_2\right),$$

and the relevant roots are given by:

(A.12) $$\lambda_i = \frac{\beta}{2}\left[\left(\phi_j + \frac{\beta+1}{\beta}\right) - \sqrt{\phi_j\left(2\frac{\beta+1}{\beta}+\phi_j\right)+\left(\frac{\beta-1}{\beta}\right)^2}\right] \quad \text{for } i = j = 1, 2,$$

where

(A.13) $$\phi_j = \frac{b_1 \pm \sqrt{b_1^2 - 4b_3\beta^2\theta}}{2b_3\beta^2} = \frac{\alpha_1}{2\gamma_2}\left(1 \pm \sqrt{1-\frac{4\gamma_2^2\theta^2}{\alpha_1\gamma_4\beta}}\right) \text{ for } j = 1, 2.$$

See Holt, Modigliani, Muth and Simon (1960, p. 100) for an explanation of why the two roots given in equation A.12 are guaranteed less than one in modulus.

Expressions for the variances and covariances are derived from the decision rules as functions of the model's parameters and the variance of sales, σ_S^2. First, pre-multiply the decision rule in equation A.1 by S_t and take expectations.

(A.14) $$E_t\begin{pmatrix} N_t \cdot S_t \\ I_t \cdot S_t \end{pmatrix} = C \cdot E_t\begin{pmatrix} N_{t-1} \cdot S_t \\ I_{t-1} \cdot S_t \end{pmatrix} + \vec{d} \cdot E_t S_t^2$$

Note that, by the definition of S_t and by the fact that the variables are covariance stationary, equation A.14 may be rewritten and solved for σ_{NS} and σ_{IS} :

(A.15) $$\begin{pmatrix} \sigma_{NS} \\ \sigma_{IS} \end{pmatrix} = \left[I - \rho \cdot C \right]^{-1} \vec{d} \cdot \sigma_S^2 .$$

Second, taking the variance of both sides of equation A.1 yields the following expression:

(A.16) $$\Omega - C\Omega C' = \rho d \begin{pmatrix} \sigma_{NS} \\ \sigma_{IS} \end{pmatrix}' C' + \rho C \begin{pmatrix} \sigma_{NS} \\ \sigma_{IS} \end{pmatrix} d' + d\sigma_S^2 d' \;\; ; \;\; \Omega = \begin{bmatrix} \sigma_N^2 & \sigma_{NI} \\ \sigma_{NI} & \sigma_I^2 \end{bmatrix} .$$

Equation A.16, after substituting equation A.15, provides sufficient information to solve for σ_N^2, σ_I^2 and σ_{NI} .