Supplementary Appendices

Appendix C considers some special cases of Proposition 6 in Section VI, while Appendix B supplements the empirical application in Section VII, explaining how the QUAIDS demand systems were estimated and how the data were calculated.

Appendix C: Implications of Proposition 6

Generalized Linearity is an extremely general specification of preferences, which nests many of the most widely-used demand systems. Proposition 6 is significantly strengthened when we specialise to some of these sub-cases:

C.1 Price-Independent Generalized Linear ("PIGL") Preferences

In this case, also due to Muellbauer (1975), the expenditure function specialises to a CES form:

$$e(p,u) \quad = \quad [(1-u)\, a(p)^{\alpha} \; + \; u\,.\,b(p)^{\alpha}\,]^{\frac{1}{\alpha}}\,, \qquad\qquad \alpha > 0\,, \qquad \textbf{(A27)}$$

and the income function in the budget shares is independent of prices (whence the name):

$$\omega_i^{PIGL}(p,z) \quad = \quad z^{-\alpha}.A_i(p) \; + \; B_i(p)\,. \qquad \textbf{(A28)}$$

It follows that the average income level which generates world spending patterns at the GAIA prices is also independent of prices and equals a CES mean of individual countries' incomes:

$$\bar{\omega}_i^* \quad = \quad \omega_i^{PIGL}(\Pi, \tilde{z}^*) \qquad \text{where:} \qquad \tilde{z}^* \quad = \quad \left\{ \sum_j \; \theta_j^*\,(z_j^*)^{-\alpha} \right\}^{-\frac{1}{\alpha}}. \qquad \textbf{(A29)}$$

C.2 Price-Independent Generalized Logarithmic ("PIGLOG") Preferences

This system is the limit of the PIGL system as α approaches zero. A special case, due to

Lewbel (1989), nests in turn both the exactly aggregable translog of Christensen, Jorgenson and Lau (1975) and the AIDS ("Almost Ideal Demand System") model of Deaton and Muellbauer (1980). The expenditure function takes a Cobb-Douglas form:

$$\ln e(p,u) \quad = \quad (1-u)\ln a(p) \;+\; u \ln b(p)\,, \qquad \textbf{(A30)}$$

and the budget shares depend linearly on lnz. Hence the average world income which generates world budget shares at the GAIA prices is a weighted geometric mean of individual countries' incomes:

$$\bar{\omega}_i^{*} \quad = \quad \omega_i^{PIGLOG}(\Pi, \tilde{z}^{*}) \qquad \text{where:} \qquad \ln\tilde{z}^{*} \quad = \quad \sum_j \theta_j^{*} \ln z_j^{*}\,. \qquad \textbf{(A31)}$$

C.3 The Gorman Polar Form

A different special case of PIGL, obtained by setting α equal to one, is the Gorman (1961) Polar Form, which nests the Linear Expenditure System corresponding to the Stone-Geary utility function. The expenditure function is now:

$$e(p,u) \quad = \quad (1-u)\, a(p) \;+\; u\,.\,b(p)\,, \qquad \textbf{(A32)}$$

and the budget shares are linear in the reciprocal of income:

$$\omega_i^{GPF}(p,z) \quad = \quad z^{-1}\,.\,A_i(p) \;+\; B_i(p)\,. \qquad \textbf{(A33)}$$

Now, the simple average of world incomes generates world consumption patterns at the GAIA prices:

$$\bar{\omega}_i^* \quad = \quad \omega_i^{GPF}(\Pi, \bar{z}^*) \qquad \text{where:} \qquad \bar{z}^* \quad = \quad \frac{1}{m} \sum_j z_j^* . \qquad \textbf{(A34)}$$

In addition, the demand patterns aggregate not just in the sense of yielding the same budget shares but in the much stronger sense of yielding the same *levels* of world expenditure on each commodity:

$$\Pi_i \sum_j q_{ij} \quad = \quad \sum_j z_j^* \omega_{ij} \quad = \quad m\bar{z}^* \omega_i^{GPF}(\Pi, \bar{z}^*) . \qquad \textbf{(A35)}$$

Thus, when preferences exhibit the Gorman Polar Form, the GAIA world prices would generate actual world demands if world income was equally distributed (or, since expenditure is linear in utility from (A32), if world utility was equally distributed).

Appendix D: Details of the Empirical Application

D.1 Imposing Negativity

The homogeneity restrictions are easily imposed in the estimation procedure by dropping the budget share equation for the *n*th good and expressing all prices relative to that good's price. However, the negativity restriction is more tricky. To impose it, I extend to the QUAIDS model an approach pioneered by Lau (1978) and applied to the AIDS model by Moschini (1998).

For any demand system, the typical term in the Slutsky substitution matrix equals:

$$e_{ih} \quad = \quad \frac{z}{p_i p_h} \left\{ \frac{\partial^2 \ln e(p,u)}{\partial \ln p_i \, \partial \ln p_h} - \delta_{ih}\omega_i + \omega_i \omega_h \right\}, \qquad \textbf{(A36)}$$

where δ_{ih} is the Kronecker delta. For the QUAIDS system, the first term on the right-hand side becomes:

$$\frac{\partial^2 \ln e(p,u)}{\partial \ln p_i \, \partial \ln p_h} = \gamma_{ih} + \beta_i \beta_h \ln y + \frac{\beta_i \lambda_h + \beta_h \lambda_i}{\beta} (\ln y)^2 + \frac{2 \lambda_i \lambda_h}{\beta^2} (\ln y)^3. \qquad \textbf{(A37)}$$

To impose negativity at a point, choose units so that $z=p_i=1$, $\forall i$ (and so $\ln y=0$ and, from (18), $\omega_i=\alpha_i$, $\forall i$) at that point. The Slutsky substitution term (A36) then reduces to a simple function of parameters only:

$$\bar{e}_{ih} = \gamma_{ih} - \delta_{ih}\alpha_i + \alpha_i \alpha_h. \qquad \textbf{(A38)}$$

(This is the same as in the AIDS case, though because of the additional quadratic and cubic terms in (A37), the restriction is more likely to be violated away from the point where it is imposed.) Finally, negativity is imposed by estimating not $\bar{E}$, the $(n-1)$-by-$(n-1)$ matrix of the $\bar{e}_{ih}$ from (A38), but rather its Cholesky decomposition, the upper-triangular matrix T, where $\bar{E}=-T'T$. In practice, I chose units to impose negativity at the sample mean, estimated the α_i, β_i and λ_i along with the elements of T, and then used (A38) to recover the implied estimates of γ_{ih}.[1]

D.2 Restricting Price Responsiveness: The Semiflexible QUAIDS

Even with the restrictions of homogeneity and negativity imposed, the QUAIDS model has $\frac{1}{2}(n-1)(n+6)$ parameters ($n-1$ each of the α_i, β_i and λ_i; and $\frac{1}{2}n(n-1)$ of the γ_{ih}).[2] With 11 commodity groups, this gives a total of 85 parameters. More seriously, most of the parameters

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[1] Moschini gives detailed formulae for the γ_{ih} in the AIDS case. In the present application, the estimation was programmed in a matrix language, which avoided the need to use such formulae.

[2] This does not include α_0. In practice it has proved difficult to estimate this parameter with precision and previous writers have used one of two methods to impose its value. Deaton and Muellbauer (1980) and Banks, Blundell and Lewbel (1997) set α_0 just below the lowest expenditure level in the sample, while Moschini (1998) sets it equal to zero. Given the range of variation in the data, these two methods are equivalent in the present application.

(all but $n-2$ of the λ_i) enter every budget share equation, so there are 76 [$=\frac{1}{2}(n^2+3n-2)$] parameters in each equation but only 60 observations. The conventional wisdom is that this makes maximum likelihood infeasible.[3] Previous authors have either resorted to approximations (replacing $\alpha(p)$ by an empirical price index such as the "Stone" index, $\ln P^S = \Sigma_i\ \omega_i \ln p_i$, following Deaton and Muellbauer) or used the two-step (and hence less efficient) minimum distance estimator rather than maximum likelihood. However, the fact that utility maximisation is a maintained hypothesis in the present application allows this problem to be overcome in a more satisfactory manner.

The estimation method I use extends to the QUAIDS model an approach developed by Diewert and Wales (1988) and applied to the AIDS model by Moschini (1998). This involves restricting the degree of price responsiveness of a flexible functional form, leading to a "semiflexible" system. Using the Cholesky decomposition, this is accomplished by setting the last $n-k-1$ rows of T to zero. As a result, the rank of T (and hence of $\bar{E}$) is k, and so the number of parameters to be estimated is reduced (to $3(n-1)+\frac{1}{2}k(2n-k-1)$ for the whole system and $\frac{1}{2}(2n-1)(k+2)-\frac{1}{2}k^2$ per equation).

Previous authors have used the semiflexible approach to estimate systems of reduced rank only. However, it has a further computational advantage. For any value of k above zero, the coefficient estimates from the *preceding* value of k can be used as starting values.[4] This

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[3] Deaton and Muellbauer (1980) and Deaton (1986, p. 1784) attribute this piece of oral tradition to Teun Kloek.

[4] Of course, estimating the system for $k=k_0$ gives estimates of only some of the τ_i parameters needed in the $k=k_0+1$ case. Starting values of 0.1 were assigned to the remaining τ_i: starting values of zero did not work, presumably because, since all the τ_i are squared, the numerical derivatives are all zero.

procedure allowed maximum likelihood estimation of both the AIDS and QUAIDS systems for *all* values of k (including the full rank case of $k=n-1$), without the need to resort to approximations.[5]

This procedure has the added attraction in the present context that restricting price responsiveness is of interest in itself. Recalling from Proposition 5 that with zero price responsiveness the Geary method is identical to the GAIA, it seems reasonable to conjecture that increasing the degree of price responsiveness should make the Geary index less attractive.

B.3 Econometric Estimates

In other respects, the estimation procedure was standard. Thirty-three different specifications were estimated, one for each value of k from 0 to 10 and for each of the HAIDS, AIDS and QUAIDS systems. For each specification, budget share equations for the first ten commodity groups were estimated by maximum likelihood. The budget share equations (18) are linear given $\beta(p)$ and $\alpha(p)$, so an iterative approach was used. The starting values were used to construct estimates of $\beta(p)$ and $\alpha(p)$, and the resulting parameter values were then used to recalculate these functions for the next iteration.

Table A.1 gives the values of the log likelihood for the different specifications estimated and

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[5] The fact that, for high values of k, estimation was only possible using the parameter values from the previous value of k as starting values might cause concern that a global maximum of the likelihood function has not been found. Some reassurance on this point is provided by the results of a serendipitous programming error. In preliminary runs, the equations given by (18) were estimated with both $\ln y$ and (erroneously) λ_i deflated by the Stone price index. This converged to estimates similar to those in Figure A.1 from arbitrary starting values (where, because of the difficulty noted in footnote 4, the "arbitrary" starting values used were typically $\alpha_i=\beta_i=\lambda_i=0$ and $\tau_i=0.1$).

Figure A.1 illustrates them in a "likelihood tree", with the number of parameters in each specification given on the horizontal axis. Each "branch" of the tree corresponds to a different specification of the relationship between budget shares and real expenditure: none in the case of HAIDS, linear in the case of AIDS and quadratic in the case of QUAIDS. Each "leaf" on a given branch corresponds to a different value of k, the rank of the Slutsky substitution matrix, as the subsistence price index $\alpha(p)$ varies between a Cobb-Douglas ($k=0$) and a translog ($k=n-1$) specification.[6] Hence, moving to a higher branch implies a greater degree of responsiveness to income, while moving rightwards along a branch implies a greater degree of responsiveness to price.

As well as allowing a convenient visual presentation of the results, the likelihood tree diagram has an additional advantage that various test criteria for comparing pairs of likelihood values can be illustrated directly. The three loci in the lower right-hand corner of Figure A.1 show how this can be done for a hypothetical log likelihood value given by point A. The three criteria illustrated are the Akaike information criterion (AIC) and the χ^2 likelihood ratio test at the 5% and 10% significance levels. The AIC attaches equal weight to identical increments in the log likelihood and the number of parameters, whereas the likelihood ratio typically trades off increases in the log likelihood more generously against increases in the number of parameters. All points below a given locus can be rejected in favour of point A by the test criterion in question. Hence, the significance of any particular specification against alternatives can be assessed visually by moving the three loci so that point A coincides with the point representing

[6] The HAIDS specification with $k=0$ (so all $\gamma_{ih}=0$) corresponds to the Cobb-Douglas utility function, with budget shares independent of price as well as income. The AIDS specification with $k=0$ is considered and estimated by Deaton (1978).

the specification of interest. Of course, some of the possible comparisons between branches are non-nested. (Specifically, for any given point, this is true for all comparisons with points that are on a *higher* branch but have a *lower* value of *k*, and conversely). Even for the nested comparisons, the significance levels cannot be taken too seriously, both because they are asymptotic only and because of the sheer number of possible comparisons: with 32 independent bilateral nested comparisons, we would expect some to be significant even with random data. Nevertheless, the results have a degree of regularity which seems to justify some tentative conclusions.[7]

The first conclusion to be drawn is the overwhelming importance of allowing for some relationship between the budget shares and income. For each value of *k*, the HAIDS specification is clearly dominated by AIDS. However, the pay-off to including quadratic terms in income is not major: QUAIDS does not do much better than AIDS for given *k*. Next, at least in the AIDS and QUAIDS cases, allowing for additional price responsiveness does not contribute very much to the likelihood. Indeed, on purely statistical grounds, the single specification which is to be preferred overall is the AIDS case with $k=0$ (henceforth abbreviated in an obvious way as "A0").

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[7] I concentrate on likelihood function comparisons for choosing between systems. Of course, every "leaf" in Figure A.1 represents a complete system of demand equations, each with its own parameter estimates, implied income and price elasticities, equation-by-equation diagnostics and observation-by-observation checks for negativity. Interested readers can calculate most of these this information themselves using GAUSS programs available at the web site mentioned in Appendix B.5.

D.4 The Data

The raw data are taken from Phase IV (1980) of the United Nations International Comparisons Project (ICP), available in hard copy as United Nations (1986). For the 60 countries in 1980, data on 11 categories of personal consumption expenditure were used: food; beverages; tobacco; clothing and footwear; gross rents; fuel and power; house furnishings, appliances and operations; medical care; transport and communication; recreation and education; and miscellaneous goods and services.

Table 6 of United Nations (1986) gives data on per capita expenditure in national currencies, $z_{ij} = p_{ij} . q_{ij}$ where p_{ij} is the price of good i in country j and q_{ij} is the quantity of good i in country j. Table 8 gives the purchasing power parities which are the national currency expenditures from Table 6 divided by expenditure in international prices. These international prices are produced by the Geary method of aggregation used in the ICP. Therefore the entries in Table 8 are: $(p_{ij} . q_{ij}) / (\pi_i . q_{ij}) = p_{ij} / \pi_i$, where π_i is the international price of good i. Dividing each entry by the corresponding entry for the United States, p_{i1} / π_i, gives prices in country j relative to prices in the United States: p_{ij} / p_{i1}. Dividing each entry in Table 6, $p_{ij} . q_{ij}$, by the corresponding relative price, p_{ij} / p_{i1}, gives quantities in country j measured in U.S. prices, $p_{i1} . q_{ij}$. This gives p_{ij} / p_{i1} and $p_{i1} . q_{ij}$, which are the price and quantity data required to calculate the various real income indexes.

D.5 Sources

The data used in the study, as well as the GAUSS files used to estimate the demand systems and calculate the indices, are available at:

http://www.ucd.ie/~economic/staff/pneary/gaia/gaia.htm

Supplementary References

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Table A1: Number of Parameters and Value of Log Likelihood for Different Specifications

k	HAIDS		AIDS		QUAIDS	
	# Params.	LF Value	# Params.	LF Value	# Params.	LF Value
0	10	435.464	20	457.913	30	464.094
1	20	437.432	30	460.174	40	464.756
2	29	448.354	39	468.280	49	473.098
3	37	452.837	47	474.616	57	479.289
4	44	455.592	54	477.625	64	483.785
5	50	456.738	60	478.922	70	485.071
6	55	457.408	65	479.861	75	485.987
7	59	457.416	69	480.054	79	486.567
8	62	457.416	72	480.054	82	486.567
9	64	457.416	74	480.054	84	486.567
10	65	457.416	75	480.054	85	486.567

Fig. A1: Value of Log Likelihood for Different Specifications, 1980